Partitioning the triangles of the cross polytope into surfaces

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Abstract

We present a constructive proof, that there exists a decomposition of the 2-skeleton of the k-dimensional cross polytope β^k into closed surfaces of genus $\leqslant 1$, each with a transitive automorphism group given by the vertex transitive $\mathbb{Z}_{2k}$-action on β^k. Furthermore we show, that for each $k \equiv 1, 5(6)$ the 2-skeleton of the $(k-1)$-simplex is a union of highly symmetric tori and Möbius strips.

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1 Introduction

Surfaces as subcomplexes of polytopes have already been studied by Altshuler who discovered triangulated tori in the 2-skeleton of the family of cyclic 4-polytopes $C_4(n)$ [1] and surfaces in the 2-skeleton of stacked polytopes [2]. Later Betke, Schulz and Wills proved that every orientable 2-manifold is contained in the 2-skeleton of infinitely many 4-polytopes [4].

A priori we can state, that every *triangulated surface*, i. e. every 2-dimensional simplicial complex with closed circuits as vertex links, with k vertices is a sub-complex of the $(k-1)$-simplex Δ^{k-1} and that every centrally symmetric surface with $2k$ vertices lies in the 2-skeleton of the k-dimensional *cross polytope* β^k, i. e. the convex hull of $2k$ points $x_i^{\pm} = (0, \ldots, 0, \pm 1, 0, \ldots, 0) \in \mathbb{R}^k$, $1 \leqslant i \leqslant k$. Hence, of particular interest are surfaces which, in addition, contain the full edge graph of an ambient polytope P. These surfaces are then referred to as 1-Hamiltonian in P.

On the other hand embeddings of certain graphs into triangulated surfaces (so-called *triangular embeddings*) were extensively studied in the course of the proof of Heawood's Map Color Theorem in graph theory (cf. [12] or chapter 4 of [13]). If a Graph G is embeddable into a surface of genus g but not into a surface of smaller genus, then g is called the *genus of* G. Here, as well as in the prior case, we are interested in graph embeddings that cover the full edge graph of an ambient triangulated surface. For the complete graph or, equivalently, the n-simplex these are exactly the 2-neighborly surfaces.

For the complete n-partite graph with two vertices in each partition or the k-dimensional cross polytope respectively, Jungerman and Ringel were able to show the following:

Theorem 1.1 (Regular cases in [7])**.** *For any orientable surface M and any $k \not\equiv 2(3)$ satisfying the equality $2(k-1)(k-3) = 3(2-\chi(M))$ there exists a triangulation of M whose 1-skeleton equals the 1-skeleton of β^k.*

There is a series of centrally symmetric 1-Hamiltonian surfaces S_n, $n \geqslant 0$, with $12n+8$ vertices in β^{6n+4} and, thus, of genus $g(S_n) = 12n^2 + 8n + 1$ (cf. [7] and Example 3.6 in [8] for a concrete list of triangles). In particular, S_0 is the 8-vertex Altshuler torus in the decomposition of β^4 described below.

A closely related question is whether or not the i-skeleton $\mathrm{skel}_i(P)$, $1 \leqslant i \leqslant (d-2)$, of a d-polytope P is *decomposable*, i. e. if there exist two (possibly bounded) PL i-manifolds M_1 and M_2 with $M_1 \cup M_2 = \mathrm{skel}_i(P)$ such that $M_1 \cap M_2 \subset \mathrm{skel}_{i-1}(P)$.

Grünbaum and Malkevitch [6] as well as Martin [11] treated the case $i = 1$. The case $i = 2$ was settled in the case of simplicial polytopes by Betke, Schulz and Wills as follows:

Theorem 1.2 (Betke, Schulz, Wills, [4])**.** *There are exactly 5 simplicial polytopes with decomposable 2-skeletons:*

1. *The 2-skeleton of the 4-simplex Δ^4 is decomposable into 2 Möbius strips with cyclic symmetry, each with the minimum number of 5 vertices, 10 edges and 5 triangles,*
2. *the 18 triangles of the cyclic 4-polytope $C_4(6)$ with 6 vertices form the union of two Möbius strips where the triangulations equal the 6-vertex real projective plane with one triangle removed,*
3. *the triangles of the double pyramid over the 3-simplex can be partitioned into two Möbius strips on 6 vertices and 8 triangles each,*
4. *the 2-skeleton of the 4-dimensional cross polytope (i. e. the double pyramid over the octahedron) equals the union of two 8-vertex Altshuler tori and*
5. *the 20 triangles of the 5-simplex Δ^5 decompose into two copies of the 6-vertex real projective plane.*

Note, that 1., 4. and 5. are highly symmetric. 1. and 4. occur as a part of the two series of decompositions presented below.

The proof of Theorem 1.2 relies on the fact, that a decomposition of the 2-skeleton of a d-Polytope P into two surfaces is only possible if each edge of P is contained in at most 4 triangles. Thus, $4 \leqslant d \leqslant 5$ and the number of vertices has to be bounded.

The idea of the proof of Theorem 1.2 shows that in general decompositions of a polytope P with more than 2 components are not as restrictive towards the local combinatorial structure of P. In this article we will focus on highly symmetric decompositions of the 2-skeleton of β^k and Δ^{k-1} with arbitrary many components. Therefore we define

Definition 1.3 (Difference cycle)**.** Let $a_i \in \mathbb{N}\backslash\{0\}$, $0 \leqslant i \leqslant d$, $n := \sum_{i=0}^{d} a_i$ and $\mathbb{Z}_n = \langle(0, 1, \ldots, n-1)\rangle$. The set

$$(a_0 : \ldots : a_d) := \mathbb{Z}_n \cdot \{0, a_0, \ldots, \Sigma_{i=0}^{d-1} a_i\},$$

where $\cdot$ is the induced cyclic $\mathbb{Z}_n$-action on subsets of $\mathbb{Z}_n$, is called *difference cycle of dimension d on n vertices*. The number of its elements is referred to as the *length* of the difference cycle. If C is a union of difference cycles of dimension d on n vertices and λ is a unit of $\mathbb{Z}_n$ such that the complex λC (obtained by multiplying all vertex labels modulo n by λ) equals C, then λ is called a *multiplier* of C.

Note, that for any unit $\lambda \in \mathbb{Z}_n^{\times}$ the complex λC is *combinatorially isomorphic* to C, i. e. C and λC are equal up to a relabeling of the vertices. In particular all $\lambda \in \mathbb{Z}_n^{\times}$ are multipliers of the complex $\bigcup_{\lambda \in \mathbb{Z}_n^{\times}} \lambda C$.

The definition of a difference cycle above is similar to the one given in [9]. For a more thorough introduction into the field of the more general difference sets and their multipliers see Chapter VI and VII in [3].

Throughout this article we will look at difference cycles as simplicial complexes with a transitive automorphism group given by the cyclic $\mathbb{Z}_n$-action on its elements: Every $(d+1)$-tuple $\{x_0, \ldots, x_d\}$ is interpreted as a d-simplex $\Delta^d = \langle x_0, \ldots, x_d\rangle$. A simplicial complex C is called *transitive*, if its group of automorphisms acts transitively on the set of vertices. In particular any union of difference cycles is a transitive simplicial complex.

Remark 1.4. It follows from the definition that the set of difference cycles of dimension d on k vertices defines a partition of the d-skeleton of the $(k-1)$-simplex. Two $(d+1)$-tuples $(a_0, \ldots, a_d)$ and $(b_0, \ldots, b_d)$ both of sum k define the same difference cycle if and only if for a fixed $j \in \mathbb{Z}$ we have $a_{(i+j) \mod (d+1)} = b_i$ for all $0 \leqslant i \leqslant d$.

Proposition 1.5. *Let $(a_0 : \ldots : a_d)$ be a difference cycle of dimension d on n vertices and $1 \leqslant k \leqslant d+1$ the smallest integer such that $k \mid (d+1)$ and $a_i = a_{i+k}$, $0 \leqslant i \leqslant d-k$. Then $(a_0 : \ldots : a_d)$ is of length $\sum_{i=0}^{k-1} a_i = \frac{nk}{d+1}$.*

Proof. We set $m := \frac{nk}{d+1}$ and compute

$$\begin{aligned} \left\langle 0+m, a_0+m, \ldots, (\Sigma_{i=0}^{d-1} a_i)+m \right\rangle &= \left\langle \Sigma_{i=0}^{k-1} a_i, \Sigma_{i=0}^{k} a_i, \ldots, \Sigma_{i=0}^{d-1} a_i, 0, a_1, \ldots, \Sigma_{i=0}^{k-2} a_i \right\rangle \\ &= \left\langle 0, a_0, \ldots, \Sigma_{i=0}^{d-1} a_i \right\rangle \end{aligned}$$

(all entries modulo n). Hence, the length of $(a_0 : \ldots : a_d)$ is $\leqslant \frac{nk}{d+1}$ and since k is minimal with $k \mid (d+1)$ and $a_i = a_{i+k}$, the upper bound is attained. $\square$

2 The decomposition of $\mathrm{skel}_2(\beta^k)$ into closed surfaces

In the sequel we will look at the boundary of the k-dimensional cross polytope in terms of the abstract simplicial complex

$$\partial\beta^k = \{\langle a_1,\dots,a_k\rangle \,|\, a_i \in \{0,\dots,2k-1\}, \{i,k+i\} \nsubseteq \{a_1,\dots,a_k\}, \forall\, 0 \leqslant i \leqslant k-1\}. \quad (2.1)$$

In particular, the diagonals of β^k are precisely the edges $\langle i, k+i\rangle$, $1 \leqslant i \leqslant k$, and, thus, coincide with the difference cycle $(k:k)$. We can now state our main result:

Theorem 2.1. *The 2-skeleton of the k-dimensional cross polytope β^k can be decomposed into triangulated vertex transitive closed surfaces.*

More precisely, if $k \equiv 1,2(3)$, $\mathrm{skel}_2(\beta^k)$ decomposes into $\frac{(k-1)(k-2)}{3}$ triangulated vertex transitive closed surfaces of Euler characteristic 0 on $2k$ vertices and, if $k \equiv 0(3)$, into $\frac{k}{3}$ disjoint copies of $\partial\beta^3$ (on 6 vertices each) and $\frac{k(k-3)}{3}$ triangulated vertex transitive closed surfaces of Euler characteristic 0 on $2k$ vertices otherwise.

In this section, we will explicitly construct the transitive surfaces and determine their topological types for any given integer $k \geqslant 3$. The proof will consist of a number of consecutive lemmata.

Lemma 2.2. *The 2-skeleton of β^k can be written as the following set of difference cycles:*

$$(l : j : 2k-l-j), (l : 2k-l-j : j)$$

for $0 < l < j < 2k-l-j$, $k \notin \{l, j, l+j\}$, and

$$(j : j : 2(k-j))$$

for $0 < j < k$ with $2j \neq k$. If $k \not\equiv 0(3)$ all of them are of length $2k$, if $k \mod 3 = 0$ the difference cycle $(\frac{2k}{3} : \frac{2k}{3} : \frac{2k}{3})$ has length $\frac{2k}{3}$.

Proof. Let β^k be the k-dimensional cross polytope with vertices $\{0,\dots,2k-1\}$ and diagonals $\{j, k+j\}$, $0 \leqslant j \leqslant k-1$. It follows from the recursive construction of β^k as the double pyramid over β^{k-1} that it contains all 3-tuples of vertices as triangles except the ones including a diagonal. Thus, a difference cycle of the form $(a : b : c)$ lies in $\mathrm{skel}_2(\beta^k)$ if and only if $k \notin \{a, b, a+b\}$. In particular $\mathrm{skel}_2(\beta^k)$ is a union of difference cycles.

Note, that each ordered 3-tuple $0 < l < j < 2k-l-j$ defines exactly two distinct difference cycles on the set of $2k$ vertices, namely

$$(l : j : 2k-l-j) \text{ and } (l : 2k-l-j : j)$$

and it follows immediately that there is no other difference cycle $(a : b : c)$, $k \notin \{a, b, a+b\}$ on $2k$ vertices with a, b, c pairwise distinct.

For any positive integer $0 < j < k$ with $2j \neq k$ there is exactly one difference cycle

$$(j : j : 2k-2j)),$$

and since j must fulfill $0 < 2j < 2k$ there are no further difference cycles without diagonals with at most two different entries.

The length of the difference cycles follows directly from Proposition 1.5 with $d = 2$ and $n = 2k$. ◻

Lemma 2.3. *A closed 2-dimensional pseudomanifold S defined by m difference cycles of full length on the set of n vertices has Euler characteristic $\chi(S) = (1 - \frac{m}{2})n$.*

Proof. Since all difference cycles are of full length, S consists of n vertices and $m \cdot n$ triangles. Additionally, the pseudo manifold property asserts that S has $\frac{3}{2}m \cdot n$ edges and, thus,

$$\chi(S) = n - \frac{3}{2}m \cdot n + m \cdot n = n(1 - \frac{m}{2}).$$

$\square$

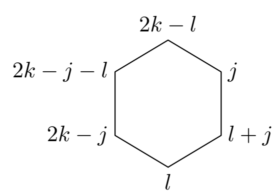
Lemma 2.4. *Let $0 < l < j < 2k - l - j$, $k \notin \{l, j, l + j\}$ and $m := \gcd(l, j, 2k)$. Then*

$$S_{l,j,2k} := \{(l : j : 2k - l - j), (l : 2k - l - j : j)\} \cong \{1, \ldots, m\} \times \mathbb{T}^2,$$

where all connected components of $S_{l,j,2k}$ are combinatorially isomorphic to each other.

Proof. The link of vertex 0 in $S_{l,j,2k}$ is equal to the cycle

$\mathrm{lk}_{S_{l,j,2k}}(0) =$



Since $0 < l < j < 2k - l - j$ and $k \notin \{l, j, l + j\}$ all vertices are distinct and $\mathrm{lk}_{S_{l,j,2k}}(0)$ is the boundary of a hexagon. By the vertex transitivity all other links are cycles and $S_{l,j,2k}$ is a surface.

Since l, j and $2k - l - j$ are pairwise distinct both $(l : j : 2k - l - j)$ and $(l : 2k - l - j : j)$ have full length and by Lemma 2.3 the surface has Euler characteristic 0.

In order to see that $S_{l,j,2k}$ is oriented we look at the (oriented) boundary of the triangles in $S_{l,j,2k}$ in terms of 1-dimensional difference cycles:

$$\begin{aligned} \partial(l : j : 2k - l - j) &= (j : 2k - j) - (l + j : 2k - l - j) + (l : 2k - l) \\ \partial(l : 2k - l - j : j) &= (2k - l - j : l + j) - (2k - j : j) + (l : 2k - l) \\ &= (j : 2k - j) - (l + j : 2k - l - j) + (l : 2k - l) \end{aligned}$$

and, thus $\partial(l : j : 2k - l - j) - \partial(l : 2k - l - j : j) = 0$ and $S_{l,j,2k}$ is oriented.

Now consider

$$(l : j : 2k - l - j) = \mathbb{Z}_{2k} \cdot \langle 0, l, l + j \rangle$$

Clearly $\langle (0 + i) \bmod 2k, (l + i) \bmod 2k, (l + j + i) \bmod 2k \rangle$ share at least one vertex if $i \in \{0, l, 2k - l, j, 2k - j, 2k - l - j, l + j\}$. For any other value of $i < 2k$ the intersection of the triangles is empty. By iteration it follows, that $(l : j : 2k - l - j)$ has exactly $\gcd(0, l, 2k - l, j, 2k - j, 2k - l - j, l + j) = \gcd(l, j, 2k) = m$ connected components. The same holds for $(l : 2k - l - j : j)$ and $(0, \ldots, (2k - 1))^i \cdot \langle 0, l, l + j \rangle$ is disjoint to $\langle 0, l, 2k - j \rangle$ for $i \notin \{0, l, 2k - l, j, 2k - j, 2k - l - j, l + j\}$. Together with the fact that $\mathrm{star}_{S_{l,j,2k}}(0)$ consists of triangles of both $(l : j : 2k - l - j)$ and $(l : 2k - l - j : j)$ it follows that $S_{l,j,2k}$ has m connected components and by a shift of the indices one can see that all of them must be combinatorially isomorphic.

As a consequence it follows that $S_{l,j,2k} \cong \{1, \dots, m\} \times \mathbb{T}^2$. □

Remark 2.5. Some of the connected components of the surfaces presented above are combinatorially isomorphic to the so-called Altshuler tori

$$\{(1 : n - 3 : 2), (1 : 2 : n - 3)\}$$

with $n = \frac{2k}{m} \geqslant 7$ vertices mentioned above (cf. proof of Theorem 4 in [2]). However, other triangulations of transitive tori are part of the decomposition as well: in the case $k = 6$ there are four different combinatorial types of tori. This is in fact the total number of combinatorial types of transitive tori on 12 vertices (cf. Table 1).

Lemma 2.6. *Let*

$$\begin{aligned} M &:= \Big\{(j : j : 2(k - j)) \mid 0 < j < k; 2j \neq k\Big\}, \\ M_1 &:= \Big\{(l : l : 2(k - l)) \mid 1 \leqslant l \leqslant \Big\lfloor \frac{k-1}{2} \Big\rfloor\Big\} \text{ and} \\ M_2 &:= \Big\{(k - l : k - l : 2l)) \mid 1 \leqslant l \leqslant \Big\lfloor \frac{k-1}{2} \Big\rfloor\Big\}. \end{aligned}$$

For all $k \geqslant 3$ the triple (M, M_1, M_2) defines a partition

$$M = M_1 \dot{\cup} M_2$$

into two sets of equal order. In particular we have $\mid M \mid \mod 2 = 0$.

Proof. From $1 \leqslant l \leqslant \lfloor \frac{k-1}{2} \rfloor$ it follows that $k - l > l$ and $2l < k < 2(k - l)$, thus $M_1 \cap M_2 = \varnothing$ and $M_1 \cup M_2 \subseteq M$.

On the other hand let $\lfloor \frac{k-1}{2} \rfloor < j < k - \lfloor \frac{k-1}{2} \rfloor$. If k is odd then $\frac{k-1}{2} < j < \frac{k+1}{2}$ which is impossible for $j \in \mathbb{N}$. If k is even, then $\frac{k}{2} - 1 < j < \frac{k}{2} + 1$, hence $j = k$ which is excluded in the definition of M. All together $M_1 \cup M_2 = M$ holds and

$$|M| = 2 \left\lfloor \frac{k-1}{2} \right\rfloor = \begin{cases} k - 1 & \text{if } k \text{ is odd} \\ k - 2 & \text{else.} \end{cases}$$

□

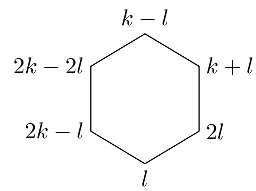
Lemma 2.7. *The complex*

$$S_{l,2k} := \{(l : l : 2(k - l)), (k - l : k - l : 2l)\},$$

$1 \leqslant l \leqslant \lfloor \frac{k-1}{2} \rfloor$, is a disjoint union of $\frac{k}{3}$ copies of $\partial\beta^3$ if $3 \mid k$ and $l = \frac{k}{3}$ and a surface of Euler characteristic 0 otherwise.

Proof. We proof that $S_{l,2k}$ is a surface by looking at the link of vertex 0:

$\mathrm{lk}_{S_{j,2k}}(0) =$



where $2l = k - l$ and $2k - 2l = k + l$ if and only if $l = \frac{k}{3}$. Thus, $\mathrm{lk}_{S_{l,j,2k}}(0)$ is either the boundary of a hexagon or, in the case $l = \frac{k}{3}$, the boundary of a quadrilateral and $S_{l,2k}$ is a surface.

Furthermore, if $l \neq \frac{k}{3}$ the surface $S_{l,2k}$ is a union of two difference cycles of full length and by Lemma 2.3 we have $\chi(S_{l,2k}) = 0$. If $l = \frac{k}{3}$, $(\frac{k}{3} : \frac{k}{3} : \frac{k}{3})$ is of length $\frac{2k}{3}$ and it follows

$$\chi(S_{\frac{k}{3},2k}) = 2k - \frac{8}{2}k + \frac{8}{3}k = \frac{2}{3}k.$$

By a calculation analogue to the one in the proof of Lemma 2.4 one obtains that $S_{\frac{k}{3},2k}$ consists of $\gcd(l, 2k) = \frac{k}{3}$ isomorphic connected components of type $\{3, 4\}$. Hence, $S_{\frac{k}{3},2k}$ is a disjoint union of $\frac{k}{3}$ copies of $\partial\beta^3$. □

Theorem 2.8 (Lutz [10])**.** *Let $n = 8 + 2m$ for $m \geqslant 0$. Then the complex*

$$A_6(n) := \{(1 : 1 : (n-2)), (2 : (\frac{n}{2} - 1) : (\frac{n}{2} - 1)\}$$

is a torus for m even and a Klein bottle for m odd.

Lemma 2.9. *Let $k \geqslant 3$, $1 \leqslant l \leqslant \lfloor \frac{k-1}{2} \rfloor$, $l \neq \frac{k}{3}$ and $n := \gcd(l, k)$. Then $S_{l,2k}$ is isomorphic to n copies of $A_6(\frac{2k}{n})$.*

Proof. Since $n = \gcd(l, k) = \gcd(l, k - l)$ we have $n = \min\{\gcd(l, 2(k - l)), \gcd(2l, k - l)\}$ and either $\frac{l}{n} \in \mathbb{Z}^{\times}_{\frac{2k}{n}}$ or $\frac{k-l}{n} \in \mathbb{Z}^{\times}_{\frac{2k}{n}}$ holds. It follows by mulitplying with l or $k - l$ that $A_6(\frac{2k}{n})$ is isomorphic to $S_{\frac{l}{n},\frac{2k}{n}} = S_{\frac{k-l}{n},\frac{2k}{n}}$. By the monomorphism

$$\mathbb{Z}_{\frac{2k}{n}} \to \mathbb{Z}_{2k} \quad j \mapsto (lj \mod 2k)$$

we have a relabeling of $S_{\frac{l}{n},\frac{2k}{n}}$ and a small computation shows that the relabeled complex is equal to the connected component of $S_{l,2k}$ containing 0. By a shift of the vertex labels we see that all other connected components of $S_{l,2k}$ are isomoprhic to the one containing 0 what states the result. □

Proof of Theorem 2.1. Lemma 2.2 and Lemma 2.6 describe $\mathrm{skel}_2(\beta^k)$ in terms of 2 series of pairs of difference cycles

$$\{(l : j : 2k - l - j), (l : 2k - l - j : j)\} \text{ and } \{(l : l : 2(k - l)), (k - l : k - l : 2l)\}$$

for certain parameters j and l. Lemma 2.4 determines the topological type of the first and Theorem 2.8 together with Lemma 2.7 and 2.9 verifies the type of the second series.

Since $|\mathrm{skel}_2(\beta^k)| = \binom{2k}{3} - k(2k-2)$ and for $k \not\equiv 0(3)$ all surfaces have exactly $4k$ triangles we get an overall number of $\frac{(k-1)(k-2)}{3}$ surfaces. If $k = 0(3)$ all surfaces but one have $4k$ triangles, the last one has $\frac{8k}{3}$ triangles. All together this implies that there are $\frac{k(k-3)}{3}$ of Euler characteristic 0 and $\frac{k}{3}$ copies of $\partial\beta^3$. □

Table 1 shows the decomposition of $\mathrm{skel}_2(\beta^k)$ for $3 \leqslant k \leqslant 10$. The table was computed using the `GAP` package `simpcomp` [5]. For a complete list of the decomposition for $k \leqslant 90$ see [14].

3 The decomposition of $\mathrm{skel}_2(\Delta^{k-1})$

First note, that $\mathrm{skel}_2(\Delta^{k-1})$, $k \geqslant 3$, equals the set of all triangles on k vertices. By looking at its vertex links we can see that in the case of $2k$ vertices the complex $\{(l : k-l : k)\}$ can not be part of a triangulated surface for any $0 < l < k$. Thus, the decomposition of $\mathrm{skel}_2(\beta^k)$ can not be extended to a decomposition of $\mathrm{skel}_2(\Delta^{2k-1})$ in an obvious manner. However, for other numbers of vertices the situation is different:

Theorem 3.1. *Let $k > 1$, $k \equiv 1, 5(6)$. Then the 2-skeleton of Δ^{k-1} decomposes into $\frac{k-1}{2}$ collections of Möbius strips*

$$M_{l,k} := \{(l : l : k-2l)\},$$

$1 \leqslant l \leqslant \frac{k-1}{2}$ each with $n := \gcd(l,k)$ isomorphic connected components on $\frac{k}{n}$ vertices and $\frac{k^2-6k+5}{12}$ collections of tori

$$S_{l,j,k} := \{(l : j : k-l-j), (l : k-l-j : j)\},$$

$1 \leqslant l < j < k-l-j$, with $m := \gcd(l,j,k)$ connected components on $\frac{k}{m}$ vertices each.

We first prove the following

Lemma 3.2. *$M_{l,k}$ with $k \geqslant 5$, $k \not\equiv 0(3)$ and $k \not\equiv 0(4)$ is a triangulation of $n := \gcd(l,k)$ cylinders $[0,1] \times \partial\Delta^2$ if $\frac{k}{n}$ is even and of n Möbius strips if $\frac{k}{n}$ is odd.*

Proof. We first look at

$$M_{1,k} = \{\langle 0,1,2\rangle, \langle 1,2,3\rangle, \ldots, \langle k-2, k-1, 0\rangle, \langle k-1, 0, 1\rangle\}$$

for $k \geqslant 5$ (see Figure 3.1). Every triangle has exactly two neighbors. Thus, the alternating sum

$$+\langle 0,1,2\rangle - \langle 1,2,3\rangle + \ldots - +(-1)^{k-1}\langle k-1, 0, 1\rangle$$

induces an orientation if and only if k is even and for any $l \in \mathbb{Z}_k^\times$ the complex $M_{l,k}$ is a cylinder if k is even and a Möbius strip if k is odd. Now suppose that $n = \gcd(l,k) > 1$. Since $k \not\equiv 0(3)$ and $k \not\equiv 0(4)$ we have $\frac{k}{n} \geqslant 5$ and by a relabeling we see, that the connected components of $M_{l,k}$ are combinatorially isomorphic to $M_{\frac{l}{n},\frac{k}{n}} \cong M_{1,\frac{k}{n}}$. □

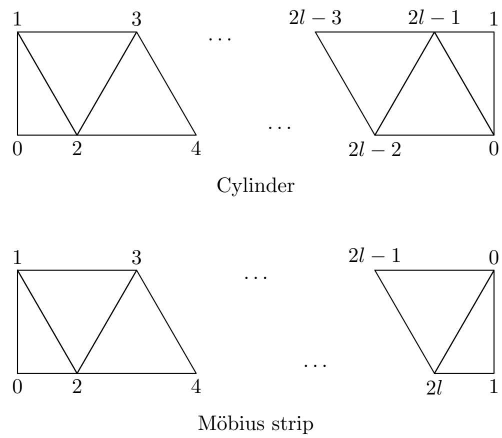


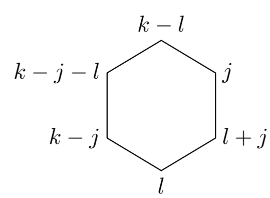
Figure 3.1: The cylinder $(1:1:2l-2)$ and the Möbius strip $(1:1:2l-3)$. The vertical boundary components ($\langle 0,1\rangle$)are identified.

Remark 3.3. If $k \equiv 0(4)$ the connected components of $M_{\frac{k}{4},k} = \{(\frac{k}{4} : \frac{k}{4} : \frac{k}{2})\}$ equal $\{(1:1:2)\}$ which is the boundary of Δ^3. If $k \equiv 0(3)$ then $M_{\frac{k}{3},k}$ is a collection of disjoint triangles (isomorphic to $\{(1:1:1)\}$).

Lemma 3.4. *$S_{l,j,k}$, $0 < l < j < k$, $k \not\equiv 0(2)$, is a triangulation of $m := \gcd(l,j,k)$ connected components of isomorphic tori on $\frac{k}{m}$ vertices.*

Proof. The link of vertex 0 equals

$$\mathrm{lk}_{S_{l,j,k}}(0) =$$



(cf. proof of Lemma 2.4). Since $0 < l < j < k-l-j$ and $k \not\equiv 0(2)$ the link is the boundary of a hexagon, $\frac{k}{m} \geqslant 7$ and $S_{l,j,k}$ is a surface. By Lemma 2.3 $S_{l,j,k}$ is of Euler characteristic 0. The proof of the orientability and the number of connected components is analogue to the one given in the proof of Lemma 2.4. It follows that

$$S_{l,j,k} \cong \{1,\ldots,m\} \times \mathbb{T}^2.$$

□

Together with Lemma 3.2 and Lemma 3.4 it suffices to show that the two series presented above contain all triangles of Δ^{k-1} in order to proof Theorem 3.1:

Proof. Let $\langle a,b,c\rangle \in \mathrm{skel}_2(\Delta^{k-1})$, $a < b < c$. Then $\langle a,b,c\rangle \in (b-a : c-b : k-(c-a))$. Now if $b-a$, $c-b$ and $k-(c-a)$ are pairwise distinct we have

- $\langle a,b,c\rangle \in S_{b-a,c-b,k} = S_{b-a,k-(c-a),k}$ if $b-a < c-b, k-(c-a)$,

- $\langle a,b,c\rangle \in S_{c-b,b-a,k} = S_{c-b,k-(c-a),k}$ if $c-b < b-a, k-(c-a)$ or
- $\langle a,b,c\rangle \in S_{k-(c-a),b-a,k} = S_{k-(c-a),c-b,k}$ if $k-(c-a) < c-b, b-a$.

If on the other hand at least two of the entries are equal, then $(b-a : c-b : k-(c-a)) = (l : l : k-2l)$ for $1 \leqslant l \leqslant \frac{k-1}{2}$. Thus, the decomposition of $\mathrm{skel}_2(\Delta^{k-1})$ is complete. $\square$

Table 2 shows the decomposition of $\mathrm{skel}_2(\Delta^{k-1})$ for $k \in \{5,7,11,13,35\}$.

A complete list of the decomposition of $\mathrm{skel}_2(\beta^k)$ and $\mathrm{skel}_2(\Delta^{k-1})$ for $k \leqslant 100$ can be found in [14].

Table 1: The decomposition of the 2-skeleton of β^k ($k \leqslant 10$) into transitive surfaces.

k	$f_2(\beta^k)$	**topological type**	**difference cycles**
3	8	$\mathbb{S}^2$	$\{(1{:}1{:}4),(2{:}2{:}2)\}$
4	32	$\mathbb{T}^2$	$\{(1{:}2{:}5),(1{:}5{:}2)\}$, $\{(1{:}1{:}6),(3{:}3{:}2)\}$
5	80	$\mathbb{T}^2$	$\{(1{:}2{:}7),(1{:}7{:}2)\}$, $\{(1{:}3{:}6),(1{:}6{:}3)\}$
		$\mathbb{K}^2$	$\{(1{:}1{:}8),(4{:}4{:}2)\}$, $\{(2{:}2{:}6),(3{:}3{:}4)\}$
6	160	$\{1,2\}\times\mathbb{S}^2$	$\{(2{:}2{:}8),(4{:}4{:}4)\}$
		$\mathbb{T}^2$	$\{(1{:}2{:}9),(1{:}9{:}2)\}$, $\{(1{:}3{:}8),(1{:}8{:}3)\}$, $\{(1{:}4{:}7),(1{:}7{:}4)\}$, $\{(2{:}3{:}7),(2{:}7{:}3)\}$, $\{(3{:}4{:}5),(3{:}5{:}4)\}$, $\{(1{:}1{:}10),(5{:}5{:}2)\}$
7	280	$\mathbb{T}^2$	$\{(1{:}2{:}11),(1{:}11{:}2)\}$,$\{(1{:}3{:}10),(1{:}10{:}3)\}$,$\{(1{:}4{:}9),(1{:}9{:}4)\}$, $\{(1{:}5{:}8),(1{:}8{:}5)\}$, $\{(2{:}3{:}9),(2{:}9{:}3)\}$, $\{(3{:}5{:}6),(3{:}6{:}5)\}$
		$\{1,2\}\times\mathbb{T}^2$	$\{(2{:}4{:}8),(2{:}8{:}4)\}$
		$\mathbb{K}^2$	$\{(1{:}1{:}12),(6{:}6{:}2)\}$, $\{(2{:}2{:}10),(5{:}5{:}4)\}$, $\{(3{:}3{:}8),(4{:}4{:}6)\}$
8	448	$\mathbb{T}^2$	$\{(1{:}2{:}13),(1{:}13{:}2)\}$,$\{(1{:}3{:}12),(1{:}12{:}3)\}$,$\{(1{:}4{:}11),(1{:}11{:}4)\}$, $\{(1{:}5{:}10),(1{:}10{:}5)\}$,$\{(1{:}6{:}9),(1{:}9{:}6)\}$, $\{(2{:}3{:}11),(2{:}11{:}3)\}$, $\{(2{:}5{:}9),(2{:}9{:}5)\}$, $\{(3{:}4{:}9),(3{:}9{:}4)\}$, $\{(3{:}6{:}7),(3{:}7{:}6)\}$, $\{(4{:}5{:}7),(4{:}7{:}5)\}$, $\{(1{:}1{:}14),(7{:}7{:}2)\}$, $\{(3{:}3{:}10),(5{:}5{:}6)\}$
		$\{1,2\}\times\mathbb{T}^2$	$\{(2{:}4{:}10),(2{:}10{:}4)\}$,$\{(2{:}2{:}12),(6{:}6{:}4)\}$
9	672	$\{1,2,3\}\times\mathbb{S}^2$	$\{(3{:}3{:}12),(6{:}6{:}6)\}$
		$\mathbb{T}^2$	$\{(1{:}2{:}15),(1{:}15{:}2)\}$,$\{(1{:}3{:}14),(1{:}14{:}3)\}$,$\{(1{:}4{:}13),(1{:}13{:}4)\}$, $\{(1{:}5{:}12),(1{:}12{:}5)\}$,$\{(1{:}6{:}11),(1{:}11{:}6)\}$,$\{(1{:}7{:}10),(1{:}10{:}7)\}$, $\{(2{:}3{:}13),(2{:}13{:}3)\}$,$\{(2{:}5{:}11),(2{:}11{:}5)\}$,$\{(3{:}4{:}11),(3{:}11{:}4)\}$, $\{(3{:}5{:}10),(3{:}10{:}5)\}$,$\{(3{:}7{:}8),(3{:}8{:}7)\}$, $\{(5{:}6{:}7),(5{:}7{:}6)\}$
		$\{1,2\}\times\mathbb{T}^2$	$\{(2{:}4{:}12),(2{:}12{:}4)\}$,$\{(2{:}6{:}10),(2{:}10{:}6)\}$,$\{(4{:}6{:}8),(4{:}8{:}6)\}$
		$\mathbb{K}^2$	$\{(1{:}1{:}16),(8{:}8{:}2)\}$, $\{(2{:}2{:}14),(7{:}7{:}4)\}$, $\{(4{:}4{:}10),(5{:}5{:}8)\}$
10	960	$\mathbb{T}^2$	$\{(1{:}2{:}17),(1{:}17{:}2)\}$,$\{(1{:}3{:}16),(1{:}16{:}3)\}$,$\{(1{:}4{:}15),(1{:}15{:}4)\}$, $\{(1{:}5{:}14),(1{:}14{:}5)\}$,$\{(1{:}6{:}13),(1{:}13{:}6)\}$,$\{(1{:}7{:}12),(1{:}12{:}7)\}$, $\{(1{:}8{:}11),(1{:}11{:}8)\}$,$\{(2{:}3{:}15),(2{:}15{:}3)\}$,$\{(2{:}5{:}13),(2{:}13{:}5)\}$, $\{(2{:}7{:}11),(2{:}11{:}7)\}$,$\{(3{:}4{:}13),(3{:}13{:}4)\}$,$\{(3{:}5{:}12),(3{:}12{:}5)\}$, $\{(3{:}6{:}11),(3{:}11{:}6)\}$,$\{(3{:}8{:}9),(3{:}9{:}8)\}$, $\{(4{:}5{:}11),(4{:}11{:}5)\}$, $\{(4{:}7{:}9),(4{:}9{:}7)\}$, $\{(5{:}6{:}9),(5{:}9{:}6)\}$, $\{(5{:}7{:}8),(5{:}8{:}7)\}$, $\{(1{:}1{:}18),(9{:}9{:}2)\}$, $\{(3{:}3{:}14),(7{:}7{:}6)\}$
		$\{1,2\}\times\mathbb{T}^2$	$\{(2{:}4{:}14),(2{:}14{:}4)\}$,$\{(2{:}6{:}12),(2{:}12{:}6)\}$
		$\{1,2\}\times\mathbb{K}^2$	$\{(2{:}2{:}16),(8{:}8{:}4)\}$, $\{(4{:}4{:}12),(6{:}6{:}8)\}$

Table 2: The decomposition of the 2-skeleton of Δ^{k-1} ($k \in \{5, 7, 11, 13, 35\}$) by topological types.

k	**topological type**	**difference cycles**		
5	$\mathbb{M}^2$	{(1:1:3)},	{(2:2:1)}	
7	$\mathbb{M}^2$	{(1:1:5)},	{(2:2:3)},	{(3:3:1)}
	$\mathbb{T}^2$	{(1:2:4), (1:4:2)}		
11	$\mathbb{M}^2$	{(1:1:9)},	{(2:2:7)},	{(3:3:5)},
		{(4:4:3)},	{(5:5:1)}	
	$\mathbb{T}^2$	{(1:2:8), (1:8:2)},	{(1:3:7), (1:7:3)},	{(1:4:6), (1:6:4)},
		{(2:3:6), (2:6:3)},	{(2:4:5), (2:5:4)}	
13	$\mathbb{M}^2$	{(1:1:11)},	{(2:2:9)},	{(3:3:7)},
		{(4:4:5)},	{(5:5:3)},	{(6:6:1)}
	$\mathbb{T}^2$	{(1:2:10), (1:10:2)},	{(1:3:9), (1:9:3)},	{(1:4:8), (1:8:4)},
		{(1:5:7), (1:7:5)},	{(2:3:8), (2:8:3)},	{(2:4:7), (2:7:4)},
		{(2:5:6), (2:6:5)},	{(3:4:6), (3:6:4)}	
35	$\mathbb{M}^2$	{(1:1:33)},	{(2:2:31)},	{(3:3:29)},
		{(4:4:27)},	{(6:6:23)},	{(8:8:19)},
		{(9:9:17)},	{(11:11:13)},	{(12:12:11)},
		{(13:13:9)},	{(16:16:3)},	{(17:17:1)}
	$\{1,\ldots,5\} \times \mathbb{M}^2$	{(5:5:25)},	{(10:10:15)},	{(15:15:5)}
	$\{1,\ldots,7\} \times \mathbb{M}^2$	{(7:7:21)},	{(14:14:7)}	
	$\mathbb{T}^2$	{(1:2:32), (1:32:2)},	{(1:3:31), (1:31:3)},	{(1:4:30), (1:30:4)},
		{(1:5:29), (1:29:5)},	{(1:6:28), (1:28:6)},	{(1:7:27), (1:27:7)},
		{(1:8:26), (1:26:8)},	{(1:9:25), (1:25:9)},	{(1:10:24), (1:24:10)},
		{(1:11:23), (1:23:11)},	{(1:12:22), (1:22:12)},	{(1:13:21), (1:21:13)},
		{(1:14:20), (1:20:14)},	{(1:15:19), (1:19:15)},	{(1:16:18), (1:18:16)},
		{(2:3:30), (2:30:3)},	{(2:4:29), (2:29:4)},	{(2:5:28), (2:28:5)},
		{(2:6:27), (2:27:6)},	{(2:7:26), (2:26:7)},	{(2:8:25), (2:25:8)},
		{(2:9:24), (2:24:9)},	{(2:10:23), (2:23:10)},	{(2:11:22), (2:22:11)},
		{(2:12:21), (2:21:12)},	{(2:13:20), (2:20:13)},	{(2:14:19), (2:19:14)},
		{(2:15:18), (2:18:15)},	{(2:16:17), (2:17:16)},	{(3:4:28), (3:28:4)},
		{(3:5:27), (3:27:5)},	{(3:6:26), (3:26:6)},	{(3:7:25), (3:25:7)},
		{(3:8:24), (3:24:8)},	{(3:9:23), (3:23:9)},	{(3:10:22), (3:22:10)},
		{(3:11:21), (3:21:11)},	{(3:12:20), (3:20:12)},	{(3:13:19), (3:19:13)},
		{(3:14:18), (3:18:14)},	{(3:15:17), (3:17:15)},	{(4:5:26), (4:26:5)},
		{(4:6:25), (4:25:6)},	{(4:7:24), (4:24:7)},	{(4:8:23), (4:23:8)},
		{(4:9:22), (4:22:9)},	{(4:10:21), (4:21:10)},	{(4:11:20), (4:20:11)},
		{(4:12:19), (4:19:12)},	{(4:13:18), (4:18:13)},	{(4:14:17), (4:17:14)},
		{(4:15:16), (4:16:15)},	{(5:6:24), (5:24:6)},	{(5:7:23), (5:23:7)},
		{(5:8:22), (5:22:8)},	{(5:9:21), (5:21:9)},	{(5:11:19), (5:19:11)},
		{(5:12:18), (5:18:12)},	{(5:13:17), (5:17:13)},	{(5:14:16), (5:16:14)},
		{(6:7:22), (6:22:7)},	{(6:8:21), (6:21:8)},	{(6:9:20), (6:20:9)},
		{(6:10:19), (6:19:10)},	{(6:11:18), (6:18:11)},	{(6:12:17), (6:17:12)},
		{(6:13:16), (6:16:13)},	{(6:14:15), (6:15:14)},	{(7:8:20), (7:20:8)},
		{(7:9:19), (7:19:9)},	{(7:10:18), (7:18:10)},	{(7:11:17), (7:17:11)},
		{(7:12:16), (7:16:12)},	{(7:13:15), (7:15:13)},	{(8:9:18), (8:18:9)},
		{(8:10:17), (8:17:10)},	{(8:11:16), (8:16:11)},	{(8:12:15), (8:15:12)},
		{(8:13:14), (8:14:13)},	{(9:10:16), (9:16:10)},	{(9:11:15), (9:15:11)},
		{(9:12:14), (9:14:12)},	{(10:11:14), (10:14:11)},	{(10:12:13), (10:13:12)}
	$\{1,\ldots,5\} \times \mathbb{T}^2$	{(5:10:20), (5:20:10)}		

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