HALF-BALANCED BRAIDED MONOIDAL CATEGORIES AND TEICHMÜLLER GROUPOIDS IN GENUS ZERO

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Abstract. We introduce the notions of a half-balanced braided monoidal category and of its contraction. These notions give rise to an explicit description of the action of the Galois group of $\mathbb{Q}$ on Teichmüller groupoids in genus 0, equivalent to that of L. Schneps. We also show that a prounipotent version of this action is equivalent to a graded action.

Introduction and main results

Let $M_{g,n}^{\mathbb{Q}}$ be the moduli space of curves of genus g with n marked points. Its fundamental groupoid with respect to the set of maximally degenerate curves is called the Teichmüller groupoid $T_{g,n}$. One of the main features of Grothendieck's geometric approach to the Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ is the study of its action on the profinite completions $\widehat{T}_{g,n}$; according to this philosophy, $G_{\mathbb{Q}}$ could be characterized as the group of automorphisms of the tower of all $\widehat{T}_{g,n}$, compatible with natural operations, such as the Knudsen morphisms. It is therefore important to describe explicitly the action of $G_{\mathbb{Q}}$ on the collection of all the $\widehat{T}_{0,n}$. Such a description was obtained in [Sch]. More precisely, an explicit profinite group $\widehat{\mathrm{GT}}$ was introduced in [Dr], together with a morphism $G_{\mathbb{Q}} \to \widehat{\mathrm{GT}}$. The following was then proved in [Sch]:

Theorem 1. *There exists a morphism $\widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{T}_{0,n})$, such that the morphism $G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{T}_{0,n})$ factors as $G_{\mathbb{Q}} \to \widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{T}_{0,n})$.*

The first purpose of this paper is to present a variant of the proof of [Sch]. This variant relies on the notion of a half-balanced braided monoidal category (b.m.c.), which appeared implicitly recently in [ST] and is here made explicit. We introduce the notion of a (half-)balanced contraction of such a category $\mathcal{C}$: it consists of a functor $\mathcal{C} \to \mathcal{O}$, satisfying certain properties. Whereas a balanced b.m.c. gives rise to representations of the framed braid group on the plane $\tilde{B}_n$ (for $n \geq 0$), which is an abelian extension of the braid group B_n, a (half-)balanced contraction gives rise to representations of quotients of $\tilde{B}_n$. This quotient is an abelian extension of the quotient $B_n/Z(B_n)$ of B_n by its center in the case of a balanced contraction, and is an abelian extension of the mapping class group in genus zero $\Gamma_{0,n}$ (another quotient of B_n) in the case of a half-balanced contraction.

To each set S, we associate an object $\widehat{\mathbf{PaB}}_S^{hbal} \to \widehat{\mathbf{PaDih}}_S$ in the category whose objects are contractions of profinite half-balanced b.m. categories, enjoying universal properties. These contractions may be viewed as the analogues of the universal b.m. categories appearing in [JS]. We show that the action of $\widehat{\mathrm{GT}}$ on such categories may be lifted to the half-balanced setup. This defines in particular an action of $\widehat{\mathrm{GT}}$ on $\widehat{\mathbf{PaDih}}_S$, from which it is is easy to derive an action of $\widehat{T}_{0,n}$.

The above profinite theory admits a prounipotent version. The group $\widehat{\mathrm{GT}}$ and the Teichmüller groupoid $\widehat{T}_{0,n}$ admit proalgebraic versions $\mathbf{k} \mapsto \mathrm{GT}(\mathbf{k}), T_{0,n}(\mathbf{k})$, where $\mathbf{k}$ is a $\mathbb{Q}$-ring. We then have morphisms $\widehat{\mathrm{GT}} \to \mathrm{GT}(\mathbb{Q}_l)$, $\widehat{T}_{0,n} \to (T_{0,n})_l \to T_{0,n}(\mathbb{Q}_l)$, where l is a prime number and $(T_{0,n})_l$ is the pro-l completion of $T_{0,n}$. We construct a group scheme $\underline{\mathrm{Aut}\,T_{0,n}}(-)$, together with a morphism $\underline{\mathrm{Aut}\,T_{0,n}}(\mathbf{k}) \to \mathrm{Aut}(T_{0,n}(\mathbf{k}))$, a group scheme morphism $\mathrm{GT}(-) \to \underline{\mathrm{Aut}\,T_{0,n}}(-)$, and a group $\mathrm{Aut}((T_{0,n})_l, T_{0,n}(\mathbb{Q}_l))$, equipped with morphisms

$$\mathrm{Aut}((T_{0,n})_l) \leftarrow \mathrm{Aut}((T_{0,n})_l, T_{0,n}(\mathbb{Q}_l)) \to \underline{\mathrm{Aut}\,T_{0,n}}(\mathbb{Q}_l).$$

Theorem 2. *The morphism $G_{\mathbb{Q}} \to \mathrm{Aut}((T_{0,n})_l)$ factors as $G_{\mathbb{Q}} \to \mathrm{Aut}((T_{0,n})_l, T_{0,n}(\mathbb{Q}_l)) \to \mathrm{Aut}((T_{0,n})_l)$, and there exists a morphism $\mathrm{GT}(-) \to \underline{\mathrm{Aut}\,T_{0,n}}(-)$, such that the following diagram commutes*

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r] \arrow[dr] \arrow[dd] & \mathrm{Aut}((T_{0,n})_l) \\
 & \mathrm{Aut}((T_{0,n})_l, T_{0,n}(\mathbb{Q}_l)) \arrow[u] \arrow[d] \\
\mathrm{GT}(\mathbb{Q}_l) \arrow[r] & \underline{\mathrm{Aut}\,T_{0,n}}(\mathbb{Q}_l)
\end{tikzcd}$$

We say that an algebraic (resp., prounipotent) group over $\mathbb{Q}$ is graded iff its Lie algebra is graded by $\mathbb{Z}_{\geq 0}$ (resp., by $\mathbb{Z}_{>0}$). We say that a groupoid $\mathcal{G}$ is graded prounipotent if for any $s \in \mathrm{Ob}\,\mathcal{G}$, $\mathrm{Aut}_{\mathcal{G}}(s)$ is graded prounipotent. In [Dr], a graded $\mathbb{Q}$-algebraic group $\mathrm{GRT}(-)$ was constructed, together with an isomorphism $\mathrm{GT}(-) \to \mathrm{GRT}(-)$.

Theorem 3. *There exists a graded prounipotent groupoid $T_{0,n}^{gr}(-)$ and a graded morphism $\mathrm{GRT}(-) \to \underline{\mathrm{Aut}\,T_{0,n}^{gr}}(-)$, such that the diagram* $\begin{array}{ccc} \mathrm{GT}(-) & \to & \underline{\mathrm{Aut}\,T_{0,n}}(-) \\ \downarrow & & \downarrow \\ \mathrm{GRT}(-) & \to & \underline{\mathrm{Aut}\,T_{0,n}^{gr}} \end{array}$ *commutes.*

1. Teichmüller groupoids in genus 0

1.1. **Quotient categories.** Let $\mathcal{C}$ be a small category and let G be a group. We define an action of G on $\mathcal{C}$ as the data of: (a) a group morphism $G \to \mathrm{Perm}(\mathrm{Ob}\,\mathcal{C})$, (b) for any $g \in G$, an assignment $\mathrm{Ob}\,\mathcal{C} \in X \mapsto i_X^g \in \mathrm{Iso}_{\mathcal{C}}(X, gX)$, such that $i_X^{gh} = i_{gX}^h i_X^g$.

We then get a group morphism $G \to \mathrm{Aut}\,\mathcal{C} = \{\text{autofunctors of } \mathcal{C}\}$, where the autofunctor induced by $g \in G$ is the action of g at the level of objects, and $g\phi := i_Y^g \phi (i_X^g)^{-1}$ for $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Lemma 4. *1) For any $\alpha, \beta \in (\mathrm{Ob}\,\mathcal{C})/G$, there is a unique action of $G \times G$ on $\mathcal{X}(\alpha, \beta) := \sqcup_{X \in \alpha, Y \in \beta} \mathrm{Hom}_{\mathcal{C}}(X, Y)$, such that $(g, h) \mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(gX, hY)$ and $(g, h)\phi = i_Y^h \phi (i_X^g)^{-1}$.*

2) Set $\mathcal{X}(X, \beta) := \sqcup_{Y \in \beta} \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $\mathcal{X}(\alpha, Y) := \sqcup_{X \in \alpha} \mathrm{Hom}_{\mathcal{C}}(X, Y)$, then G acts on these sets (by permutation of β in the first case and of α in the second one) and we have a well-defined map $\mathcal{X}(X, \beta)^G \times \mathcal{X}(\beta, Z)^G \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ compatible all the maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$. Taking the product of these maps over $X \in \alpha$, $Z \in \gamma$ and further the quotient by $G \times G$, we obtain a map $\mathcal{X}(\alpha, \beta)^{G \times G} \times \mathcal{X}(\beta, \gamma)^{G \times G} \to \mathcal{X}(\alpha, \gamma)^{G \times G}$, which is associative.

The proof is straightforward. We then define the quotient category $\mathcal{C}/G$ by $\mathrm{Ob}(\mathcal{C}/G) := (\mathrm{Ob}\,\mathcal{C})/G$ and $(\mathcal{C}/G)(\alpha, \beta) := \mathcal{X}(\alpha, \beta)^{G \times G}$.

Remark 5. If $X \in \alpha$ and $Y \in \beta$, then $(\mathcal{C}/G)(\alpha, \beta) \simeq \mathcal{C}(X, Y)^{G_X \times G_Y}$, where $G_X = \{g \in G | gX = X\}$.

Proposition 6. *If $\mathcal{D}$ is a small category, then a functor $\mathcal{C}/\Gamma \to \mathcal{D}$ is the same as a functor $F : \mathcal{C} \to \mathcal{D}$, such that $F(gX) = F(X)$ and $F(i_X^g) = \mathrm{id}_{F(X)}$ for any $g \in G$, $X \in \mathrm{Ob}\,\mathcal{C}$.*

The proof is immediate.

1.2. **Quotients of the braid group.** Let B_n be the braid group of n strands in the plane. It is presented by generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the Artin relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, \ldots, n-2$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$. Its center $Z_n := Z(B_n)$ is isomorphic to $\mathbb{Z}$ and is generated by $(\sigma_1 \cdots \sigma_{n-1})^n$. There is a morphism $B_n \to S_n$, uniquely determined by $\sigma_i \mapsto s_i := (i, i+1)$; it factors through a morphism $B_n/Z_n \to S_n$.

Lemma 7. *Let $C_n := \langle g | g^n = 1\rangle$ be the cyclic group of order n. We have an injection $C_n \hookrightarrow S_n$ via $g \mapsto \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \end{smallmatrix}\right)$, which admits a lift $C_n \to B_n/Z_n$, given by $g \mapsto \sigma_1 \cdots \sigma_{n-1}$.*

Let $\Gamma_{0,n} := B_n/((\sigma_1 \cdots \sigma_{n-1})^n, \sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1)$ be the mapping class group of type $(0,n)$ (see [Bi]). The relation $\sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1 = 1$ is called the sphere relation as the quotient $B_n/(\sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1)$ is isomorphic to the braid group of n points on the sphere. In this group, the relation $(\sigma_1 \cdots \sigma_{n-1})^{2n} = 1$ holds. The morphism $B_n \to S_n$ factors through a morphism $\Gamma_{0,n} \to S_n$.

The dihedral group $D_n := \langle r, s | r^n = s^2 = (rs)^2 = 1\rangle$ may be viewed as a subgroup of S_n via $r \mapsto \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \end{smallmatrix}\right)$, $s \mapsto \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{smallmatrix}\right)$.

Lemma 8. *There exists a unique morphism $D_n \to \Gamma_{0,n}$, $r \mapsto \sigma_1 \cdots \sigma_{n-1}$, $s \mapsto \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1)$, lifting the injection $D_n \hookrightarrow S_n$.*

Proof. One knows that $h_n := \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1) \in B_n$ is the half-twist, so that $h_n^2 = z_n = (\sigma_1 \cdots \sigma_{n-1})^n = \rho^n$, where $\rho = \sigma_1 \cdots \sigma_{n-1}$ and z_n is the full twist, generating $Z(B_n)$. Moreover, $h_n\rho^{-1} = \mathrm{im}(h_{n-1} \in B_{n-1} \to B_n)$, so $(h_n\rho^{-1})^2 = z_{n-1} = z_n(\sigma_{n-1} \cdots \sigma_1^2 \cdots \sigma_{n-1})^{-1}$, where we identify z_{n-1} with its image under $B_{n-1} \to B_n$. The images of h_n, ρ in $\Gamma_{0,n}$ therefore satisfy $\bar{h}_n^2 = \bar{\rho}^n = (\bar{h}_n\bar{\rho}^{-1})^2 = 1$, which are equivalent to a presentation of D_n. $\square$

1.3. **Teichmüller groupoids.** Let G be a group and $\Gamma \subset S_n$ be a subgroup. Assume that $G \to S_n$ is a group morphism and let $\Gamma \to G$ be such that
$$\begin{tikzcd} G \arrow[r] & S_n \\ & \Gamma \arrow[ul] \arrow[u] \end{tikzcd}$$
commutes. Let S be a set, with $|S| = n$.

Define a category $\mathcal{C}_{G,S}$ by $\mathrm{Ob}\,\mathcal{C}_{G,S} := \mathrm{Bij}([n], S)$; for $\sigma, \sigma' \in \mathrm{Ob}\,\mathcal{C}_{G,S}$, $\mathrm{Hom}(\sigma, \sigma') := G \times_{S_n} \{(\sigma')^{-1}\sigma\}$; the composition of morphisms is induced by the product in G.

Define an action of Γ on $\mathcal{C}_{G,S}$ as follows. For $\gamma \in \Gamma$, $\sigma \in \mathrm{Bij}([n], S)$, $\gamma \cdot \sigma := \sigma\gamma^{-1}$, and $i_\sigma^\gamma \in \mathrm{Hom}(\sigma, \sigma\gamma^{-1}) = G \times_{S_n} \{\gamma\}$ is $\mathrm{im}(\gamma \in \Gamma \to G)$. We then obtain a quotient category $\mathcal{C}_{\Gamma,G,S} := \mathcal{C}_{G,S}/\Gamma$.

Example 9. When $G = B_n/Z_n$ and $\Gamma = C_n$, we set $\underline{\mathrm{Cyc}}(S) := \mathcal{C}_{\Gamma,G,S}$; its set of objects is $\mathrm{Cyc}(S) := \mathrm{Bij}([n], S)/C_n$ (the set of cyclic orders on S).

Example 10. When $G = \Gamma_{0,n}$ and $\Gamma = D_n$, we set $\underline{\mathrm{Dih}}(S) := \mathcal{C}_{\Gamma,G,S}$; its set of objects is $\mathrm{Dih}(S) := \mathrm{Bij}([n], S)/D_n = \mathrm{Cyc}(S)/\{\pm 1\}$, which we call the set of dihedral orders on S.

Definition 11. *If $\mathcal{C}$ is a small category and $T \overset{\pi}{\to} \mathrm{Ob}\,\mathcal{C}$ is a map, we define the category $\pi^*\mathcal{C}$ by $\mathrm{Ob}\,\pi^*\mathcal{C} := T$ and $\pi^*\,\mathcal{C}(t, t') := \mathcal{C}(\pi(t), \pi(t'))$ for $t, t' \in T$.*

We have natural maps

$$\{\text{planar 3-valent trees with leaves bijectively indexed by } S\} \overset{\pi_{cyc}}{\to} \mathrm{Cyc}(S)$$

and

$$\{\text{planar 3-valent trees with leaves bijectively indexed by } S\}/(\text{mirror symmetry}) \overset{\pi_{dih}}{\to} \mathrm{Dih}(S).$$

We then set $T'_{0,S} := \pi^*_{cyc}\underline{\mathrm{Cyc}}(S)$, $T_{0,S} := \pi^*_{dih}\underline{\mathrm{Dih}}(S)$.

When $S = [n]$, $T_{0,S}$ identifies with the fundamental groupoid to the moduli stack $M^{\mathbb{Q}}_{0,n}$ with respect to the set of maximally degenerate real curves (see [Sch]).

2. Contractions on (half-)balanced categories

2.1. **(Half-)balanced categories.** Recall that a braided monoidal category (b.m.c.) is a set $(\mathcal{C}, \otimes, \mathbf{1}, \beta_{XY}, a_{XYZ})$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is a bifunctor, $\beta_{XY} : X \otimes Y \to Y \otimes X$ and $a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ are natural constraints, $\mathbf{1} \in \mathrm{Ob}\,\mathcal{C}$ and $X \otimes \mathbf{1} \stackrel{\sim}{\to} X \stackrel{\sim}{\leftarrow} \mathbf{1}\otimes X$ are natural isomorphisms, satisfying the hexagon and pentagon conditions (see e.g. [Ka]).

A balanced structure on the small b.m.c. $\mathcal{C}$ is the datum of a natural assignment $\mathrm{Ob}\,\mathcal{C} \ni X \mapsto \theta_X \in \mathrm{Aut}_{\mathcal{C}}(X)$, such that

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y)\beta_{YX}\beta_{XY}$$

for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$ (see [JS]); the naturality condition is $\theta_{X'}\phi = \phi\theta_X$ for any $X, X' \in \mathrm{Ob}\,\mathcal{C}$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, X')$.

Similarly, a half-balanced structure on $\mathcal{C}$ is the data of: (a) an involutive autofunctor $* : \mathcal{C} \to \mathcal{C}$, $X \mapsto X^*$, such that $(X \otimes Y)^* = Y^* \otimes X^*$, $(f \otimes g)^* = g^* \otimes f^*$ for any $X, \ldots, Y' \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$, $a^*_X = a_{X^*}$, $\beta^*_{XY} = \beta_{Y^*X^*}$, $a^*_{XYZ} = a_{Z^*Y^*X^*}$; (b) a natural assignment $\mathrm{Ob}\,\mathcal{C} \in X \mapsto a_X \in \mathrm{Iso}_{\mathcal{C}}(X, X^*)$, such that

$$a_{X\otimes Y} = (a_Y \otimes a_X)\beta_{XY}$$

for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$; here naturality means that $a_Y\phi = \phi^* a_X$ for any $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Note that a half-balanced structure gives rise to a balanced structure by $\theta_X := a_{X^*}a_X$.

2.2. **Contractions.**

Definition 12. *A contraction on the small balanced category $\mathcal{C}$ is a functor $\langle - \rangle : \mathcal{C} \to \mathcal{O}$, $X \mapsto \langle X \rangle$, such that:*

1) for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\langle Y \otimes X\rangle = \langle X \otimes Y\rangle (=: \langle X, Y\rangle)$, and $\langle(\theta_Y \otimes \mathrm{id}_X)\beta_{XY}\rangle = \mathrm{id}_{\langle X,Y\rangle}$;

2) for any $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$, $\langle(X \otimes Y) \otimes Z\rangle = \langle X \otimes (Y \otimes Z)\rangle (=: \langle X, Y, Z\rangle)$ and $\langle a_{XYZ}\rangle = \mathrm{id}_{\langle X,Y,Z\rangle}$.

When needed, we will call such a contraction a "balanced contraction".

Remark 13. These axioms imply $\langle\theta_{X\otimes Y}\rangle = \mathrm{id}_{\langle X,Y\rangle}$ for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$, and therefore $\langle\theta_X\rangle = \mathrm{id}_{\langle X\rangle}$ by taking $Y = \mathbf{1}$.

Definition 14. *A contraction on the small half-balanced category $\mathcal{C}$ is a functor $\langle - \rangle : \mathcal{C} \to \mathcal{O}$, such that:*

1) $\langle - \rangle$ is a balanced contraction on $\mathcal{C}$;

2) for any $X \in \mathrm{Ob}\,\mathcal{C}$, $\langle X\rangle = \langle X^*\rangle$ and $\langle a_X\rangle = \mathrm{id}_{\langle X\rangle}$.

When needed, we will call such a contraction a "half-balanced contraction".

Lemma 15. *If $\langle - \rangle : \mathcal{C} \to \mathcal{O}$ is a contraction on a half-balanced category, then for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\langle\theta_X \otimes \theta_Y^{-1}\rangle = \mathrm{id}_{\langle X,Y\rangle} = \langle(\theta_X^2 \otimes \mathrm{id}_Y)\beta_{YX}\beta_{XY}\rangle$.*

Proof. We have

$$\begin{aligned}
&\beta^{-1}_{XY}(\theta_Y^{-1} \otimes \mathrm{id}_X)a_{X^*\otimes Y^*}\beta^{-1}_{X^*Y^*}(\theta^{-1}_{Y^*} \otimes \mathrm{id}_{X^*})a_{X\otimes Y}\\
&= \beta^{-1}_{XY}(\theta_Y^{-1} \otimes \mathrm{id}_X)a_{X^*\otimes Y^*}\beta^{-1}_{X^*Y^*}a_{X\otimes Y}(\mathrm{id}_X \otimes\theta_Y^{-1})\\
&= \beta^{-1}_{XY}(\theta_Y^{-1} \otimes \mathrm{id}_X)(a_{Y^*} \otimes a_{X^*})a_{X\otimes Y}(\mathrm{id}_X \otimes\theta_Y^{-1})\\
&= \beta^{-1}_{XY}(\theta_Y^{-1} \otimes \mathrm{id}_X)(a_{Y^*}a_Y \otimes a_{X^*}a_X)\beta_{XY}(\mathrm{id}_X \otimes\theta_Y^{-1})\\
&= \beta^{-1}_{XY}(\mathrm{id}_Y \otimes\theta_X)\beta_{XY}(\mathrm{id}_X \otimes\theta_Y^{-1}) = \theta_X \otimes \theta_Y^{-1}.
\end{aligned}$$

Now $\langle a_{X\otimes Y}\rangle = \langle a_{X^*\otimes Y^*}\rangle = \mathrm{id}_{\langle X,Y\rangle}$ by the half-balanced contraction axiom, and $\langle \beta_{XY}^{-1}(\theta_Y^{-1}\otimes \mathrm{id}_X)\rangle = \langle \beta_{X^*Y^*}^{-1}(\theta_{Y^*}^{-1}\otimes \mathrm{id}_{X^*})\rangle = \mathrm{id}_{\langle X,Y\rangle}$ by the balanced contraction axiom. It follows that $\langle \theta_X\otimes\theta_Y^{-1}\rangle = \mathrm{id}_{\langle X,Y\rangle}$. The second statement follows from $(\theta_X^2\otimes \mathrm{id}_Y)\beta_{YX}\beta_{XY} = (\theta_X\otimes\theta_Y^{-1})\theta_{X\otimes Y}$ and $\langle\theta_{X\otimes Y}\rangle = \mathrm{id}_{\langle X,Y\rangle}$. □

2.3. **Relation with braid group representations.** Set $\tilde B_n := \mathbb{Z}^n \rtimes B_n$, where the action of B_n is $\mathbb{Z}^n$ is via $B_n \to S_n \to \mathrm{Aut}(\mathbb{Z}^n)$; $\tilde B_n$ is usually called the framed braid group of the plane. If $\mathcal{C}$ is a balanced b.m.c. and $X \in \mathrm{Ob}\,\mathcal{C}$, then there is a morphism $\tilde B_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})$ (a parenthesization of the nth fold tensor product being chosen), given in the strict case by

$$\delta_i \mapsto \mathrm{id}_{X^{\otimes i-1}}\otimes\theta_X\otimes \mathrm{id}_{X^{\otimes n-i}}, \quad \sigma_i \mapsto \mathrm{id}_{X^{\otimes i-1}}\otimes\beta_{X,X}\otimes \mathrm{id}_{X^{\otimes n-i-1}}.$$

Here δ_i is the ith generator of $\mathbb{Z}^n \subset \tilde B_n$.

We now define $\widetilde{B_n/Z_n}$ to be the quotient of $\tilde B_n$ by its central subgroup (isomorphic to $\mathbb{Z}$) generated by $(\prod_{i=1}^n \delta_i)z_n$ (recall that z_n is a generator of $Z_n = Z(B_n)$; the product in $\mathbb{Z}^n$ is denoted multiplicatively). One can prove that there is a (generally non-split) exact sequence $1 \to \mathbb{Z}^n \to \widetilde{B_n/Z_n} \to B_n/Z_n \to 1$.

Proposition 16. *Let $\mathcal{C} \overset{\langle - \rangle}{\to} \mathcal{O}$ be a balanced contraction of $\mathcal{C}$, then we have a commutative diagram*

$$\begin{array}{ccccc} B_n & \leftarrow & \tilde B_n & \rightarrow & \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n}) \\ \downarrow & & \downarrow & & \downarrow\langle-\rangle \\ B_n/Z_n & \leftarrow & \widetilde{B_n/Z_n} & \rightarrow & \mathrm{Aut}_{\mathcal{O}}(\langle X^{\otimes n}\rangle) \end{array}$$

Proof. We have $\mathrm{im}((\prod_{i=1}^n \delta_i)z_n \in \tilde B_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})) = \theta_{X^{\otimes n}}$, so according to Remark 13, the image of this in $\mathrm{Aut}_{\mathcal{O}}(\langle X^{\otimes n}\rangle)$ is $\mathrm{id}_{\langle X^{\otimes n}\rangle}$. The factorization implied in the right square follows. The left square obviously commutes. □

Set now $\tilde\Gamma_{0,n}$ be the quotient of $\tilde B_n$ by the normal subgroup generated by $(\prod_{i=1}^n \delta_i)z_n$ and $\delta_1^2\sigma_1\cdots\sigma_{n-1}^2\cdots\sigma_1$. Then we have an exact sequence $1 \to \mathbb{Z}^n \to \tilde\Gamma_{0,n} \to \Gamma_{0,n} \to 1$.

Proposition 17. *Let $\mathcal{C}$ be a half-balanced b.m.c., let $\mathcal{C} \overset{\langle - \rangle}{\to} \mathcal{O}$ be a half-balanced contraction and let $X \in \mathrm{Ob}\,\mathcal{C}$. Then we have a comutative diagram*

$$\begin{array}{ccccc} B_n & \leftarrow & \tilde B_n & \rightarrow & \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n}) \\ \downarrow & & \downarrow & & \downarrow\langle-\rangle \\ \Gamma_n & \leftarrow & \tilde\Gamma_n & \rightarrow & \mathrm{Aut}_{\mathcal{O}}(\langle X^{\otimes n}\rangle) \end{array}$$

Proof. We have $\mathrm{im}(\delta_1^2\sigma_1\cdots\sigma_{n-1}^2\cdots\sigma_1 \in \tilde B_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})) = (\theta_X^2\otimes \mathrm{id}_Y)\beta_{YX}\beta_{XY}$ ($Y = X^{\otimes n-1}$), whose image in $\mathrm{Aut}_{\mathcal{O}}(\langle X^{\otimes n}\rangle)$ is $\mathrm{id}_{X^{\otimes n}}$ by Lemma 15. □

3. Universal (half-)balanced categories

3.1. **Universal (strict) braided monoidal categories.** Recall that the small b.m.c. $\mathcal{C}$ is called strict iff $(X\otimes Y)\otimes Z = X\otimes(Y\otimes Z)(= X\otimes Y\otimes Z)$ and $a_{X,Y,Z} = \mathrm{id}_{X\otimes Y\otimes Z}$ for any $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$. Following [JS], we associate a universal strict b.m.c. $\mathbf{B}_S$ to each set S. Its set of objects is $\mathrm{Ob}\,\mathbf{B}_S := \sqcup_{n\geq 0} S^n$; the tensor product is defined by $\underline{s}\otimes\underline{s}' = (s_1,\ldots,s_n,s'_1,\ldots,s'_{n'}) \in S^{n+n'}$ for $\underline{s} = (s_1,\ldots,s_n) \in S^n$, $\underline{s}' = (s'_1,\ldots,s'_{n'}) \in S^{n'}$. If $\underline{s} \in S^n$, $\underline{s}' \in S^{n'}$, then $\mathrm{Hom}_{\mathbf{B}_S}(\underline{s},\underline{s}') = \emptyset$ if $n \neq n'$, and $\mathrm{Hom}_{\mathbf{B}_s}(\underline{s},\underline{s}') = B_n \times_{S_n} \{f \in S_n | \underline{s}' f = \underline{s}\}$ if $n = n'$. The tensor product of morphisms is induced by restriction from the group morphism $B_n \times B_{n'} \to B_{n+n'}$, $(b, b') \mapsto b * b'$, uniquely determined by $\sigma_i * 1 = \sigma_i$, $1 * \sigma_{i'} = \sigma_{n-1+i'}$ (which corresponds to the juxtaposition of braids). The braiding is $\beta_{\underline{s},\underline{s}'} = b_{nn'}$, where $b_{nn'} \in B_{n+n'}$ is given by

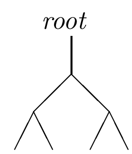
$$b_{nn'} = (\sigma_{n'} \cdots \sigma_1) \cdots (\sigma_{n+n'-1} \cdots \sigma_n).$$

The universal property of $\mathbf{B}_S$ is then expressed as follows: to each strict small b.m.c. $\mathcal{C}$ and any map $S \to \mathrm{Ob}\,\mathcal{C}$, there corresponds a unique tensor functor $\mathbf{B}_S \to \mathcal{C}$, such that the diagram

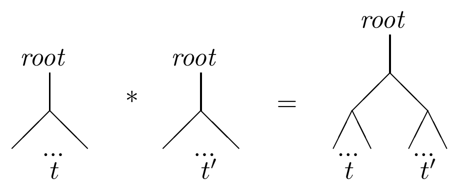
$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathrm{Ob}\,\mathcal{C} \\ & \mathrm{Ob}\,\mathbf{B}_S \arrow[u] \end{tikzcd}$$

commutes.

We now describe the universal b.m.c. $\mathbf{PaB}_S$ associated to S ([JS, Ba]). Define first T_n as the set of parenthesizations of a word in n identical letters. Equivalently, this is the set of planar 3-valent rooted trees with n leaves, e.g. the tree



corresponds to the word $(\bullet\bullet)(\bullet\bullet)$. The concatenation of words is a map $T_n \times T_m \to T_{n+m}$, $(t,t') \mapsto t * t'$ (e.g., $(\bullet\bullet, \bullet\bullet) \mapsto (\bullet\bullet)(\bullet)\bullet$); this is illustrated in terms of trees as follows



The set of objects of $\mathbf{PaB}_S$ is then defined by $\mathrm{Ob}\,\mathbf{PaB}_S := \sqcup_{n\geq 0} T_n \times S^n$; the tensor product is defined by $(t,\underline{s})\otimes(t',\underline{s}') := (t*t', \underline{s}\otimes\underline{s}')$. The morphisms are defined by $\mathrm{Hom}_{\mathbf{PaB}_S}((t,\underline{s}),(t',\underline{s}')) := \mathrm{Hom}_{\mathbf{B}_S}(\underline{s},\underline{s}')$. The tensor product of morphisms and the braiding and associativity constraints are uniquely determined by the condition that the obvious functor $\mathbf{PaB}_S \to \mathbf{B}_S$ is monoidal. In particular, a_{XYZ} corresponds to $1 \in B_{|X|+|Y|+|Z|}$, where $|(\underline{s},t)| = n$ for $(\underline{s},t) \in T_n \times S^n$. Then $\mathbf{PaB}_S$ has a universal property with respect to non-necessarily strict braided monoidal categories, analogous to that of $\mathbf{B}_S$.

3.2. **Universal balanced categories.** For $\underline{s} \in \mathrm{Ob}\,\mathbf{B}_S$, set $\theta_{\underline{s}} := z_{|\underline{s}|} \in \mathrm{Aut}_{\mathbf{B}_S}(\underline{s}) \subset B_{|\underline{s}|}$. The assignment $\underline{s} \mapsto \theta_{\underline{s}}$ equips $\mathbf{B}_S$ with a balanced structure. We denote by $\mathbf{B}_S^{bal}$ the resulting balanced strict b.m.c. One checks that it has the following universal property:

Lemma 18. *To any balanced strict small b.m.c. $\mathcal{C}$ and any map $S \xrightarrow{f} \mathrm{Ob}\,\mathcal{C}$, such that $\theta_{f(s)} = \mathrm{id}_{f(s)}$ for any $s \in S$, there corresponds a unique functor $\mathbf{B}_S^{bal} \to \mathcal{C}$ compatible with the balanced and monoidal structures, such that the diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathrm{Ob}\,\mathcal{C} \\ & \mathrm{Ob}\,\mathbf{B}_S^{bal} \arrow[u] \end{tikzcd}$$

commutes.

If now $X = (\underline{s},t) \in \mathrm{Ob}\,\mathbf{PaB}_S$, we set $\theta_X := \theta_{\underline{s}} \in \mathrm{Aut}_{\mathbf{B_S}}(\underline{s}) = \mathrm{Aut}_{\mathbf{PaB}_S}(X)$. The assignment $X \mapsto \theta_X$ equips $\mathbf{PaB}_S$ with the structure of a balanced b.m.c., denoted $\mathbf{PaB}_S^{bal}$ and enjoying a universal property with respect to maps $S \to \mathrm{Ob}\,\mathcal{C}$, where $\mathcal{C}$ is a balanced braided monoidal category such that $\theta_{f(s)} = \mathrm{id}_{f(s)}$ similar to Lemma 18.

3.3. **Universal half-balanced categories.** We define an involution $* : \mathbf{B}_S \to \mathbf{B}_S$ as follows. It is given at the level of objects by $\underline{s}^* := (s_n, \dots, s_1)$ for $\underline{s} = (s_1, \dots, s_n)$ and the level of morphisms by restriction of the automorphism $\sigma_i \mapsto \sigma_{n-i}$ of B_n. For $\underline{s} \in \mathrm{Ob}\,\mathbf{B}_S$, we set $a_{\underline{s}} := h_{|\underline{s}|} \in \mathrm{Iso}_{\mathbf{B}_S}(\underline{s}, \underline{s}^*) \subset B_{|\underline{s}|}$. This defines the structure of a half-balanced category on $\mathbf{B}_S$, denoted $\mathbf{B}_S^{hbal}$, whose balanced structure is that described in Subsection 3.2. It has the following universal property:

Lemma 19. *For each strict half-balanced small b.m.c. $\mathcal{C}$ and each map $S \xrightarrow{f} \mathrm{Ob}\,\mathcal{C}$ such that for any $s \in S$, $f(s)^* = f(s)$ and $a_{f(s)} = \mathrm{id}_{f(s)}$, there exists a unique functor $\mathbf{B}_S^{hbal} \to \mathcal{C}$, compatible with the monoidal and half-balanced structures, and such that the diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathrm{Ob}\,\mathcal{C} \\ & \mathrm{Ob}\,\mathbf{B}_S^{bal} \arrow[u] \end{tikzcd}$$

commutes.

We now define an involution $*$ of $\mathbf{PaB}_S$ as follows. At the level of objects, it is given by $X^* = (t^*, \underline{s}^*)$ for $X = (t, \underline{s})$, where t^* is the parenthesized word t, read in the reverse order (in terms of trees, this is the mirror image of t). At the level of morphisms, it coincides with the involution $*$ of $\mathbf{B}_S$. We define the assignment $\mathrm{Ob}\,\mathbf{PaB}_S \ni X \mapsto a_X$ by $a_X := a_{\underline{s}} \in \mathrm{Iso}_{\mathbf{B}_S}(\underline{s}, \underline{s}^*) = \mathrm{Iso}_{\mathbf{PaB}_S}(X, X^*)$ for $X = (t, \underline{s})$. This equips $\mathbf{PaB}_S$ with a half-balanced structure; the resulting half-balanced b.m.c. is denoted $\mathbf{PaB}_S^{hbal}$. Its underlying balanced b.m.c. is $\mathbf{PaB}_S^{bal}$. It has a universal property with respect to half-balanced small braided monoidal categories $\mathcal{C}$ and maps $S \xrightarrow{f} \mathrm{Ob}\,\mathcal{C}$, such that $f(s)^* = f(s)$ and $a_{f(s)} = \mathrm{id}_{f(s)}$, similar to that of Lemmas 18 and 19.

4. Universal contractions for balanced categories

We will construct categories $(\mathbf{Pa})\mathbf{Cyc}_S$ and a diagram $\begin{array}{ccc} \mathbf{PaB}_S^{bal} & \to & \mathbf{PaCyc}_S \\ \downarrow & & \downarrow \\ \mathbf{B}_S^{bal} & \to & \mathbf{Cyc}_S \end{array}$ in which the horizontal functors are contractions and the left vertical functor is the canonical monoidal functor.

We construct $\mathbf{Cyc}_S$ as follows. Define first $\widetilde{\mathbf{Cyc}}_S$ as the category with the same objects as $\mathbf{B}_S^{bal}$, and B_n replaced by B_n/Z_n) in the definition of morphisms. Define an action of $\mathbb{Z}$ on $\widetilde{\mathbf{Cyc}}_S$ by $1 \cdot (s_1, \cdots, s_n) := (s_n, s_1, \dots, s_{n-1})$ and $i^1_{\underline{s}} \in \mathrm{Iso}(\underline{s}, 1 \cdot \underline{s}) \subset B_n/Z_n$ is the class of $\sigma_1 \cdots \sigma_{n-1}$. We then set $\mathbf{Cyc}_S := \widetilde{\mathbf{Cyc}}_S/\mathbb{Z}$. Note that $\mathrm{Ob}\,\mathbf{Cyc}_S = \sqcup_{n \geq 0} \mathrm{Cyc}_n(S)$, where $\mathrm{Cyc}_n(S) = S^n/C_n$. We then define a functor $\mathbf{B}_S^{bal} \to \mathbf{Cyc}_S$ as the composite functor $\mathbf{B}_S^{bal} \to \widetilde{\mathbf{Cyc}}_S \to \mathbf{Cyc}_S$.

Let us show that the functor $\langle - \rangle : \mathbf{B}_S^{bal} \to \mathbf{Cyc}_S$ satisfies the balanced contraction condition. If $\underline{s}, \underline{s}' \in \mathrm{Ob}\,\mathbf{B}_S$, with $\underline{s} = (s_1, \dots, s_n)$ and $\underline{s}' = (s'_1, \dots, s'_{n'})$, then $\underline{s}' \otimes \underline{s} = (s'_1, \dots, s_n) = n' \cdot (\underline{s} \otimes \underline{s}')$, which implies that $\langle \underline{s} \otimes \underline{s}' \rangle = \langle \underline{s}' \otimes \underline{s} \rangle$. Then $(\theta_{\underline{s}'} \otimes \mathrm{id}_{\underline{s}})\beta_{\underline{s},\underline{s}'} \in \mathrm{Iso}_{\mathbf{B}_S^{bal}}(\underline{s} \otimes \underline{s}', \underline{s}' \otimes \underline{s}) = B_{n+n'}$ corresponds to $(z_{n'} * \mathrm{id}_n) b_{nn'} = (\sigma_1 \cdots \sigma_{n+n'-1})^{n'}$. Its image in $\widetilde{\mathbf{Cyc}}_S$ is then $i^{n'}_{\underline{s} \otimes \underline{s}'} \in \widetilde{\mathbf{Cyc}}_S(\underline{s} \otimes \underline{s}', n' \cdot (\underline{s} \otimes \underline{s}'))$, whose image in $\mathbf{Cyc}_S$ is $\mathrm{id}_{\langle \underline{s}, \underline{s}' \rangle}$.

We now prove the universality of this contraction.

Proposition 20. *Let $\mathcal{C}$ be a strict small balanced b.m.c., equipped with a map $S \xrightarrow{f} \mathrm{Ob}\,\mathcal{C}$ and a balanced contraction $\mathcal{C} \to \mathcal{O}$. Then there is a functor $\mathbf{Cyc}_S \to \mathcal{O}$, such that the diagram* $\begin{array}{ccc} \mathbf{B}_S^{bal} & \to & \mathbf{Cyc}_S \\ \downarrow & & \downarrow \\ \mathcal{C} & \to & \mathcal{O} \end{array}$ *commutes.*

Proof. First note that since $\langle\theta_X\rangle = \mathrm{id}_{\langle X\rangle}$ for $X = f(s_1)\otimes\cdots\otimes f(s_n)$ and any $(s_1,\ldots,s_n)\in \mathrm{Ob}\,\mathbf{B}_S^{bal}$, we have a functor $F:\widetilde{\mathbf{Cyc}}_S\to\mathcal{O}$, such that the diagram $\begin{array}{ccc}\mathbf{B}_S^{bal} & \to & \widetilde{\mathbf{Cyc}}_S\\ \downarrow & & \downarrow\\ \mathcal{C} & \to & \mathcal{O}\end{array}$ commutes.

If $(s_1,\ldots,s_n)\in\mathrm{Ob}\,\widetilde{\mathbf{Cyc}}_S=\mathrm{Ob}\,\mathbf{B}_S^{bal}$, then $F(s_1,\ldots,s_n)=\langle f(s_1)\otimes\cdots\otimes f(s_n)\rangle = \langle f(s_n)\otimes f(s_1)\otimes\cdots\otimes f(s_{n-1})\rangle = F(s_n,\ldots,s_{n-1})$, therefore $F(gX)=F(X)$ for any $X\in\mathrm{Ob}\,\widetilde{\mathbf{Cyc}}_S$ and any $g\in\mathbb{Z}$. Moreover, we have

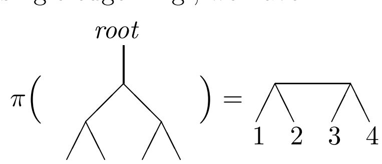
$$\begin{aligned}F(i^1_{(s_1,\ldots,s_n)}) &= F(\sigma_1\cdots\sigma_{n-1}) = \langle(\theta_{f(s_n)}\otimes\mathrm{id}_{f(s_1)\otimes\cdots\otimes f(s_{n-1})})\beta_{f(s_1)\otimes\cdots\otimes f(s_{n-1}),f(s_n)}\rangle\\ &= \mathrm{id}_{F(s_1,\cdots,s_n)}\end{aligned}$$

by the balanced contraction property.

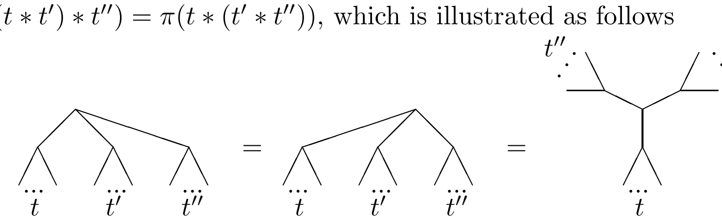
According to Proposition 6, this implies that we have a factorization
$$\begin{tikzcd}\widetilde{\mathbf{Cyc}}_S \arrow[r] \arrow[dr] & \mathbf{Cyc}_S \arrow[d]\\ & \mathcal{O}\end{tikzcd}$$
$\square$

We now construct the category $\mathbf{PaCyc}_S$ as follows. Let $PlT_n :=$ {planar 3-valent trees equipped with a bijection {leaves} $\to [n]$, compatible with the cyclic orders}. We first define the category $\widetilde{\mathbf{PaCyc}}_S$ by $\mathrm{Ob}\,\widetilde{\mathbf{PaCyc}}_S=\sqcup_{n\geq 0}PlT_n\times S^n$, $\mathrm{Hom}_{\widetilde{\mathbf{PaCyc}}_S}((t,\sigma),(t',\sigma'))=\mathrm{Hom}_{\widetilde{\mathbf{Cyc}}_S}(\sigma,\sigma')$. We define an action of $\mathbb{Z}$ on $\mathbf{PaCyc}_S$ by $1\cdot(t,(s_1,\ldots,s_n)):=(t',(s_n,s_1,\ldots,s_{n-1}))$, where if $t=(T,\{\text{leaves of }T\}\xrightarrow{\alpha}[n])$, then $t':=(T,\{\text{leaves of }T\}\xrightarrow{\alpha}[n]\overset{+1 \bmod n}{\to}[n])$, and $i^1_{(t,\sigma)}:=i^1_\sigma$; we then set $\mathbf{PaCyc}_S:=\widetilde{\mathbf{PaCyc}}_S/\mathbb{Z}$, so in particular $\mathrm{Ob}\,\mathbf{PaCyc}_S=\{$(a planar 3-valent tree, a map {leaves} $\to S$)$\}$.

We define a map $T_n\to PlT_n$, $t\mapsto\pi(t)$ as the operation of (a) assigning labels $1,\ldots,n$ to the vertices of the tree t, numbered from left to right; (b) replacing the root and the edges connected to it, by a single edge. E.g., we have



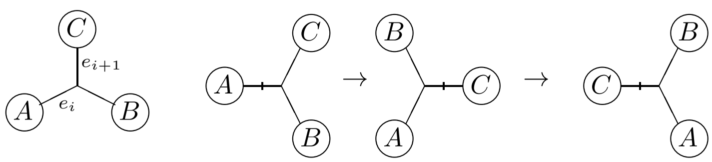
We define a functor $\mathbf{PaB}_S^{bal}\to\mathbf{PaCyc}_S$ by the condition that (a) at the level of objects, it is given by the map $\sqcup_{n\geq 0}T_n\times S^n\to\sqcup_{n\geq 0}(PlT_n\times S^n)/C_n$ and by projection, and (b) the diagram $\begin{array}{ccc}\mathbf{PaB}_S^{bal} & \to & \mathbf{PaCyc}_S\\ \downarrow & & \downarrow\\ \mathbf{B}_S^{bal} & \to & \mathbf{Cyc}_S\end{array}$ commutes. Let us check that this defines a contraction.
$\langle X\otimes Y\rangle=\langle Y\otimes X\rangle$ follows from the fact that for $t\in T_n$, $t'\in T_{n'}$, $\pi(t\otimes t')$ and $\pi(t'\otimes t)$ can be obtained from one another by cyclic permutation of $[n+n']$; here we recall that $(t,t')\mapsto t*t'$ is the concatenation map $T_n\times T_{n'}\to T_{n+n'}$. The fact that $\langle(X\otimes Y)\otimes Z\rangle=\langle X\otimes(Y\otimes Z)\rangle$ follows from $\pi((t*t')*t'')=\pi(t*(t'*t''))$, which is illustrated as follows



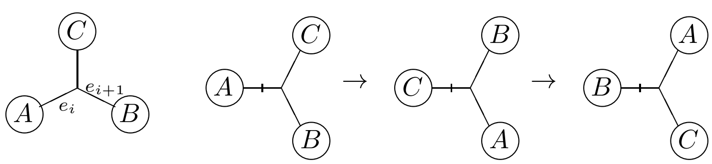
It is then clear that $\langle a_{XYZ}\rangle = \mathrm{id}_{\langle X,Y,Z\rangle}$. The proof of $\langle(\theta_Y \otimes \mathrm{id}_X)\beta_{XY}\rangle = \mathrm{id}_{\langle X,Y\rangle}$ is as above. We now prove the universality of the contraction $\langle - \rangle : \mathbf{PaB}_S^{bal} \to \mathbf{PaCyc}_S$.

Proposition 21. *Let $\mathcal{C}$ be a balanced small b.m.c., equipped with a contraction $\mathcal{C} \to \mathcal{O}$ and a map $S \to \mathrm{Ob}\,\mathcal{C}$. Then there exists a functor $\mathbf{PaCyc}_S \to \mathcal{O}$, such that the diagram* $\begin{array}{ccc} \mathbf{PaB}_S^{bal} & \to & \mathbf{PaCyc}_S \\ \downarrow & & \downarrow \\ \mathcal{C} & \to & \mathcal{O} \end{array}$ *commutes.*

Proof. We first construct a functor $\widetilde{\mathbf{PaCyc}}_S \to \mathcal{O}$, such that $\begin{array}{ccc} \mathbf{PaB}_S^{bal} & \to & \widetilde{\mathbf{PaCyc}}_S \\ \downarrow & & \downarrow \\ \mathcal{C} & \to & \mathcal{O} \end{array}$ commutes. We define a map $PlT_n \times S^n \to \mathrm{Ob}\,\mathcal{O}$ as follows. Let $(t, (s_1, \ldots, s_n)) \in PlT_n \times S^n$. Let e be an edge of t. Cutting t at e, we obtain two rooted trees t_i ($i = 1, 2$) equipped with injective maps $\{\text{leaves of } t_i\} \to [n]$. The images of these maps are of the form $\{a, a+1, \ldots, a+n_1\}$ and $\{a+n_1+1, \ldots, a+n_1+n_2\}$ (the integers being taken modulo n). We then define the image of $(t, (s_1, \ldots, s_n))$ to be $\langle(\otimes^{t_1}_{i\in a+[n_1]} f(s_i)) \otimes (\otimes^{t_2}_{i\in a+n_1+[n_2]} f(s_i))\rangle$. The axioms then imply that this object do not depend on e. Indeed, if e' is another edge, then to the shortest path $e = e_1 \to e_2 \to \cdots \to e_k = e'$ from e to e' there corresponds a sequence of isomorphisms of the corresponding objects; each isomorphism has the form $\langle A \otimes (B \otimes C)\rangle \xrightarrow{\langle a^{-1}_{ABC}\rangle} \langle (A \otimes B) \otimes C\rangle \xrightarrow{\langle \beta^{-1}_{C,A\otimes B}(\theta_C^{-1} \otimes \mathrm{id}_{A\otimes B})\rangle} \langle C \otimes (A \otimes B)\rangle$, see



or $\langle A \otimes (B \otimes C)\rangle \to \langle C \otimes (A \otimes B)\rangle \to \langle B \otimes (C \otimes A)\rangle$, see



One then proves as before that we have a functor $\widetilde{\mathbf{PaCyc}}_S \to \mathcal{O}$, which factors through the action of $\mathbb{Z}$. $\square$

5. Universal contractions for half-balanced categories

We now construct categories $(\mathbf{Pa})\mathbf{Dih}_S$ and a commutative diagram $\begin{array}{ccc} \mathbf{PaB}_S^{hbal} & \to & \mathbf{PaDih}_S \\ \downarrow & & \downarrow \\ \mathbf{B}_S^{hbal} & \to & \mathbf{Dih}_S \end{array}$ where the horizontal functors are contractions.

We first construct $\mathbf{Dih}_S$ as follows. Define first $\widetilde{\mathbf{Dih}}_S$ as the category with the same objects as $\mathbf{B}_S^{hbal}$, with B_n replaced by its quotient $\Gamma_{0,n}$. Let $D := \mathbb{Z} \rtimes (\mathbb{Z}/2)$ be the infinite dihedral group presented as $D := \langle r, s | s^2 = (rs)^2 = 1\rangle$. We define an action of D on $\widetilde{\mathbf{Dih}}_S$ as follows. The action on objects is defined by $r\cdot(s_1,\ldots,s_n) := (s_n,s_1,\ldots,s_{n-1})$, $s\cdot(s_1,\ldots,s_n) := (s_n,\ldots,s_1)$, and $i^r_{\underline{s}} = \sigma_1\cdots\sigma_{n-1}$, $i^s_{\underline{s}} = h_n$. We then set $\mathbf{Dih}_S = \widetilde{\mathbf{Dih}}_S/D$.

Note that $\mathrm{Ob}\,\mathbf{Dih}_S = \sqcup_{n\geq 0}\mathrm{Dih}_n(S)$, where $\mathrm{Dih}_n(S) = S^n/D_n$, and D_n is the quotient of D by the relation $r^n = 1$. We define a functor $\mathbf{B}^{hbal}_S \overset{\langle-\rangle}{\to} \mathbf{Dih}_S$ as the composite functor $\mathbf{B}^{hbal}_S \to \widetilde{\mathbf{Dih}}_S \to \mathbf{Dih}_S$. Let us show that it satisfies the half-balanced contraction conditions.

We have a commutative diagram $\begin{array}{ccc} \mathbf{B}^{bal}_S & \overset{\langle-\rangle}{\to} & \mathbf{Cyc}_S \\ \downarrow & & \downarrow \\ \mathbf{B}^{hbal}_S & \overset{\langle-\rangle}{\to} & \mathbf{Dih}_S \end{array}$ Since the left vertical functor is surjective on objects and the bottom functor is a balanced contraction, the upper functor satisfies the balanced contraction condition. If now $\underline{s} = (s_1,\ldots,s_n) \in \mathrm{Ob}\,\mathbf{B}^{hbal}_S$, then $\underline{s}^* = (s_n,\ldots,s_1) = s\cdot\underline{s}$, so the classes of $\underline{s}$ and $\underline{s}^*$ are the same in $\mathbf{Dih}_S = \widetilde{\mathbf{Dih}}_S/D$, hence $\langle\underline{s}\rangle = \langle\underline{s}^*\rangle$. Then $\langle a_{\underline{s}}\rangle = \langle i^s_{\underline{s}}\rangle = \mathrm{id}_{\langle\underline{s}\rangle}$. All this shows that $\mathbf{B}^{hbal}_S \overset{\langle-\rangle}{\to} \mathbf{Dih}_S$ is a half-balanced contraction. We now prove the universality of this contraction.

Proposition 22. *Let $\mathcal{C}$ be a strict half-balanced b.m.c., equipped with a map $S \overset{f}{\to} \mathrm{Ob}\,\mathcal{C}$, such that $f(s)^* = f(s)$ for any $s\in S$, and with a half-balanced contraction $\mathcal{C}\to\mathcal{O}$. Then there exists a functor $\mathbf{Dih}_S \to \mathcal{O}$, such that the diagram* $\begin{array}{ccc} \mathbf{B}^{hbal}_S & \to & \mathbf{Dih}_S \\ \downarrow & & \downarrow \\ \mathcal{C} & \to & \mathcal{O} \end{array}$ *commutes.*

Proof. We define a functor $\widetilde{\mathbf{Dih}}_S \to \mathcal{O}$ by the following conditions: it coincides at the level of objects with the functor $\mathbf{B}^{hbal}_S \to \mathcal{C} \to \mathcal{O}$; since the images by this functor of $z_n, \sigma_1\cdots\sigma^2_{n-1}\cdots\sigma \in \mathrm{Aut}_{\mathbf{B}^{hbal}_S}(s_1,\ldots,s_n) \subset B_n$ are respectively $\langle\theta_{f(s_1)\otimes\cdots\otimes f(s_n))}\rangle$ and

$$\langle(\theta^2_{f(s_1)}\otimes\mathrm{id}_{\otimes^n_{i=2}f(s_i)}\,\beta_{\otimes^n_{i=2}f(s_i),f(s_1)}\beta_{f(s_1),\otimes^n_{i=2}f(s_i)}\rangle \in \mathrm{Aut}_{\mathcal{O}}(\langle f(s_1)\otimes\cdots\otimes f(s_n)\rangle),$$

which are the identity by Remark 13 and Lemma 15, the composite functor $\mathbf{B}^{hbal}_S \to \mathcal{C}\to\mathcal{O}$ factorizes as $\begin{array}{ccc} \mathbf{B}^{hbal}_S & \to & \widetilde{\mathbf{Dih}}_S \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{C} & \to & \mathcal{O} \end{array}$ We now show as above that F factorizes as $\begin{array}{ccc} \widetilde{\mathbf{Dih}}_S & \to & \mathbf{Dih}_S \\ & {\scriptstyle F}\searrow & \downarrow \\ & & \mathcal{O} \end{array}$

Indeed, for $\underline{s} = (s_1,\ldots,s_n) \in \mathrm{Ob}\,\widetilde{\mathbf{Dih}}_S$, then $F(\underline{s}) = \langle f(s_1)\otimes\cdots\otimes f(s_n)\rangle$. Then $F(r\cdot\underline{s}) = \langle f(s_n)\otimes\cdots\otimes f(s_{n-1})\rangle = F(\underline{s})$, using the axiom $\langle X\otimes Y\rangle = \langle Y\otimes X\rangle$ of balanced contractions, and $F(s\cdot\underline{s}) = \langle f(s_n)\otimes\cdots\otimes f(s_1)\rangle = \langle(f(s_1)\otimes\cdots\otimes f(s_n))^*\rangle = \langle f(s_1)\otimes\cdots\otimes f(s_n)\rangle = F(\underline{s})$ using the axiom $\langle X^*\rangle = \langle X\rangle$ of half-balanced contraction. If now $\underline{s} = (s_1,\cdots,s_n) \in \mathrm{Ob}\,\widetilde{\mathbf{Dih}}_S$, then $F(i^r_{\underline{s}}) = \mathrm{id}_{\langle X\rangle}$ by the same argument as in Proposition 20, and

$$\begin{aligned} F(i^s_{\underline{s}}) = f(h_n) = \quad & (a_{f(s_n)}\otimes\cdots\otimes a_{f(s_1)})(\mathrm{id}_{f(s_1)\otimes\cdots\otimes f(s_{n-2})}\otimes\beta_{f(s_{n-1}),f(s_n)}) \\ & (\mathrm{id}_{f(s_1)\otimes\cdots\otimes f(s_{n-3})}\otimes\beta_{f(s_{n-2}),f(s_{n-1})\otimes f(s_n)})\cdots\beta_{f(s_1),f(s_2)\otimes\cdots\otimes f(s_n)} \\ & = a_{f(s_1)\otimes\cdots\otimes f(s_n)}. \end{aligned}$$

Hence $F(i^s_{\underline{s}}) = \langle a_{f(s_1)\otimes\cdots\otimes f(s_n)}\rangle = \mathrm{id}_{\langle\underline{s}\rangle}$. So we have the desired factorization of F. $\square$

We now construct the category $\mathbf{PaDih}_S$ as follows. We first define the category $\widetilde{\mathbf{PaDih}}_S$ by $\mathrm{Ob}\,\widetilde{\mathbf{PaDih}}_S = \sqcup_{n\geq 0} PlT_n\times S^n$, $\widetilde{\mathbf{PaDih}}_S((t,\underline{s}),(t',\underline{s}')) = \widetilde{\mathbf{Dih}}_S(\underline{s},\underline{s}')$. The group D acts on $\widetilde{\mathbf{PaDih}}_S$ as follows. The action on objects is $g\cdot(t,\underline{s}) = (g\cdot t, g\cdot\underline{s})$, where for $t = (T,\{\text{leaves of } T\} \overset{\alpha}{\to} [n])$, $r\cdot t = (T,\{\text{leaves of } T\} \overset{\alpha}{\to} [n] \overset{+1 \bmod n}{\to} [n])$, $s\cdot t = (T,\{\text{leaves of } T\} \overset{\alpha}{\to} [n] \overset{x\mapsto n+1-x}{\to} [n])$, and $i^g_{(t,\underline{s})} = i^g_{\underline{s}} \in \mathrm{Iso}_{\widetilde{\mathbf{Dih}}_S}(\underline{s}, g\cdot\underline{s})$ for $(t,\underline{s}) \in \mathrm{Ob}\,\widetilde{\mathbf{PaDih}}_S$. We then set $\mathbf{PaDih}_S := \widetilde{\mathbf{PaDih}}_S/D$.

We have $\mathrm{Ob}\,\mathbf{PaDih}_S = \{(\text{a planar 3-valent tree, a map } \{\text{leaves}\} \to S)\}/(\text{mirror symmetry}) = \sqcup_{n\geq 0}(PlT_n \times S^n)/D_n$. We define a functor $\mathbf{PaB}_S^{hbal} \to \mathbf{PaDih}_S$ by the condition that: (a) at the level of objects, it is given by the canonical map $T_n \times S^n \to (PlT_n \times S^n)/D_n$, (b) the diagram $\begin{array}{ccc} \mathbf{PaB}_S^{hbal} & \to & \mathbf{PaDih}_S \\ \downarrow & & \downarrow \\ \mathbf{B}_S^{hbal} & \to & \mathbf{Dih}_S \end{array}$ commutes. One proves as above that this is a half-balanced contraction. Using the arguments of the proofs of Propositions 20, 21 and 22, one proves:

Proposition 23. *Let $\mathcal{C}$ be a half-balanced braided monoidal category, equipped with a map $S \xrightarrow{f} \mathrm{Ob}\,\mathcal{C}$ such that $f(s)^* = f(s)$ for any $s \in S$, and a balanced contraction $\mathcal{C} \to \mathcal{O}$. Then there exists a functor $\mathbf{PaDih}_S \to \mathcal{O}$, such that the diagram* $\begin{array}{ccc} \mathbf{PaB}_S & \to & \mathbf{PaDih}_S \\ \downarrow & & \downarrow \\ \mathbf{B}_S^{hbal} & \to & \mathbf{Dih}_S \end{array}$ *commutes.*

We then have natural diagrams

$$\begin{array}{ccccc} \mathbf{B}_S & \to & \mathbf{B}_S^{bal} & \to & \mathbf{B}_S^{hbal} \\ & & \downarrow & & \downarrow \\ & & \mathbf{Cyc}_S & \to & \mathbf{Dih}_S \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathbf{PaB}_S & \to & \mathbf{PaB}_S^{bal} & \to & \mathbf{PaB}_S^{hbal} \\ & & \downarrow & & \downarrow \\ & & \mathbf{PaCyc}_S & \to & \mathbf{PaDih}_S \end{array}$$

These diagrams fit in a bigger diagram, with a collection of functors from the left to the right-hand side diagram.

6. Completions

Let $G \to S_n$ be a group morphism. One can define the relative pro-l and relative prounipotent completions G_l and $G(-)$ of $G \to S_n$. They fit in exact sequences $1 \to U_l \to G_l \to S_n \to 1$ and $1 \to U(-) \to G(-) \to S_n \to 1$, where U_l and $U(-)$ are pro-l and $\mathbb{Q}$-prounipotent. We have a morphism $G_l \to G(\mathbb{Q}_l)$ ([HM], Lemma A.7), fitting in a sequence of morphisms $G \to \widehat{G} \to G_l \to G(\mathbb{Q}_l)$, where $\widehat{G}$ is the profinite completion of G. Applying this to B_n are any of this quotients B_n/Z_n, $\Gamma_{0,n}$ considered above, we obtain for each of the categories $\mathcal{C} = (\mathbf{Pa})\mathbf{B}_S^{(h)(bal)}$, $(\mathbf{Pa})\mathbf{Cyc}_S$, $(\mathbf{Pa})\mathbf{Dih}_S$, completed categories $\widehat{\mathcal{C}}$, $\mathcal{C}_l$, $\mathcal{C}(-)$, and functors $\mathcal{C} \to \widehat{\mathcal{C}} \to \mathcal{C}_l \to \mathcal{C}(\mathbb{Q}_l)$.

Let us say that a pro-l (resp., prounipotent) b.m.c. is a b.m.c. $\mathcal{C}$, equipped with an assignment $\mathrm{Ob}\,\mathcal{C} \ni X \mapsto \mathcal{U}_X \lhd \mathrm{Aut}_{\mathcal{C}}(X)$, such that $\mathcal{U}_X$ is pro-l (resp., prounipotent) for any X, and for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Iso}_{\mathcal{C}}(X,Y)$, $f\mathcal{U}_X f^{-1} = \mathcal{U}_Y$ and $\mathrm{im}(P_n \to \mathrm{Aut}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n)) \subset \mathcal{U}_{X_1 \otimes \cdots \otimes X_n}$ (here $P_n = \mathrm{Ker}(B_n \to S_n)$ is the pure braid group with n strands). Similarly, $\mathcal{C}$ is called profinite if $\mathrm{Aut}_{\mathcal{C}}(X)$ is profinite for any $X \in \mathrm{Ob}\,\mathcal{C}$.

Then the completions $\widehat{(\mathbf{Pa})\mathbf{B}}_S$, $(\mathbf{Pa})\mathbf{B}_{S,l}$ and $(\mathbf{Pa})\mathbf{B}_S(-)$ are profinite, pro-l and prounipotent (strict) braided monoidal categories and are universal for such braided monoidal categories $\mathcal{C}$, equipped with a map $S \to \mathrm{Ob}\,\mathcal{C}$.

7. Actions of the Grothendieck–Teichmüller group

7.1. Grothendieck-Teichmüller semigroups. Recall that the Grothendieck–Teichmüller semigroup is defined ([Dr]) as

$$\underline{\mathrm{GT}} = \{(\lambda, f) \in (1+2\mathbb{Z}) \times F_2 | f(Y,X) = f(X,Y)^{-1},$$
$$f(X_3,X_1)X_3^m f(X_2,X_3)X_2^m f(X_1,X_2)X_1^m = 1, \quad \partial_3(f)\partial_1(f) = \partial_0(f)\partial_2(f)\partial_4(f)\},$$

where F_2 is the free group with two generators X, Y, $\partial_0, \ldots, \partial_4 : F_2 \to P_4$ are simplicial morphisms, $X_1X_2X_3 = 1$, $m = (\lambda-1)/2$. It is a semigroup with $(\lambda, f)(\lambda', f') = (\lambda'', f'')$, where $\lambda'' = \lambda\lambda'$ and $f'' = \theta_{(\lambda',f')}(f)f'$, where $\theta_{(\lambda',f')} \in \mathrm{End}(F_2)$ is given by $(X,Y) \mapsto (f'X^{\lambda'}f'^{-1}, Y^{\lambda'})$. Then $\underline{\mathrm{GT}} \to \mathrm{End}(F_2)^{op}$, $(\lambda, f) \mapsto \theta_{(\lambda,f)}$ is a semigroup morphism. The profinite, pro-l and prounipotent analogues $\widehat{\underline{\mathrm{GT}}}$, $\underline{\mathrm{GT}}_l$ and $\underline{\mathrm{GT}}(-)$ of $\underline{\mathrm{GT}}$ are defined by replacing $(\mathbb{Z}, F_2)$ by $(\widehat{\mathbb{Z}}, \widehat{F}_2)$, $(\mathbb{Z}_l, (F_2)_l)$, and $\mathbf{k} \mapsto (\mathbf{k}, F_2(\mathbf{k}))$ where $\mathbf{k}$ is a $\mathbb{Q}$-ring. We then have morphisms of semigroups $\underline{\mathrm{GT}} \to \widehat{\underline{\mathrm{GT}}} \to \underline{\mathrm{GT}}_l \to \underline{\mathrm{GT}}(\mathbb{Q}_l)$; the associated groups are denoted $\mathrm{GT}, \widehat{\mathrm{GT}}, \mathrm{GT}_l, \mathrm{GT}(-)$.

7.2. **Action on (half-)braided monoidal categories.** The semigroup $\underline{\mathrm{GT}}$ acts on {braided monoidal categories} as follows: $(\lambda, f) * (\mathcal{C}, \otimes, \beta_{XY}, a_{XYZ}) = (\mathcal{C}, \otimes, \beta'_{XY}, a'_{XYZ})$, where $\beta'_{XY} = \beta_{XY}(\beta_{YX}\beta_{XY})^m$ and

$$a'_{XYZ} = a_{XYZ} f(\beta_{YX}\beta_{XY} \otimes \mathrm{id}_Z, a^{-1}_{XYZ}(\mathrm{id}_X \otimes \beta_{ZY}\beta_{YZ})a_{XYZ}).$$

In the same way, $\widehat{\underline{\mathrm{GT}}}$ acts on {braided monoidal categories $\mathcal{C}$, such that $\mathrm{Aut}_{\mathcal{C}}(X)$ is finite for any $X \in \mathrm{Ob}\,\mathcal{C}$}, $\underline{\mathrm{GT}}_l$ acts on {pro-l braided monoidal categories} and $\underline{\mathrm{GT}}(\mathbf{k})$ acts on {$\mathbf{k}$-prounipotent braided monoidal categories}.

We have natural functors {half-balanced braided monoidal categories} $\to$ {balanced braided monoidal categories} $\to$ {braided monoidal categories}.

Proposition 24. *The action of* $\underline{\mathrm{GT}}$ *on {braided monoidal categories} lifts to compatible actions on {(half-)balanced braided monoidal categories}. Similarly, the actions of* $\widehat{\underline{\mathrm{GT}}}, \ldots, \underline{\mathrm{GT}}(\mathbf{k})$ *lift to compatible actions on {(half-)balanced finite braided monoidal categories}, ..., {(half-)balanced* $\mathbf{k}$*-prounipotent braided monoidal categories}.*

Proof. This lift is given by $(\lambda, f) * (\mathcal{C}, \otimes, \beta_{XY}, a_{XYZ}, \theta_X) := (\mathcal{C}, \otimes, \beta'_{XY}, a'_{XYZ}, \theta'_X)$, where $\theta'_X := \theta_X^\lambda$ and $(\lambda, f) * (\mathcal{C}, \otimes, \beta_{XY}, a_{XYZ}, a_X) := (\mathcal{C}, \otimes, \beta'_{XY}, a'_{XYZ}, a'_X)$, where $a'_X := (a_{X^*}a_X)^m a_X$, where $m = (\lambda - 1)/2$. □

Proposition 25. *Let* $\mathcal{C}$ *be a half-balanced category and let* $\mathcal{C} \overset{\langle - \rangle}{\to} \mathcal{O}$ *be a half-balanced contraction. Then for any* $(\lambda, f) \in \underline{\mathrm{GT}}$*, the composite functor* $(\lambda, f) * \mathcal{C} \overset{\sim}{\to} \mathcal{C} \overset{\langle - \rangle}{\to} \mathcal{O}$ *is a half-balanced contraction on* $(\lambda, f) * \mathcal{C}$*. Here* $(\lambda, f) * \mathcal{C} \overset{\sim}{\to} \mathcal{C}$ *is the identity functor (which is not tensor). Same statements with* $\mathcal{C}$ *finite, ...,* $\mathbf{k}$*-unipotent and* $\underline{\mathrm{GT}}$ *replaced by* $\widehat{\underline{\mathrm{GT}}}$*, ...,* $\underline{\mathrm{GT}}(\mathbf{k})$.

Proof. Assume that $(\mathcal{C}, \beta_{XY}, a_X)$ is half-balanced; we set $\theta_X := a_{X^*}a_X$. Then $(\mathcal{C}, \beta_{XY}, \theta_X)$ is balanced and $\theta'_X = \theta_X^\lambda$. Then $(\theta'_Y \otimes \mathrm{id}_X)\beta'_{XY} = (\theta_Y \otimes \mathrm{id}_X)\beta_{XY}(\theta_X^{-1} \otimes \theta_Y)^m \theta^m_{X \otimes Y}$. The identities $\langle \theta_X \rangle = \mathrm{id}_{\langle X \rangle}$, $\langle \theta_X^{-1} \otimes \theta_Y \rangle = \mathrm{id}_{\langle X,Y \rangle}$ (see Lemma 15) and $\langle (\theta_Y \otimes \mathrm{id}_X)\beta_{XY} \rangle = \mathrm{id}_{\langle X,Y \rangle}$ (as $\langle - \rangle$ is a half-balanced contraction) imply that $\langle (\theta'_Y \otimes \mathrm{id}_X)\beta'_{XY} \rangle = \mathrm{id}_{\langle X,Y \rangle}$, so $\langle - \rangle$ is a balanced contraction for $(\lambda, f) * \mathcal{C}$. Moreover, $a'_X = a_X(a_{X^*}a_X)^m = a_X\theta_X^m$, so $\langle \theta_X \rangle = \mathrm{id}_{\langle X \rangle}$ implies $\langle a'_X \rangle = \langle a_X \rangle = \mathrm{id}_{\langle X \rangle}$. □

7.3. **Action on $\mathbf{PaDih}_S$.** For $(\lambda, f) \in \underline{\mathrm{GT}}$, let $i_{(\lambda,f)}$ be the endomorphism of $\mathbf{PaB}_S^{(h)bal}$ defined as the composite functor $\mathbf{PaB}_S^{(h)bal} \overset{\alpha_{(\lambda,f)}}{\to} (\lambda, f) * \mathbf{PaB}_S^{(h)bal} \overset{\sim}{\to} \mathbf{PaB}_S^{(h)bal}$, where the first functor is the unique (half-)balanced monoidal functor which is the identity on objects, and the second functor is the identity functor (which is not monoidal). As in [E], Proposition 80, one shows that $(\lambda, f) \mapsto i_{(\lambda,f)}$ is a morphism $\underline{\mathrm{GT}} \to \mathrm{End}(\mathbf{PaB}_S^{(h)bal})^{op}$. One similarly defines morphisms $\widehat{\underline{\mathrm{GT}}} \to \mathrm{End}(\widehat{\mathbf{PaB}}_S^{(h)bal})^{op}$, ..., $\underline{\mathrm{GT}}(\mathbf{k}) \to \mathrm{End}(\mathbf{PaB}_{S,\mathbf{k}}^{(h)bal})^{op}$.

For $(\lambda, f) \in \underline{\mathrm{GT}}$, we define an endofunctor $j_{(\lambda,f)}$ of $\mathbf{PaDih}_S$ as follows: according to Proposition 25, the composite functor $(\lambda, f) * \mathbf{PaB}_S^{hbal} \overset{\sim}{\to} \mathbf{PaB}_S^{hbal} \overset{\langle - \rangle}{\to} \mathbf{PaDih}_S$ is a half-balanced contraction. By universality of the contraction $\mathbf{PaB}_S^{hbal} \overset{\langle - \rangle}{\to} \mathbf{PaDih}_S$, there exists a unique endofunctor $j_{(\lambda,f)}$ of $\mathbf{PaDih}_S$, such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{PaB}_S^{hbal} \arrow[r, "\alpha_{(\lambda,f)}"'] \arrow[d, "\langle - \rangle"'] \arrow[rr, bend left, "i_{(\lambda,f)}"] & (\lambda,f)*\mathbf{PaB}_S^{hbal} \arrow[r, "\sim"] \arrow[d] & \mathbf{PaB}_S^{hbal} \arrow[dl, "\langle - \rangle"] \\
\mathbf{PaDih}_S \arrow[r, "j_{(\lambda,f)}"'] & \mathbf{PaDih}_S &
\end{tikzcd}$$

Proposition 26. *The map* $(\lambda, f) \mapsto j_{(\lambda,f)}$ *defines a morphism* $\underline{\mathrm{GT}} \to \mathrm{End}(\mathbf{PaDih}_S)^{op}$*; one similarly defines morphisms* $\widehat{\underline{\mathrm{GT}}} \to \mathrm{End}(\widehat{\mathbf{PaDih}}_S)^{op}$*, etc.*

Proof. We have a commutative diagram

(1)
$$\begin{tikzcd}
\mathbf{PaB}_S^{hbal} \arrow[r, "\alpha_{(\lambda',f')}"] \arrow[dd, "\langle - \rangle"'] & (\lambda',f')*\mathbf{PaB}_S^{hbal} \arrow[d, "\sim"] \\
& \mathbf{PaB}_S^{hbal} \arrow[d, "\langle - \rangle"] \\
\mathbf{PaDih}_S \arrow[r, "j_{(\lambda',f')}"'] & \mathbf{PaDih}_S
\end{tikzcd}$$

which gives rise to

$$\begin{tikzcd}
(\lambda,f)*\mathbf{PaB}_S^{hbal} \arrow[r, "(\lambda,f)*\alpha_{(\lambda',f')}"] \arrow[dd, "\langle - \rangle"'] & (\lambda,f)(\lambda',f')*\mathbf{PaB}_S^{hbal} \arrow[d, "\sim"] \\
& \mathbf{PaB}_S^{hbal} \arrow[d, "\langle - \rangle"] \\
\mathbf{PaDih}_S \arrow[r, "j_{(\lambda',f')}"'] & \mathbf{PaDih}_S
\end{tikzcd}$$

Composing it with the analogue of (1) with (λ', f') replaced by (λ, f), we get a commutative diagram

$$\begin{tikzcd}
\mathbf{PaB}_S^{hbal} \arrow[r, "((\lambda,f)*\alpha_{(\lambda',f')})\circ\alpha_{(\lambda,f)}"] \arrow[dd, "\langle - \rangle"'] & (\lambda,f)(\lambda',f')*\mathbf{PaB}_S^{hbal} \arrow[d, "\sim"] \\
& \mathbf{PaB}_S^{hbal} \arrow[d, "\langle - \rangle"] \\
\mathbf{PaDih}_S \arrow[r, "j_{(\lambda',f')}\circ j_{(\lambda,f)}"'] & \mathbf{PaDih}_S
\end{tikzcd}$$

On the other hand, both $((\lambda,f)*\alpha_{(\lambda',f')})\alpha_{(\lambda,f)}$ and $\alpha_{(\lambda,f)(\lambda',f')}$ are tensor functors $\mathbf{PaB}_S^{hbal} \to (\lambda,f)(\lambda',f')*\mathbf{PaB}_S^{hbal}$ of half-balanced braided monoidal categories, inducing the identity at the level of objects, and by the uniqueness of such functors, they coincide. The above diagram may therefore be rewritten as

$$\begin{tikzcd}
\mathbf{PaB}_S^{hbal} \arrow[r, "\alpha_{(\lambda,f)(\lambda',f')}"] \arrow[dd, "\langle - \rangle"'] & (\lambda,f)(\lambda',f') * \mathbf{PaB}_S^{hbal} \arrow[d, "\sim"] \\
& \mathbf{PaB}_S^{hbal} \arrow[d, "\langle - \rangle"] \\
\mathbf{PaDih}_S \arrow[r, "j_{(\lambda',f')} \circ j_{(\lambda,f)}"'] & \mathbf{PaDih}_S
\end{tikzcd}$$

which may be viewed as a functor between half-balanced categories with a contraction.

On the other hand, another such a functor is

$$\begin{tikzcd}
\mathbf{PaB}_S^{hbal} \arrow[r, "\alpha_{(\lambda,f)(\lambda',f')}"] \arrow[dd, "\langle - \rangle"'] & (\lambda,f)(\lambda',f') * \mathbf{PaB}_S^{hbal} \arrow[d, "\sim"] \\
& \mathbf{PaB}_S^{hbal} \arrow[d, "\langle - \rangle"] \\
\mathbf{PaDih}_S \arrow[r, "j_{(\lambda,f)(\lambda',f')}"'] & \mathbf{PaDih}_S
\end{tikzcd}$$

By the universality of the contraction $\mathbf{PaB}_S^{hbal} \overset{\langle - \rangle}{\to} \mathbf{PaDih}_S$, we then have $j_{(\lambda,f)(\lambda',f')} = j_{(\lambda',f')} j_{(\lambda,f)}$. $\square$

7.4. **Action on Teichmüller groupoids and proof of Theorem 1.** $T_{0,S}$ may be viewed as the full subcategory of $\mathbf{PaDih}_S$ whose objects are the classes modulo D of $PlT_{|S|} \times \mathrm{Bij}(|S|, S)$. The action of $\underline{\mathrm{GT}}$ then restricts to $T_{0,S}$, and similarly in the completed cases. In the profinite case, one checks that that resulting action coincides with that defined in in [Sch]. This proves Theorem 1

7.5. **Proof of Theorem 2.** We define $T_{0,n}(\mathbf{k})$ by $\mathrm{Ob}\, T_{0,n}(\mathbf{k}) = \mathrm{Ob}\, T_{0,n}$ and for $b, c \in \mathrm{Ob}\, T_{0,n}$, $\mathrm{Hom}_{T_{0,n}(\mathbf{k})}(b,c) = \mathrm{Aut}_{T_{0,n}}(b)(\mathbf{k}) \times_{\mathrm{Aut}_{T_{0,n}}(b)} \mathrm{Hom}_{T_{0,n}}(b,c)$, where for G a finitely generated group, $G(\mathbf{k})$ is its prounipotent completion.

If π is a finitely generated group, we define the group scheme $\underline{\mathrm{Aut}\,\pi}(-)$ by $\underline{\mathrm{Aut}\,\pi}(\mathbf{k}) := \mathrm{Aut}((\mathrm{Lie}\,\pi)^{\mathbf{k}})$, where for $\mathrm{Lie}\,\pi$ is the Lie algebra of the prounipotent completion of π, $\mathfrak{g}^{\mathbf{k}} = \lim_{\leftarrow}(\mathfrak{g}/\mathfrak{g}_n) \otimes \mathbf{k}$, and $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$. We then have a morphism $\underline{\mathrm{Aut}\,\pi}(\mathbf{k}) \to \mathrm{Aut}(\pi(\mathbf{k}))$, $\theta \mapsto \theta_*$. $\mathrm{Aut}(\pi_l, \pi(\mathbb{Q}_l))$ is then defined as $\{(\theta, \theta_l) \in \underline{\mathrm{Aut}\,\pi}(\mathbb{Q}_l) \times \mathrm{Aut}(\pi_l) | \theta_* i = i\theta_l\}$, where i is the morphism $\pi_l \to \pi(\mathbb{Q}_l)$.

If G is a groupoid such that $\mathrm{Iso}_G(b,c) \neq \emptyset$ for any $b, c \in \mathrm{Ob}\, G$, then the choice of $b \in \mathrm{Ob}\, G$ gives rise to an isomorphism $\mathrm{Aut}\, G \simeq \pi^{\mathrm{Ob}\, G - \{b\}} \rtimes \mathrm{Aut}\,\pi$, where $\pi = \mathrm{Aut}_G(b)$; we then define the group scheme $\underline{\mathrm{Aut}\, G}(-)$ by $\underline{\mathrm{Aut}\, G}(\mathbf{k}) := \pi(\mathbf{k})^{\mathrm{Ob}\, G - \{b\}} \rtimes \underline{\mathrm{Aut}\,\pi}(\mathbf{k})$. We define as above $\mathrm{Aut}(G_l, G(\mathbb{Q}_l))$ and the morphisms $\mathrm{Aut}(G_l) \leftarrow \mathrm{Aut}(G_l, G(\mathbb{Q}_l)) \to \underline{\mathrm{Aut}\, G}(\mathbb{Q}_l)$.

We have morphisms $G_{\mathbb{Q}} \to \mathrm{GT}_l \to \mathrm{GT}(\mathbb{Q}_l)$ and a functor $\mathbf{PaB}_{S,l} \to \mathbf{PaB}_S(\mathbb{Q}_l)$. Theorem 2 follows from the fact that this functor is compatible with the actions of $\mathrm{GT}_l, \mathrm{GT}(\mathbb{Q}_l)$ on $\mathbf{PaB}_{S,l}$, $\mathbf{PaB}_S(\mathbb{Q}_l)$.

8. Graded aspects

Let $\mathfrak{t}_n$ be the graded Lie algebra with generators t_{ij}, $i \neq j \in [n]$ and relations $t_{ji} = t_{ij}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$, $[t_{ij}, t_{kl}] = 0$ for i, j, k, l distinct. Let $\mathfrak{p}_n$ be the quotient of $\mathfrak{t}_n$ by the relations $\sum_{j|j\neq i} t_{ij} = 0$, for any $i \in [n]$. Equivalently, $\mathfrak{p}_n$ is presented by generators t_{ij} are relations $t_{ji} = t_{ij}$, $\sum_{j|j\neq i} t_{ij} = 0$ for any i, and $[t_{ij}, t_{kl}] = 0$ for i, j, k, l distinct.

Let $\mathbf{k}$ be a $\mathbb{Q}$-ring, then the set $M(\mathbf{k})$ of Drinfeld associators defined over $\mathbf{k}$ is the set of pairs $(\mu, \Phi) \in \mathbf{k} \times \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$, satisfying the duality, hexagon and pentagon conditions[1] (see [Dr]). The data of $t \in T_n$ and $(\mu, \Phi) \in M(\mathbf{k})$ gives rise to a morphism $B_n \stackrel{i_{t,\Phi}}{\to} \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes S_n$, which extends to an isomorphism $B_n(\mathbf{k}) \stackrel{\sim}{\to} \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes S_n$ (see e.g. [AET]) if $\mu \in \mathbf{k}^\times$.

Proposition 27. *There exists a unique morphism $\Gamma_{0,n} \to \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}}) \rtimes S_n$, such that the diagram*
$$\begin{array}{ccc} B_n & \stackrel{i_{t,\Phi}}{\to} & \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes S_n \\ \downarrow & & \downarrow \\ \Gamma_{0,n} & \to & \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}}) \rtimes S_n \end{array}$$
commutes. It gives rise to an isomorphism $\Gamma_{0,n}(\mathbf{k}) \stackrel{\sim}{\to} \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}}) \rtimes S_n$.

Proof. One checks that $i_{t,\Phi}$ takes z_n to $\exp(\mu \sum_{1\leq i<j\leq n} t_{ij})$ and $\sigma_i \cdots \sigma_{n-1}^2 \cdots \sigma_0^2 \cdots \sigma_{i-1}$ to a conjugate of $\exp(\mu \sum_{j|j\neq i} t_{ij})$. This implies the announced commutative diagram. Let $\Gamma_{0,[n]} := \mathrm{Ker}(\Gamma_{0,n} \to S_n)$ and $\Gamma_{0,[n]}(\mathbf{k})$ be its $\mathbf{k}$-prounipotent completion. The morphism $\Gamma_{0,n} \to \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}}) \rtimes S_n$ gives rise to a morphism $\Gamma_{0,[n]}(\mathbf{k}) \to \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}})$; let us show that this is an isomorphism. We have a morphism $\mathfrak{t}_n \to \mathrm{gr}\,\mathrm{Lie}\, P_n$, where $P_n := \mathrm{Ker}(B_n \to S_n)$, given by $t_{ij} \mapsto$ class of $\log(\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}^2(\sigma_i \cdots \sigma_{j-2})^{-1}$. We then have a commutative diagram
$$\begin{array}{ccc} \mathfrak{t}_n & \to & \mathrm{gr}\,\mathrm{Lie}\, P_n \\ \downarrow & & \downarrow \\ \mathfrak{p}_n & \to & \mathrm{gr}\,\mathrm{Lie}\,\Gamma_{0,[n]} \end{array}$$
where the horizontal maps are surjective and the Lie algebras in the right side are generated in degree 1. The Lie algebra morphism corresponding to the group morphism $\Gamma_{0,[n]}(\mathbf{k}) \to \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}})$ is a Lie algebra morphism $\mathrm{Lie}\,\Gamma_{0,[n]}(\mathbf{k}) \to \hat{\mathfrak{p}}_n^{\mathbf{k}}$, whose associated graded morphism is a graded Lie algebra morphism $\mathrm{gr}\,\mathrm{Lie}\,\Gamma_{0,[n]}(\mathbf{k}) \to \mathfrak{p}_n^{\mathbf{k}}$. The composite map $\mathfrak{p}_n^{\mathbf{k}} \to \mathrm{gr}\,\mathrm{Lie}\,\Gamma_{0,[n]} \otimes \mathbf{k} \to \mathfrak{p}_n^{\mathbf{k}}$ is a graded isomorphism, as it can be checked on the degree 1 part of $\mathfrak{p}_n$. It follows that the morphism $\mathfrak{p}_n \to \mathrm{gr}\,\mathrm{Lie}\,\Gamma_{0,[n]}$ is injective as well, therefore it is an isomorphism of Lie algebras. So $\Gamma_{0,[n]}(\mathbf{k}) \to \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}})$ is an isomorphism. $\square$

We define a category $\mathbf{PaDih}_S^{gr}$ similarly to $\mathbf{PaDih}_S$, i.e., as the quotient by D of an intermediate category $\widetilde{\mathbf{PaDih}}_S^{gr}$ obtained from $\widetilde{\mathbf{PaDih}}_S$ by replacing $\Gamma_{0,n}$ by $\exp(\hat{\mathfrak{p}}_n^{\mathbf{k}}) \rtimes S_n$, and the morphism $D \to D_n \to \Gamma_{0,n}$ by $D \to D_n \to S_n \to \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}}) \rtimes S_n$.

If $(\mu, \Phi) \in M(\mathbf{k})$, recall that a braided monoidal category $\mathbf{PaCD}_S^\Phi$ may be defined as follows: $\mathrm{Ob}\,\mathbf{PaCD}_S^\Phi = \mathrm{Ob}\,\mathbf{PaB}_S$; $\mathrm{Hom}_{\mathbf{PaCD}_S^\Phi}((\underline{s}, t), (\underline{s}', t'))$ is empty if $|\underline{s}| \neq |\underline{s}'|$, and is equal to $\exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \{f \in S_n | \underline{s}' f = \underline{s}\}$; the composition is induced by the product in $\exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes S_n$; and the tensor product is obtained by restriction from the group morphism $(\exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes S_n) \times (\exp(\hat{\mathfrak{t}}_{n'}^{\mathbf{k}}) \rtimes S_{n'}) \to \exp(\hat{\mathfrak{t}}_{n+n'}^{\mathbf{k}}) \rtimes S_{n+n'}$, induced by the Lie algebra morphism $\hat{\mathfrak{t}}_n^{\mathbf{k}} \times \hat{\mathfrak{t}}_{n'}^{\mathbf{k}} \to \hat{\mathfrak{t}}_{n+n'}^{\mathbf{k}}$, $(t_{ij}, 0) \mapsto t_{ij}$, $(0, t_{ij}) \mapsto t_{n+i,n+j}$, and the group morphism $S_n \times S_{n'} \to S_{n+n'}$, $(\sigma, \sigma') \mapsto \sigma * \sigma'$, such that $(\sigma * \sigma')(i) = \sigma(i)$ for $i \in [n]$, and $(\sigma * \sigma')(n + i) = n + \sigma'(i)$ for $i \in [n']$. The braiding constraint is defined by $\beta_{XY} = (e^{\mu t_{12}/2})^{[n],n+[n']} s_{n,n'}$ and the associativity constraint is defined by $a_{XYZ} = (\Phi(t_{12}, t_{23}))^{[n],n+[n'],n+n'+[n'']}$ for $|X| = n$, $|Y| = n'$, $|Z| = n''$, $s_{n,n'} \in S_{n+n'}$ is defined by $s_{n,n'}(i) = n' + i$ for $i \in [n]$ and $s_{n,n'}(n + i) = i$ for $i \in [n']$, and for $I_1, \ldots, I_n \subset [m]$ disjoint subsets, the morphism $\mathfrak{t}_n \to \mathfrak{t}_m$, $x \mapsto x^{I_1,\ldots,I_n}$ is defined by $t_{ij} \mapsto \sum_{\alpha \in I_i, \beta \in I_j} t_{\alpha\beta}$. Then

[1] If $\mathfrak{g}$ is a graded Lie algebra, then $\hat{\mathfrak{g}}^{\mathbf{k}}$ is the degree completion of $\mathfrak{g} \otimes \mathbf{k}$.

$\mathbf{PaCD}_S^\Phi$ is a braided monoidal category; it follows that there is a unique monoidal functor $j_\Phi : \mathbf{PaB}_S \to \mathbf{PaCD}_S^\Phi$, which induces the identity on objects.

We then define a functor $\mathbf{PaCD}_S^\Phi \to \mathbf{PaDih}_S^{gr}$ as the composite functor $\mathbf{PaCD}_S^\Phi \to \widetilde{\mathbf{PaDih}}_S^{gr} \to \mathbf{PaDih}_S^{gr}$, where the first functor is induced by the projection morphisms $\mathfrak{t}_n \to \mathfrak{p}_n$ and the second functor is the quotient functor $\widetilde{\mathbf{PaDih}}_S^{gr} \to \widetilde{\mathbf{PaDih}}_S^{gr}/D \simeq \mathbf{PaDih}_S^{gr}$.

Proposition 28. *The functor $\mathbf{PaCD}_S^\Phi \to \mathbf{PaDih}_S^{gr}$ is a half-balanced contraction.*

Proof. We first show:

Lemma 29. *Let $X \in \mathrm{Ob}\,\mathbf{PaB}_{\{\bullet\}}$ be of degree n, then*

$$\mathrm{im}(h_n \in \mathbf{PaB}_{\{\bullet\}}(X,X^*) \to \mathbf{PaCD}^\Phi_{\{\bullet\}}(X,X^*)) = \exp(\frac{\mu}{2}\sum_{1\le i<j\le n} t_{ij})\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right) \in \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}})\rtimes S_n.$$

Proof. It suffices to prove this for a particular $X_0 \in \mathrm{Ob}\,\mathbf{PaB}_{\{\bullet\}}$ of degree n, say $X_0 = \bullet(\bullet(\cdots(\bullet\bullet)))$. Indeed, if we denote by $h_n^X \in \mathbf{PaB}_{\{\bullet\}}(X,X^*)$ the element corresponding to h_n and if we have $\mathrm{im}(h_n^{X_0}) = \exp(\frac{\mu}{2}\sum_{1\le i<j\le n} t_{ij})\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right)$, then if X is another object with the same degree, then $\mathrm{im}(h_n^X) = \Phi_{X_0^*,X^*}\,\mathrm{im}(h_n^{X_0})\Phi_{X,X_0}$, where $\Phi_{X,Y} = \mathrm{im}(1 \in \mathbf{PaB}_{\{\bullet\}}(X,Y) \to \mathbf{PaCD}^\Phi_{\{\bullet\}}(X,Y) = \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}})\rtimes S_n)$. As $\Phi_{X_0^*,X^*} = \left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right)\Phi_{X_0,X}\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right)$, $\sum_{1\le i<j\le n} t_{ij} \in \mathfrak{t}_n$ is central and $\Phi_{X_0^*,X^*} = \Phi^{-1}_{X^*,X_0^*}$, $\mathrm{im}(h_n^X) = \exp(\frac{\mu}{2}\sum_{1\le i<j\le n} t_{ij})\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right)$.

We now prove that statement for $X_0 = \bullet(\bullet(\cdots(\bullet\bullet)))$ of degree n, which we redenote X_n. The proof is by induction on n. The statement is clear for $n = 1, 2$. Assume it at order $n-1$. Then $h_n^{X_n} \in \mathbf{PaB}_{\{\bullet\}}(X_n, X_n^*)$ may be viewed as the composite morphism $X_n = \bullet \otimes X_{n-1} \overset{\mathrm{id}_\bullet \otimes h_{n-1}}{\to} \bullet \otimes X_{n-1}^* \overset{\beta_{\bullet,X_{n-1}^*}}{\to} X_{n-1}^* \otimes \bullet = X_n^*$, whose image in $\mathbf{PaCD}^\Phi_{\{\bullet\}}$ is $s\exp(\frac{\mu}{2}\sum_{i=2}^n t_{1i})\exp(\frac{\mu}{2}\sum_{2\le i<j\le n} t_{ij})\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ 1 & n & \cdots & 2\end{smallmatrix}\right)$, where $s = \left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & 1 & \cdots & n-1\end{smallmatrix}\right)$. So this image is $\exp(\frac{\mu}{2}\sum_{1\le i<j\le n} t_{ij})\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right)$. □

We then show:

Lemma 30. *Let $X, Y \in \mathrm{Ob}\,\mathbf{PaB}_{\{\bullet\}}$ be of degrees n, m, then*

$$\begin{aligned}&\mathrm{im}((\theta_Y \otimes \mathrm{id}_X)\beta_{XY} \in \mathbf{PaB}_{\{\bullet\}}(X\otimes Y, Y\otimes X) \to \mathbf{PaCD}^\Phi_{\{\bullet\}}(X\otimes Y, Y\otimes X))\\ &= \left(\begin{smallmatrix}1 & \cdots & n & n+1 & \cdots & n+m\\ m+1 & \cdots & m+n & 1 & \cdots & m\end{smallmatrix}\right)\exp(\frac{\mu}{2}\sum_{j\in n+[m]}\ \sum_{\alpha\in[n+m]-\{j\}} t_{\alpha j}).\end{aligned}$$

Proof. The image of β_{XY} is $\left(\begin{smallmatrix}1 & \cdots & n & n+1 & \cdots & n+m\\ m+1 & \cdots & m+n & 1 & \cdots & m\end{smallmatrix}\right)\exp(\frac{\mu}{2}\sum_{i\in[n], j\in n+[m]} t_{ij})$, while the image of $\theta_Y \otimes \mathrm{id}_X$ is $\exp(\mu\sum_{j<j'\in[m]} t_{jj'})$. □

End of proof of Proposition 28. If $X \in \mathrm{Ob}\,\mathbf{PaCD}_S^\Phi$ has degree n, then the image of $a_X \in \mathbf{PaCD}_S^\Phi(X,X^*)$ in $\widetilde{\mathbf{PaDih}}_S^{gr}(X,X^*) = \exp(\hat{\mathfrak{p}}_n^{\mathbf{k}}) \rtimes S_n$ is $\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right)$ as $\mathrm{im}(\sum_{i<j\in[n]} t_{ij} \in \mathfrak{t}_n \to \mathfrak{p}_n) = 0$. Now $\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{smallmatrix}\right) = i_X^s$, therefore after taking the quotient by D, $\langle a_X\rangle = \mathrm{id}_{\langle X\rangle}$ in $\mathrm{End}_{\mathbf{PaDih}_S^{gr}}(\langle X\rangle)$.

Similarly, if $X, Y \in \mathrm{Ob}(\mathbf{PaCD}_S^\Phi)$ have degrees n, m, then the image of $(\theta_Y \otimes \mathrm{id}_X)\beta_{XY} \in \mathbf{PaCD}_S^\Phi(X\otimes Y, Y\otimes X)$ in $\widetilde{\mathbf{PaDih}}_S^{gr}(X\otimes Y, Y\otimes X) = \exp(\hat{\mathfrak{p}}^{\mathbf{k}}_{n+m}) \rtimes S_{n+m}$ is c^m, where $c = \left(\begin{smallmatrix}1 & 2 & \cdots & n+m\\ 2 & 3 & \cdots & 1\end{smallmatrix}\right)$ as $\mathrm{im}(\sum_{\alpha\in[n+m]-\{j\}} t_{\alpha j} \in \mathfrak{t}_{n+m} \to \mathfrak{p}_{n+m}) = 0$ for any j. It follows that this image coincides with $i^{r^m}_{X\otimes Y}$, whose image in $\mathrm{Aut}_{\mathbf{PaDih}_S^{gr}}(\langle X\otimes Y\rangle)$ is $\mathrm{id}_{\langle X\otimes Y\rangle}$. It follows that $\langle(\theta_Y \otimes \mathrm{id}_X)\beta_{XY}\rangle = \mathrm{id}_{\langle X\otimes Y\rangle} \in \mathrm{Aut}_{\mathbf{PaDih}_S^{gr}}(\langle X\otimes Y\rangle)$. All this implies that $\mathbf{PaCD}_S^\Phi \to \mathbf{PaDih}_S^{gr}$ satisfies the half-balanced contraction conditions. □

Proposition 28 immediately implies:

Corollary 31. *There exists a unique functor* $\mathbf{PaDih}_S \stackrel{k_\Phi}{\to} \mathbf{PaDih}_S^{gr}$*, with is the identity on objects and such that the diagram*

$$\begin{array}{ccc} \mathbf{PaB}_S & \xrightarrow{j_\Phi} & \mathbf{PaCD}_S^\Phi \\ \downarrow{\scriptstyle\langle - \rangle} & & \downarrow{\scriptstyle\langle - \rangle} \\ \mathbf{PaDih}_S & \xrightarrow{k_\Phi} & \mathbf{PaDih}_S^{gr} \end{array}$$

commutes.

Recall that the graded Grothendieck-Teichmüller group $\mathrm{GRT}(\mathbf{k})$ is defined as $\mathrm{GRT}(\mathbf{k}) = \mathrm{GRT}_1(\mathbf{k}) \rtimes \mathbf{k}^\times$, where $\mathrm{GRT}_1(\mathbf{k})$ is the set of all $g \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}}) \subset \exp(\hat{\mathfrak{t}}_3^{\mathbf{k}})$ ($\mathfrak{f}_2 \subset \mathfrak{t}_3$ being the Lie subalgebra generated by t_{12}, t_{23}), such that

$$g^{3,2,1} = g^{-1}, \quad t_{12} + \mathrm{Ad}(g^{1,2,3})^{-1}(t_{23}) + \mathrm{Ad}(g^{2,1,3})^{-1}(t_{13}) = t_{12} + t_{23} + t_{13},$$
$$g^{2,3,4} g^{1,23,4} g^{1,2,3} = g^{1,2,34} g^{12,3,4},$$

equipped with the group law $(g_1 * g_2)(A,B) := g_1(\mathrm{Ad}(g_2(A,B))(A), B) g_2(A,B)$, on which $\mathbf{k}^\times$ acts by $(c \cdot g)(A,B) := g(c^{-1}A, c^{-1}B)$.

We now construct an action of this group on $\mathbf{PaDih}_S^{gr}$. For this, we recall from [E] the notion of infinitesimally braided monoidal category (i.b.m.c.).

Definition 32. *An i.b.m.c. is a braided monoidal category* $(\mathcal{C}, \otimes, c_{XY}, a_{XYZ})$*, which is*

(1) symmetric, i.e., such that $c_{YX} c_{XY} = \mathrm{id}_{X\otimes Y}$ *for any* $X, Y \in \mathrm{Ob}\,\mathcal{C}$*,*

(2) prounipotent (see Section 6), i.e., equipped with an assignment $\mathrm{Ob}\,\mathcal{C} \ni X \mapsto \mathcal{U}_X \triangleleft \mathrm{Aut}_\mathcal{C}(X)$*, such that* $f\mathcal{U}_X f^{-1} = \mathcal{U}_Y$ *for* $f \in \mathrm{Iso}_\mathcal{C}(X,Y)$*,*

(3) equipped with a functorial assignment $(\mathrm{Ob}\,\mathcal{C})^2 \ni (X,Y) \mapsto t_{XY} \in \mathrm{Lie}\mathcal{U}_{X\otimes Y}$*, such that* $t_{YX} = c_{YX} t_{YX} c_{XY}$ *and*

$$t_{X\otimes Y,Z} = a_{XYZ}(\mathrm{id}_X \otimes t_{YZ}) a_{XYZ}^{-1} + (c_{YX} \otimes \mathrm{id}_Z) a_{YXZ} (\mathrm{id}_Y \otimes t_{XZ}) a_{YXZ}^{-1} (c_{YX} \otimes \mathrm{id}_Z)^{-1}.$$

According to [Dr], $\mathrm{GRT}(\mathbf{k})$ acts on {i.m.b. categories} from the right as follows: $g \in \mathrm{GRT}_1(\mathbf{k}) \subset \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$ acts by $(\mathcal{C}, \otimes, c_{XY}, a_{XYZ}, t_{XY}) \cdot g := (\mathcal{C}, \otimes, c_{XY}, a'_{XYZ}, t_{XY})$, where $a'_{XYZ} := g(t_{XY} \otimes \mathrm{id}_Z, a_{XYZ}(\mathrm{id}_X \otimes t_{YZ}) a_{XYZ}^{-1}) a_{XYZ}$ and $c \in \mathbf{k}^\times$ acts by $(\mathcal{C}, \ldots) \cdot g := (\mathcal{C}, \otimes, c_{XY}, a_{XYZ}, c t_{XY})$. Moreover, $\mathbf{PaCD}_S$, equipped with $c_{XY} := s_{|X|,|Y|}$, $a_{XYZ} := \mathrm{id}_{|X|+|Y|+|Z|}$ and $t_{XY} := t_{12}^{[|X|],|X|+[|Y|]}$ is universal among i.b.m.cs $\mathcal{C}$, equipped with a map $S \to \mathrm{Ob}\,\mathcal{C}$. We derive from this, as in [E], Proposition 80, a morphism $\mathrm{GRT}(\mathbf{k}) \to \mathrm{Aut}(\mathbf{PaCD}_S)$.

We now introduce the notion of a balanced i.b.m.c.

Definition 33. *A balanced structure on the i.b.m.c.* $\mathcal{C}$ *is a functorial assignment* $\mathrm{Ob}\,\mathcal{C} \ni X \mapsto t_X \in \mathrm{Lie}\mathcal{U}_X$*, such that for any* $X, Y \in \mathrm{Ob}\,\mathcal{C}$*,* $t_{X\otimes Y} - t_X \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes t_Y = t_{XY}$*.*

Definition 34. *A contraction on the small balanced i.b.m.c.* $\mathcal{C}$ *is a functor* $\mathcal{C} \stackrel{\langle - \rangle}{\to} \mathcal{O}$*, such that for any* $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$*,* $\langle X \otimes Y \rangle = \langle Y \otimes X \rangle (=: \langle X, Y \rangle)$*,* $\langle (X\otimes Y) \otimes Z \rangle = \langle X \otimes (Y \otimes Z) \rangle (=: \langle X, Y, Z \rangle)$*,* $\langle c_{XY} \rangle = \mathrm{id}_{\langle X,Y \rangle}$*,* $\langle a_{XYZ} \rangle = \mathrm{id}_{\langle X,Y,Z \rangle}$*, and* $\langle t_{XY} + 2\,\mathrm{id}_X \otimes t_Y \rangle = 0$*.*

Remark 35. We derive from the latter condition that $\langle t_X \rangle = 0$ for any $X \in \mathrm{Ob}\,\mathcal{C}$. Indeed, it gives by symmetrization $\langle t_{X\otimes Y} \rangle = 0$, and therefore $\langle t_X \rangle = 0$ by taking $Y = \mathbf{1}$. By antisymmetrization, this condition also implies $\langle t_X \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes t_Y \rangle = 0$.

We now construct a universal contraction on balanced i.b.m. categories.

Proposition 36. *The i.b.m.c.* $\mathbf{PaCD}_S$ *is equipped with a balanced structure given by* $t_X = \sum_{1\le i<j\le n} t_{ij}$ *for* $|X| = n$*. Then the functor* $\mathbf{PaCD}_S \to \mathbf{PaDih}_S^{gr}$ *is a contraction.*

Proof. For $|X| = n$, $|Y| = m$, $t_{XY} + 2\,\mathrm{id}_X \otimes t_Y = \sum_{j\in n+[m]} \sum_{\alpha \in [n+m]-\{j\}} t_{j\alpha}$, so $\langle t_{XY} + 2\,\mathrm{id}_X \otimes t_Y \rangle = 0$ as $\sum_{\alpha\in[n+m]-\{j\}} t_{j\alpha} = 0$ in $\mathfrak{p}_{n+m}$ for any $j \in n + [m]$. $\square$

Proposition 37. *Let $\mathcal{C}$ be a balanced i.b.m.c. and let $\mathcal{C} \overset{\langle - \rangle}{\to} \mathcal{O}$ be a contraction. Let $S \overset{f}{\to} \mathrm{Ob}\mathcal{C}$ be a map such that for any $s \in S$, $t_{f(s)} = 0$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathbf{PaCD}_S & \to & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathbf{PaDih}_S^{gr} & \to & \mathcal{O} \end{array}$$

Proof. As $\mathcal{C}$ is an i.b.m.c., there exists a unique functor $\mathbf{PaCD}_S \to \mathcal{C}$ of i.b.m. categories, extending f. As $t_{f(s)} = 0$ for $s \in S$, it is compatible with the balanced structures. The construction of the commutative diagram is similar to the proof of Propositions 21, 22. $\square$

Proposition 38. *1) The action of* $\mathrm{GRT}(\mathbf{k})$ *on* $\{$*i.b.m. categories*$\}$ *lifts to* $\{$*balanced i.b.m. categories*$\}$ *as follows: for* $(\mathcal{C}, \otimes, c_{XY}, a_{XYZ}, t_{XY}, t_X)$ *a balanced i.b.m.c., and* $g \in \mathrm{GRT}(\mathbf{k})$, $\mathcal{C} \cdot g = (\mathcal{C}, \ldots, t'_X)$, *where* $t'_X = ct_X$ *and* $c = \mathrm{im}(g \in \mathrm{GRT}(\mathbf{k}) \to \mathbf{k}^\times)$.

2) If $\mathcal{C} \overset{F}{\to} \mathcal{O}$ *is a contraction of the balanced i.b.m.c.* $\mathcal{C}$, *then* $\mathcal{C} \cdot g \overset{\sim}{\to} \mathcal{C} \overset{F}{\to} \mathcal{O}$ *is a contraction of the balanced i.b.m.c.* $\mathcal{C}$ *(where* $\mathcal{C} \cdot g \overset{\sim}{\to} \mathcal{C}$ *is the identity of the underlying categories).*

The proof is immediate.

We now construct an action of $\mathrm{GRT}(\mathbf{k})$ on $\mathbf{PaCD}_S \to \mathbf{PaDih}_S^{gr}$. A morphism $\mathrm{GRT}(\mathbf{k}) \to \mathrm{Aut}(\mathbf{PaCD}_S)$, $g \mapsto a_g$ is defined by $a_g : \mathbf{PaCD}_S \to \mathbf{PaCD}_S * g \overset{\sim}{\to} \mathbf{PaCD}_S$, where the first morphism is the unique functor of i.m.b. categories, inducing the identity on objects, and the second morphism is the identification of the underlying categories.

We define a morphism $\mathrm{GRT}(\mathbf{k}) \to \mathrm{Aut}(\mathbf{PaDih}_S^{gr})$, $g \mapsto j_g$ by the condition that the diagram $\begin{array}{ccc} \mathbf{PaCD}_S & \to & \mathbf{PaCD}_S * g \\ \langle - \rangle \downarrow & & \downarrow \langle - \rangle \\ \mathbf{PaDih}_S^{gr} & \overset{j_g}{\to} & \mathbf{PaDih}_S^{gr} \end{array}$ is a functor of balanced i.b.m. categories with contractions. We then have a commutative diagram $\begin{array}{ccc} \mathbf{PaCD}_S & \overset{a_g}{\to} & \mathbf{PaCD}_S \\ \langle - \rangle \downarrow & & \downarrow \langle - \rangle \\ \mathbf{PaDih}_S^{gr} & \overset{j_g}{\to} & \mathbf{PaDih}_S^{gr} \end{array}$

Let now $(\mu, \Phi) \in M(\mathbf{k})$, where $\mu \in \mathbf{k}^\times$, be an associator. It gives rise to an isomorphism $i_\Phi : \mathrm{GT}(\mathbf{k}) \to \mathrm{GRT}(\mathbf{k})$, defined by the condition that $g * \Phi = \Phi * i_\Phi(g)$ for any $g \in \mathrm{GT}(\mathbf{k})$. In the diagram

$$\begin{tikzcd}
\mathbf{PaB}_S \arrow[r, "j_\Phi"'] \arrow[rr, bend left, "\langle - \rangle"] \arrow[d, "g"'] & \mathbf{PaCD}_S \arrow[rr, bend left, "\langle - \rangle"] \arrow[d, "i_\Phi(g)"'] & \mathbf{PaDih}_S \arrow[r, "k_\Phi"'] \arrow[d, "g"'] & \mathbf{PaDih}_S^{gr} \arrow[d, "i_\Phi(g)"] \\
\mathbf{PaB}_S \arrow[r, "j_\Phi"] \arrow[rr, bend right, "\langle - \rangle"'] & \mathbf{PaCD}_S \arrow[rr, bend right, "\langle - \rangle"'] & \mathbf{PaDih}_S \arrow[r, "k_\Phi"] & \mathbf{PaDih}_S^{gr}
\end{tikzcd}$$

all the squares except perhaps the rightmost one commute. But this last square has to commute by the uniqueness of the morphism $\mathbf{PaDih}_S \to \mathcal{O}$ in Proposition 23 (the existence in this proposition implies uniqueness by abstract nonsense).

All this implies that the isomorphism $\mathbf{PaDih}_S \overset{k_\Phi}{\to} \mathbf{PaDih}_S^{gr}$ gives rise to a commutative diagram $\begin{array}{ccc} \mathrm{GT}(\mathbf{k}) & \to & \mathrm{Aut}\,\mathbf{PaDih}_S \\ i_\Phi \downarrow & & \downarrow \\ \mathrm{GRT}(\mathbf{k}) & \to & \mathrm{Aut}\,\mathbf{PaDih}_S^{gr} \end{array}$

The isomorphism $\mathbf{PaDih}_S \overset{k_\Phi}{\to} \mathbf{PaDih}_S^{gr}$ and the actions of $\mathrm{G(R)T}(\mathbf{k})$ on these categories induce the identity at the level of objects. We then define $T_{0,n}^{gr}$ to be the full subcategory of

$\mathbf{PaDih}^{gr}_{[n]}$, whose set of objects is $(PlT_n \times \mathrm{Bij}([n],[n]))/D_n$, and obtain this way an isomorphism $T_{0,n}(\mathbf{k}) \to T^{gr}_{0,n}$ inducing a commutative diagram $\begin{array}{ccc} \mathrm{GT}(\mathbf{k}) & \to & \mathrm{Aut}\, T_{0,n}(\mathbf{k}) \\ \downarrow & & \downarrow \\ \mathrm{GRT}(\mathbf{k}) & \to & \mathrm{Aut}\, T^{gr}_{0,n}(\mathbf{k}) \end{array}$

This proves Theorem 3.

Remark 39. $T^{gr}_{0,n}$ could alternatively be defined as $\pi^*_{dih}\mathcal{C}_{\Gamma,G,S}$, where $\Gamma = D_n$, $G = \exp(\hat{\mathfrak{p}}^{\mathbf{k}}_n) \rtimes S_n$, and $S = [n]$ (see Section 1).

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