Derivation of an integral of Boros and Moll via convolution of Student t-densities

Christian Berg and Christophe Vignat

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Abstract

We show that the evaluation of an integral considered by Boros and Moll is a special case of a convolution result about Student t-densities obtained by the authors in 2008.

Keywords Quartic integral, Student t-density.
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1 Introduction

In a series of papers [4],[1],[5],[6],[2] Moll and his coauthors have considered the integral

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, a > -1, m = 0, 1, \ldots. \tag{1}$$

It was evaluated first by George Boros, who gave the identity

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2}\frac{P_m(a)}{[2(a+1)]^{m+1/2}}, \tag{2}$$

where

$$P_m(a) = \sum_{j=0}^{m} d_{j,m} a^j \tag{3}$$

and

$$d_{j,m} = 2^{-2m} \sum_{i=j}^{m} 2^i \binom{2m-2i}{m-i}\binom{m+i}{m}\binom{i}{j}. \tag{4}$$

The paper [1] gives a survey of different proofs of the formula (2).

The purpose of the present paper is to point out that the evaluation can be considered as a special case of a convolution result about Student t-densities, thereby adding yet another proof to the list of [1].

For $\nu > 0$ the probability density on $\mathbb{R}$

$$f_\nu(x) = \frac{A_\nu}{(1+x^2)^{\nu+\frac{1}{2}}}, \quad A_\nu = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\nu)} \tag{5}$$

is called a Student t-density with $f = 2\nu$ degrees of freedom.

It is the special case $\nu = m + 1/2$ which is relevant in connection with the integral (1).

The relevant convolution result from [3] is

$$\frac{1}{a} f_{n+\frac{1}{2}}\left(\frac{x}{a}\right) * \frac{1}{1-a} f_{m+\frac{1}{2}}\left(\frac{x}{1-a}\right) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) f_{k+\frac{1}{2}}(x), \tag{6}$$

where $0 < a < 1$, n, m are nonnegative integers and $*$ is the ordinary convolution of densities.

The important issue in [3] is to prove that the coefficients $\beta_k^{(n,m)}(a)$ are nonnegative for $0 < a < 1$. This follows from explicit formulas for these coefficients in two cases: (I): $n = m$, (II): n arbitrary, $m = 0$, combined with the symmetry relation

$$\beta_k^{(n,m)}(a) = \beta_k^{(m,n)}(1-a) \tag{7}$$

and a recursion formula

$$\frac{1}{2k+1}\beta_{k+1}^{(n,m)}(a) = \frac{a^2}{2n-1}\beta_k^{(n-1,m)}(a) + \frac{(1-a)^2}{2m-1}\beta_k^{(n,m-1)}(a). \tag{8}$$

We do not know an explicit formula for $\beta_k^{(n,m)}(a)$ when n, m are arbitrary. The formula when $m = n$ is given in [3, Theorem 2.2] and reads

$$\beta_{m+i}^{(m,m)}(a) = (4a(1-a))^i \left(\frac{m!}{(2m)!}\right)^2 2^{-2m} \frac{(2m-2i)!(2m+2i)!}{(m-i)!(m+i)!} \tag{9}$$

$$\times \sum_{j=0}^{m-i} \binom{2m+1}{2j}\binom{m-j}{i}(2a-1)^{2j}, \quad i = 0, \ldots, m. \tag{10}$$

The case $a = 1/2$ leads to

$$\beta_{m+i}^{(m,m)}(1/2) = \left(\frac{m!}{(2m)!}\right)^2 2^{-2m} \frac{(2m-2i)!(2m+2i)!}{(m-i)!(m+i)!}\binom{m}{i}. \tag{11}$$

Let us consider $n = m$ and $a = 1/2$ in (6), where we replace x by $x/2$ and multiply by $1/2$ on both sides:

$$f_{m+1/2} * f_{m+1/2}(x) = \sum_{k=m}^{2m} \frac{1}{2}\beta_k^{(m,m)}(1/2) f_{k+1/2}(x/2). \tag{12}$$

The left-hand side is equal to

$$L := A_{m+1/2}^2 \int_{-\infty}^{\infty} \frac{dy}{[(1+y^2)(1+(x-y)^2)]^{m+1}}$$

$$= A_{m+1/2}^2 \int_{-\infty}^{\infty} \frac{dt}{[(1+(t+x/2)^2)(1+(t-x/2)^2)]^{m+1}},$$

where we have used the substitution $t = y - x/2$. Clearly

$$L = A_{m+1/2}^2 \left(1+x^2/4\right)^{-2(m+1)} \int_{-\infty}^{\infty} \frac{dt}{\left[1+2t^2\frac{1-x^2/4}{(1+x^2/4)^2}+\frac{t^4}{(1+x^2/4)^2}\right]^{m+1}}.$$

Finally, substituting $t = \sqrt{1+x^2/4}\, s$ we get

$$L = 2A_{m+1/2}^2 \left(1+x^2/4\right)^{-2m-3/2} \int_0^{\infty} \frac{ds}{[1+2as^2+s^4]^{m+1}},$$

where $a = (1-x^2/4)/(1+x^2/4)$.

The right-hand side of (12) is equal to

$$R := \sum_{i=0}^{m} \frac{1}{2} \beta_{m+i}^{(m,m)}(1/2) \frac{A_{m+i+1/2}}{\left(1+x^2/4\right)^{m+i+1}}. \tag{13}$$

Combining this gives

$$\int_0^{\infty} \frac{dx}{(x^4+2ax^2+1)^{m+1}} =$$
$$\frac{\pi}{2^{2m+2}} \frac{(1/2)_m}{((2m)!)^2} \sum_{i=0}^{m} \frac{(2m-2i)!(2m+2i)!}{(m+1/2)_i (m-i)!} \binom{m}{i} \left(1+x^2/4\right)^{m+1/2-i}.$$

Using that $2(a+1) = 4/(1+x^2/4)$ we get

$$\int_0^{\infty} \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+1/2}},$$

where

$$P_m(a) = \frac{(1/2)_m}{((2m)!)^2} \sum_{i=0}^{m} \frac{(2m-2i)!(2m+2i)!}{(m+1/2)_i (m-i)!} \binom{m}{i} [(a+1)/2])^i.$$

Using the binomial formula for $(a+1)^i$ and interchanging the summations, we finally get

$$P_m(a) = \sum_{j=0}^{m} d_{j,m} a^j$$

with

$$d_{j,m} = \frac{(1/2)_m}{((2m)!)^2} \sum_{i=j}^{m} \frac{(2m-2i)!(2m+2i)!}{(m+1/2)_i (m-i)! 2^i} \binom{m}{i} \binom{i}{j},$$

which can easily be reduced to (4).

References

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Christian Berg, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen, Denmark
email: berg@math.ku.dk

Christophe Vignat, Laboratoire des Signaux et Systèmes, Université d'Orsay, France
email: christophe.vignat@u-psud.fr