SOME PROPERTIES AND APPLICATIONS OF F-FINITE F-MODULES

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1. Introduction

The purpose of this paper is to describe several applications of finiteness properties of F-finite F-modules recently discovered by M. Hochster in [H] to the study of Frobenius maps on injective hulls, Frobenius near-splittings and to the nature of morphisms of F-finite F-modules.

Throughout this paper (R, m) shall denote a complete regular local ring of prime characteristic p. At the heart of everything in this paper is the Frobenius map $f : R \to R$ given by $f(r) = r^p$ for $r \in R$. We can use this Frobenius map to define a new R-module structure on R given by $r \cdot s = r^p s$; we denote this R-module F_*R. We can then use this to define the *Frobenius functor* from the category of R-modules to itself: given an R-module M we define $F(M)$ to be $F_*R \otimes_R M$ with R-module structure given by $r(s \otimes m) = rs \otimes m$ for $r, s \in R$ and $m \in M$..

Let $R[\Theta; f]$ be the skew polynomial ring which is a free R-module $\oplus_{i=0}^{\infty} R\Theta^i$ with multiplication $\Theta r = r^p \Theta$ for all $r \in R$. As in [K1], $\mathcal{C}$ shall denote the category $R[\Theta; f]$-modules which are Artinian as R-modules. For any two such modules M, N, we denote the morphisms between them in $\mathcal{C}$ with $\operatorname{Hom}_{R[\Theta;f]}(M, N)$; thus an element $g \in \operatorname{Hom}_{R[\Theta;f]}(M, N)$ is an R-linear map such that $g(\Theta a) = \Theta g(a)$ for all $a \in M$. The first main result of this paper (Theorem 3.3) shows that under some conditions on N, $\operatorname{Hom}_{R[\Theta;f]}(M, N)$ is a finite set.

An *F-module* (cf. the seminal paper [L] for an introduction to F-modules and their properties) over the ring R is an R-module $\mathcal{M}$ together with an R-module isomorphism $\theta_{\mathcal{M}} : \mathcal{M} \to F(\mathcal{M})$. This isomorphism $\theta_{\mathcal{M}}$ is the *structure morphism* of $\mathcal{M}$.

A *morphism of F-modules* $\mathcal{M} \to \mathcal{N}$ is an R-linear map g which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\;g\;} & \mathcal{N} \\ \downarrow{\scriptstyle \theta_{\mathcal{M}}} & & \downarrow{\scriptstyle \theta_{\mathcal{N}}} \\ F(\mathcal{M}) & \xrightarrow{F(g)} & F(\mathcal{N}) \end{array}$$

where $\theta_{\mathcal{M}}$ and $\theta_{\mathcal{N}}$ are the structure isomorphisms of $\mathcal{M}$ and $\mathcal{N}$, respectively. We denote $\operatorname{Hom}_{\mathcal{F}}(\mathcal{M}, \mathcal{N})$ the R-module of all *morphism of F-modules* $\mathcal{M} \to \mathcal{N}$

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Given any finitely generated R-module M and R-linear map $\beta : M \to F(M)$ one can obtain an R-module

$$\mathcal{M} = \varinjlim \left(M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} \dots \right).$$

Since

$$F(\mathcal{M}) = \varinjlim \left(F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} F^3(M) \xrightarrow{F^3(\beta)} \dots \right) = \mathcal{M}$$

we obtain an isomorphism $\mathcal{M} \cong F(\mathcal{M})$, and hence $\mathcal{M}$ is an F-module. Any F-module which can be constructed as a direct limit as $\mathcal{M}$ above is called an *F-finite* F-module with *generating morphism* β.

There is a close connection between $R[\Theta; f]$-modules and F-finite F-modules given by *Lyubeznik's Functor* from $\mathcal{C}$ to the category of F-finite F-modules which is defined as follows (see section 4 in [L] for the details of the construction.) Given an $R[\Theta; f]$-module M one defines the R-linear map $\alpha : F(M) \to M$ by $\alpha(r\Theta \otimes m) = r\Theta m$; an application of Matlis duality then yields an R-linear map $\alpha^\vee : M^\vee \to F(M)^\vee \cong F(M^\vee)$ and one defines

$$\mathcal{H}(M) = \varinjlim \left(M^\vee \xrightarrow{\alpha^\vee} F(M^\vee) \xrightarrow{F(\alpha^\vee)} F^2(M^\vee) \xrightarrow{F^2(\alpha^\vee)} \dots \right).$$

Since M is an Artinian R-module, $M^\vee$ is finitely generated and $\mathcal{H}(M)$ is an F-finite F-module with generating morphism $M^\vee \xrightarrow{\alpha^\vee} F(M^\vee)$. This construction is functorial and results in an exact covariant functor from $\mathcal{C}$ to the category of F-finite F-modules.

The main result in [H] is the surprising fact that for F-finite F-modules $\mathcal{M}$ and $\mathcal{N}$, $\operatorname{Hom}_{\mathcal{F}}(\mathcal{N}, \mathcal{M})$ is a finite set. In section 3 of this paper we exploit this fact to prove the second main result in this paper (Theorem 3.4) to show the following. Let $\gamma : M \to F(M)$ and $\beta : N \to F(N)$ be generating morphisms for $\mathcal{N}$ and $\mathcal{M}$. Given an R-linear map g which makes the following diagram commute,

$$\begin{array}{ccc} N & \xrightarrow{\beta} & F(N) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} \\ M & \xrightarrow{\gamma} & F(M) \end{array}$$

one can extend that diagram to

$$\begin{array}{ccccccc} N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} & \cdots \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} & & \downarrow{\scriptstyle F(g)} & & \\ M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} & \cdots \end{array}$$

and obtain a map between the direct limits of the horizontal sequences, i.e., an element in $\operatorname{Hom}_{\mathcal{F}}(\mathcal{N}, \mathcal{M})$. We prove that all elements in $\operatorname{Hom}_{\mathcal{F}}(\mathcal{N}, \mathcal{M})$ arise in this way (cf. Theorem 3.4), thus morphisms of F-finite F-modules have a particularly simple form. This answers a question implicit in [L, Remark 1.10(b)].

Finally, in section 4 we consider the module $\operatorname{Hom}_R(F_*R^n, R^n)$ of *near-splittings* of F_*R^n. We establish a correspondence between these near-splittings and Frobenius actions on E^n which enables us to prove the third main result in this paper (Theorem 4.5) which asserts that given a near-splitting ϕ corresponding to a injective Frobenius actions, there are finitely many F_*R-submodules $V \subseteq F_*R^n$ such that $\phi(V) \subseteq V$. This generalizes a similar result in [BB] to the case where R is not F-finite.

Our study of Frobenius near-splittings is based on the study of its dual notion, i.e., Frobenius maps on the injective hull $E = E_R(R/m)$ of the residue field of R. This injective hull is given explicitly as the module of inverse polynomials $\mathbb{K}[\![x_1^-, \dots, x_d^-]\!]$ where $x_1, \dots, x_d$ are minimal generators of the maximal ideal of R (cf. [BS, §12.4].) Thus E has a natural $R[T; f]$-module structure extending $T\lambda x_1^{-\alpha_1} \dots x_1^{-\alpha_d} = \lambda^p x_1^{-p\alpha_1} \dots x_1^{-p\alpha_d}$ for $\lambda \in \mathbb{K}$ and $\alpha_1, \dots, \alpha_d > 0$. We can further extend this to a natural $R[T; f]$-module structure on E^n given by

$$T \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Ta_1 \\ \vdots \\ Ta_n \end{pmatrix}.$$

The results of section 4 will follow from the fact that there is a dual correspondence between Frobenius near-splittings and sets of $R[\Theta; f]$-module structures on E^n.

2. Frobenius maps of Artinian modules and their stable submodules

Given an Artinian R-module M we can embed M in E^α for some $\alpha \geq 0$ and extend this inclusion to an exact sequence

$$0 \to M \to E^\alpha \xrightarrow{A^t} E^\beta \to \dots$$

where $A^t \in \operatorname{Hom}_R(E_R^\alpha, E_R^\beta) \cong \operatorname{Hom}_R(R^\alpha, R^\beta)$ is a $\beta \times \alpha$ matrix with entries in R. Henceforth in this section we will describe certain properties of Artinian R-modules in terms of their representations as kernels of matrices with entries in R. We shall denote $\mathbf{M}_{\alpha,\beta}$ the set of $\alpha \times \beta$ matrices with entries in R.

In this section and the next we will need the following constructions. Following [K1] we shall denote the category of Artinian $R[\Theta; f]$-modules $\mathcal{C}$. We denote $\mathcal{D}$ the category of R-linear maps $M \to F_R(M)$ where M is a finitely generated R-module, $F_R(-)$ denotes the Frobenius functor, and where a morphism between $M \xrightarrow{a} F_R(M)$ and $N \xrightarrow{b} F_R(N)$ is a commutative diagram of R-linear maps

$$\begin{array}{ccc} M & \xrightarrow{\mu} & N \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ F_R(M) & \xrightarrow{F_R(\mu)} & F_R(N) \end{array}$$

Section 3 of [K1] constructs a pair of functors $\Delta : \mathcal{C} \to \mathcal{D}$ and $\Psi : \mathcal{D} \to \mathcal{C}$ with the property that for all $A \in \mathcal{C}$, the $R[\Theta; f]$-module $\Psi \circ \Delta(A)$ is canonically isomorphic to A and for all $D = (B \xrightarrow{u} F_R(B)) \in \mathcal{D}$, $\Delta \circ \Psi(D)$ is canonically isomorphic to D. The functor Δ amounts to the "first step" in the construction of Lyubeznik's functor $\mathcal{H}$: for $A \in \mathcal{C}$ we define the R-linear map $\alpha : F(A) \to A$ to be the one given by $\alpha(r\Theta \otimes a) = r\Theta a$ and we let $\Delta(A)$ to be the map $\alpha^\vee : A^\vee \to F(A)^\vee \cong F(A^\vee)$ (cf. section 3 in [K1] for the details of the construction.)

Proposition 2.1. *Let $M = \ker A^t \subseteq E^\alpha$ be an Artinian R-module where $A \in \mathbf{M}_{\alpha,\beta}$. Let $\mathbf{B} = \left\{B \in \mathbf{M}_{\alpha,\alpha} \mid \operatorname{Im} BA \subseteq \operatorname{Im} A^{[p]}\right\}$. For any $R[\Theta; f]$-module structure on M, $\Delta(M)$ can be identified with an element in $\operatorname{Hom}_R(\operatorname{Coker} A, \operatorname{Coker} A^{[p]})$ and thus represented by multiplication by some $B \in \mathbf{B}$. Conversely, any such B defines an $R[\Theta; f]$-module structure on M which is given by the restriction to M of the Frobenius map $\phi : E^\alpha \to E^\alpha$ defined by $\phi(v) = B^t T(v)$ where T is the natural Frobenius map on E^α.*

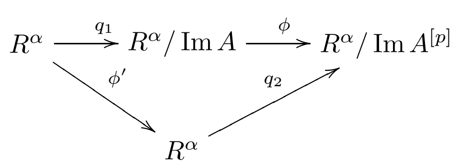
Proof. Matlis duality gives an exact sequence $R^\beta \xrightarrow{A} R^\alpha \to M^\vee \to 0$ hence

$$\Delta(M) \in \operatorname{Hom}_R(M^\vee, F_R(M^\vee)) \cong \operatorname{Hom}_R(\operatorname{Coker} A, \operatorname{Coker} A^{[p]}).$$

Let $\Delta(M)$ be the map $\phi : \operatorname{Coker} A \to \operatorname{Coker} A^{[p]}$.

In view of Theorem 3.1 in [K1] we only need to show that any such R-linear map is given by multiplication by an $B \in \mathbf{B}$, and that any such B defines an element in $\Delta(M)$.

We can find a map ϕ' which makes the following diagram



commute, where q_1 and q_2 are quotient maps. The map ϕ' is given by multiplication by some $\alpha \times \alpha$ matrix $B \in \mathbf{B}$. Conversely, any such matrix B defines a map ϕ making the diagram above commute, and $\Psi(\phi)$ gives a $R[\Theta; f]$-module structure on M as described in the last part of the theorem. □

Notation 2.2. We shall henceforth describe Artinian R-modules with a given $R[\Theta; f]$-module structure in terms of the two matrices in the statement of Proposition 2.1 and talk about Artinian R-modules $M = \operatorname{Ker} A^t \subseteq E^\alpha$ where $A \in \mathbf{M}_{\alpha\beta}$ with $R[\Theta; f]$-module structure given by $B \in \mathbf{M}_{\alpha\alpha}$.

3. Morphisms in $\mathcal{C}$

In this section we raise two questions. The first of these asks when for given $R[\Theta; f]$-modules M, N, the set $\operatorname{Hom}_{R[\Theta;f]}(M, N)$ is finite; later in this section we prove that this holds when N has no Θ-torsion. The following two examples illustrate why this set is not finite in general, and why it is finite in a special simple case.

Example 3.1. Let $\mathbb{K}$ be an infinite field of prime characteristic p and let $R = \mathbb{K}[\![x]\!]$. Let $M = \operatorname{ann}_E xR$ and fix an $R[\Theta; f]$-module structure on M given by $\Theta a = x^p T a$ where T is the standard Frobenius action on E. Note that $\Theta M = 0$ and that for all $\lambda \in \mathbb{K}$ the map

$\mu_\lambda : M \to M$ given by multiplication by λ is in $\operatorname{Hom}_{R[\Theta;f]}(M,M)$, and hence this set is infinite.

Example 3.2. Let $I, J \subseteq R$ be ideals, and fix $u \in (I^{[p]} : I)$ and $v \in (J^{[p]} : J)$. Endow $\operatorname{ann}_E I$ and $\operatorname{ann}_E J$ with $R[\Theta; f]$-module structures given by $\Theta a = uTa$ and $\Theta b = vTb$ for $a \in \operatorname{ann}_E I$ and $b \in \operatorname{ann}_E J$ where T is the standard Frobenius map on E.

If $g : \operatorname{ann}_E I \to \operatorname{ann}_E J$ is R-linear, an application of Matlis duality yields $g^\vee : R/J \to R/I$ and we deduce that g is given by multiplication by an element in $w \in (I : J)$. If in addition $g \in \operatorname{Hom}_{R[\Theta;f]}(\operatorname{ann}_E I, \operatorname{ann}_E J)$, we must have $wuTa = g(\Theta a) = \Theta g(a) = vTwa = vw^pTa$, for all $a \in \operatorname{ann}_E I$, hence $(vw^p - uw)T \operatorname{ann}_E I = 0$ and $vw^p - uw \in I^{[p]}$. The finiteness of $\operatorname{Hom}_{R[\Theta;f]}(\operatorname{ann}_E I, \operatorname{ann}_E J)$ translates in this setting to the finiteness of the set of solutions for the variable w of the equation above, and it is not clear why this set should be finite. However, if we simplify to the case where $I = 0$, the set of solutions of $vw^p - uw = 0$ over the the fraction field of R has at most p elements, and in this case we can deduce that $\operatorname{Hom}_{R[\Theta;f]}(E, \operatorname{ann}_E J)$ has at most p elements.

As in [L], for any $R[\Theta; f]$-module M we define the *submodule of nilpotent elements* to be $\operatorname{Nil}(M) = \{a \in M \,|\, \Theta^e a = 0 \text{ for some } e \geq 0\}$. We recall that when M is an Artinian R-module there exists an $\eta \geq 0$ such that $\Theta^\eta M = 0$ (cf. [HS, Proposition 1.11] and [L, Proposition 4.4].) We also define $M_{\text{red}} = M/\operatorname{Nil}(M)$ and $M^* = \cap_{e\geq 0} R\Theta^e M$ where $R\Theta^e M$ denotes the R-module generated by $\{\Theta^e a \,|\, a \in M\}$. We also note that when M is an $R[\Theta; f]$-module which is Artinian as an R-module, there exists an $e \geq 0$ such that $M^* = R\Theta^e M$ and also $(M_{\text{red}})^* = (M^*)_{\text{red}}$ (cf. section 4 in [K2].)

Theorem 3.3. *Let M, N be $R[\Theta; f]$-modules and let $\phi \in \operatorname{Hom}_{R[\Theta;f]}(M,N)$. We have $\mathcal{H}(\operatorname{Im}\phi) = 0$ if and only if $\phi(M) \subseteq \operatorname{Nil}(N)$ and, consequently, if $\operatorname{Nil}(N) = 0$, the map $\mathcal{H} : \operatorname{Hom}_{R[\Theta;f]}(M,N) \to \operatorname{Hom}_{\mathcal{F}_R}(\mathcal{H}(N), \mathcal{H}(M))$ is an injection and $\operatorname{Hom}_{R[\Theta;f]}(M,N)$ is a finite set.*

Proof. We apply $\mathcal{H}$ to the commutative diagram

$$\begin{array}{ccc} M & & \\ {\scriptstyle\phi}\downarrow & \searrow^{\phi} & \\ \operatorname{Im}\phi & \hookrightarrow & N \end{array}$$

to obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{H}(N) & \twoheadrightarrow & \mathcal{H}(\operatorname{Im}\phi) \\ & {\scriptstyle\mathcal{H}(\phi)}\searrow & \downarrow \\ & & \mathcal{H}(M) \end{array}.$$

Now $\mathcal{H}(\phi) = 0$ if and only if $\mathcal{H}(\operatorname{Im}\phi) = 0$, and by [L, Theorem 4.2] this is equivalent to $(\operatorname{Im}\phi)^*_{\text{red}} = 0$.

Choose $\eta \geq 0$ such that $\Theta^\eta \operatorname{Nil}(N) = 0$ and choose $e \geq 0$ such that $(\operatorname{Im}\phi)^* = R\Theta^e \operatorname{Im}\phi$.

Now

$$\begin{aligned}(\operatorname{Im}\phi)^*_{\mathrm{red}} = 0 \quad &\Leftrightarrow \quad R\Theta^{\eta} R\Theta^{e} \phi(M) = 0\\ &\Leftrightarrow \quad R\Theta^{\eta+e}\phi(M) = 0\\ &\Leftrightarrow \quad \operatorname{Im}\phi \subseteq \operatorname{Nil}(N)\end{aligned}$$

The second statement now follows immediately. □

The second main result in this section, Theorem 3.4 shows that all morphisms of F-finite F-modules arise as images of maps of $R[\Theta; f]$-modules under Lyubeznik's functor $\mathcal{H}$.

Theorem 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be F-finite F-modules. For every $\phi \in \operatorname{Hom}_{\mathcal{F}_R}(\mathcal{N}, \mathcal{M})$ there exist generating morphisms $\gamma : M \to F(M) \in \mathcal{D}$ and $\beta : N \to F(N) \in \mathcal{D}$ for $\mathcal{M}$ and $\mathcal{N}$, respectively, and a morphism (in the category $\mathcal{D}$)*

$$\begin{array}{ccc} N & \xrightarrow{\beta} & F(N) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} \\ M & \xrightarrow{\gamma} & F(M) \end{array}$$

such that $\phi = \mathcal{H}(\Psi(g))$.

Proof. Choose any generating morphisms

$$\mathcal{N} = \varinjlim \left(N \xrightarrow{\beta} F(N) \xrightarrow{F(\beta)} F^2(N) \xrightarrow{F^2(\beta)} \dots \right)$$

and

$$\mathcal{M} = \varinjlim \left(M \xrightarrow{\gamma} F(M) \xrightarrow{F(\gamma)} F^2(M) \xrightarrow{F^2(\gamma)} \dots \right)$$

and fix any $\phi \in \operatorname{Hom}_{\mathcal{F}_R}(\mathcal{N}, \mathcal{M})$.

For all $j \geq 0$ let ϕ_j be the restriction of ϕ to the image of $F^j(N)$ in $\mathcal{N}$.

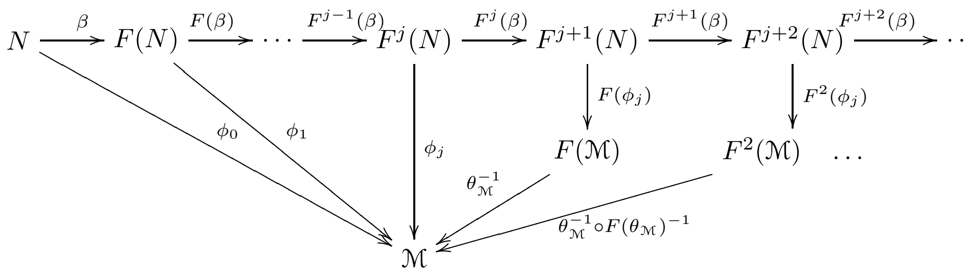
The fact that ϕ is a morphism of F-modules implies that for every $j \geq 0$ we have a commutative diagram

$$\begin{array}{ccc} F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) \\ \downarrow & & \downarrow \\ \mathcal{N} & \xrightarrow{\theta_{\mathcal{N}}} & F(\mathcal{N}) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle F(\phi)} \\ \mathcal{M} & \underset{\theta_{\mathcal{M}}}{\xrightarrow{\cong}} & F(\mathcal{M}) \end{array}$$

where $\theta_{\mathcal{M}}$ and $\theta_{\mathcal{N}}$ are the structure isomorphims of $\mathcal{M}$ and $\mathcal{N}$, respectively, and where the compositions of the vertical maps are ϕ_j and $F(\phi_j)$. Repeated applications of the Frobenius functor yields a commutative diagram

$$\begin{array}{ccccc}
F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) & \xrightarrow{F^{j+1}(\beta)} & \cdots \\
\downarrow{\scriptstyle \phi_j} & & \downarrow{\scriptstyle F(\phi_j)} & & \\
\mathcal{M} & \xrightarrow[\theta_{\mathcal{M}}]{\cong} & F(\mathcal{M}) & \xrightarrow[F(\theta_{\mathcal{M}})]{\cong} & \cdots
\end{array}$$

and we can now extend this commutative diagram to the left to obtain



This commutative diagram defines a R-linear map $\psi_j : \mathcal{N} \to \mathcal{M}$. Furthermore, we show next that this ψ_j is a map of $\mathcal{F}$-modules, i.e., that for all $j \geq 0$, $F(\psi_j) \circ \theta_{\mathcal{N}} = \theta_{\mathcal{M}} \circ \psi_j$. Fix $j \geq 0$ and abbreviate $\psi = \psi_j$.

Pick any $a \in \mathcal{N}$ represented as an element of $F^e(N)$. If $e < j$ then the fact that ϕ is a morphism of F-modules. implies that

$$\theta_{\mathcal{M}} \circ \psi(a) = \theta_{\mathcal{M}} \circ \phi(a) = F(\phi) \circ \theta_{\mathcal{N}}(a) = F(\psi) \circ \theta_{\mathcal{N}}(a).$$

Assume now that $e \geq j$; we have

$$\begin{aligned}
\theta_{\mathcal{M}} \circ \psi(a) &= \theta_{\mathcal{M}} \circ \theta_{\mathcal{M}}^{-1} \circ F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\phi_j)(a) \\
&= F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\phi_j)(a)
\end{aligned}$$

and

$$\begin{aligned}
F(\psi) \circ \theta_{\mathcal{N}}(a) &= F\left(\theta_{\mathcal{M}}^{-1} \circ F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\phi_j)\right)(F^e(\beta)(a)) \\
&= F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e+1-j}(\phi_j)\,(F^e(\beta)(a)) \\
&= F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\theta_{\mathcal{M}}) \circ F^{e-j}(\phi_j)(a) \\
&= F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\phi_j)(a)
\end{aligned}$$

where the penultimate inequality follows from the fact that ϕ is a morphism of F-modules.

Consider now the set $\{\psi_i\}_{i\geq 0}$; it is a finite set according to Theorem 5.1 in [H], hence we can find a sequence $0 \leq i_1 < i_2 < \dots$ such that $\psi_{i_1} = \psi_{i_2} = \dots$. By replacing $\mathcal{N}$ and $\mathcal{M}$ with $F^{i_1}(\mathcal{N})$ and $F^{i_1}(\mathcal{M})$ we may assume that $i_1 = 0$.

Pick $j \geq 0$ so that $\phi(N) \subseteq F^j(M)$. Since $\mathcal{M} \cong F^j(\mathcal{M})$ we may replace $\mathcal{M}$ with $F^j(\mathcal{M})$ and assume that $\phi(N) \subseteq M$ and hence also that for all $e \geq 0$, $F^e(\phi)(F^e(N)) \subseteq F^e(M)$.

Fix now any $e \geq 0$ and pick any $i_k > e$; the fact that $\psi_0 = \psi_{i_k}$ implies that for all $a \in F^e(N)$, $F^e(\phi_0)(a) = \psi_0(a) = \psi_{i_k}(a) = \phi(a)$ and since this holds for all $e \geq 0$ we deduce that ϕ is induced from the commutative diagram

$$\begin{array}{ccccccc} N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} & \cdots \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle F(\phi_0)} & & \downarrow{\scriptstyle F^2(\phi_0)} & & \\ M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} & \cdots \end{array}$$

An application of the functor Ψ to the leftmost square in the commutative diagram above yields a morphism of $R[\Theta; f]$-modules $g : M \to N$ and $\phi = \mathcal{H}(g)$. □

4. Applications to Frobenius splittings

For any R-module M let F_*M denote the additive Abelian group M with R-module structure given by $r \cdot a = r^p a$ for all $r \in R$ and $a \in M$. In this section we study the module $\operatorname{Hom}_R(F_*R^n, R^n)$ of *near-splittings* of F_*R^n. Given such an element $\phi \in \operatorname{Hom}_R(F_*R^n, R^n)$ we will describe the submodules $V \subseteq F_*R^n$ for which $\phi(V) \subseteq V$. These submodules in the case $n = 1$, known as *ϕ-compatible ideals*, are of significant importance in algebraic geometry (cf. [BK] for a study of applications of Frobenius splittings and their compatible submodules in algebraic geometry.) We will prove that under some circumstances these form a finite set and thus generalize a result in [BB] obtained in the F-finite case.

We first exhibit the following easy implication of Matlis duality necessary for the results of this section.

Lemma 4.1. *For any (not necessarily finitely generated) R-module M,* $\operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_R(R^\vee, M^\vee)$.

Proof. For all $a \in E$ let $h_a \in \operatorname{Hom}_R(R, E)$ denote the map sending 1 to a.

For any $\phi \in \operatorname{Hom}_R(M, R)$, $\phi^\vee \in \operatorname{Hom}_R(R^\vee, M^\vee)$ is defined as $(\phi^\vee(h_a))(m) = \phi(m)a$ for any $m \in M$ and $a \in E$. For any $\psi \in \operatorname{Hom}_R(R^\vee, M^\vee)$ we define $\widetilde{\psi} \in \operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_R(M, E^\vee)$ as $\left(\widetilde{\psi}(m)\right)(a) = (\psi(h_a))(m)$ for all $a \in E$ and $m \in M$. Note that the function $\psi \mapsto \widetilde{\psi}$ is R-linear.

It is now enough to show that for all $\phi \in \operatorname{Hom}_R(M, R)$, $\widetilde{\phi^\vee} = \phi$, and indeed for all $a \in E$ and $m \in M$

$$\left(\widetilde{\phi^\vee}(m)\right)(a) = (\phi^\vee(h_a))(m) = \phi(m)a,$$

i.e., $\left(\widetilde{\phi^\vee}(m)\right) \in \operatorname{Hom}_R(E, E)$ is given by multiplication by $\phi(m)$ and so under the identification of $\operatorname{Hom}_R(E, E)$ with R, $\widetilde{\phi^\vee}$ is identified with ϕ. □

We can now prove a generalization Lemma 1.6 in [F] in the form of the next two theorems.

Theorem 4.2. (a) *The F_*R-module* $\operatorname{Hom}_R(F_*R, E)$ *is injective of the form* $\oplus_{\gamma \in \Gamma} F_*E \oplus H$ *where Γ is non-empty, $H = \bigoplus_{\lambda \in \Lambda} F_*E(R/P_\lambda)$, Λ is a (possibly empty) set, P_λ*

is a non-maximal prime ideal of R for all $\lambda \in \Lambda$ and $E(R/P_\lambda)$ denotes the injective hull of R/P_λ.

(b) *Write $\mathcal{B} = \operatorname{Hom}_{F_*R}(E, \oplus_{\gamma\in\Gamma} F_*E) \subseteq \prod_{\gamma\in\Gamma} \operatorname{Hom}_{F_*R}(E, F_*E)$. We have*

$$\operatorname{Hom}_R(F_*R, R) \cong \mathcal{B} \subseteq \prod_{\gamma\in\Gamma} \operatorname{Hom}_{F_*R}(E, F_*E) \cong \prod_{\gamma\in\Gamma} F_*RT$$

where T is the standard Frobenius map on E.

(c) *The set Γ is finite if and only if $F_*\mathbb{K}$ is a finite extension of $\mathbb{K}$, in which case $\#\Gamma = 1$.*

Proof. The functors $\operatorname{Hom}_R(-, E) = \operatorname{Hom}_R(- \otimes_{F_*R} F_*R, E)$ and $\operatorname{Hom}_{F_*R}(-, \operatorname{Hom}_R(F_*R, E))$ from the category of F_*R-modules to itself are isomorphic by the adjointness of Hom and $\otimes$, and since $\operatorname{Hom}_R(-, E)$ is an exact functor, so is $\operatorname{Hom}_{F_*R}(-, \operatorname{Hom}_R(F_*R, E))$, thus $\operatorname{Hom}_R(F_*R, E)$ is an injective F_*R-module and hence of the form $G \oplus H$ where G is a direct sum of copies of F_*E and H is as in the statement of the Theorem. Write $G = \oplus_{\gamma\in\Gamma} F_*E$. To finish establishing (a) we need only verify that $\Gamma \neq \emptyset$ and we do this below.

Pick any $h \in \operatorname{Hom}_R\left(E, \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right)$. For any $a \in E$, $h(a)$ can be written as a finite sum $b_{\lambda_1} + \cdots + b_{\lambda_s}$ where $\lambda_1, \ldots, \lambda_s \in \Lambda$ and $b_{\lambda_1} \in F_*E(R/P_{\lambda_1}), \ldots, b_{\lambda_s} \in F_*E(R/P_{\lambda_s})$. Use prime avoidance to pick a $z \in m \setminus \cup_{i=1}^s P_{\lambda_i}$; now z and its powers act invertibly on each of $F_*E(R/P_{\lambda_1}), \ldots, F_*E(R/P_{\lambda_s})$ while a power of z kills a, and so we must have $h(a) = 0$. We deduce that $\operatorname{Hom}_R\left(E, \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right) = 0$ and

$$\begin{aligned}
\operatorname{Hom}_R(E, \operatorname{Hom}_R(F_*R, E)) &\cong \operatorname{Hom}_R\left(E, G \oplus \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right) \\
&\cong \operatorname{Hom}_R(E, G) \oplus \operatorname{Hom}_R\left(E, \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right) \\
&\cong \operatorname{Hom}_R(E, G) \\
&\cong \operatorname{Hom}_R(E, \oplus_{\gamma\in\Gamma} F_*E) \\
&= \mathcal{B}.
\end{aligned}$$

Now $\operatorname{Hom}_R(E, F_*E)$ is the R-module of Frobenius maps on E which is isomorphic as an F_*R module to F_*RT and we conclude that $\operatorname{Hom}_R(E, \operatorname{Hom}_R(F_*R, E)) \subseteq \prod_{\gamma\in\Gamma} F_*RT$.

An application of the Matlis dual and Lemma 4.1 now gives

$$\operatorname{Hom}_R(F_*R, R) \cong \operatorname{Hom}_R(E, \operatorname{Hom}_R(F_*R, E))$$

and (b) follows.

Write $\mathbb{K} = R/m$ and note that $F_*\mathbb{K}$ is the field extension of $\mathbb{K}$ obtained by adding all pth roots of elements in $\mathbb{K}$. We next compute the cardinality of Γ as the $F_*\mathbb{K}$-dimension of $\operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, G)$. A similar argument to the one above shows that

$$\operatorname{Hom}_{F_*\mathbb{K}}\left(F_*\mathbb{K}, \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right) = 0$$

hence $\operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, G) = \operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, \operatorname{Hom}_R(F_*R, E))$.

We may identify $\operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, \operatorname{Hom}_R(F_*R, E))$ and $\operatorname{Hom}_{F_*R}(F_*\mathbb{K}, \operatorname{Hom}_R(F_*R, E))$. Another application of the adjointness of Hom and $\otimes$ gives

$$\operatorname{Hom}_{F_*R}(F_*\mathbb{K}, \operatorname{Hom}_R(F_*R, E)) \cong \operatorname{Hom}_R(F_*\mathbb{K} \otimes_{F_*R} F_*R, E) \cong \operatorname{Hom}_R(F_*\mathbb{K}, E)$$

Since $mF_*\mathbb{K} = 0$, we see that the image of any $\phi \in \operatorname{Hom}_R(F_*\mathbb{K}, E)$ is contained in $\operatorname{ann}_E m \cong \mathbb{K}$ and we deduce that $\operatorname{Hom}_R(F_*\mathbb{K}, E) \cong \operatorname{Hom}_R(F_*\mathbb{K}, \mathbb{K})$. We can now conclude that the cardinality of Γ is the $F_*\mathbb{K}$-dimension of $\operatorname{Hom}_R(F_*\mathbb{K}, \mathbb{K})$. In particular Γ cannot be empty and (a) follows.

If $\mathcal{U}$ is a $\mathbb{K}$-basis for $F_*\mathbb{K}$ containing $1 \in F_*\mathbb{K}$,

$$\operatorname{Hom}_{\mathbb{K}}(F_*\mathbb{K}, \mathbb{K}) \cong \prod_{b \in \mathcal{U}} \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}b, \mathbb{K}) \tag{1}$$

and when $\mathcal{U}$ is finite, this is a one-dimensional $F_*\mathbb{K}$-vector space spanned by the projection onto $\mathbb{K}1 \subset F_*\mathbb{K}$. If $\mathcal{U}$ is not finite, the dimension as $\mathbb{K}$-vector space of (1) is at least $2^{\#\mathcal{U}}$ hence $\operatorname{Hom}_{\mathbb{K}}(F_*\mathbb{K}, \mathbb{K})$ cannot be a finite-dimensional $F_*\mathbb{K}$-vector space. ☐

Theorem 4.3. *Let $G = \oplus_{\gamma\in\Gamma} F_*E$ and $\mathcal{B}$ be as in Theorem 4.2. Let $B \in \operatorname{Hom}_R(F_*R^n, R^n)$ be represented by $(B_\gamma T)_{\gamma\in\Gamma} \in \mathcal{B}$. For all $\gamma \in \Gamma$ consider E^n as an $R[\Theta_\gamma; f]$-module with $\Theta_\gamma v = B_\gamma^t T v$ for all $v \in E^n$. Let V be an R-submodule of R^n and fix a matrix A whose columns generate V. If $B(F_*V) \subseteq V$, then $\operatorname{ann}_{E^n} A^t$ is a $R[\Theta_\gamma; f]$ submodule of E^n for all $\gamma \in \Gamma$.*

Proof. Apply the Matlis dual to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_*A & \longrightarrow & F_*R^n & \longrightarrow & F_*R^n/F_*A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle \overline{B}} & & \\
0 & \longrightarrow & A & \longrightarrow & R^n & \longrightarrow & R^n/A & \longrightarrow & 0
\end{array}$$

where the rightmost vertical map is induced by the middle map to obtain

$$\begin{array}{ccccc}
0 & \longrightarrow & (R^n/A)^\vee & \longrightarrow & E^n \\
 & & \downarrow{\scriptstyle B^\vee} & & \downarrow{\scriptstyle B^\vee} \\
0 & \longrightarrow & (F_*R^n/F_*A)^\vee & \longrightarrow & \operatorname{Hom}_R(F_*R^n, E)
\end{array}$$

Note that $B^\vee \in \operatorname{Hom}_R(E^n, \oplus_{\gamma\in\Gamma} E^n)$ is given by $(B_\gamma^t)_{\gamma\in\Gamma}$.

Using the presentation $F_*R^m \xrightarrow{F_*A} F_*R^n \to F_*R^n / \operatorname{Im} F_*A \to 0$ we obtain the exact sequence

$$0 \to (F_*R^n/F_*A)^\vee \to \operatorname{Hom}_R(F_*R^n, E) \xrightarrow{F_*A^t} \operatorname{Hom}_R(F_*R^m, E)$$

thus

$$(F_*R^n/F_*A)^\vee = \operatorname{ann}_{\operatorname{Hom}(F_*R^n, E)} F_*A^t.$$

We obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{ann}_{E^n} A^t & \longrightarrow & E^n \\ & & \Big\downarrow {\scriptstyle (B_\gamma^t T)_{\gamma\in\Gamma}} & & \Big\downarrow {\scriptstyle (B_\gamma^t T)_{\gamma\in\Gamma}} \\ 0 & \longrightarrow & \oplus_{\gamma\in\Gamma} \operatorname{ann}_{F_* E^n} F_* A^t & \longrightarrow & \oplus_{\gamma\in\Gamma} F_* E^n \end{array}$$

and we deduce that $\operatorname{ann}_{E^n} A^t$ is a $R[\Theta_\gamma; f]$-module for all $\gamma \in \Gamma$. □

Theorem 4.4. *Let M be an $R[\Theta; f]$-module with no nilpotents and assume M is an Artinian R-module. Then M has finitely many $R[\Theta; f]$-submodules. (Cf. Corollary 4.18 in* [BB]*.)*

Proof. Write $\mathcal{M} = \mathcal{H}(M)$. In view of [L, Theorem 4.2], there is an injection between the set of inclusions of $R[\Theta; f]$-submodules $N \subseteq M$ and the set of surjections of F-finite F-modules $\mathcal{M} \to \mathcal{N}$ hence it is enough to show that there are finitely many such surjections. By [L, Theorem 2.8] the kernels of these surjections are F-finite F-submodules of $\mathcal{M}$ hence it is enough to show that $\mathcal{M}$ has finitely many submodules. Assume this statement is false and choose a counterexample $\mathcal{M}$ with infinitely many submodules.

All objects in the category of F-finite F-modules have finite length (cf. [L, Theorem 3.2]) hence we may assume that among all counterexamples $\mathcal{M}$ has minimal length. By [H, Corollary 5.2] the isomorphism class of any simple F-finite F-module is a finite set and the set of simple submodules of $\mathcal{M}$ belong to finitely many of these isomorphism classes, namely those occurring as factors in a composition series for $\mathcal{M}$. We deduce that there are finitely many simple F-finite F-submodules of $\mathcal{M}$. Since $\mathcal{M}$ has infinitely many F-finite F-submodules, there must be a simple F-finite F-submodule $\mathcal{P} \subsetneq \mathcal{M}$ contained in infinitely many F-finite F-submodules of $\mathcal{M}$. The infinite set of images of these in the quotient $\mathcal{M}/\mathcal{P}$ exhibit a counterexample of smaller length. □

Corollary 4.5. *Let $B \in \operatorname{Hom}_R(F_* R^n, R)$ be represented by $(B_\gamma^t T)_{\gamma\in\Gamma} \in \mathcal{B}$, and assume that $(B_\gamma^t T) : E \to \oplus_{\gamma\in\Gamma} E$ is injective. Then there are finitely many B-compatible submodules of $F_* R^n$.*

Proof. For all $\gamma \in \Gamma$ write $Z_\gamma = \{v \in E^n \mid B_\gamma^t T v\}$ and let C_γ be a matrix with columns in R^n be such that $Z_\gamma = \operatorname{ann}_{E^n} C_\gamma^t$. If $\operatorname{Im} C_\gamma \subseteq mR^n$ for all $\gamma \in \Gamma$, then $\sum_{\gamma\in\Gamma} \operatorname{Im} C_\gamma$ is not the whole of R^n, and if C is a matrix whose columns generate $\sum_{\gamma\in\Gamma} \operatorname{Im} C_\gamma$, for any non-zero $v \in \operatorname{ann}_{E^n} C^t \neq 0$, we have $(B_\gamma)^t T v = 0$ for all $\gamma \in \Gamma$. We conclude that there exists a $\gamma \in \Gamma$ such that, $\operatorname{Im} C_\gamma = R^n$, i.e., that the Frobenius map $B_\gamma^t T$ on E^n has no nilpotents. For this $\gamma \in \Gamma$, Theorem 4.4 shows that E^n has finitely many $R[\Theta; f]$-submodules where the action of Θ is given by $B_\gamma^t T$.

Let V be an R-submodule of R^n and fix a matrix A whose columns generate V. Theorem 4.3 implies that if $F_* V \subseteq F_* R^n$ is B-compatible then $\operatorname{ann}_{E^n} A^t \subseteq E^n$ is an $R[\Theta; f]$-submodule of E^n with the Frobenius action given by $B_\gamma^t T$ for all $\gamma \in \Gamma$, and hence there are finitely many such B-compatible submodules. □

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