TRANSIENCE IN DYNAMICAL SYSTEMS

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ABSTRACT. We extend the theory of transience to general dynamical systems with no Markov structure assumed. This is linked to the theory of phase transitions. We also provide examples of new kinds of transient behaviour.

1. Introduction

Given a metric space X and a dynamical system $f : X \to X$, the set of f-invariant probability measures, which we denote by $\mathcal{M}_f$, can be an extremely large an complicated simplex. Indeed, in such a simple setting as the full-shift on two symbols (Σ_2, σ), the set $\mathcal{M}_\sigma$ is a Poulsen simplex (see [GW, LOS]), that is, an infinite dimensional, convex and compact set for which the extreme points are dense on the whole set. It is, therefore, an important problem to find criteria to choose *relevant* invariant measures. Here is where thermodynamic formalism comes into play. Given a continuous function $\varphi : X \to \mathbb{R}$ (the *potential*) the *topological pressure* is a number $P(\varphi)$ that can be defined using $(n, \epsilon)-$separated sets (see [W3, Chapter 9]). This definition makes use of the metric in the space X. It can be shown, see [W3, p.171], that if the space X is compact then the definition of pressure is independent of the metric (as long as they generate the same topology). The situation is more subtle if the space is no longer compact (see Section 3.2) or if the potential is no longer continuous (see Section 4.4). A major result in the field is that the pressure satisfies the following *Variational Principle*:

$$P(\varphi) = \sup \left\{ h(\mu) + \int \varphi \, d\mu : \mu \in \mathcal{M}_f \text{ and } -\int \varphi \, d\mu < \infty \right\}, \tag{1}$$

where $h(\mu)$ denotes the entropy of the measure μ. A measure $\nu \in \mathcal{M}_f$ attaining the above supremum is called an *equilibrium measure/state*. Proving existence and uniqueness of equilibrium measures is one of the major problems in the theory of thermodynamic formalism. The main tool used to study the pressure and the equilibrium measures is the *Transfer (or Ruelle) operator*, which is defined by

$$(L_\varphi g)(x) = \sum_{Ty=x} g(y) \exp(\varphi(y)) \tag{2}$$

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for $x \in X$. Constructing suitable Banach spaces where this operator acts and behaves well is an important line of research in the field.

In the context of finite state Markov shifts (Σ, σ), equilibrium measures always exist. Moreover, if the system is topologically mixing (see Section 3 for a precise definition) and for regular potentials (e.g. Hölder) these measures are unique (see [Ru] or Theorem 3.1 below). In order to prove this result, one strategy is to show that there exists a constant $\lambda > 0$, a positive continuous function $h : \Sigma \to \mathbb{R}^+$ and a Borel probability measure ν such that

$$L_\varphi h = \lambda h \ , \ L_\varphi^* \nu = \lambda \nu \text{ and } \int h \ d\nu < \infty,$$

where L_φ^* is the dual operator of L_φ. If φ is Hölder it is possible to prove that $\log \lambda = P(\varphi)$ and that the normalisation of the measure $h\nu$ is the unique equilibrium measure of φ. Moreover, for every continuous function $g : \Sigma \to \mathbb{R}$ we have that

$$\frac{(L_\varphi^n g)(x)}{\lambda^n} \rightrightarrows h(x) \int g \ d\nu,$$

where $\rightrightarrows$ denotes uniform convergence in Σ (see [W1] for more details).

When the phase space is no longer compact, as in the case of countable Markov shifts, there are obstructions to the existence of equilibrium measures even when the potential $\varphi : X \to \mathbb{R}$ is Hölder continuous (even for locally constant potentials). The main cause of such obstructions is so-called *transience* (see Definition 3.3 for precise statements). This is a property of the triple (X, f, φ), and implies that there is no conservative (see Definition 2.2) Borel measure ν, finite and positive on cylinders satisfying the equation

$$L_\varphi^* \nu = e^{P(\varphi)} \nu. \tag{3}$$

Transient phenomena for countable Markov shifts have been studied by Sarig and Cyr in [S2, S3, S5, CS, C1, C2].

Returning to the compact setting, it is possible, if for example the potential φ is not Hölder, that a system may still have the same kind of transient behaviour seen in the countable Markov shift setting. A classical example of this is provided by Hofbauer [H].

In this paper, after a fairly extensive review of transience in the symbolic setting, we formulate a definition of transience for general maps and compare it, in the compact interval case, with the phenomena observed in the symbolic case. One of our principal aims is to remove the need for a Markov structure when checking for transience. We show how our notion of transience applies to quadratic maps and to Manneville-Pomeau examples.

We next go on to discuss what types of behaviour are possible for systems satisfying our definition of transience. In doing this, we consider uniformly hyperbolic interval maps $f : [0,1] \to [0,1]$ with transient potentials $\varphi : [0,1] \to \mathbb{R}$ for which the pressure function, $t \mapsto P(t\varphi)$, exhibits a behaviour which is new in this setting. It not only has *phase transitions* (see Section 3 for precise definitions) of positive entropy, but there exists finite interval $[t_1, t_2]$ where the pressure function is constant and $t\varphi$ is transient. In the complement $\mathbb{R} \setminus [t_1, t_2]$ the potentials $t\varphi$ are recurrent. Olivier

[O], constructed an example similar to ours that also exhibits a phase transition of positive entropy. The main differences in our results are that we obtain very precise information on the behaviour at the phase transition, and that we can set up the system so that the support of the relevant equilibrium states jumps from the whole space to a proper invariant subset and then back out to the whole space. Between the phase transitions we have transience, so our example shows that systems $(X, f, t\varphi)$ can move into transience and back out again as t increases, even for a topologically transitive system (X, f). (We comment on non-topologically transitive systems in Section 3.3.)

We conclude this section with some comments on the notation. Unless it is explicitly given in another way, the topological pressure will be defined as in (1) (by the Variational Principle, this will be equal to other definitions of pressure). For sequences $(A_n)_n$ and $(B_n)_n$, the notation $A_n \asymp B_n$ denotes that there exists $C \geqslant 1$ such that $1/C \leqslant A_n/B_n \leqslant C$ for all $n \in \mathbb{N}$, and $A_n \sim B_n$ denotes that $A_n/B_n \to 1$ as $n \to \infty$.

2. Conformal and Conservative measures

This section is devoted to a discussion of a special class of measures that will be of importance to us, because of their connection with potentials. Recall that a Borel measure μ is called a *Radon measure* if every point is contained in a ball of finite measure. For Borel spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$, a map $f : X \to Y$ is said to be *Borel* if it preserves the Borel structure, that is if $B \in \mathcal{B}_Y$ then $f^{-1}(B) \in \mathcal{B}_X$.

Definition 2.1. *Let $f : X \to X$ be a Borel function and $\varphi : X \to [-\infty, \infty]$ be a Borel potential. A Radon measure ν on X, which satisfies the equation*

$$L_\varphi^* \nu = \nu \tag{4}$$

is called φ-conformal *measure for (X, f).*

Remark 2.1. *The following are properties of conformal measures that depend upon the potential and the space:*

(a) If Σ is a finite state Markov shift (for a precise definitions see Section 3) then any conformal measure is a finite measure. This is not necessarily the case when considering countable Markov shifts.
(b) Observe that we don't allow shifts in the potential in our definition of conformal: for example, if ν is a Borel measure on X which is finite on cylinders and $L_\varphi^* \nu = \lambda \nu$ then ν is $(\varphi - \log \lambda)$-conformal.

For the remainder of this section we discuss conformal measures: for example their existence and their conservativity.

The following proposition shows that in many cases the existence of a $(\varphi - P(\varphi))$-conformal measure is guaranteed, although we note that this measure may not be conservative. The statement is only a minor modification of [MP, Proposition 2.2], but we include a proof for completeness.

Proposition 2.1. *Let X be a compact metric space and $f : X \to X$ a continuous function such that the cardinality of the sets $f^{-1}(x) := \{y \in [0,1] : f(y) = x\}$ is uniformly bounded. Suppose that there exists a dense subset of the space of continuous functions $D \subset C(X)$ such for every $\psi \in C(X)$ there exists a probability measure $\eta = \eta_\psi$ satisfying*

$$L_\psi^* \eta = e^{P(\psi)} \eta.$$

Then for each continuous potential $\varphi : X \to \mathbb{R}$ there exists a probability measure $\nu = \nu_\varphi$ such that

$$L_\varphi^* \nu = e^{P(\varphi)} \nu.$$

Note that given a finite state Markov shift, the set of locally constant potentials can play the role of D. So, for example, the above proposition is satisfied if $f : [0,1] \to [0,1]$ is a continuous dynamical system that is topologically conjugated to a finite state Markov shift.

Proof. Let $(\psi_n)_n$ be a sequence of elements in D converging in the supremum norm to φ. Recall that the pressure is a continuous function on $C(X)$ (see [W3, Theorem 9.7 (iv)]), therefore $\lim_{n\to\infty} e^{P(\psi_n)} = e^{P(\varphi)}$. Let $\nu_n \in \mathcal{M}_X$ be a probability measure such that $L_{\psi_n}^* \nu_n = e^{P(\psi_n)} \nu_n$. Assume that the sequence $(\nu_n)_n$ converges to the measure $\nu \in \mathcal{M}_X$. Then given $h \in C(X)$, we have

$$\begin{aligned}
&\left| \int L_\varphi h \, d\nu - e^{P(\varphi)} \int h \, d\nu \right| = \lim_{n\to\infty} \left| \int L_\varphi h \, d\nu_n - e^{P(\varphi)} \int h \, d\nu_n \right| \\
&\leq \lim_{n\to\infty} \left| \int L_\varphi h \, d\nu_n - \int L_{\psi_n} h \, d\nu_n \right| + \lim_{n\to\infty} \left| \int L_{\psi_n} h \, d\nu_n - e^{P(\varphi)} \int h \, d\nu_n \right| \\
&= \lim_{n\to\infty} \left| \int (L_\varphi h - L_{\psi_n} h) \, d\nu_n \right| + \lim_{n\to\infty} \left| e^{P(\psi_n)} - e^{P(\varphi)} \right| \left| \int h \, d\nu_n \right| \\
&\leq \lim_{n\to\infty} \|L_\varphi h - L_{\psi_n} h\|_\infty + \left| \int h \, d\mu \right| \lim_{n\to\infty} \left| e^{P(\psi_n)} - e^{P(\varphi)} \right| \\
&= \lim_{n\to\infty} \|L_\varphi h - L_{\psi_n} h\|_\infty .
\end{aligned}$$

Since the cardinality of the sets $f^{-1}(x) := \{y \in [0,1] : f(y) = x\}$ is uniformly bounded, the transfer operator is continuous and therefore $\lim_{n\to\infty} \|L_\varphi h - L_{\psi_n} h\|_\infty = 0$. In particular,

$$\left| \int L_\varphi h \, d\nu - e^{P(\varphi)} \int h \, d\nu \right| = 0.$$

Hence $\int L_\varphi h \, d\nu = e^{P(\varphi)} \int h \, d\nu$ and so $L_\varphi^* \nu = e^{P(\varphi)} \nu$ as required. □

Remark 2.2. *We stress that there are other ways of constructing conformal measures. We would like to single out the Patterson-Sullivan construction (see for example* [PU, Section 12]*) where the conformal measure is obtained as a weak* limit of appropriately weighted atomic measures.*

Recall that a dynamical system $f : X \to X$ is called *topologically exact* if for every pair of open sets $A, B \subset X$ there exists $n_0 \in N$ such that $B \subset f^{n_0} A$. There exist a measure theoretical counterpart of this definition, indeed we say that a f-invariant measure μ is *exact* if for every Borel set $A \in \mathcal{B}$ of positive measure we have $\lim_{n\to\infty} \mu(f^n(A)) = 1$.

Lemma 2.1. *Let $f : X \to X$ be a topologically exact dynamical system, $\varphi : X \to [-\infty, \infty]$ and $\mu \in \mathcal{M}_f$ a φ-conformal measure for which $\mu(\{\varphi = -\infty\}) = 0$. Then for any open set $A \subset X$, we have $\mu(A) > 0$.*

Proof. Let $B \subset X$ be an open set such that $\mu(B) > 0$. Since f is topologically exact there exits $n_0 \in \mathbb{N}$ such that $f^{n_0}A \cap B \neq \emptyset$. Consider the set

$$AB := \{x \in X : x \in A \text{ and } f^{n_0}x \in B\}$$

then

$$\mu(AB) = \int \mathbb{1}_{AB} \, d\mu = \int L_{\varphi}^{n_0} \mathbb{1}_{AB}(x) \, d\mu.$$

Since the integrand is positive on the set B we have that $\mu(AB) > 0$. Therefore $\mu(A) \geqslant \mu(AB) > 0$. □

Another important property of dynamically relevant measures is that of being conservative. Let $f : X \to X$ be a dynamical system. A measure μ on X is called *non-singular* if $\mu(A) = 0$ if and only if $\mu(f^{-1}A) = 0$. A set $W \subset X$ is called *wandering* if the sets $\{f^{-n}W\}_{n=0}^{\infty}$ are disjoint.

Definition 2.2. *Let $f : X \to X$ be a dynamical system. An f-non-singular measure μ is called* conservative *if every wandering set W is such that $\mu(W) = 0$.*

A conservative measure satisfies the Poincaré Recurrence Theorem (see [Aa, p.17], or [S6, p.30]). Indeed, the following was proved by Halmos [Ha, p.10].

Proposition 2.2. *Let f be a non-singular map on a sigma-finite measure space (X, μ). Then f is conservative if and only if for each measurable set E and for μ-almost every $x \in E$ we have that $f^n x \in E$ for infinitely many values of $n \in \mathbb{N}$.*

In the rest of the paper, we will apply these ideas to more specific dynamical systems.

3. Symbolic spaces

In this section we discuss thermodynamic formalism in the context of Markov shifts. We review several results concerning the existence and uniqueness of equilibrium measures. Understanding these results will enable us to generalise to other dynamical systems in later sections. We also discuss the regularity properties of the pressure function. We emphasise that the properties of Markov shifts defined in finite alphabets are different to those for a countable alphabet. The lack of compactness of the latter shifts is a major obstruction for the existence of equilibrium measures. Symbolic spaces are of particular importance, not only because of their intrinsic interest, but also because they provide models for uniformly and non-uniformly hyperbolic dynamical systems (see for example [Bo1, Ra]).

Let $S \subset \mathbb{N}$ be the *alphabet* and $\mathcal{T}$ be a matrix $(t_{ij})_{S \times S}$ of zeros and ones (with no row and no column made entirely of zeros). The corresponding *symbolic space* is defined by

$$\Sigma := \{x \in S^{\mathbb{N}_0} : t_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N}_0\},$$

and the shift map is defined by $\sigma(x_0x_1\cdots) = (x_1x_2\cdots)$. If the alphabet S is finite we say that (Σ, σ) is a *finite Markov shift*, if S is (infinite) countable we say that (Σ, σ) is a *countable Markov shift*. Given $n \geqslant 0$, the *word* $x_0\cdots x_{n-1} \in S^n$ is called *admissible* if $t_{x_ix_{i+1}} = 1$ for every $0 \leqslant i \leqslant n-2$. We will always assume that (Σ, σ) is *topologically mixing*, except in Section 3.3 where the consequences of not having this hypothesis are discussed. This is equivalent to the following property: for each pair $a, b \in S$ there exists $N \in \mathbb{N}$ such that for every $n > N$ there is an admissible word $\underline{a} = (a_0 \ldots a_{n-1})$ of length n such that $a_0 = a$ and $a_{n-1} = b$. If the alphabet S is finite this is equivalent to the existence of an integer $N \in \mathbb{N}$ such that every entry of the matrix $\mathcal{T}^N$ is positive.

We equip Σ with the topology generated by the n-cylinder sets:

$$C_{i_0\cdots i_{n-1}} := \{x \in \Sigma : x_j = i_j \text{ for } 0 \leqslant j \leqslant n-1\}.$$

Given a function $\varphi \colon \Sigma \to \mathbb{R}$, for each $n \geqslant 1$ we set

$$V_n(\varphi) := \sup\left\{|\varphi(x) - \varphi(y)| : x, y \in \Sigma,\ x_i = y_i \text{ for } 0 \leqslant i \leqslant n-1\right\}.$$

Note that $\varphi : \Sigma \to \mathbb{R}$ is continuous, that is $\varphi \in C(\Sigma)$, if and only if $V_n(\varphi) \to 0$. The regularity of the potentials that we consider is fundamental when it comes to proving existence of equilibrium measures.

Definition 3.1. *We say that $\varphi : \Sigma \to \mathbb{R}$ has* summable variations *if $\sum_{n=2}^{\infty} V_n(\varphi) < \infty$. Clearly, if φ has summable variations then it is continuous. We say that $\varphi : \Sigma \to \mathbb{R}$ is* weakly Hölder continuous *if $V_n(\varphi)$ decays exponentially, that is there exists $C > 0$ and $\theta \in (0, 1)$ such that $V_n(\varphi) < C\theta^n$. If this is the case then clearly it has summable variations.*

Note that in this symbolic context, given any symbolic metric, the notions of Hölder and Lipschitz function are essentially the same (see [PP, p.16]).

We say that μ is a *Gibbs measure* on Σ if there exist $K, P \in \mathbb{R}$ such that for every $n \geqslant 1$, given an n-cylinder $C_{i_0\cdots i_{n-1}}$,

$$\frac{1}{K} \leqslant \frac{\mu(C_{i_0\cdots i_{n-1}})}{e^{S_n\varphi(x) - nP}} \leqslant K$$

for any $x \in C_{i_0\cdots i_{n-1}}$. We will usually have $P = P(\varphi)$.

3.1. Compact case.

When the alphabet S is finite, the space Σ is compact. Moreover, the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous. Therefore, continuous potentials have equilibrium measures. In order to prove uniqueness of such measures, regularity assumptions on the potential and a transitivity/mixing assumption on the system are required. The following is the Ruelle-Perron-Frobenius Theorem.

Theorem 3.1. *Let (Σ, σ) be a topologically mixing finite Markov shift and let $\varphi : \Sigma \to \mathbb{R}$ be a Hölder potential. Then*

(a) there exists a $(\varphi - P(\varphi))$-conformal measure m_φ;
(b) there exists a unique equilibrium measure μ_φ for φ;
(c) there exists a positive function $h_\varphi \in L^1(m_\varphi)$ such that $L_\varphi h_\varphi = e^{P(\varphi)} h_\varphi$ and $\mu_\varphi = h_\varphi m_\varphi$;

(d) For every $\psi \in C(\Sigma)$ *we have*

$$\lim_{n\to\infty} \left\| e^{-nP(\varphi)} L_\varphi^n(\psi) - \left(\int \psi \, d\mu_\varphi \right) h \right\|_\infty = 0.$$

(e) m_φ *and* μ_φ *are Gibbs measures;*
(f) there exist $\epsilon > 0$ *such that the pressure function* $t \mapsto P(t\varphi)$ *is real analytic for* $t \in (1-\epsilon, 1+\epsilon)$.

Remark 3.1. *Since in the above theorem, the potential* $\varphi : \Sigma \to \mathbb{R}$ *is assumed to be Hölder, so is the potential* $t\varphi$ *for every* $t \in \mathbb{R}$*. Therefore, by virtue of item* (f) *of Theorem* 3.1*, we can conclude that the pressure function* $t \mapsto P(t\varphi)$ *is real analytic for* $t \in \mathbb{R}$.

Throughout this paper, we will be particularly interested in systems (X, f, φ) where the potential $\varphi : X \to \mathbb{R}$ is such that $(X, f, t\varphi)$ is recurrent for some values of t, and transient for others, see Subsection 3.2 for precise definitions. This is linked to the smoothness of the pressure function

$$p_\varphi(t) := P(t\varphi).$$

When finite, this function is continuous in t. We say that the pressure function p_φ has a *phase transition* at $t = t_0$ if p_φ is not analytic at $t = t_0$. Moreover we say that the pressure function has a *first order phase transition* at $t = t_0$ if the function p_φ is not differentiable at $t = t_0$. By virtue of Theorem 3.1, if φ is Hölder there are no phase transitions. On the other hand, Hofbauer [H] showed that for a particular class of non-Hölder potentials, phase transition occur and equilibrium states are not unique. We describe this example in Section 4.1. We are led to the following natural question:

Question: how much can we relax the regularity assumption on the potential and still have uniqueness of the equilibrium measure?

This question is related to the existence and uniqueness of conformal measures. In order to give a partial answer to this question, Walters [W2] introduced the following class of functions.

Definition 3.2. *Let* $\varphi : \Sigma \to \mathbb{R}$ *and*

$$S_n\varphi(x) := \varphi(x) + \cdots + \varphi \circ \sigma^{n-1}(x).$$

We say that $\varphi : \Sigma \to \mathbb{R}$ *is a* Walters function *if for every* $p \in \mathbb{N}$ *we have* $\sup_{n\geqslant 1} V_{n+p}(S_n\varphi) < \infty$ *and*

$$\lim_{p\to\infty} \sup_{n\geqslant 1} V_{n+p}(S_n\varphi) = 0.$$

We say that $\varphi : \Sigma \mapsto \mathbb{R}$ *is a* Bowen function *if*

$$\sup_{n\geqslant 1} V_n(S_n\varphi) < \infty.$$

Note that if φ is of summable variations then it is a Walters function. Walters showed that if a potential φ is Walters then it satisfies the Ruelle-Perron-Frobenius Theorem. In particular it has a unique equilibrium measure. Bowen introduced the class of functions we call Bowen in [Bo2]. Note that every Walters function is a

Bowen function and that there exist Bowen functions which are not Walters [W5]. Bowen functions satisfy conditions (a)-(e) of Theorem 3.1, but not necessarily (f). The following result was proven by Bowen [Bo1] and Walters [W4].

Theorem 3.2 (Bowen-Walters). *If $\varphi : \Sigma \to \mathbb{R}$ is a Bowen function then there exists a unique equilibrium measure μ for φ. Moreover, there exists a $(\varphi - P(\varphi))-$conformal measure and the measure μ is exact.*

Bowen [Bo1] showed the existence of a unique equilibrium measure and Walters [W4] described the convergence properties of the Ruelle operator of a Bowen function. Recently, Walters [W5] defined a new class of functions that he called 'Ruelle functions' which includes potentials having more than one equilibrium measure. He also characterised Ruelle functions having unique equilibrium measures.

3.2. **Non-compact case.** The definition of pressure in the case that the alphabet S is finite (compact case) was introduced by Ruelle [Ru1]. In the (non-compact) case when the alphabet S is infinite the situation is more complicated because the definition of pressure using $(n, \epsilon)-$ separated sets depends upon the metric and can be different even for two equivalent metrics. Mauldin and Urbański [MU1] gave a definition of pressure for symbolic systems close to the full-shift. Later, Sarig [S1], generalising previous work by Gurevich [Gu2, Gu1], gave a definition of pressure that satisfies the Variational Principle for any topologically mixing countable Markov shift. This definition and the one given by Mauldin and Urbański coincide for systems close to the full-shift.

Let (Σ, σ) be a topologically mixing countable Markov shift, fix a symbol i_0 in the alphabet S and let $\varphi \colon \Sigma \to \mathbb{R}$ be a Walters potential. We let

$$Z_n(\varphi, C_{i_0}) := \sum_{x:\sigma^n x = x} e^{S_n\varphi(x)} \mathbb{1}_{C_{i_0}}(x) \tag{5}$$

where $\mathbb{1}_{C_{i_0}}$ is the characteristic function of the cylinder $C_{i_0} \subset \Sigma$. The so-called *Gurevich pressure* of φ is defined by

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, C_{i_0}).$$

This limit is proved to exist by Sarig [S1, Theorem 1]. Since (Σ, σ) is topologically mixing, one can show that $P(\varphi)$ does not depend on i_0. This notion of pressure satisfies the Variational Principle and it coincides with the usual definition of pressure when the alphabet S is finite (see [S1]).

Since the phase space Σ is no longer compact, not every continuous potential has an equilibrium measure. Indeed, Sarig [S1] proved that exactly three different types of behaviour are possible for a Walters potential[1] φ:

(a) there exists an equilibrium measure for $(\Sigma, \sigma, \varphi)$ absolutely continuous with respect to a conservative $(\varphi - P(\varphi))$-conformal measure (in which case we say that φ is *positive recurrent*);

[1] Actually, he considered potentials of summable variations but the proofs of his results need no changes if it is assumed that the potential is a Walters function, see [S6].

(b) there exists a conservative $(\varphi - P(\varphi))$-conformal measure and no finite equilibrium measure absolutely continuous to it (*null recurrent*);
(c) there is no conservative $(\varphi - P(\varphi))$-conformal measure (*transient*).

In this paper, we are interested in the final case. We now give a precise definition of transience in the symbolic setting (we give an alternative definition of transience in Section 5).

Definition 3.3. *Let $\varphi : \Sigma \to \mathbb{R}$ be a Walters potential of finite pressure $P(\varphi)$. We say that φ is* transient *if and only if*

$$\sum_{n \geqslant 1} e^{-nP(\varphi)} Z_n(\varphi, C_{i_0}) < \infty,$$

where C_{i_0} is any 1-cylinder.

Let us stress that, since the system is assumed to be topologically mixing, the definition does not depend on the 1−cylinder we choose.

Remark 3.2. *It was shown by Sarig* [S3, Theorem 1] *that a Walters potential φ is transient by this definition if and only if there is no conservative $(\varphi - P(\varphi))$-conformal measure.*

Remark 3.3. *If a potential φ is transient then it either has no conformal measure or a dissipative conformal measure. Examples of both cases have been constructed by Cyr* [C2, Section 5]*. Moreover, examples are also given where there is more than one φ-conformal measure in the transient setting.*

Recently Cyr and Sarig [CS] gave a characterisation of transient potentials which involves a phase transition of some pressure function, indeed they proved that

Proposition 3.1 (Cyr and Sarig)**.** *The potential $\varphi : \Sigma \to \mathbb{R}$ is transient if and only if for each $a \in S$ there exists $t_0 \in \mathbb{R}$ such that $P(\varphi + t1_{[a]}) = P(\varphi)$ for every $t \leqslant t_0$ and $P(\varphi + t1_{[a]}) > P(\varphi)$ for $t > t_0$.*

Moreover, Cyr [C1] proved that, in a precise sense, most countable Markov shifts have at least one transient potential.

Remark 3.4. *In the context of countable Markov shifts, (Σ, σ), the main issue is to prove existence (rather than uniqueness) of equilibrium measures. Indeed, it was proved by Mauldin and Urbański* [MU2] *and by Buzzi and Sarig* [BuS] *that if an equilibrium measure exists for a potential $\varphi : \Sigma \to \mathbb{R}$ then it is unique. Conditions which guarantee that this measure is Gibbs are given in* [MU2, S4].

We conclude this section with a very important example of a countable Markov shift, the so called *renewal shift*. Let $S = \{0, 1, 2, \dots\}$ be a countable alphabet. Consider the transition matrix $A = (a_{ij})_{i,j \in S}$ with $a_{0,0} = a_{0,n} = a_{n,n-1} = 1$ for each $n \geqslant 1$ and with all other entries equal to zero. The *renewal shift* is the (countable) Markov shift (Σ_R, σ) defined by the transition matrix A, that is, the shift map σ on the space

$$\Sigma_R := \left\{ (x_i)_{i \geqslant 0} : x_i \in S \text{ and } a_{x_i x_{i+1}} = 1 \text{ for each } i \geqslant 0 \right\}.$$

The *induced system* (Σ_I, σ) is defined as the full-shift on the new alphabet given by $\{C_{0n(n-1)(n-2)\cdots 1} : n \geqslant 1\}$. The first return map to the cylinder C_0 is defined by

$$r(x) := \mathbb{1}_{C_0}(x) \inf\{n \geqslant 1 : \sigma^n x \in C_0\}.$$

Given a function $\varphi \colon \Sigma_R \to \mathbb{R}$ with summable variation we define a new function, the *induced potential*, $\Phi \colon \Sigma_I \to \mathbb{R}$ by

$$\Phi(x) := \sum_{k=0}^{r(x)-1} \varphi(\sigma^k x).$$

Sarig [S3] proved that if $\varphi : \Sigma_R \to \mathbb{R}$ is a potential of summable variations, bounded above, with finite pressure and such that the induced potential Φ is weakly Hölder continuous then there exists $t_c > 0$ such that

$$P(t\varphi) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t \in [0, t_c), \\ At & \text{if } t > t_c, \end{cases}$$

where $A = \sup\{\int \varphi \, d\mu : \mu \in \mathcal{M}\}$. This result is important since several of the examples known to exhibit phase transitions can be modelled by the renewal shift. Indeed, this is the case for the interval examples discussed in Sections 4.1–4.3.

3.3. **Non topologically mixing systems.** All the results we have discussed so far are under the assumption that the systems are topologically mixing. This is a standard irreducibility hypothesis. Moreover, as we show below, it is easy to construct counterexamples to all the previous theorems when there is no mixing assumption.

Consider the dynamical system $(\Sigma_{0,1} \sqcup \Sigma_{2,3}, \sigma)$, where $\Sigma_{i,j}$ is the full-shift on the alphabet $\{i, j\}$. It is easy to see that the topological entropy of this system is equal to $\log 2$. Moreover, there exist two invariant measures of maximal entropy: the $(1/2, 1/2)$-Bernoulli measure supported in $\Sigma_{0,1}$ and the $(1/2, 1/2)$-Bernoulli measure supported in $\Sigma_{2,3}$. Therefore, the constant (and hence Hölder) potential $\varphi(x) = 0$ has two equilibrium measures. Actually, it is possible to construct a locally constant (and hence Hölder) potential exhibiting phase transitions. Let

$$\psi(x) = \begin{cases} -1 & \text{if } x \in \Sigma_{0,1}, \\ -2 & \text{if } x \in \Sigma_{2,3}. \end{cases}$$

The pressure function has the following form

$$p_\psi(t) = \begin{cases} -t + \log 2 & \text{if } t \geqslant 0; \\ -2t + \log 2 & \text{if } t < 0. \end{cases}$$

Therefore the pressure exhibits a phase transition at $t = 0$. For $t > 0$ the equilibrium state for $t\psi$ is the $(1/2, 1/2)$-Bernoulli measure supported on $\Sigma_{0,1}$ and for $t < 0$ the equilibrium state for $t\psi$ is the $(1/2, 1/2)$-Bernoulli measure supported on $\Sigma_{2,3}$. For $t = 0$ these measures are both equilibrium states. Note that in both cases these measures are also $(t\psi - p_\psi(t))$-conformal, so the phase transitions here are not linked to transience.

Phase transitions caused by the non mixing structure of the system also appear in the case of interval maps. Indeed, the Chebyshev polynomials which are discussed in Section 4.4 and the renormalisable examples studied by Dobbs [D] are examples of this type.

4. The interval case

In this section we describe examples of systems of interval maps and potentials with phase transitions. In the following sections we will define transience in this setting and then show that phase transitions occur at the onset of transience.

The situation in the compact interval context is different from that of the compact symbolic case in that rather smooth potentials can have more than one equilibrium measure. All the examples we consider are such that entropy map is upper semi-continuous. Since the interval is compact, weak* compactness of the space of invariant probability measures implies that every continuous potential has (at least) one equilibrium measure. The study of phase transitions in the context of topologically mixing interval maps is far less developed that in the case of Markov shifts. Indeed, almost all of the examples where the pressure function has phase transitions that we are aware of exhibit the same type of behaviour. That is, the pressure function has one of the following two forms:

$$p_\varphi(t) = \begin{cases} \text{strictly convex and differentiable} & \text{if } t \in [0, t_0), \\ At & \text{if } t > t_0, \end{cases} \tag{6}$$

where $A \in \mathbb{R}$ is a constant. The regularity at the point $t = t_0$ varies depending on the examples. The other possibility is

$$p_\varphi(t) = \begin{cases} Bt + C & \text{if } t \in [0, t_0), \\ At & \text{if } t > t_0, \end{cases} \tag{7}$$

where $A, B, C \in \mathbb{R}$ are constants.

Remark 4.1. *Note that it also possible for the pressure function to have the 'reverse form' to the one given in equation* (7)*: i.e., there are interval maps and potentials for which the pressure function has the form $p_\varphi(t) = At$ in an interval $(-\infty, t_0]$ and $p_\varphi(t) = Bt + C$ for $t > t_0$. The same 'reverse form' exists in the case that the pressure function is given as in equation* (6)*.*

Remark 4.2. *We stress that the symbolic examples constructed by Olivier* [O] *can be easily constructed in the interval as well. These examples have phase transitions of positive entropy and are included in the class we construct in Section* 7*.*

Heuristically what happens is that the dynamics can be divided into an hyperbolic part and a non-hyperbolic part (the latter having zero entropy, for example a parabolic fixed point or the post-critical set).

Remark 4.3. *As in Section* 3.3*, the situation can be completely different if the map is not assumed to be topologically mixing.*

We review some of these examples.

4.1. **Hofbauer-Keller.** The following example was constructed by Hofbauer and Keller [HK] based on previous work in the symbolic setting by Hofbauer [H]. We will present it defined in a half open interval, but it is possible to define it in a compact set, namely the circle.

The dynamical system considered is the angle doubling map $f : [0,1) \mapsto [0,1)$ defined by $f(x) = 2x$ (mod 1). Let $b < -\log 2$ and $K \geqslant 0$, we define the potential by

$$\varphi = \sum_{k=0}^{\infty} a_k \cdot \mathbb{1}_{[2^{-k-1}, 2^{-k})},$$

where

$$a_k := \begin{cases} b & \text{if } 0 \leqslant k < K, \\ 2\log\left(\frac{k+1}{k+2}\right) & \text{if } k \geqslant K. \end{cases}$$

Let F be the first return map to $X = [1/2, 1)$ with return time τ. So for $X_n := \{\tau = n\}$, the induced potential Φ (see Section 3.2) takes the value $s_n := \sum_{k=0}^{n-1} a_k$. Figure 1 summarises the possible behaviours of the thermodynamic formalism depending on the sums s_n. Note that there was mistake in the original paper corrected by Walters in [W4, p.1329]. In the final column, even though we haven't yet defined recurrence and transience in the non-symbolic setting, we will use conditions (a), (b) and (c) on p8 to determine these notions.

		Pressure $P(\varphi)$	μ_φ is a Gibbs measure	φ has a unique equilibrium state	φ is +ve recurrent/ transient
$\sum_k e^{s_k} > 1$	$\sum_k a_k < \infty$	$P(\varphi) > 0$	yes	yes	+ve recurrent
	$\sum_k a_k = \infty$	$P(\varphi) > 0$	no	yes	+ve recurrent
$\sum_k e^{s_k} = 1$	$\sum_k (k+1)e^{s_k} < \infty$	$P(\varphi) = 0$	yes	no	+ve recurrent
	$\sum_k (k+1)e^{s_k} = \infty$	$P(\varphi) = 0$	no	yes	null recurrent
$\sum_k e^{s_k} < 1$		$P(\varphi) = 0$	no	yes	transient

FIGURE 1. Summary of results in [H]: Equation (2.6) and Section 5. The final column is added for clarity.

It follows from the above table that we can choose K and b for which the pressure function has the form:

$$p_\varphi(t) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t \in [0, t_0), \\ 0 & \text{if } t > t_0. \end{cases}$$

The pressure is not analytic at $t = t_0$. Moreover, for some choices of $(a_n)_n$, the map $t \mapsto P(t\varphi)$ is differentiable at $t = t_0$ where $t_0\varphi$ has only one equilibrium measure (the Dirac delta at zero); and for some choices of $(a_n)_n$, the map $t \mapsto P(t\varphi)$ is not differentiable at $t = t_0$ and $t_0\varphi$ has two equilibrium states (one is the Dirac delta at zero and the other can be seen as the projection of the Gibbs measure $\mu_{t_0\Phi}$, the equilibrium state for $t_0\Phi$).

Remark 4.4. *The setting described above provides a relatively transparent example of a triple (X, f, φ) where there exists a φ-conformal measure, but where any such φ-conformal measure is dissipative:*

Let us assume that $\sum_{k=0}^{\infty} a_k \cdot \mathbb{1}_{[2^{-k-1},2^{-k})}$ is defined so that $\sum_k e^{s_k} < 1$. Since φ gives rise to a continuous potential for the full shift on two symbols, Proposition 2.1 gives us a conformal measure ν such that $L_\varphi^*\nu = e^{P(\varphi)}\nu = \nu$, since $P(\varphi) = 0$. In other words, ν is a φ-conformal measure. We will show that ν must be dissipative.

Conformality implies that for $k \geqslant 1$, we have

$$1 = \nu([0,1)) = \nu(f^k(X_k)) = \int_{X_k} e^{-s_k} \, d\nu,$$

so $\nu(X_k) = e^{s_k}$ which implies

$$1 = \nu([0,1)) = \nu(\{0\}) + \sum_k e^{s_k}. \tag{8}$$

Therefore $\nu(\{0\}) > 0$. Moreover, since we can similarly show that $\nu(\{1/2\}) = \nu(\{0\})e^{a_0} > 0$, and since $\{f^{-n}(1/2)\}_{n\geqslant 0}$ is a wandering set, it follows that ν is dissipative.

We also note here that for the induced potential Φ, we can show that $P(\Phi) < 0$, from which we can give an alternative proof that any φ-conformal measure must be supported on $\{f^{-n}(0)\}_{n\geqslant 0}$ using the techniques in Section 6.

4.2. **Manneville-Pomeau.** The following example was introduced by Manneville and Pomeau in [MP]. It is one of the simplest examples of a non-uniformly hyperbolic map. It is expanding and it has a parabolic fixed point at $x = 0$. We give the form studied in [LSV]. For $\alpha > 0$, the map is defined by

$$f(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, 1/2) \\ 2x - 1 & \text{if } x \in [1/2, 1) \end{cases} \tag{9}$$

The pressure function of the potential $-\log|f'|$ has the following form (see, for example, [S3]),

$$p(t) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t \in [0, 1), \\ 0 & \text{if } t > 1. \end{cases}$$

where, for brevity we let

$$p(t) := P(-t \log|f'|).$$

(We use this notation throughout for this particular kind of potential.) If $\alpha \in (0, 1)$ then the map f has a probability invariant measure absolutely continuous with respect to the Lebesgue measure, which is also an equilibrium state for $-\log|Df|$, and the pressure function is not differentiable at $t = 1$. On the other hand if

$\alpha \geqslant 1$ then there is no absolutely continuous invariant probability measure and the pressure function is differentiable at $t = 1$.

The value of α determines the class of differentiability of the map f and determines the amount of time 'typical' orbits spend near the parabolic fixed point. For $\alpha \in (0,1)$, the amount of time spent near the point $x = 0$ by Lebesgue-typical points is not long enough to force the relevant invariant measure (the equilibrium state for $-\log|Df|$) to be infinite. But if $\alpha \geqslant 1$ then the map has a sigma-finite (but infinite) invariant measure absolutely continuous with respect to the Lebesgue measure.

Remark 4.5. *Note that the Dirac delta measure on* 0 *is a conformal measure for* $-t\log|Df|$ *with* $t > 1$ *if we remove all preimages of 0.*

4.3. **Pesin-Zhang.** The following example was studied by Pesin and Zhang in [PZ]. As in the case of Hofbauer and Keller the dynamical system is uniformly hyperbolic. Let $I_1 = [0, a]$ and $I_2 = [b, 1]$ be two disjoint intervals contained in $[0, 1]$ and consider the map

$$f(x) = \begin{cases} \frac{x}{a} & \text{if } x \in [0, a], \\ \frac{b}{b-1}(x-1) & \text{if } x \in [b, 1]. \end{cases}$$

In order to define the potential, consider $t \in \mathbb{R}$ and $\alpha \in (0, 1]$. Let

$$\varphi(x) := \begin{cases} -(1-\log x)^{-\alpha} & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

The potential φ is continuous on $[0, 1]$ but is not Hölder continuous at zero. Pesin and Zhang proved that there exist $t_c > 0$ such that

$$p_\varphi(t) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t \in [0, t_c), \\ 0 & \text{if } t > t_c. \end{cases}$$

4.4. **Chebyshev.** A simple example of a transitive map exhibiting a phase transition in the quadratic family is the Chebyshev polynomial $f(x) := 4x(1-x)$ defined on $[0, 1]$ (see for example [D]). Note that in this setting the phase space is compact, but the potential is not continuous. It is well known that the equilibrium states for the potentials $\{-t\log|Df| : t \in \mathbb{R}\}$ are the absolutely continuous (with respect to Lebesgue) invariant probability measure μ_1, which has $\int \log|f'|\, d\mu_1 = \log 2$, and the Dirac measure δ_0 on the fixed point at 0, which has $\int \log|f'|\, d\delta_0 = \log 4$. So, there exists a phase transition at $t_0 = -1$ and

$$p(t) = \begin{cases} -t\log 4 & \text{if } t < -1, \\ (1-t)\log 2 & \text{if } t \geqslant -1. \end{cases}$$

4.5. **Multimodal maps.** Let $\mathcal{F}$ be the collection of C^2 multimodal interval maps $f : I \to I$, where $I = [0, 1]$, satisfying:

a) the critical set $\mathcal{C}r = \mathcal{C}r(f)$ consists of finitely many critical points c with critical order $1 < \ell_c < \infty$, i.e., there exists a neighbourhood U_c of c and a C^2 diffeomorphism $g_c : U_c \to g_c(U_c)$ with $g_c(c) = 0$ $f(x) = f(c) \pm |g_c(x)|^{\ell_c}$;
b) f has negative Schwarzian derivative, i.e., $1/\sqrt{|Df|}$ is convex;

c) f is topologically transitive on I;
d) $f^n(\mathcal{C}r) \cap f^m(\mathcal{C}r) = \emptyset$ for $m \neq n$.

For $f \in \mathcal{F}$ and $\mu \in \mathcal{M}_f$, let us define,

$$\lambda(\mu) := \int \log |f'| \, d\mu \qquad \text{and} \qquad \lambda_m := \inf\{\lambda(\mu) : \mu \in \mathcal{M}_f\}.$$

It was proved in [IT1] that there exists $t^+ > 0$ such that the pressure function of the (discontinuous) potential $\log |f'|$ satisfies,

$$p(t) = \begin{cases} \text{strictly convex and } C^1 & \text{if } t \in (-\infty, t^+), \\ At & \text{if } t > t^+. \end{cases}$$

In the case $\lambda_m = 0$, $t^+ \leqslant 1$ and $A = 0$; while in the case $\lambda_m > 0$, $t^+ > 1$ and $A < 0$.

Remark 4.6. *The number of equilibrium measures at the phase transition can be large. Indeed, Cortez and Rivera-Letelier* [CRL] *proved that given $\mathcal{E}$ a non-empty, compact, metrisable and totally disconnected topological space then there exists a parameter $\gamma \in (0, 4]$ such that set of invariant probability measures of $x \mapsto \gamma x(1-x)$, supported on the omega-limit set of the critical point is homeomorphic to $\mathcal{E}$. Examples of quadratic maps having multiple ergodic measures supported on the omega-limit set of the critical point were first constructed in* [Br].

5. Definition of Transience/Recurrence

In this section we propose a definition of transience which holds beyond the Markov shift case (in which case, Definition 3.3 is sufficient) and indeed requires no Markov structure.

We first need to define the notion of weakly expanding measures.

5.1. **Weakly expanding measures.**

We suppose that $f : X \to X$ for X a compact metric space.

Definition 5.1. *We say that $x \in X$* goes to ε-large scale *at time n if the ball $B_\varepsilon(f^n(x))$ can be pulled back bijectively by the branch of f^{-n} corresponding to the orbit of x. We say that x* goes to ε-large scale infinitely often *if there exists $\varepsilon > 0$ such that x goes to ε-large scale for infinitely many times $n \in \mathbb{N}$. Let $LS_\varepsilon \subset X$ denote the set of points which go to ε-large scale infinitely often.*

In the topologically mixing Markov shift (Σ, σ) case, every point goes to large scale infinitely often, even in the countable Markov shift case. However, in the non-Markov shift case, we will often need to assume that the points of interest go to some ε-large scale infinitely often.

Definition 5.2. *Given a Borel measure μ on X we say that μ is* weakly expanding *if there exist $\varepsilon > 0$ such that $\mu(LS_\varepsilon) > 0$.*

We use the term 'weakly expanding' for our measures to distinguish from the expanding measures in [Pi] (note that those measures go to large scale with positive frequency).

Question: can a conformal measure be conservative and not have a.e. point going to large scale infinitely often?

5.2. **The definition of transience in the general case.**

Definition 5.3. *Let $f : X \to X$ be a dynamical system and $\varphi : X \to [-\infty, \infty]$ a Borel potential. Then (X, f, φ) is called* recurrent *if there is a weakly expanding conservative $(\varphi - P(\varphi))$-conformal measure. Otherwise (X, f, φ) is called* transient.

Our definition raises the following questions:

- How do our definitions apply to the examples in Section 4? This is addressed in Section 6.
- Must the onset of transience always give pressure functions of the type in (6) or (7) (i.e., the onset of transience occurs 'at zero entropy' and once a potential is transient for some t_0 is either transient for all $t < t_0$ or $t > t_0$)? This is shown to be false in Section 7.
- What does the existence of a dissipative $(t_0\varphi - p_\varphi(t_0))$-conformal measure tell us about a phase transition at t_0?

6. No conservative conformal measure

In this section we will show that the systems considered in Section 4 are transient past the phase transition. We focus on multimodal maps $f \in \mathcal{F}$ defined in Section 4.5. We will show that for a certain range of values of $t \in \mathbb{R}$ the potential $-t \log |f'|$ has no conservative conformal measure and hence is transient. The results described here also hold for the Manneville-Pomeau map, but since the proof is essentially the same, but simpler, we only discuss the former case. The first result deals with the recurrent case.

Theorem 6.1. *Suppose that $f \in \mathcal{F}$. If $t < t^+$ then there is a weakly expanding conservative $(-t \log |Df| - p(t))$-conformal measure.*

This is proved in the appendix of [T], where it is referred to as Proposition 7'.

Proposition 6.1. *Suppose that $f : I \to I$ belongs to $\mathcal{F}$ and $\lambda_m = 0$. Then for any $t > 1$, $(I, f, -t \log |Df|)$ is transient.*

This proposition covers the case when $t^+ = 1$. We expect this to also hold when $t^+ \neq 1$, but we do not prove this. As in Sections 4.1 and 4.2, the strategy used to study multimodal maps $f \in \mathcal{F}$, and indeed to prove Proposition 6.1, considering that they lack Markov structure and uniform expansivity, is to consider a generalisation of the first return map. These maps are expanding and are Markov (although over a countable alphabet). The idea is to study the inducing scheme through the theory of Countable Markov Shifts and then to translate the results into the original system.

We say that $(X, \{X_i\}_i, F, \tau) = (X, F, \tau)$ is an *inducing scheme* for (I, f) if

- X is an interval containing a finite or countable collection of disjoint intervals X_i such that F maps each X_i diffeomorphically onto X, with bounded distortion (i.e. there exists $K > 0$ so that for all i and $x, y \in X_i$, $1/K \leqslant DF(x)/DF(y) \leqslant K$);
- $\tau|_{X_i} = \tau_i$ for some $\tau_i \in \mathbb{N}$ and $F|_{X_i} = f^{\tau_i}$. If $x \notin \cup_i X_i$ then $\tau(x) = \infty$.

The function $\tau : \cup_i X_i \to \mathbb{N}$ is called the *inducing time*. It may happen that $\tau(x)$ is the first return time of x to X, but that is certainly not the general case. We denote the set of points $x \in I$ for which there exists $k \in \mathbb{N}$ such that $\tau(F^n(f^k(x))) < \infty$ for all $n \in \mathbb{N}$ by $(X, F, \tau)^\infty$.

The space of $F-$invariant measures is related to the space of f-invariant measures. Indeed, given an f-invariant measure μ, if there is an F-invariant measure μ_F such that for a subset $A \subset I$,

$$\mu(A) = \frac{1}{\int \tau \, d\mu_F} \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \mu_F\left(f^{-k}(A) \cap X_i\right) \tag{10}$$

where $\frac{1}{\int \tau \, d\mu_F} < \infty$, we call μ_F the *lift* of μ and say that μ is a *liftable* measure. Conversely, given a measure μ_F that is F-invariant we say that μ_F *projects* to μ if (10) holds.

Remark 6.1. *Let μ be a liftable measure and be ν be its lift. A classical result by Abramov* [A] *allow us to relate the entropy of both measures. Further results obtained in* [PS, Z] *allow us to de the same with the integral of a given potential. Indeed, we have that*

$$h(\mu) = \frac{h(\nu)}{\int \tau \, d\nu} \text{ and } \int \varphi \, d\mu = \frac{\int \Phi \, d\nu}{\int \tau \, d\nu}.$$

For $f \in \mathcal{F}$ we choose the domains X to be n-cylinders coming from the so-called *branch partition*: the set $\mathcal{P}_1^f$ consisting of maximal intervals on which f is monotone. So if two domains $C_1^i, C_1^j \in \mathcal{P}_1^f$ intersect, they do so only at elements of $\mathcal{C}r$. The set of corresponding n-cylinders is denoted $\mathcal{P}_n^f := \vee_{k=1}^n f^{-k}\mathcal{P}_1$. We let $\mathcal{P}_0^f := \{I\}$. For an inducing scheme (X, F, τ) we use the same notation for the corresponding n-cylinders $\mathcal{P}_n^F$.

The following result, proved in [T] (see also [BT2]) proves that useful inducing schemes exist for maps $f \in \mathcal{F}$.

Theorem 6.2. *Let $f \in \mathcal{F}$. There exist a countable collection $\{(X^n, F_n, \tau_n)\}_n$ of inducing schemes with $\partial X^n \notin (X^n, F_n, \tau_n)^\infty$ such that any ergodic invariant probability measure μ with $\lambda(\mu) > 0$ is compatible with one of the inducing schemes (X^n, F_n, τ_n). Moreover, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $LS_\varepsilon \subset \cup_{n=1}^N (X^n, F_n, \tau_n)^\infty$.*

We are now ready to apply this theory to the question of transience, building up to proving Proposition 6.1.

Lemma 6.1. *Suppose that $f \in \mathcal{F}$. If, for $t \notin (t^-, t^+)$, there is a conservative weakly expanding $(-t\log|Df| - s)$-conformal measure $m_{t,s}$ for some $s \in \mathbb{R}$, then $s \leqslant P(-t\log|Df|)$. Moreover, there is an inducing scheme (X, F, τ) such that*

$$P(-t\log|DF| - \tau s) = 0$$

and

$$m_{t,s}\left(\left\{x \in X : \tau^k(x) \text{ is defined for all } k \geqslant 0\right\}\right) = m_{t,s}(X).$$

Proof. We prove the second part of the lemma first.

Suppose that $m_{t,s}$ is a weakly expanding $(-t\log|Df| - s)$-conformal measure. We introduce an inducing scheme (X, F). Since $m_{t,s}$ is weakly expanding, by Theorem 6.2 there exists an inducing scheme (X, F, τ) such that

$$m_{t,s}\left(\left\{x \in X : \tau^k(x) \text{ is defined for all } k \geqslant 0\right\}\right) = m_{t,s}(X) > 0. \tag{11}$$

By the distortion control for the inducing scheme, for any n-cylinder $\mathbf{C}_{n,i} \in \mathcal{P}_n^F$ and since $m_{t,s}(X) = \int_{\mathbf{C}_{n,i}} |DF^n|^t e^{s\tau^n}\, dm_t$, there exists $K \geqslant 1$ such that

$$|\mathbf{C}_{n,i}|^t e^{s\tau^n} = K^{\pm t}|X|^t m_{t,s}(\mathbf{C}_{n,i}). \tag{12}$$

Since the inducing scheme is the full shift, and because of this distortion property, the pressure of $-t\log|DF| - s\tau$ can be computed as

$$\lim_{n\to\infty} \frac{\log\left(\sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|^t e^{s\tau^n}\right)}{n}.$$

However, using first (12) and then (11), we have

$$\sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|^t e^{s\tau^n} = K^{\pm t}|X|^t \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} m_{t,s}(\mathbf{C}_{n,i}) = K^{\pm t}|X|^t m_{t,s}(X)$$

for all $n \geqslant 1$. This implies that $P(-t\log|DF| - s\tau) = 0$, proving the second part of the lemma.

We prove the first part of the lemma by applying the Variational Principle to the inducing scheme. Since $P(-t\log|DF| - s\tau) = 0$, by [S1, Theorem 2], there exists a sequence $(\mu_{F,n})_n$ each supported on a finite number of cylinders in $\mathcal{P}_1^F$ and with

$$\lim_{n\to\infty}\left(h(\mu_{F,n}) + \int -t\log|DF| - s\int \tau\, d\mu_{F,n}\right) = 0.$$

Therefore, by the Abramov Theorem (see Remark 6.1), for the projected measures μ_n we have

$$h(\mu_n) - \int t\log|Df|\, d\mu_n \to s.$$

Hence the definition of pressure implies that $s \leqslant p(t)$. $\square$

Proof of Proposition 6.1. Suppose that there exists a weakly expanding conservative $-t\log|Df|$-conformal measure m_t. Let (X,F) be the inducing scheme in Lemma 6.1, with distortion $K\geqslant 1$. Then $P(\Psi_t)=0$ and

$$\begin{aligned} m_t(X) &= \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} m_t(\mathbf{C}_{n,i}) = K^t|X|^t \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|^t \\ &= K^t|X|^t \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}||\mathbf{C}_{n,i}|^{t-1} \\ &\leqslant K^t|X|^t \left(\sup_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|\right)^{t-1} \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}| \\ &\leqslant K^t|X|^{t+1} \left(\sup_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|\right)^{t-1}. \end{aligned}$$

Since by choosing n large, we can make this is arbitrarily small, we are led to a contradiction. □

7. Possible transient behaviours

In this section we address some of the questions raised about the possible behaviours of transient systems in Section 5. In particular, we present an example which gives us a range of possible behaviours for a pressure function which has one or two phase transitions. This example is very similar to that presented by Olivier in [O, Section 4] in which he extended the ideas of Hofbauer [H] to produce a system with hyperbolic dynamics, but with a potential φ which was sufficiently irregular to produce a phase transition: the support of the relevant equilibrium states $t\varphi$ jumping from the whole space to an invariant subset as t moved through the phase transition. We follow the same kind of argument, with slightly simpler potentials. In our case, we are able to obtain very precise information on the pressure function and on the measures at the phase transition. Moreover, we can arrange our system so that the support of the relevant equilibrium state for $t\varphi$ jumps from the whole space, to an invariant subset, and then back out to the whole space as t increases from $-\infty$ to ∞. Between the phase transitions we have transience.

Definition 7.1. *For a dynamical system (X,f) with a potential φ, let us consider conditions i) $\lim_{t\to-\infty} p_\varphi(t)=\infty$; ii) there exist $t_1<t_2$ such that $p_\varphi(t)$ is constant on $[t_1,t_2]$; iii) $\lim_{t\to\infty} p_\varphi(t)=\infty$. We say that p_φ is* DF *(for down-flat) if i) and ii) hold; that p_φ is* DU *(for down-up) if i) and iii) hold; that p_φ is* DFU *(for down-flat-up) if i), ii) and iii) hold.*

In this section we describe a situation with pressure which is DFU. The system is the full-shift on three symbols (Σ_3,σ). (Note that we could instead consider $\{I_i\}_{i=1}^3$, three pairwise disjoint intervals contained in $[0,1]$, and the map $f:\bigcup_{i=1}^3 I_i\subset[0,1]\to[0,1]$, where $f(I_i)=[0,1]$ which is topologically (semi-)conjugated to (Σ_3,σ).) The construction we will use can be thought of as a generalisation of the renewal shift (see Section 3.2). Let (Σ_3,σ) be the full shift on three symbols $\{1,2,3\}$. A point $x\in\Sigma_3$ can be written as $x=(x_0x_1x_2\dots)$, where

$x_i \in \{1,2,3\}$. Our *bad set* (the we will denote by B) will be the full shift on two symbols $\{1,3\}$ and the *renewal vertex* will be $\{2\}$.

For $N \geqslant 1$, for $(x_0, x_1, \dots, x_{N-1}) \in \{1,2,3\}^N$, let $[x_0x_1 \dots x_{N-1}]$ denote the cylinder $C_{x_0x_1\dots x_{N-1}}$. We set X_0 to be the cylinder $[2]$ and define the *first return time* on X_0 as the function $\tau : [2] \to \mathbb{N}$ defined by $\tau(x) = \inf\{n \in \mathbb{N} : \sigma^n x \in [2]\}$. The set of points for which the first return time is equal to $n \geqslant 1$ will be denoted by X_n. It consists of 2^{n-1} cylinders. Indeed, we list the first three sets from which this assertion is already clear,

$$X_1 = [22]$$

$$X_2 = [212] \cup [232]$$

$$X_3 = [2112] \cup [2132] \cup [2312] \cup [2332].$$

Remark 7.1. *Note that the system $F : X_n \mapsto \bigcup_{n=1}^{\infty} X_0$ is a 2^{n-1}-to-one map for each $n \geqslant 1$.*

The class of potentials is given as follows.

Definition 7.2. *A function $\varphi : \Sigma_3 \to \mathbb{R}$ is called a* grid function *if it is of the form*

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot \mathbb{1}_{X_n}(x),$$

where $\mathbb{1}_{X_n}(x)$ is the characteristic function of the set M_n and $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$. Note that $\varphi|_B = 0$.

Grid functions were introduced by Markley and Paul [MP] as a generalisation of those functions by Hofbauer [H] which we described in Section 4.1. They can be thought of as weighted distance functions.

We are now ready to state our result concerning the thermodynamic formalism for grid functions.

Theorem 7.1. *Let (Σ_3, σ) be the full-shift on three symbols and let $\varphi : \Sigma_3 \mapsto \mathbb{R}$ be a grid function defined by a sequence $(a_n)_n$. Then*

1. *there exist $(a_n)_n$ so that $D^-p_\varphi(1) < 0$, but $p_\varphi(t) = \log 2$ for all $t \geqslant 1$;*
2. *there exist $(a_n)_n$ and $t_1 > 1$ so that $D^-p_\varphi(1) < 0$, $p_\varphi(t) = \log 2$ for all $t \in [1, t_1]$ and $Dp_\varphi(t) > 0$ for all $t > t_1$;*
3. *there exist $(a_n)_n$ so that $Dp_\varphi(t) < 0$ for $t < 1$, but p_φ is C^1 at $t = 1$ and $p_\varphi(t) = \log 2$ for all $t \geqslant 1$;*
4. *there exist $(a_n)_n$ and $t_1 > 1$ so that $Dp_\varphi(t) < 0$ for $t < 1$, but p_φ is C^1 at $t = 1$, and $p_\varphi(t) = \log 2$ for all $t \in [1, t_1]$ and $Dp_\varphi(t) > 0$ for all $t > t_1$;*

We comment further on the systems $(\Sigma_3, \sigma, t\varphi)$ with reference to Table 1 (note that the only aspects which don't follow more or less immediately from the construction of our sequences $(a_n)_n$ are the null recurrent parts, which follow from Lemmas 7.3 and 7.4):

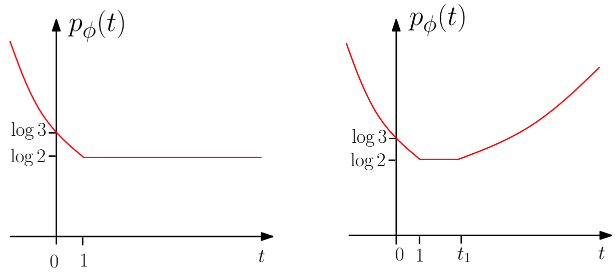

FIGURE 2. Sketch of cases (1) and (2) of Theorem 7.1.

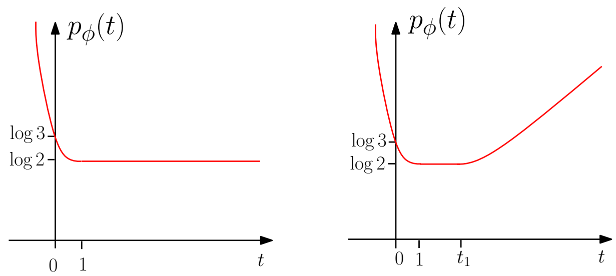

FIGURE 3. Sketch of cases (3) and (4) of Theorem 7.1.

- In case (1) of the theorem, the system is positive recurrent for $t \leqslant 1$ and transient for $t > 1$. The pressure function p_φ is DF. See the left hand side of Figure 2.
- In case (2) of the theorem, the system is positive recurrent for $t \in (-\infty, 1] \cup [t_1, \infty)$ and transient for $t \in (1, t_1)$. The pressure function is DFU. See the right hand side of Figure 2.
- In case (3) of the theorem, the system is positive recurrent for $t < 1$, null recurrent for $t = 1$ and transient for $t > 1$. The pressure function is DF. See the left hand side of Figure 3.
- In case (4) of the theorem, the system is positive recurrent for $t \in (-\infty, 1) \cup (t_1, \infty)$, null recurrent for $t = 1, t_1$ and transient for $t \in (1, t_1)$. The pressure is DFU. See the right hand side of Figure 3.
- In the cases above where $t\varphi$ is transient, as in Remark 4.4, there is a dissipative $t\varphi$-conformal measure.

The first and third parts of this theorem follow easily from the second and fourth parts, so we omit their proof. The proof of this theorem occupies the rest of this section.

Note that in the case of Hofbauer's example, described in 4.1, the modes of recurrence of the potential and behaviour of the pressure function are determined by the sums of the sequence $(a_n)_n$ (see Table 1). This is also the case in our example.

7.1. **The inducing scheme.** The *first return map*, denoted by F, is defined by

$$F(x) = \sigma^n(x) \text{ if } x \in X_n.$$

Note that the bad set B, on which F is not defined, can be thought of as a coding for the middle third Cantor set.

The induced potential for this first return map is given by $\Phi(x) = S_{\tau(x)}\varphi(x)$. For $n \geqslant 1$ we let

$$s_n := a_0 + \cdots + a_{n-1}.$$

Then by definition of φ, for any $x \in X_n$ we have $\Phi(x) = s_n$.

The definitions of liftability of measures and aspects of inducing schemes for the case considered here are directly analogous to the setting considered in Section 6, so we do not give them here.

Remark 7.2. *Note that the potential Φ is a locally constant over the countable Markov partition $\bigcup_{n=1}^{\infty} X_n$. Therefore with the inducing procedure we have gained regularity on our potential. Observe that if there is a Φ-conformal measure μ, it is also a Gibbs measure, and indeed for any $x \in X_n$, $\mu(X_n) = e^{\Phi(x)} = e^{s_n}$. Since X_n consists of 2^{n-1} connected components, each has mass $2^{-(n-1)}e^{s_n}$.*

Given a grid function (our potential) φ defined on Σ_3, to discuss equilibrium states for the induced system, as in Section 6, it is convenient to shift the original potential to ensure that its induced version will have pressure zero. Therefore, since we will be interested in the family of potentials $t\varphi$ with $t \in \mathbb{R}$, we set

$$\psi_t := t\varphi - p_\varphi(t) \text{ and } \Psi_t := t\Phi - \tau p_\varphi(t).$$

We denote $\Psi_t|_{X_i}$ by $\Psi_{t,i}$.

In this setting, the properties of the pressure function will directly depend on the choice of the sequence $(a_n)_n$.

Our first restriction on the sequence $(a_n)_n$ comes from a normalisation requirement. Indeed, for $t = 1$ we want $p_\varphi(t) = P(\varphi) = \log 2$. If this is the case, we have that the measure of maximal entropy μ_B on B is an equilibrium measure since $h(\mu_B) + \int \varphi \, d\mu_B = h(\mu_B) = \log 2$. We choose $(a_n)_n$ so that

$$1 = \sum_i e^{\Psi_{1,i}} = \sum_{n \geqslant 1} 2^{n-1} e^{s_n - n \log 2} = \frac{1}{2} \sum_{n \geqslant 1} e^{s_n}. \tag{13}$$

As in [Ba, p25] for example, since Ψ_t is locally constant, $e^{P(\Psi_1)} = \sum_i e^{\Psi_{1,i}}$, so (13) implies $P(\Psi_1) = 0$. Similarly,

$$e^{P(\Psi_t)} = \sum_{i \geqslant 1} e^{\Psi_{t,i}} = \sum_{i \geqslant 1} e^{t\Phi_i - \tau p_\varphi(t)} = \sum_{n \geqslant 1} 2^{n-1} e^{ts_n - np_\varphi(t)}. \tag{14}$$

We have the following result, similarly to Lemma 6.1.

Lemma 7.1. *The existence of a conservative ψ_t-conformal measure m_t implies $P(\Psi_t) = 0$.*

Proof. To apply the argument of Lemma 6.1, we only need to show that

$$m_t\left(\{x : f^n(x) \in X_1 \text{ for infinitely many } n\}\right) > 0.$$

If not then for

$$A_k := \left\{x \in X_1 : f^{k+n}(x) \in X_0 \cup X_2 \text{ for all } n \geqslant 1\right\},$$

$m_t(\cup_k A_k) > 0$. In particular, we have $m_t(A_0) > 0$. Observe that A_0 is a wandering set since $f^{-p}(A_0) \cap f^{-q}(A_0) = \emptyset$ for positive $q \neq p$. Therefore m_t is not conservative. □

7.2. **Down-flat-up pressure occurs.** We first show that the DF case occurs and then show DFU occurs too. First we set $a_0 = 0$ and suppose that for every $n > 0$ the numbers a_n are chosen so that $a_n < 0$ and (13) holds.

Since $\varphi \leqslant 0$; the pressure function p_φ is decreasing in t; $p_\varphi(1) = \log 2$; and $p_\varphi(t) \geqslant \log 2$; this means that $p_\varphi(t) = \log 2$ for all $t \geqslant 1$, so the DF case occurs. The transience of $(\Sigma, \sigma, t\varphi)$ for $t > 1$ follows as below. In this case, $s_1 = 0$, and (13) can be rewritten as

$$1 = \frac{1}{2}\sum_{n\geqslant 1} e^{s_n} = \frac{1}{2}\left(1 + \sum_{n\geqslant 2} e^{s_n}\right). \tag{15}$$

Now to show that the DFU case occurs, let us first set $(a_n)_n$ as above. Next we replace a_0 by $\tilde{a}_0 := \delta \in (0, \log 2)$, and a_1 by $\tilde{a}_1 := a_1 + \delta'$, where $\delta' < 0$ is such that (15) still holds when $(s_n)_n$ is replaced by $(\tilde{s}_n)_n$. The rest of the a_n are kept fixed. So (15) implies that

$$\frac{1}{2}e^{\delta} + \frac{1}{2}e^{\delta+\delta'} = 1,$$

so $P(\Psi_1) = 0$. Using Taylor series, we have $2\delta + \delta' < 0$. We now replace φ, Φ and Ψ_t by the adjusted potentials $\tilde{\varphi}$, $\tilde{\Phi}$, $\tilde{\Psi}_t$.

Lemma 7.2. *There exists $t_1 > 1$ such that $P(\tilde{\Psi}_t) < 0$ for all $t \in (1, t_1)$.*

Since $p_{\tilde{\varphi}}(t) \geqslant \log 2$, the lemma implies that $p_{\tilde{\varphi}}(t) = \log 2$ for $t \in [1, t_1]$, so the DF property of the pressure function persists under our perturbation of φ to $\tilde{\varphi}$. Moreover, Lemma 7.1 implies that $(\Sigma, \sigma, t\tilde{\varphi})$ is transient for $t \in [1, t_1]$.

The 'up' part of the DFU property, must hold for $p_{\tilde{\varphi}}$ since $\tilde{a}_0 > 0$: indeed the graph of $p_{\tilde{\varphi}}$ must be asymptotic to $t \mapsto \tilde{a}_0 t$, and the equilibrium measures for $t\tilde{\varphi}$ denoted by μ_t must tend to the Dirac measure on the fixed point in [2].

Proof of Lemma 7.2. As above, since $\tilde{\Psi}_t$ is locally constant, $e^{P(\tilde{\Psi}_t)}$ can be computed as

$$e^{P(\tilde{\Psi}_t)} = \sum_{i\geqslant 1} e^{t\tilde{\Phi}_i - \tau_i p_{\tilde{\varphi}}(t)} = \sum_{n\geqslant 1} 2^{n-1} e^{t\tilde{s}_n - np_{\tilde{\varphi}}(t)}. \tag{16}$$

Since $p_{\tilde{\varphi}}(t) \geqslant \log 2$ and $s_n < 0$ for $n \geqslant 2$ for $t > 1$ close to 1 we have

$$\sum_{i\geqslant 1} e^{t\tilde{\Phi}_i - \tau_i p_{\tilde{\varphi}}(t)} \leqslant \frac{1}{2}\sum_{n\geqslant 1} e^{t\tilde{s}_n} = \frac{1}{2}\left(e^{t\delta} + e^{t(\delta+\delta')}\sum_{n\geqslant 2} e^{ts_n}\right) < \frac{1}{2}\left(e^{t\delta} + e^{t(\delta+\delta')}\right) < 1,$$

where the final inequality follows from a Taylor series expansion and the fact that $2\delta + \delta' < 0$. This implies that $P(\tilde{\Psi}_t) < 0$ for $t > 1$ close to 1. We let $t_1 > t'' > 1$ be minimal such that $t > t_1$ implies $p_{\tilde{\varphi}}(t) > \log 2$. □

For brevity, from here on we will drop the tildes from our notation when discussing the potentials above.

7.3. **Tails and smoothness.** So far we have not made any assumptions on the precise form of a_n for large n. In this section we will make our assumptions precise in order to distinguish cases (1) from case (3) in Theorem 7.1, as well as case (2) from case (4). That is to say, we will address the question of the smoothness of p_φ at 1 and t_1 by defining different forms that a_n, and hence s_n, can take as $n \to \infty$. In fact, it is only the form of a_n for large n which separates the cases we consider. As in [H, Section 4], see also [BT, Section 6], let us assume that for all large n, for some $\gamma > 1$ we have

$$a_n = \gamma \log \left(\frac{n}{n+1} \right).$$

We will see that we have a first order phase transition in the pressure function p_φ whenever $\gamma > 2$, but not when $\gamma \in (1, 2]$.

Clearly, there is some $\kappa \in \mathbb{R}$ so that $s_n \sim \kappa - \gamma \log n$. So applying the computation in (13),

$$\sum_i e^{\Psi_{1,i}} = \frac{1}{2} \sum_n e^{s_n} = (1 + O(1)) \sum_n \frac{1}{n^\gamma}.$$

Since we assumed that $\gamma > 1$, we can ensure that this is finite, and indeed we can choose $(a_n)_n$ in such a way that $\sum_i e^{\Psi_{1,i}} = 1$ as in (13).

We now show that the graph of the pressure in the case that the pressure is DFU is either C^1 everywhere or only non-C^1 at both $t = 1$ and $t = t_1$. The issue of smoothness of p_φ in the DF case follows as in the DFU case, so Theorem 7.1 then follows from Lemma 7.2 and the following proposition.

Proposition 7.1. *For potential φ chosen as above, there exists $t_1 > 1$ such that $p_\varphi(t) = \log 2$ for all $t \in [1, t_1]$. Moreover, if $\gamma \in (1, 2]$ then p_φ is everywhere C^1, while if $\gamma > 2$ then p_φ fails to be differentiable at both $t = 1$ and $t = t_1$.*

The first part of the proposition follows from Lemma 7.2, while the second follows directly from the following two lemmas. We will use the fact that if p_φ is C^1 at t then $Dp_\varphi(t) = \int \varphi \, d\mu_t$ (see [PU, Chapter 4]).

Lemma 7.3. *If $\gamma \in (1, 2]$ then $Dp_\varphi(1) = 0$.*

Proof. Since by Lemma 7.2, for $t \in [1, t_1]$, p_φ is constant $\log 2$, we have $Dp_\varphi^+(1) = 0$, so to prove $Dp_\varphi(1) = 0$ we must show $Dp_\varphi^-(1) = 0$.

Suppose that $t < 1$. Then by the Abramov formula (see Remark 6.1),

$$\int \varphi \, d\mu_t = \frac{\int \Phi \, d\mu_{\Psi_t}}{\int \tau \, d\mu_{\Psi_t}} = \frac{\sum_n s_n e^{ts_n - n(p_\varphi(t) - \log 2)}}{2 \sum_n n e^{ts_n - n(p_\varphi(t) - \log 2)}}. \tag{17}$$

As above, for large n, $s_n \sim \kappa - \gamma \log n$, which is eventually much smaller, in absolute value, than n. Since also, $\sum_n n e^{s_n - n(p_\varphi(t) - \log 2)} \to \infty$ as $t \to 1$, we can make $\int \varphi \, d\mu_t$ arbitrarily small by taking $t < 1$ close enough to 1. Since when p_φ is C^1 at t then $Dp_\varphi(t) = \int \varphi \, d\mu_t$, this completes the proof. □

Lemma 7.4. *Suppose that φ is a grid function as above and the pressure p_φ is DFU. Then p_φ is C^1 at $t = 1$ if and only if p_φ is C^1 at $t = t_1$.*

Proof. Using the argument in the proof of Lemma 7.3, in particular (17), if $\int \tau \, d\mu_{\Psi_t} = \infty$, we can make $Dp_\varphi(t')$ arbitrarily close to 0 by taking t' close enough to t. Similarly if this integral is finite at t then the derivative $Dp_\varphi(t)$ is non-zero. So to prove the lemma, we need to show that the finiteness or otherwise of $\int \tau \, d\mu_{\Psi_t}$ is the same at both $t = 1$ and $t = t_1$.

As in the proof of Lemma 7.2, $s_n < 0$ for $n \geqslant 2$. So since $p_\varphi(t) = p_\varphi(t_1)$ and $t_1 > 1$,

$$\int \tau \, d\mu_{\Psi_1} > \sum_{i \geqslant 2} \tau_i e^{\Phi_i - \tau_i p_\varphi(1)} > \sum_{i \geqslant 2} \tau_i e^{t_1 \Phi_i - \tau_i p_\varphi(t_1)}.$$

Therefore if $\int \tau \, d\mu_{\Psi_1} < \infty$ then $\int \tau \, d\mu_{\Psi_{t_1}} < \infty$. Similarly, if $\int \tau \, d\mu_{\Psi_{t_1}} = \infty$ then $\int \tau \, d\mu_{\Psi_1} = \infty$. Hence either $Dp_\varphi(1)$ and $Dp_\varphi(t_1)$ are both 0 or are both non-zero. □

Remark 7.3. *In the case $\gamma \in (1, 2]$, the measure μ_{Ψ_1} is not regarded as an equilibrium state for the system (M_0, F, Ψ_1) since*

$$\int \Psi_1 \, d\mu_{\Psi_1} = -\infty.$$

This follows since

$$\int \Psi_1 \, d\mu_{\Psi_1} = \sum_n (s_n - np_\varphi(t)) e^{s_n} \asymp \sum_n \frac{a - \gamma \log n - np_\varphi(t)}{n^\gamma},$$

so for all large n the summands are dominated by the terms $-p_\varphi(t) n^{1-\gamma}$ which are not summable.

Remark 7.4. *If we wanted the limit of μ_t as $t \to \infty$ to be a measure with positive entropy, then one way would be to choose our dynamics to be $x \mapsto 5x \mod 1$ and the set M_0 to correspond to the interval $[0, 2/5]$ for example.*

Note that for our examples, we can not produce more than two equilibrium states simultaneously. One can see this as following since we are essentially working with two intermingled systems.

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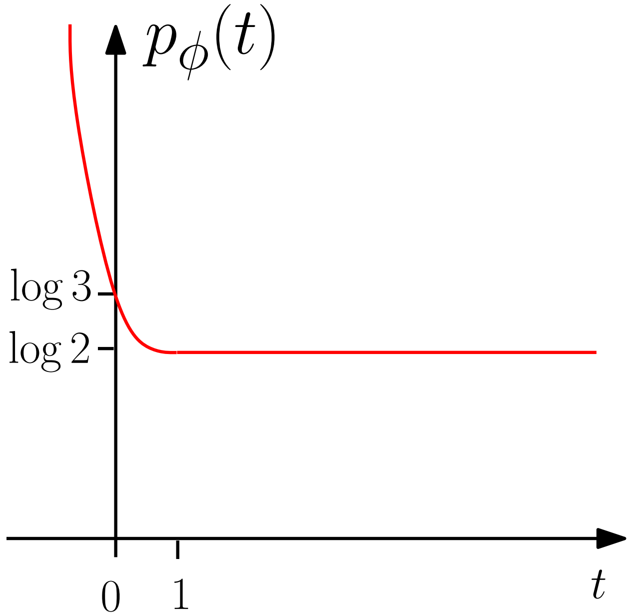
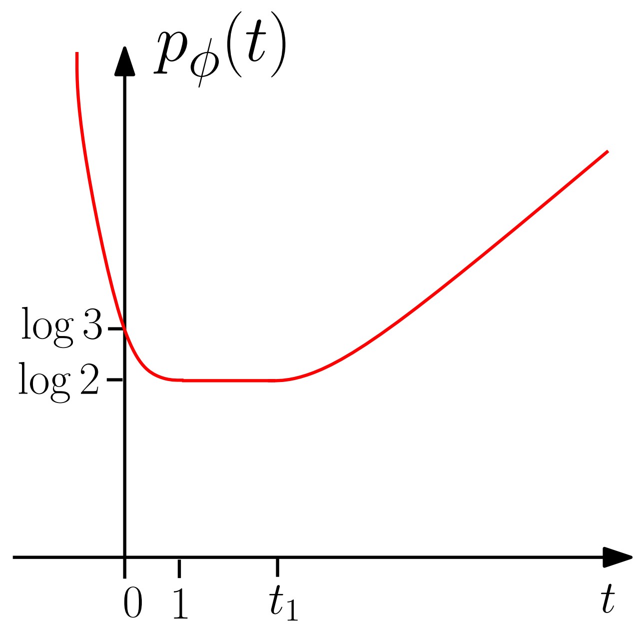


$p_\phi(t)$
$\log 3$
$\log 2$
0
1
t



$p_\phi(t)$
$\log 3$
$\log 2$
0
1
t_1
t