

SOME IMPROVEMENTS ON THE CONSTANTS FOR THE REAL BOHNENBLUST-HILLE INEQUALITY

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ABSTRACT. A classical inequality due to Bohnenblust and Hille states that for every $N \in \mathbb{N}$ and every m -linear mapping $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \rightarrow \mathbb{C}$ we have

$$\left(\sum_{i_1, \dots, i_m=1}^N |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|U\|,$$

where $C_m = 2^{\frac{m-1}{2}}$. The result is also true for real Banach spaces. In this note we show that an adequate use of a recent new proof of Bohnenblust-Hille inequality, due to Defant, Popa and Schwarting, combined with the optimal constants of Khinchine's inequality (due to Haagerup) provides quite better estimates for the constants involved in the real Bohnenblust-Hille inequality. For instance, for $2 \leq m \leq 14$, we show that the constants $C_m = 2^{\frac{m-1}{2}}$ can be replaced by $2^{\frac{m^2+6m-8}{8m}}$ if m is even and by $2^{\frac{m^2+6m-7}{8m}}$ if m is odd, which substantially improve the known values of C_m . We also show that the new constants present a better asymptotic behavior.

1. PRELIMINARIES AND BACKGROUND

In 1931, Bohnenblust and Hille ([2], or the more recent [6, 7]) asserted that for every positive integer N and every m -linear mapping $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \rightarrow \mathbb{C}$ we have

$$\left(\sum_{i_1, \dots, i_m=1}^N |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|U\|,$$

where $C_m = 2^{\frac{m-1}{2}}$ (actually this result also holds for real Banach spaces). The case $m = 2$ is a famous result known as Littlewood's 4/3-inequality. It seems that the Bohnenblust-Hille inequality was overlooked and was only re-discovered several decades later by Davie [4] and Kaijser [11]. While the exponent $\frac{2m}{m+1}$ is optimal, the constant $C_m = 2^{\frac{m-1}{2}}$ is not. Very recently, A. Defant and P. Sevilla-Peris [6, Section 4] indicated that by using Sawa's estimate for the constant of the complex Khinchine's inequality in Steinhaus variables (see [14]) it is possible to prove that $C_m \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ in the complex case (this is a strong improvement of the previous constants and it seems that these are the best known estimates for the complex case).

The (complex and real) Bohnenblust-Hille inequality can be re-written in the context of multiple summing multilinear operators, as we will see below.

Multiple summing multilinear mappings between Banach spaces is a recent, very important and useful nonlinear generalization of the concept of absolutely summing linear operators.

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This class was introduced, independently, by Matos [13] (under the terminology fully summing multilinear mappings) and Bombal, Pérez-García and Villanueva [3].

Throughout this paper X_1, \dots, X_m and Y will stand for Banach spaces over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and X' stands for the dual of X . By $\mathcal{L}(X_1, \dots, X_m; Y)$ we denote the Banach space of all continuous m -linear mappings from $X_1 \times \dots \times X_m$ to Y with the usual sup norm.

For x_1, \dots, x_n in X , let

$$\|(x_j)_{j=1}^n\|_{w,1} := \sup\{\|(\varphi(x_j))_{j=1}^n\|_1 : \varphi \in X', \|\varphi\| \leq 1\}.$$

If $1 \leq p < \infty$, an m -linear mapping $U \in \mathcal{L}(X_1, \dots, X_m; Y)$ is multiple $(p; 1)$ -summing (denoted $\Pi_{(p;1)}(X_1, \dots, X_m; Y)$) if there is a constant $L_m \geq 0$ such that

$$(1.1) \quad \left(\sum_{j_1, \dots, j_m=1}^N \|U(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)})\|^p \right)^{\frac{1}{p}} \leq L_m \prod_{k=1}^m \|(x_j^{(k)})_{j=1}^N\|_{w,1}$$

for every $N \in \mathbb{N}$ and any $x_{j_k}^{(k)} \in X_k$, $j_k = 1, \dots, N$, $k = 1, \dots, m$. The infimum of the constants satisfying (1.1) is denoted by $\|U\|_{\pi(p;1)}$. For $m = 1$ we have the classical concept of absolutely $(p; 1)$ -summing operators (see, e.g. [5, 8]).

A simple reformulation of Bohnenblust-Hille inequality asserts that every continuous m -linear form $T : X_1 \times \dots \times X_m \rightarrow \mathbb{K}$ is multiple $(\frac{2m}{m+1}; 1)$ -summing with $L_m = C_m = 2^{\frac{m-1}{2}}$ (or $L_m = (\frac{2}{\sqrt{\pi}})^{m-1}$ for the complex case, using the estimates of Defant and Sevilla-Peris, [6]).

However, in the real case the best constants known seem to be $C_m = 2^{\frac{m-1}{2}}$.

The main goal of this note is to obtain better constants for the Bohnenblust-Hille inequality in the real case. For this task we will use a recent proof of a general vector-valued version of Bohnenblust-Hille inequality ([7, Theorem 5.1]). The inequality of Bohnenblust-Hille is stated in [7, Corollary 2] as a consequence of [7, Theorem 5.1]. The procedure of the proof of [7, Corollary 2] allows us to obtain quite better values than $C_m = 2^{\frac{m-1}{2}}$. However, in this note we explore the ideas of [7] in a different way, in order to obtain even better estimates for the constants that can be derived from [7, Corollary 2]. The constants we obtain can be derived from [7, Theorem 5.1] through an adequate choice of variables.

Let us recall some results that we will need in this note. The first result is a well-known inequality due to Khinchine (see [8]):

Theorem 1.1 (Khinchine's inequality). *For all $0 < p < \infty$, there are constants A_p and B_p such that*

$$(1.2) \quad A_p \left(\sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left(\sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}$$

for all positive integer N and scalars a_1, \dots, a_n (here, r_n denotes the n -Rademacher function).

Above, it is clear that $B_2 = 1$. From (1.2) it follows that

$$(1.3) \quad \left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p A_r^{-1} \left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^r dt \right)^{\frac{1}{r}}$$

and the product of the constants $B_p A_r^{-1}$ will appear later on Theorem 1.3.

The notation of A_p and B_p will be kept along the paper. Now, let us recall a variation of an inequality due to Blei (see [7, Lemma 3.1]).

Theorem 1.2 (Blei, Defant et al.). *Let A and B be two finite non-void index sets, and $(a_{ij})_{(i,j) \in A \times B}$ a scalar matrix with positive entries, and denote its columns by $\alpha_j = (a_{ij})_{i \in A}$ and its rows by $\beta_i = (a_{ij})_{j \in B}$. Then, for $q, s_1, s_2 \geq 1$ with $q > \max(s_1, s_2)$ we have*

$$\left(\sum_{(i,j) \in A \times B} a_{ij}^{w(s_1, s_2)} \right)^{\frac{1}{w(s_1, s_2)}} \leq \left(\sum_{i \in A} \|\beta_i\|_q^{s_1} \right)^{\frac{f(s_1, s_2)}{s_1}} \left(\sum_{j \in B} \|\alpha_j\|_q^{s_2} \right)^{\frac{f(s_2, s_1)}{s_2}},$$

with

$$w : [1, q]^2 \rightarrow [0, \infty), \quad w(x, y) := \frac{q^2(x+y) - 2qxy}{q^2 - xy},$$

$$f : [1, q]^2 \rightarrow [0, \infty), \quad f(x, y) := \frac{q^2x - qxy}{q^2(x+y) - 2qxy}.$$

The following theorem is a particular case of [7, Lemma 2.2], for $Y = \mathbb{K}$, using that the cotype 2 constant of \mathbb{K} is 1, i.e., $C_2(\mathbb{K}) = 1$ (following the notation from [7]):

Theorem 1.3 (Defant et al.). *Let $1 \leq r \leq 2$, and let $(y_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^N$ be a matrix in \mathbb{K} . Then*

$$\left(\sum_{i_1, \dots, i_m=1}^N |y_{i_1 \dots i_m}|^2 \right)^{1/2} \leq (A_{2,r})^m \left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^N r_{i_1}(t_1) \dots r_{i_m}(t_m) y_{i_1 \dots i_m} \right|^r dt_1 \dots dt_m \right)^{1/r},$$

where

$$A_{2,r} \leq A_r^{-1} B_2 = A_r^{-1} \quad (\text{because } B_2 = 1).$$

The meaning of $A_{2,r}$, w and f from the above theorems will also be kept in the next section. Moreover, K_G will denote the complex Grothendieck constant.

2. IMPROVED CONSTANTS FOR THE BOHNENBLUST-HILLE THEOREM

The main results from [7], Theorem 5.1 and Corollary 5.2, are very interesting vector-valued generalizations of Bohnenblust-Hille inequality. In this note we explore the proof of [7, Theorem 5.1] in such a way that the constants obtained are better than those that can be derived from [7, Corollary 5.2]. We present here the proof of [7, Corollary 5.2] for the particular case of the Bohnenblust-Hille inequality, with some changes (inspired in [7, Theorem 5.1]) which improve the constants (this task makes the proof clearer and avoids technicalities from the arguments from [7, Theorem 5.1] in our particular case, with our particular choice of the variables selected).

Following the proof of [7, Corollary 5.2], and using the optimal values for the constants of Khinchine inequality (due to Haagerup) and K_G for $C_{\mathbb{C},2}$ (see [1]), the following estimates can be calculated for C_m :

$$C_{\mathbb{C},2} = K_G \leq 1,40491 < \sqrt{2},$$

$$C_{\mathbb{C},m} = 2^{\frac{m-1}{2m}} \left(\frac{C_{\mathbb{C},m-1}}{A_{\frac{2m-2}{m}}} \right)^{1-\frac{1}{m}} \quad \text{for } m \geq 3,$$

and

$$(2.1) \quad C_{\mathbb{R},2} = \sqrt{2},$$

$$(2.2) \quad C_{\mathbb{R},m} = 2^{\frac{m-1}{2m}} \left(\frac{C_{\mathbb{R},m-1}}{A^{\frac{2m-2}{m}}} \right)^{1-\frac{1}{m}} \quad \text{for } m \geq 3.$$

In particular, if $2 \leq m \leq 13$,

$$(2.3) \quad \begin{aligned} C_{\mathbb{C},m} &\leq 2^{\frac{m^2+m-6}{4m}} K_G^{2/m} \\ C_{\mathbb{R},m} &\leq 2^{\frac{m^2+m-2}{4m}}. \end{aligned}$$

Remark 2.1. *It worths to mention that the above constants are not explicitly calculated in [7]. Since our procedure (below) will provide better constants for the real case we will not detail the above estimates.*

For the complex case the estimates (2.2) are much better than $C_{\mathbb{C},m} = 2^{\frac{m-1}{2}}$ but worst than the constants $C_{\mathbb{C},m} = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ obtained by Defant and Sevilla-Peris [6]. However, a more appropriate use of some ideas from [7] can give better estimates for the real case, as we see in the following result.

Theorem 2.2. *For every positive integer m and every real Banach spaces X_1, \dots, X_m ,*

$$\Pi_{(\frac{2m}{m+1},1)}(X_1, \dots, X_m; \mathbb{R}) = \mathcal{L}(X_1, \dots, X_m; \mathbb{R}) \quad \text{and} \quad \|\cdot\|_{\pi(\frac{2m}{m+1},1)} \leq C_{\mathbb{R},m} \|\cdot\|$$

with

$$\begin{aligned} C_{\mathbb{R},2} &= 2^{\frac{1}{2}} \quad \text{and} \quad C_{\mathbb{R},3} = 2^{\frac{5}{6}}, \\ C_{\mathbb{R},m} &\leq 2^{\frac{1}{2}} \left(\frac{C_{\mathbb{R},m-2}}{A^{\frac{2m-4}{m-1}}} \right)^{\frac{m-2}{m}} \quad \text{for } m > 3. \end{aligned}$$

In particular, if $2 \leq m \leq 14$,

$$\begin{aligned} C_{\mathbb{R},m} &\leq 2^{\frac{m^2+6m-8}{8m}} \quad \text{if } m \text{ is even} \\ C_{\mathbb{R},m} &\leq 2^{\frac{m^2+6m-7}{8m}} \quad \text{if } m \text{ is odd.} \end{aligned}$$

Proof. The case $m = 2$ is Littlewood's 4/3 inequality. For $m = 3$ we have $C_3 = 2^{\frac{5}{6}}$ from (2.3)

The proof is done by induction, but the case m is obtained as a combination of the cases 2 with $m - 2$ instead of 1 and $m - 1$ as in [7, Corollary 5.2].

Suppose that the result is true for $m - 2$ and let us prove for m . Let $U \in \mathcal{L}(X_1, \dots, X_m; \mathbb{R})$ and N be a positive integer. For each $1 \leq k \leq m$ consider $x_1^{(k)}, \dots, x_N^{(k)} \in X_k$ so that $\left\| (x_j^{(k)})_{j=1}^N \right\|_{w,1} \leq 1$, $k = 1, \dots, m$.

Consider, in the notation of Theorem 1.2,

$$q = 2, \quad s_1 = \frac{4}{3}, \quad \text{and} \quad s_2 = \frac{2(m-2)}{(m-2)+1} = \frac{2m-4}{m-1}.$$

Thus,

$$w(s_1, s_2) = \frac{2m}{m+1}$$

and, from Theorem 1.2, we have

$$\begin{aligned} & \left(\sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} \leq \\ & \leq \left(\sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_{m-1}, i_m=1}^N \right\|_2^{\frac{2(m-2)}{(m-2)+1}} \right)^{f(s_2, \frac{4}{3}) / \frac{2(m-2)}{(m-2)+1}} \\ & \leq \left(\sum_{i_{m-1}, i_m=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \right)^{f(\frac{4}{3}, s_2)}. \end{aligned}$$

Now we need to estimate the two factors above. For simplicity, we write below $dt := dt_1 \dots dt_{m-2}$.

For each i_{m-1}, i_m fixed, we have (from Theorem 1.3),

$$\begin{aligned} & \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \leq \\ & \leq \left(A_{2, \frac{4}{3}}^{m-2} \right)^{4/3} \int_{[0,1]^{m-2}} \left| \sum_{i_1, \dots, i_{m-2}=1}^N r_{i_1}(t_1) \dots r_{i_{m-2}}(t_{m-2}) U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{4}{3}} dt \\ & = \left(A_{2, \frac{4}{3}}^{m-2} \right)^{4/3} \int_{[0,1]^{m-2}} \left| U \left(\sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m-2}=1}^N r_{i_{m-2}}(t_{m-2}) x_{i_{m-2}}^{(m-2)}, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)} \right) \right|^{\frac{4}{3}} dt. \end{aligned}$$

Summing over all $i_{m-1}, i_m = 1, \dots, N$ we obtain

$$\begin{aligned} & \sum_{i_{m-1}, i_m=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \leq \\ & \left(A_{2, \frac{4}{3}}^{m-2} \right)^{4/3} \int_{[0,1]^{m-2}} \sum_{i_{m-1}, i_m=1}^N \left| U \left(\sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m-2}=1}^N r_{i_{m-2}}(t_{m-2}) x_{i_{m-2}}^{(m-2)}, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)} \right) \right|^{\frac{4}{3}} dt. \end{aligned}$$

Using the case $m = 2$ we thus have

$$\begin{aligned} & \sum_{i_{m-1}, i_m=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \leq \\ & \leq \left(A_{2, \frac{4}{3}}^{m-2} \right)^{\frac{4}{3}} \int_{[0,1]^{m-2}} \left\| U \left(\sum_{i_1=1}^N r_{i_1}(t_1) x_{i_1}^{(1)}, \dots, \sum_{i_{m-2}=1}^N r_{i_{m-2}}(t_{m-2}) x_{i_{m-2}}^{(m-2)}, \dots \right) \right\|_{\pi(\frac{4}{3}; 1)}^{\frac{4}{3}} dt \\ & \leq \left(A_{2, \frac{4}{3}}^{m-2} \right)^{\frac{4}{3}} \int_{[0,1]^{m-2}} \left(\|U\| \sqrt{2} \right)^{\frac{4}{3}} dt \\ & = \left(A_{2, \frac{4}{3}}^{m-2} \right)^{\frac{4}{3}} \left(\|U\| \sqrt{2} \right)^{\frac{4}{3}}. \end{aligned}$$

Hence

$$\left(\sum_{i_{m-1}, i_m=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_1, \dots, i_{m-2}=1}^N \right\|_2^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq A_{2, \frac{4}{3}}^{m-2} \|U\| \sqrt{2}.$$

Next we obtain the other estimate. For each i_1, \dots, i_{m-2} fixed, and $dt := dt_{m-1} dt_m$, we have (from Theorem 1.3):

$$\begin{aligned} & \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_{m-1}, i_m=1}^N \right\|_2 \leq \\ & \leq A_{2, s_2}^2 \left(\int_{[0,1]^2} \left| \sum_{i_{m-1}, i_m=1}^N r_{i_{m-1}}(t_{m-1}) r_{i_m}(t_m) U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{s_2} dt \right)^{1/s_2} \\ & = A_{2, s_2}^2 \left(\int_{[0,1]^2} \left| U \left(x_{i_1}^{(1)}, \dots, x_{i_{m-2}}^{(m-2)}, \sum_{i_{m-1}=1}^N r_{i_{m-1}}(t_{m-1}) x_{i_{m-1}}^{(m-1)}, \sum_{i_m=1}^N r_{i_m}(t_m) x_{i_m}^{(m)} \right) \right|^{s_2} dt \right)^{1/s_2}. \end{aligned}$$

Summing over all $i_1, \dots, i_{m-2} = 1, \dots, N$ we get:

$$\begin{aligned} & \sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_m=1}^N \right\|_2^{s_2} \leq \\ & \leq A_{2, s_2}^{2s_2} \int_{[0,1]^2} \sum_{i_1, \dots, i_{m-2}=1}^N \left| U \left(x_{i_1}^{(1)}, \dots, x_{i_{m-2}}^{(m-2)}, \sum_{i_{m-1}=1}^N r_{i_{m-1}}(t_{m-1}) x_{i_{m-1}}^{(m-1)}, \sum_{i_m=1}^N r_{i_m}(t_m) x_{i_m}^{(m)} \right) \right|^{s_2} dt. \end{aligned}$$

We thus have, by the induction step,

$$\begin{aligned} & \sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_m=1}^N \right\|_2^{s_2} \leq \\ & \leq (A_{2, s_2}^2)^{s_2} \leq \int_{[0,1]} \left\| U \left(\cdot, \dots, \sum_{i_{m-1}=1}^N r_{i_{m-1}}(t_{m-1}) x_{i_{m-1}}^{(m-1)}, \sum_{i_m=1}^N r_{i_m}(t_m) x_{i_m}^{(m)} \right) \right\|_{\pi(s_2; 1)}^{s_2} dt \\ & \leq (A_{2, s_2}^2)^{s_2} \int_{[0,1]} C_{\mathbb{R}, m-2}^{s_2} \|U\|^{s_2} dt = (A_{2, s_2}^2)^{s_2} C_{\mathbb{R}, m-2}^{s_2} \|U\|^{s_2} \end{aligned}$$

and so

$$\left(\sum_{i_1, \dots, i_{m-2}=1}^N \left\| \left(U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right)_{i_m=1}^N \right\|_2^{s_2} \right)^{1/s_2} \leq (A_{2, s_2}^2) C_{\mathbb{R}, m-2} \|U\|.$$

Hence, combining both estimates, we obtain

$$\left(\sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} \leq \left[A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \|U\| \right]^{f(\frac{4}{3}, s_2)} \left[(A_{2, s_2}^2) C_{\mathbb{R}, m-2} \|U\| \right]^{f(s_2, \frac{4}{3})}.$$

Although,

$$f\left(\frac{4}{3}, s_2\right) = \frac{4\frac{4}{3} - 2\frac{4}{3} \left(\frac{2m-4}{m-1}\right)}{4 \left(\frac{4}{3} + \frac{2m-4}{m-1}\right) - 4\frac{4}{3} \left(\frac{2m-4}{m-1}\right)} = \frac{2}{m},$$

$$f\left(s_2, \frac{4}{3}\right) = \frac{4 \left(\frac{2m-2}{m}\right) - 2 \left(\frac{2m-2}{m}\right)}{4 \left(1 + \frac{2m-2}{m}\right) - 4 \left(\frac{2m-2}{m}\right)} = 1 - \frac{2}{m},$$

and, therefore

$$\begin{aligned} \left(\sum_{i_1, \dots, i_m=1}^N \left| U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \right|^{\frac{2m}{m+1}} \right)^{(m+1)/2m} &\leq \left[A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \|U\| \right]^{\frac{2}{m}} \left[\left(A_{2, \frac{2m-4}{m-1}}^2 \right) C_{\mathbb{R}, m-2} \|U\| \right]^{1-\frac{2}{m}} \\ &= \left[A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \right]^{\frac{2}{m}} \left(A_{2, \frac{2m-4}{m-1}}^2 \right)^{1-\frac{2}{m}} C_{\mathbb{R}, m-2}^{1-\frac{2}{m}} \|U\| \\ &= \left[A_{2, \frac{4}{3}}^{m-2} \sqrt{2} \right]^{\frac{2}{m}} \left(A_{2, \frac{2m-4}{m-1}}^2 C_{\mathbb{R}, m-2} \right)^{1-\frac{2}{m}} \|U\| \\ &= 2^{\frac{1}{m}} \left(A_{2, \frac{4}{3}} A_{2, \frac{2m-4}{m-1}} \right)^{\frac{2m-4}{m}} \left(C_{\mathbb{R}, m-2} \right)^{\frac{m-2}{m}} \|U\|. \end{aligned}$$

Now let us estimate the constants $C_{\mathbb{R}, m}$. We know that $B_2 = 1$ and, from [9], we also know that $A_p = 2^{\frac{1}{2} - \frac{1}{p}}$ whenever $p \leq 1.847$. So, for $2 \leq m \leq 14$ we have

$$A_{\frac{2m-4}{m-1}} = 2^{\frac{1}{2} - \frac{m-1}{2m-4}}.$$

Hence, from (1.3) and using the best constants of Khinchine's inequality from [9], we have

$$A_{2, \frac{4}{3}} \leq A_{\frac{4}{3}}^{-1} = 2^{\frac{3}{4} - \frac{1}{2}},$$

$$A_{2, \frac{2m-4}{m-1}} \leq A_{\frac{2m-4}{m-1}}^{-1} = 2^{\frac{m-1}{2m-4} - \frac{1}{2}},$$

and

$$C_{\mathbb{R}, m} \leq 2^{\frac{1}{m}} \left(\left(2^{\frac{3}{4} - \frac{1}{2}} \right) \left(2^{\frac{m-1}{2m-4} - \frac{1}{2}} \right) \right)^{\frac{2m-4}{m}} \left(C_{\mathbb{R}, m-2} \right)^{1-\frac{2}{m}} = 2^{\frac{m+2}{2m}} \left(C_{\mathbb{R}, m-2} \right)^{1-\frac{2}{m}}$$

and we obtain, if $2 \leq m \leq 14$,

$$C_{\mathbb{R}, m} \leq 2^{\frac{m^2+6m-8}{8m}} \text{ if } m \text{ is even,}$$

$$C_{\mathbb{R}, m} \leq 2^{\frac{m^2+6m-7}{8m}} \text{ if } m \text{ is odd.}$$

In general we easily get

$$C_{\mathbb{R}, m} \leq 2^{\frac{1}{2}} \left(\frac{C_{\mathbb{R}, m-2}}{A_{\frac{2m-4}{m-1}}^2} \right)^{\frac{m-2}{m}}.$$

The numerical values of $C_{\mathbb{R}, m}$, for $m > 14$, can be easily calculated by using the exact values of $A_{\frac{2m-4}{m-1}}$, when $m > 14$ (see [9]):

$$A_{\frac{2m-4}{m-1}} = \sqrt{2} \left(\frac{\Gamma \left(\frac{\frac{2m-4}{m-1} + 1}{2} \right)}{\sqrt{\pi}} \right)^{(m-1)/(2m-4)}.$$

□

In the below table we compare the first constants $C_m = 2^{\frac{m-1}{2}}$ and the constants that can be derived from [7, Cor. 5.2] with the *new* constants $C_{\mathbb{R},m}$:

	New Constants	Constants from [7, Cor. 5.2]	$C_m = 2^{\frac{m-1}{2}}$
$m = 3$	$2^{20/24} \approx 1.782$	$2^{5/6} \approx 1.782$	$2^{2/2} = 2$
$m = 4$	$2^{32/32} = 2$	$2^{\frac{18}{16}} \approx 2.18$	$2^{3/2} \approx 2.828$
$m = 5$	$2^{\frac{48}{40}} \approx 2.298$	$2^{\frac{36}{20}} \approx 2.639$	$2^2 = 4$
$m = 6$	$2^{\frac{64}{48}} \approx 2.520$	$2^{\frac{40}{24}} \approx 3.17$	$2^{5/2} \approx 5.656$
$m = 7$	$2^{\frac{84}{56}} \approx 2.828$	$2^{\frac{54}{28}} \approx 3.807$	$2^{6/2} = 8$
$m = 8$	$2^{\frac{104}{64}} \approx 3.084$	$2^{\frac{70}{32}} \approx 4.555$	$2^{7/2} \approx 11.313$
$m = 9$	$2^{\frac{128}{72}} \approx 3.429$	$2^{\frac{88}{36}} \approx 5.443$	$2^{8/2} = 16$
$m = 10$	$2^{\frac{152}{80}} \approx 3.732$	$2^{\frac{108}{40}} \approx 6.498$	$2^{9/2} \approx 22.627$
$m = 11$	$2^{\frac{180}{88}} \approx 4.128$	$2^{\frac{130}{44}} \approx 7.752$	$2^{10/2} = 32$
$m = 12$	$2^{\frac{208}{96}} \approx 4.490$	$2^{\frac{154}{48}} \approx 9.243$	$2^{11/2} \approx 45.254$
$m = 13$	$2^{\frac{240}{104}} \approx 4.951$	$2^{\frac{180}{52}} \approx 11.016$	$2^{12/2} = 64$
$m = 14$	$2^{\frac{272}{112}} \approx 5.383$	$2^{\frac{13}{28}} \left(\frac{2^{\frac{180}{52}}}{A_{\frac{26}{14}}^2} \right)^{1 - \frac{1}{14}} \approx 13.457$	$2^{13/2} \approx 90.509$

In the last line of the above table we have used that $A_{\frac{26}{14}} = \sqrt{2} \left(\frac{\Gamma\left(\frac{\frac{26}{14}+1}{2}\right)}{\sqrt{\pi}} \right)^{14/26} \approx 0.9736$.

3. COMPARING THE ASYMPTOTIC BEHAVIOR OF THE DIFFERENT CONSTANTS

In this final section we show that the new constants obtained present a better asymptotic behavior than the previous (including the constants derived from [7, Cor. 5.2]).

We have seen that

$$C_{\mathbb{R},m} \leq 2^{\frac{1}{2}} \left(\frac{C_{\mathbb{R},m-2}}{A_{\frac{2m-4}{m-1}}^2} \right)^{\frac{m-2}{m}}.$$

As $m \rightarrow \infty$ we know that $A_{\frac{2m-4}{m-1}}$ increases to 1. So,

$$\limsup \frac{C_{\mathbb{R},m}}{(C_{\mathbb{R},m-2})^{\frac{m-2}{m}}} \leq 2^{\frac{1}{2}}.$$

For the original constants $C_m = 2^{\frac{m-1}{2}}$ we have

$$\frac{C_m}{(C_{m-2})^{\frac{m-2}{m}}} = 2^{\frac{2m-3}{m}}$$

and thus

$$\lim \frac{C_m}{(C_{m-2})^{\frac{m-2}{m}}} = 4.$$

and for the constants from [7, Cor. 5.2] a similar calculation shows us that $2^{\frac{1}{2}}$ is replaced by 2 in (3).

In resume, the new constants, besides smaller than the others, are those which have the best asymptotic behavior.

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