

Split Quaternionic Analysis and Separation of the Series for $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})/SL(2, \mathbb{R})$

Igor Frenkel and Matvei Libine

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Abstract

We extend our previous study of quaternionic analysis based on representation theory to the case of split quaternions $\mathbb{H}_{\mathbb{R}}$. The special role of the unit sphere in the classical quaternions \mathbb{H} identified with the group $SU(2)$ is now played by the group $SL(2, \mathbb{R})$ realized by the unit quaternions in $\mathbb{H}_{\mathbb{R}}$. As in the previous work, we use an analogue of the Cayley transform to relate the analysis on $SL(2, \mathbb{R})$ to the analysis on the imaginary Lobachevski space $SL(2, \mathbb{C})/SL(2, \mathbb{R})$ identified with the one-sheeted hyperboloid in the Minkowski space \mathbb{M} . We study the counterparts of Cauchy-Fueter and Poisson formulas on $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} and show that they solve the problem of separation of the discrete and continuous series. The continuous series component on $\mathbb{H}_{\mathbb{R}}$ gives rise to the minimal representation of the conformal group $SL(4, \mathbb{R})$, while the discrete series on \mathbb{M} provides its K -types realized in a natural polynomial basis. We also obtain a surprising formula for the Plancherel measure on $SL(2, \mathbb{R})$ in terms of the Poisson integral on the split quaternions $\mathbb{H}_{\mathbb{R}}$. Finally, we show that the massless singular functions of four-dimensional quantum field theory are nothing but the kernels of projectors onto the discrete and continuous series on the imaginary Lobachevski space $SL(2, \mathbb{C})/SL(2, \mathbb{R})$. Our results once again reveal the central role of the Minkowski space in quaternionic and split quaternionic analysis as well as a deep connection between split quaternionic analysis and the four-dimensional quantum field theory.

Keywords: Harmonic analysis on $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})/SL(2, \mathbb{R})$, minimal representation of $SO(3, 3)$, Cauchy-Fueter formula, split quaternions, Minkowski space, imaginary Lobachevski space, conformal group, Cayley transform.

1 Introduction

Six years after William Rowan Hamilton's fundamental discovery of quaternions, in 1849 James Cockle introduced a related algebra of "coquaternions", which in modern language is called the algebra of split quaternions $\mathbb{H}_{\mathbb{R}}$. As an algebra, $\mathbb{H}_{\mathbb{R}}$ is isomorphic to the real 2×2 matrices and has signature $(2, 2)$ with respect to the determinant viewed as a quadratic form. Unlike Hamilton's quaternions, not every non-zero element in $\mathbb{H}_{\mathbb{R}}$ is invertible. However, there is a strong parallel between the geometry and analysis on the two algebras.

In our previous paper [FL1] we began studying analysis on the algebra of quaternions \mathbb{H} from the point of view of representation theory of the quaternionic conformal group $SL(2, \mathbb{H})$. It is well-known that the spaces of harmonic, left- and right-regular functions admit natural actions of $SL(2, \mathbb{H})$ by fractional linear transformations. We regarded \mathbb{H} as a real form of the space of complex quaternions $\mathbb{H}_{\mathbb{C}}$, and the questions of unitarity of these representations led us to another real form in $\mathbb{H}_{\mathbb{C}}$, namely the Minkowski space \mathbb{M} . The conformal group of \mathbb{M} is naturally identified with $SU(2, 2)$ and acts by unitary transformations on the spaces of harmonic, left- and right-regular functions on \mathbb{M} . By harmonic functions on \mathbb{M} we mean the

solutions of the wave equation; similarly, left- and right-regular functions on \mathbb{M} are solutions of the the left and right Weyl equations, which together form the massless Dirac equations. To relate these two real forms of $\mathbb{H}_{\mathbb{C}}$ we considered the Cayley transform which maps \mathbb{M} into $U(2)$ and the unit two-sheeted hyperboloid $H^3 \subset \mathbb{M}$ into the unit sphere $S^3 \subset \mathbb{H}$. The quaternionic picture, however, had an important advantage over \mathbb{M} – it gave a natural realization of the K -types of the unitary representations of $SU(2, 2)$ as the spaces of polynomials in quaternionic coordinates. While the group $SU(2, 2)$ does not act naturally on \mathbb{H} , its maximal compact subgroup $K = S(U(2) \times U(2))$ does act there and one can identify the basis of K -types with the matrix coefficients of the group $U(2)$ and its subgroup $SU(2)$.

In this work we consider analysis on another real form of $\mathbb{H}_{\mathbb{C}}$, namely the split quaternions $\mathbb{H}_{\mathbb{R}}$, and study the action of the conformal group $SL(2, \mathbb{H}_{\mathbb{R}}) \simeq SL(4, \mathbb{R})$. Just as the quaternionic analysis on \mathbb{H} is based on analysis on the unit sphere identified with the group $SU(2)$, the split quaternionic analysis on $\mathbb{H}_{\mathbb{R}}$ has at its foundation analysis on the unit hyperboloid identified with the group $SL(2, \mathbb{R}) \simeq SU(1, 1)$. It is a classical result that the analysis on $SL(2, \mathbb{R})$ has the discrete and continuous components corresponding to the two types of irreducible unitary representations of this group. All results in this paper will come in two flavors – for left-/right-regular functions and solutions of the appropriate wave equation. For simplicity, in this introduction we only announce the results for left-regular functions.

One of the central results of our paper is a solution of the problem of separation of the series on $SU(1, 1)$ using the main formulas of quaternionic analysis on $\mathbb{H}_{\mathbb{R}}$, namely the counterparts of the Cauchy-Fueter and Poisson integrals. Again, the relationship between the analysis on $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} is an important tool of our study. We use an analogue of the Cayley transform previously studied in [KouØ] which now maps \mathbb{M} into $U(1, 1)$ and the unit one-sheeted hyperboloid $H_{\mathbb{M}}^3 \subset \mathbb{M}$ into the unit hyperboloid $H_{\mathbb{R}}^3 \subset \mathbb{H}_{\mathbb{R}}$ (Proposition 58). There are natural identifications $H_{\mathbb{M}}^3 \simeq SL(2, \mathbb{C})/SU(1, 1)$ and $H_{\mathbb{R}}^3 \simeq SU(1, 1)$, and this Cayley transform interchanges the discrete and continuous components on $H_{\mathbb{M}}^3$ and $H_{\mathbb{R}}^3$. The study of harmonic analysis on $SL(2, \mathbb{C})/SU(1, 1)$ – also known as the imaginary Lobachevski space – goes back to Gelfand and his school [GGV] and is based on the geometry of horospheres. These methods were used for the problem of separation of the series on $SU(1, 1)$ in more recent works by Gindikin [Gi1, Gi2]. (See also references therein for related work on this problem.) In our approach we realize functions on $SL(2, \mathbb{C})/SU(1, 1)$ and $SU(1, 1)$ using harmonic extensions to the flat spaces \mathbb{M} and $\mathbb{H}_{\mathbb{R}}$. This idea was previously explored in greater generality by Strichartz [St]. In our work the methods and formulas of quaternionic analysis are naturally applied to solve the problems of harmonic analysis on $SU(1, 1)$ and $SL(2, \mathbb{C})/SU(1, 1)$.

We study first the discrete series component of split quaternionic analysis, which goes in a strong parallel with the case of (classical) quaternions following the results and constructions of our paper [FL1]. It turns out that the discrete components of the harmonic¹, left- and right-regular functions on $\mathbb{H}_{\mathbb{R}}$ yield *the same* unitary representations of the group $SU(2, 2)$ as in the (classical) quaternionic case. The unitary structure and the identification of the representations become transparent after the Cayley transform relating \mathbb{M} with $U(1, 1)$ and mapping the tube domains \mathbb{T}^{\mp} in $\mathbb{H}_{\mathbb{C}}$ (see Section 3.5 in [FL1]) into certain complex semigroups Γ^{\mp} with the Shilov boundary $U(1, 1)$. These are Ol'shanskii semigroups which were first introduced in [Vin], [Ol].

One can obtain again polynomial realizations of the discrete components of the harmonic and regular functions on $\mathbb{H}_{\mathbb{R}}$ by restricting the group $SU(2, 2)$ to another subgroup $K_{\mathbb{R}} = S(U(1, 1) \times U(1, 1))$ and identifying the polynomial basis of $K_{\mathbb{R}}$ -representations with the matrix coefficients of the discrete component of the group $U(1, 1)$ and its subgroup $SU(1, 1)$. In particular, the two irreducible representations \mathcal{V}^{\mp} of $SU(2, 2)$ realized in the space of left-regular

¹By harmonic functions on $\mathbb{H}_{\mathbb{R}}$ we mean the solutions of the ultrahyperbolic wave equation, which is in parallel with the Minkowski case \mathbb{M} .

functions on $\mathbb{H}_{\mathbb{R}}$ are decomposed with respect to the shifted degree operator $\widetilde{\text{deg}} = \text{deg} + 1$ as follows:

$$\mathcal{V}^- = \bigoplus_{n \in \mathbb{Z}} \mathcal{V}^-(n), \quad \mathcal{V}^+ = \bigoplus_{n \in \mathbb{Z}} \mathcal{V}^+(n) \quad (1)$$

where $\mathcal{V}^-(n)$ (respectively $\mathcal{V}^+(n)$), $n \neq 0$, are the irreducible representations from the holomorphic (respectively antiholomorphic) discrete series, and the case $n = 0$ corresponds to the limit of the holomorphic (respectively antiholomorphic) discrete series. Then the Fueter formula yields projections P^{\mp} of the space of left-regular functions f onto the holomorphic or antiholomorphic discrete series components depending on the domain of the variable $W \in \Gamma^{\mp}$ in

$$-\frac{1}{2\pi^2} \int_{Z \in H_{\mathbb{R}}^3} \frac{(Z - W)^{-1}}{\det(Z - W)} \cdot *dZ \cdot f(Z) = (P^{\mp} f_{>0} - P^{\mp} f_{<0})(W), \quad (2)$$

where $P^{\mp} f_{>0}$ (respectively $P^{\mp} f_{<0}$) denote the sum of the components in (1) of positive (respectively negative) shifted degree (Theorem 49). Note that the components $P^{\mp} f_{>0}$ and $P^{\mp} f_{<0}$ enter with opposite signs and the limits of the discrete series would cause the integral to be divergent. (In the case of Poisson formula the limits of the discrete series are projected out,) Since $H_{\mathbb{R}}^3 \simeq SU(1, 1)$ lies inside the Shilov boundary of Γ^{\mp} , the values of the functions on $H_{\mathbb{R}}^3$ can be recovered by taking limits. The Fueter formula for the right-regular functions and the Poisson formula for harmonic functions have similar structures. The results and formulas for the middle series of representations of $SU(2, 2)$ obtained in [FL1] have even more exact analogues for the discrete component of split quaternionic analysis. In particular, the limits of the discrete series do not occur in the restrictions of these representations to $K_{\mathbb{R}}$.

Fundamentally new features of split quaternionic analysis appear when we study the continuous series component. Now the space of harmonic functions on $\mathbb{H}_{\mathbb{R}}$ of the continuous series component gives rise to a single irreducible representation of the conformal group $SL(4, \mathbb{R})/\{\pm 1\} \simeq SO(3, 3)$, known as the minimal representation². It was studied for an arbitrary signature in [KobØ] (also see references therein for the previous work on this subject). Again, the representation theory of $SL(4, \mathbb{R})$ is crucial for studying the continuous component of split quaternionic analysis. On the other hand, the latter illuminates various aspects of the representation theory. In particular, applying the Cayley transform to the space of harmonic functions on $\mathbb{H}_{\mathbb{R}}$ spanning the minimal representation, we realize the minimal representation in a space $\mathcal{D}_{\mathbb{M}}^0$ with basis consisting of certain harmonic polynomials on \mathbb{M} . This basis appears naturally when we restrict the minimal representation of $SL(4, \mathbb{R})$ to its subgroup $K_{\mathbb{M}} = \mathbb{R}^{>0} \times SL(2, \mathbb{C})$. Decomposing $\mathcal{D}_{\mathbb{M}}^0$ with respect to the shifted degree operator $\widetilde{\text{deg}}$ on \mathbb{M} yields

$$\mathcal{D}_{\mathbb{M}}^0 = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{\mathbb{M}}^0(n),$$

where each $\mathcal{D}_{\mathbb{M}}^0(n)$, $n \in \mathbb{Z}$, is an irreducible representation of $SL(2, \mathbb{C})$ and can be decomposed further into the irreducible polynomial subspaces with respect to the compact subgroup $SU(2) \subset SL(2, \mathbb{C})$. This description also gives an explicit realization of the K -types of the minimal representation relative to the maximal compact subgroup $K' = SO(4) \subset SL(4, \mathbb{R})$. The Poisson and Cauchy-Fueter integrals provide us projections onto the discrete component on \mathbb{M} , but now the procedure is more subtle than the case of discrete series projectors in $\mathbb{H}_{\mathbb{R}}$ such as (2). The projection is obtained as a boundary value of these integrals in the sense of

²Strictly speaking, $SO(3, 3)$ does not have a minimal representation, so by “minimal representation of $SO(3, 3)$ ” we mean the representation $\varpi^{p,q}$ of $O(p, q)$ in the notations of [KobØ] for $p = q = 3$. When $p + q \geq 8$ one gets a genuine minimal representation.

hyperfunctions, namely as a limit of the difference of two integrals. Thus for the Fueter integral we obtain the following result:

$$\lim_{\varepsilon \rightarrow 0^+} (\Phi(W, i\varepsilon) - \Phi(W, -i\varepsilon)) = (\mathbf{P}^{discr(\mathbb{M})} f_{>0} - \mathbf{P}^{discr(\mathbb{M})} f_{<0})(W), \quad (3)$$

where

$$\Phi(W, i\varepsilon) = \frac{i}{2\pi^2} \int_{Y \in H_{\mathbb{M}}^3} \frac{(Y - W)^+}{(\det(Y - W) + i\varepsilon)^2} \cdot *dY \cdot f(Y) \quad (4)$$

and $\mathbf{P}^{discr(\mathbb{M})}$ is the projection of the space of left-regular functions on \mathbb{M} onto the discrete series component (Theorem 84). Note that, as in the case of the discrete series projectors on $\mathbb{H}_{\mathbb{R}}$ (2), $\mathbf{P}^{discr(\mathbb{M})} f_{>0}$ and $\mathbf{P}^{discr(\mathbb{M})} f_{<0}$ enter with opposite signs and the functions f with $\widehat{\deg} f = 0$ would cause the integrals to be divergent. (In the case of Poisson formula the functions in $\mathcal{D}_{\mathbb{M}}^0(0)$ are projected out.) Thus we can view $\mathcal{D}_{\mathbb{M}}^0(0)$ as the limit of the discrete series on $H_{\mathbb{M}}^3 \simeq SL(2, \mathbb{C})/SU(1, 1)$.

Finally, we return to the $\mathbb{H}_{\mathbb{R}}$ -setting and study the continuous series component by applying the Cayley transform to the discrete series component on \mathbb{M} . Quite surprisingly, the Poisson and Cauchy-Fueter integrals do not give the projectors as in (3), (4), but instead become diagonal operators commuting only with the subgroup $S(GL(2, \mathbb{R}) \times GL(2, \mathbb{R}))$ of the conformal group $SL(4, \mathbb{R})$, and the diagonal density is precisely the inverse of the Plancherel density on $SL(2, \mathbb{R})$! Recall that in the parameterization of the continuous series by $\chi = (l, \varepsilon)$, where $l = -\frac{1}{2} + i\lambda$, $\lambda \in \mathbb{R}$ and $\varepsilon \in \{0, \frac{1}{2}\}$, the Plancherel density is given by

$$Pl(\chi) = \begin{cases} \lambda \tanh(\pi\lambda) & \text{if } \varepsilon = 0; \\ \lambda \coth(\pi\lambda) & \text{if } \varepsilon = \frac{1}{2}. \end{cases}$$

Then we have the following identity

$$\lim_{\varepsilon \rightarrow 0^+} (\tilde{\Phi}(W, i\varepsilon) - \tilde{\Phi}(W, -i\varepsilon)) = \frac{\varphi_{\chi}(W)}{Pl(\chi)}, \quad W \in SU(1, 1), \quad (5)$$

where

$$\tilde{\Phi}(W, i\varepsilon) = \frac{1}{2\pi^2} \int_{X \in SU(1, 1)} \frac{\varphi_{\chi}(X)}{\det(X - W) + i\varepsilon} dX^3 \quad (6)$$

and $\varphi_{\chi}(X)$ belongs to the linear span of matrix coefficients of the irreducible representation corresponding to $\chi = (l, \varepsilon)$ (Theorem 94). We remark that for the discrete series of $SL(2, \mathbb{R})$ the Plancherel density is $-(2l + 1)$, and we could rewrite the Poisson integral for the discrete series projector in a form similar to (5)-(6) using the kernels $(\det(X - W) \pm i\varepsilon)^{-1}$. Thus quaternionic analysis over the split quaternions $\mathbb{H}_{\mathbb{R}}$ naturally reveals the important salient features of harmonic analysis on $SL(2, \mathbb{R})$.

To summarize our studies of quaternionic analysis on \mathbb{H} in [FL1] and split quaternionic analysis on $\mathbb{H}_{\mathbb{R}}$ in this work we would like to point out once again the special role of the Minkowski space \mathbb{M} related to the two algebras of quaternions by the two types of Cayley transform. Thus analysis on \mathbb{H} and $\mathbb{H}_{\mathbb{R}}$ is equivalent to analysis on the two-sheeted and one-sheeted hyperboloids in \mathbb{M} respectively. Moreover, doing analysis on \mathbb{M} in many ways facilitates and clarifies analysis on \mathbb{H} and $\mathbb{H}_{\mathbb{R}}$. The relation between \mathbb{H} , $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} also reveals the hierarchy

of the symmetry groups of these three spaces, which can be presented as follows:

$$\begin{array}{ccccc}
 SO(5,1) & & SO(4,2) & & SO(3,3) \\
 & \searrow & & \searrow & \\
 & & SO(4,1) & & SO(3,2) \\
 & \swarrow & & \swarrow & \\
 SO(4) & & SO(3,1) & & SO(2,2) \\
 \wr & & \wr & & \wr \\
 \mathbb{H} \simeq \mathbb{R}^4 & & \mathbb{M} \simeq \mathbb{R}^{3,1} & & \mathbb{H}_{\mathbb{R}} \simeq \mathbb{R}^{2,2}
 \end{array}$$

(In this diagram some groups are replaced with locally isomorphic ones.) Clearly, the groups at the bottom of the diagram are the metric-preserving transformations of the three real forms of $\mathbb{H}_{\mathbb{C}}$ of our interest, while the groups at the top row are the corresponding conformal groups. The groups in the middle row appear as subgroups of the conformal groups preserving the hyperboloids. Thus the Minkowski space literally plays the central role in quaternionic analysis on $\mathbb{H}_{\mathbb{C}}$ and its real forms \mathbb{H} and $\mathbb{H}_{\mathbb{R}}$.

In the same way as various results in quaternionic analysis on \mathbb{H} can be generalized to Clifford analysis on arbitrary Euclidean spaces (the book [GSp] contains a comprehensive bibliography on these subjects), split quaternionic analysis on $\mathbb{H}_{\mathbb{R}}$ as well as analysis on the Minkowski space \mathbb{M} can be further extended to Clifford analysis on real vector spaces of *arbitrary signature*. However, quaternionic analysis on \mathbb{H} , $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} is singled out because of its relation to the harmonic analysis on the simplest groups $SU(2)$, $SL(2, \mathbb{R})$ and homogeneous spaces $SL(2, \mathbb{C})/SU(2)$, $SL(2, \mathbb{C})/SL(2, \mathbb{R})$ as well as uniqueness of the space of quaternions \mathbb{H} . Moreover, we believe that the special role of the four-dimensional case will reveal itself at deeper levels in the future developments of the subject when the relation with exotic smoothness in four-dimensional topology and renormalization in four-dimensional quantum field theory becomes more transparent.

By design, analysis on the Minkowski space has many connections with the four-dimensional physics. In our previous paper [FL1] we noted several relations between representation theory underlying quaternionic analysis and Feynman diagrams. Split quaternionic analysis adds another important connection between the mathematical and physical structures. To make the connections with physics more apparent, in both cases we consider various results of quaternionic analysis over \mathbb{H} and $\mathbb{H}_{\mathbb{R}}$ in the Minkowski space realization. The projectors onto the continuous and discrete components on \mathbb{M} are expressed using the kernels:

$$\frac{1}{\det Z + i\varepsilon z^0}, \quad \frac{1}{\det Z - i\varepsilon z^0}, \quad \frac{1}{\det Z + i\varepsilon}, \quad \frac{1}{\det Z - i\varepsilon}, \quad Z = Y - W, \quad (7)$$

z^0 denotes the time coordinate of Z , in the case of Poisson integrals and their ∇_Y -derivatives in the case of Cauchy-Fueter integrals. But these are precisely the massless singular functions that form the foundation of the theory of Feynman integrals! (See, for example, [BSh].) Any other singular function appearing in physics is a linear combination of the ones listed in (7). Since there is a linear relation between these kernels:

$$\frac{1}{\det Z + i\varepsilon z^0} + \frac{1}{\det Z - i\varepsilon z^0} = \frac{1}{\det Z + i\varepsilon} + \frac{1}{\det Z - i\varepsilon},$$

there are only three essentially different kernels in the list (7), which is exactly the number of irreducible components of the space of functions on imaginary Lobachevski space $SL(2, \mathbb{C})/SU(1, 1)$! Thus the problem of separation of the series on $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})/SU(1, 1)$, which was solved by means of quaternionic analysis, inevitably leads to the heart of four-dimensional quantum field theory! Unfortunately, our representation-theoretic approach does not naturally

include the massive versions of the singular functions in (7). However, in our forthcoming work [FL2] we will show that, in spite of the enormous rigidity of quaternionic analysis, it is possible to make a one-parameter deformation that we expect to include the missing mass.

The paper is organized as follows. In Section 2 we introduce notations and overview basics of split quaternionic analysis. In particular, we derive an analogue of the Cauchy-Fueter formula for regular functions on $\mathbb{H}_{\mathbb{R}}$. We also give the bases for the spaces of harmonic, left- and right-regular on $\mathbb{H}_{\mathbb{R}}$. In Section 3 we study the discrete series component of split quaternionic analysis (over $\mathbb{H}_{\mathbb{R}}$). First we find the polynomial bases of the spaces of harmonic, left- and right-regular functions \mathcal{D} , \mathcal{V} , \mathcal{V}' and describe the action of the Lie algebra $\mathfrak{sl}(4, \mathbb{C})$ in those bases. After a brief review of Ol'shanskii semigroups, we obtain expansions of the Poisson and Fueter kernels in terms of the matrix coefficients of the discrete series of $SU(1, 1)$. From these expansions we derive the discrete series projectors. In Section 4 we prove that the Cayley transform between $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} interchanges the discrete and continuous series components on $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} . Using results of Section 3 we obtain the continuous series projectors onto the same spaces \mathcal{D} , \mathcal{V} and \mathcal{V}' in the new setting of the Minkowski space. Then we study the minimal representation of $SL(4, \mathbb{R})$ realized in the space of harmonic functions on \mathbb{M} , find a polynomial basis for the K -types and use that basis to derive the discrete series projector. In Section 5 we study the Poisson integrals and their effect on the discrete and continuous series on $\mathbb{H}_{\mathbb{R}}$. We end this section with the proof that the boundary value of the Poisson integral in the sense of hyperfunctions yields a diagonal operator with density given by the inverse of the Plancherel measure on the group $SL(2, \mathbb{R})$. In the Appendix we give a brief introduction to the special functions that we use throughout the paper and list their properties.

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2 Split Quaternionic Analysis

2.1 The Quaternionic Spaces $\mathbb{H}_{\mathbb{C}}$, $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M}

In this article we use notations established in [FL1]. In particular, e_0, e_1, e_2, e_3 denote the units of the classical quaternions \mathbb{H} corresponding to the more familiar $1, i, j, k$ (we reserve the symbol i for $\sqrt{-1} \in \mathbb{C}$). Thus \mathbb{H} is an algebra over \mathbb{R} generated by e_0, e_1, e_2, e_3 , and the multiplicative structure is determined by the rules

$$e_0 e_i = e_i e_0 = e_i, \quad (e_i)^2 = -e_0, \quad e_i e_j = -e_j e_i, \quad 1 \leq i < j \leq 3,$$

and the fact that \mathbb{H} is a division ring. Next we consider the algebra of complexified quaternions $\mathbb{H}_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{H}$. We define a complex conjugation on $\mathbb{H}_{\mathbb{C}}$ with respect to \mathbb{H} :

$$Z = z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3 \quad \mapsto \quad Z^c = \overline{z^0} e_0 + \overline{z^1} e_1 + \overline{z^2} e_2 + \overline{z^3} e_3, \quad z^0, z^1, z^2, z^3 \in \mathbb{C},$$

so that $\mathbb{H} = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^c = Z\}$. The quaternionic conjugation on $\mathbb{H}_{\mathbb{C}}$ is defined by:

$$Z = z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3 \quad \mapsto \quad Z^+ = z^0 e_0 - z^1 e_1 - z^2 e_2 - z^3 e_3, \quad z^0, z^1, z^2, z^3 \in \mathbb{C};$$

it is an anti-involution:

$$(ZW)^+ = W^+ Z^+, \quad \forall Z, W \in \mathbb{H}_{\mathbb{C}}.$$

We will also use an involution

$$Z \mapsto Z^- = -e_3 Z e_3 \quad (\text{conjugation by } e_3).$$

Then the complex conjugation, the quaternionic conjugation and the involution $Z \mapsto Z^-$ commute with each other.

In this article we will be primarily interested in the space of *split quaternions* $\mathbb{H}_{\mathbb{R}}$ which is a real form of $\mathbb{H}_{\mathbb{C}}$ defined by

$$\mathbb{H}_{\mathbb{R}} = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^{c-} = Z\} = \{\mathbb{R}\text{-span of } e_0, \tilde{e}_1 = ie_1, \tilde{e}_2 = -ie_2, e_3\}$$

and the Minkowski space \mathbb{M} which is regarded as another real form of $\mathbb{H}_{\mathbb{C}}$:

$$\mathbb{M} = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^{c+} = -Z\} = \{\mathbb{R}\text{-span of } \tilde{e}_0 = -ie_0, e_1, e_2, e_3\}.$$

On $\mathbb{H}_{\mathbb{C}}$ we have a quadratic form N defined by

$$N(Z) = ZZ^+ = Z^+Z = (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2,$$

hence $Z^{-1} = Z^+/N(Z)$. The corresponding symmetric bilinear form on $\mathbb{H}_{\mathbb{C}}$ is

$$\langle Z, W \rangle = \frac{1}{2} \text{Tr}(Z^+W) = \frac{1}{2} \text{Tr}(ZW^+), \quad Z, W \in \mathbb{H}_{\mathbb{C}}, \quad (8)$$

where $\text{Tr } Z = 2z^0 = Z + Z^+$. When this quadratic form is restricted to \mathbb{H} , $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} , it has signature $(4, 0)$, $(2, 2)$ and $(3, 1)$ respectively.

We will use the standard matrix realization of \mathbb{H} so that

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

and

$$\mathbb{H} = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^c = Z\} = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}; z_{22} = \overline{z_{11}}, z_{21} = -\overline{z_{12}} \right\}.$$

Then $\mathbb{H}_{\mathbb{C}}$ can be identified with the algebra of all complex 2×2 matrices:

$$\mathbb{H}_{\mathbb{C}} = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}; z_{ij} \in \mathbb{C} \right\},$$

the quadratic form $N(Z)$ becomes $\det Z$, and the involution $Z \mapsto Z^-$ becomes

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \mapsto Z^- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} z_{11} & -z_{12} \\ -z_{21} & z_{22} \end{pmatrix}.$$

The split quaternions $\mathbb{H}_{\mathbb{R}}$ and the Minkowski space \mathbb{M} have matrix realizations

$$\mathbb{H}_{\mathbb{R}} = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}; z_{22} = \overline{z_{11}}, z_{21} = \overline{z_{12}} \right\},$$

$$\mathbb{M} = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}; z_{11}, z_{22} \in i\mathbb{R}, z_{21} = -\overline{z_{12}} \right\}.$$

From this realization it is easy to see that the split quaternions form an algebra over \mathbb{R} isomorphic to $\mathfrak{gl}(2, \mathbb{R})$ and the invertible elements in $\mathbb{H}_{\mathbb{R}}$, denoted by $\mathbb{H}_{\mathbb{R}}^{\times}$, are nothing else but $GL(2, \mathbb{R})$. We regard $SL(2, \mathbb{C})$ as a quadric $\{N(Z) = 1\}$ in $\mathbb{H}_{\mathbb{C}}$, and we also regard $SU(1, 1) \simeq SL(2, \mathbb{R})$ as the set of real points of this quadric:

$$SU(1, 1) = \{Z \in \mathbb{H}_{\mathbb{R}}; N(Z) = 1\} = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ \overline{z_{12}} & \overline{z_{11}} \end{pmatrix} \in \mathbb{H}_{\mathbb{R}}; \det Z = |z_{11}|^2 - |z_{12}|^2 = 1 \right\}.$$

The group

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^c \end{pmatrix}; a \in \mathbb{H}_{\mathbb{C}}, N(a) = 1 \right\} \subset GL(2, \mathbb{H}_{\mathbb{C}}) \quad (9)$$

is naturally isomorphic to $SL(2, \mathbb{C})$ and acts on the unit hyperboloid of one sheet $\tilde{H} = \{Y \in \mathbb{M}; N(Y) = 1\}$ transitively. The stabilizer group of $e_3 \in \tilde{H}$ is

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^c \end{pmatrix}; a \in \mathbb{H}_{\mathbb{R}}, N(a) = 1 \right\} \simeq SU(1, 1).$$

Thus the hyperboloid \tilde{H} – also known as the imaginary Lobachevski space – is naturally identified with the homogeneous space $SL(2, \mathbb{C})/SU(1, 1)$.

The algebra of split quaternions $\mathbb{H}_{\mathbb{R}}$ is spanned over \mathbb{R} by the four matrices

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

so

$$\mathbb{H}_{\mathbb{R}} = \left\{ x^0 e_0 + x^1 \tilde{e}_1 + x^2 \tilde{e}_2 + x^3 e_3 = \begin{pmatrix} x^0 - ix^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 + ix^3 \end{pmatrix}; x^0, x^1, x^2, x^3 \in \mathbb{R} \right\}.$$

The quaternionic conjugation in this basis becomes

$$e_0^+ = e_0, \quad \tilde{e}_1^+ = -\tilde{e}_1, \quad \tilde{e}_2^+ = -\tilde{e}_2, \quad e_3^+ = -e_3.$$

The multiplication rules for $\mathbb{H}_{\mathbb{R}}$ are:

$$\begin{aligned} e_0 & \text{ commutes with all elements of } \mathbb{H}_{\mathbb{R}}, \\ \tilde{e}_1, \tilde{e}_2, e_3 & \text{ anti-commute,} \\ e_0^2 = \tilde{e}_1^2 = \tilde{e}_2^2 = e_0, \quad e_3^2 = -e_0, \\ \tilde{e}_1 \tilde{e}_2 = e_3, \quad \tilde{e}_2 e_3 = -\tilde{e}_1, \quad e_3 \tilde{e}_1 = -\tilde{e}_2. \end{aligned}$$

The elements $e_0, \tilde{e}_1, \tilde{e}_2, e_3$ are orthogonal with respect to the bilinear form (8) and $\langle e_0, e_0 \rangle = \langle e_3, e_3 \rangle = 1$, $\langle \tilde{e}_1, \tilde{e}_1 \rangle = \langle \tilde{e}_2, \tilde{e}_2 \rangle = -1$.

Similarly, the Minkowski space \mathbb{M} is a subspace of $\mathbb{H}_{\mathbb{C}}$ spanned over \mathbb{R} by the four matrices

$$\tilde{e}_0 = -ie_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

Define a norm on $\mathbb{H}_{\mathbb{C}}$ by

$$\|Z\| = \frac{1}{\sqrt{2}} \sqrt{|z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2 + |z_{22}|^2}, \quad Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}},$$

so that $\|e_i\| = 1$, $0 \leq i \leq 3$.

The (classical) quaternions \mathbb{H} are oriented so that $\{e_0, e_1, e_2, e_3\}$ is a positive basis. Let $dV = dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3$ be the holomorphic 4-form on $\mathbb{H}_{\mathbb{C}}$ determined by $dV(e_0, e_1, e_2, e_3) = 1$, then the restriction $dV|_{\mathbb{H}}$ is the Euclidean volume form corresponding to $\{e_0, e_1, e_2, e_3\}$. On the other hand, the restriction $dV|_{\mathbb{H}_{\mathbb{R}}}$ is also real-valued and hence determines an orientation on $\mathbb{H}_{\mathbb{R}}$ so that $\{e_0, \tilde{e}_1, \tilde{e}_2, e_3\}$ becomes a positively oriented basis. The orientation on \mathbb{M} was defined in [FL1] so that $\{\tilde{e}_0, e_1, e_2, e_3\}$ is a positively oriented basis.

In [FL1] we defined a holomorphic 3-form on $\mathbb{H}_{\mathbb{C}}$ with values in $\mathbb{H}_{\mathbb{C}}$

$$Dz = e_0 dz^1 \wedge dz^2 \wedge dz^3 - e_1 dz^0 \wedge dz^2 \wedge dz^3 + e_2 dz^0 \wedge dz^1 \wedge dz^3 - e_3 dz^0 \wedge dz^1 \wedge dz^2$$

characterized by the property

$$\langle Z_1, Dz(Z_2, Z_3, Z_4) \rangle = \frac{1}{2} \text{Tr}(Z_1^+, Dz(Z_2, Z_3, Z_4)) = dV(Z_1, Z_2, Z_3, Z_4)$$

for all $Z_1, Z_2, Z_3, Z_4 \in \mathbb{H}_{\mathbb{C}}$. Let $Dx = Dz|_{\mathbb{H}_{\mathbb{R}}}$ and $D\tilde{x} = Dz|_{\mathbb{H}}$.

Proposition 1 *The 3-form Dx takes values in $\mathbb{H}_{\mathbb{R}}$. If we write $X = x^0 e_0 + x^1 \tilde{e}_1 + x^2 \tilde{e}_2 + x^3 e_3 \in \mathbb{H}_{\mathbb{R}}$, $x^0, x^1, x^2, x^3 \in \mathbb{R}$, then Dx is given explicitly by*

$$Dx = e_0 dx^1 \wedge dx^2 \wedge dx^3 + \tilde{e}_1 dx^0 \wedge dx^2 \wedge dx^3 - \tilde{e}_2 dx^0 \wedge dx^1 \wedge dx^3 - e_3 dx^0 \wedge dx^1 \wedge dx^2. \quad (10)$$

Remark 2 *Clearly, the form Dx satisfies the property*

$$\langle X_1, Dx(X_2, X_3, X_4) \rangle = \frac{1}{2} \text{Tr}(X_1^+, Dx(X_2, X_3, X_4)) = dV(X_1, X_2, X_3, X_4)$$

for all $X_1, X_2, X_3, X_4 \in \mathbb{H}_{\mathbb{R}}$, which could be used to define it.

Let $U \subset \mathbb{H}_{\mathbb{R}}$ be an open region with piecewise smooth boundary ∂U . We give a canonical orientation to ∂U as follows. The positive orientation of U is determined by $\{e_0, \tilde{e}_1, \tilde{e}_2, e_3\}$. Pick a smooth point $p \in \partial U$ and let \vec{n}_p be a non-zero vector in $T_p \mathbb{H}_{\mathbb{R}}$ perpendicular to $T_p \partial U$ and pointing outside of U . Then $\{\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3\} \subset T_p \partial U$ is positively oriented in ∂U if and only if $\{\vec{n}_p, \vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3\}$ is positively oriented in $\mathbb{H}_{\mathbb{R}}$. We orient $SU(1, 1)$ and more generally hyperboloids $H_R = \{X \in \mathbb{H}_{\mathbb{R}}; N(X) = R^2\}$ as the boundaries of the open sets $\{X \in \mathbb{H}_{\mathbb{R}}; N(X) < R^2\}$.

Lemma 3 *Let $R \in \mathbb{R}$ be a constant, then we have the following restriction formula:*

$$Dx|_{H_R} = X \frac{dS}{\|X\|},$$

where dS denotes the restrictions of the Euclidean measure on $\mathbb{H}_{\mathbb{R}}$ to H_R .

We recall some notations from [FL1]. Let \mathbb{S} be the natural two-dimensional complex representation of the algebra $\mathbb{H}_{\mathbb{C}}$. We realize it as a column of two complex numbers, then $\mathbb{H}_{\mathbb{C}}$ acts on \mathbb{S} by matrix multiplication on the left. Similarly, we denote by \mathbb{S}' the dual space of \mathbb{S} , this time realized as a row of two complex numbers. We have a right action of $\mathbb{H}_{\mathbb{C}}$ on \mathbb{S}' by multiplication on the right.

2.2 Regular Functions on \mathbb{H} and $\mathbb{H}_{\mathbb{C}}$

Recall that regular functions on \mathbb{H} are defined using an analogue of the Cauchy-Riemann equations. Write $\tilde{X} \in \mathbb{H}$ as $\tilde{X} = \tilde{x}^0 e_0 + \tilde{x}^1 e_1 + \tilde{x}^2 e_2 + \tilde{x}^3 e_3$, $\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \mathbb{R}$, and factor the four-dimensional Laplacian operator \square on \mathbb{H} as a product of two Dirac operators

$$\square = \frac{\partial^2}{(\partial \tilde{x}^0)^2} + \frac{\partial^2}{(\partial \tilde{x}^1)^2} + \frac{\partial^2}{(\partial \tilde{x}^2)^2} + \frac{\partial^2}{(\partial \tilde{x}^3)^2} = \nabla \nabla^+ = \nabla^+ \nabla,$$

where

$$\begin{aligned} \nabla^+ &= e_0 \frac{\partial}{\partial \tilde{x}^0} + e_1 \frac{\partial}{\partial \tilde{x}^1} + e_2 \frac{\partial}{\partial \tilde{x}^2} + e_3 \frac{\partial}{\partial \tilde{x}^3} \\ \text{and} \quad \nabla &= e_0 \frac{\partial}{\partial \tilde{x}^0} - e_1 \frac{\partial}{\partial \tilde{x}^1} - e_2 \frac{\partial}{\partial \tilde{x}^2} - e_3 \frac{\partial}{\partial \tilde{x}^3}. \end{aligned}$$

The operators ∇^+ , ∇ can be applied to functions on the left and on the right. For an open subset $U \subset \mathbb{H}$ and a differentiable function f on U with values in \mathbb{H} , \mathbb{S} or $\mathbb{H}_{\mathbb{C}}$, we say f is *left-regular* if $(\nabla^+ f)(\tilde{X}) = 0$ for all $\tilde{X} \in U$. Similarly, a differentiable function g on U with values in \mathbb{H} , \mathbb{S}' or $\mathbb{H}_{\mathbb{C}}$, is *right-regular* if $(g \nabla^+)(\tilde{X}) = 0$ for all $\tilde{X} \in U$.

Proposition 4 For any C^1 -functions $f : U \rightarrow \mathbb{S}$ and $g : U \rightarrow \mathbb{S}'$, where $U \subset \mathbb{H}$ is an open subset,

$$d(D\tilde{x} \cdot f) = -D\tilde{x} \wedge df = (\nabla^+ f) dV|_{\mathbb{H}}, \quad d(g \cdot D\tilde{x}) = dg \wedge D\tilde{x} = (g\nabla^+) dV|_{\mathbb{H}}.$$

In particular,

$$\nabla^+ f = 0 \iff D\tilde{x} \wedge df = 0, \quad g\nabla^+ = 0 \iff dg \wedge D\tilde{x} = 0.$$

Let $U^{\mathbb{C}} \subset \mathbb{H}_{\mathbb{C}}$ be an open set. Following [FL1], we say that a differential function on $U^{\mathbb{C}}$ with values in $\mathbb{C}, \mathbb{H}_{\mathbb{C}}, \mathbb{S}$ or \mathbb{S}' is *holomorphic* if it is holomorphic with respect to the complex variables z^0, z^1, z^2, z^3 . Then we define $f^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}$ to be *holomorphic left-regular* if it is holomorphic and $\nabla^+ f^{\mathbb{C}} = 0$. Similarly, $g^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}'$ is defined to be *holomorphic right-regular* if it is holomorphic and $g^{\mathbb{C}} \nabla^+ = 0$. Finally, we call a function $\varphi : U^{\mathbb{C}} \rightarrow \mathbb{C}$ *holomorphic harmonic* if it is harmonic and $\square\varphi = 0$.

If we identify $\mathbb{H}_{\mathbb{C}}$ with complex 2×2 matrices $\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$, $z_{ij} \in \mathbb{C}$, then a function is holomorphic if and only if it is holomorphic with respect to the complex variables z_{ij} , $1 \leq i, j \leq 2$. For holomorphic functions $f^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}$ and $g^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}'$, the following derivatives are equal:

$$\begin{aligned} \nabla^+ f^{\mathbb{C}} &= e_0 \frac{\partial f^{\mathbb{C}}}{\partial z^0} + e_1 \frac{\partial f^{\mathbb{C}}}{\partial z^1} + e_2 \frac{\partial f^{\mathbb{C}}}{\partial z^2} + e_3 \frac{\partial f^{\mathbb{C}}}{\partial z^3} = 2 \begin{pmatrix} \frac{\partial}{\partial z_{22}} & -\frac{\partial}{\partial z_{21}} \\ -\frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{11}} \end{pmatrix} f^{\mathbb{C}}, \\ g^{\mathbb{C}} \nabla^+ &= \frac{\partial g^{\mathbb{C}}}{\partial z^0} e_0 + \frac{\partial g^{\mathbb{C}}}{\partial z^1} e_1 + \frac{\partial g^{\mathbb{C}}}{\partial z^2} e_2 + \frac{\partial g^{\mathbb{C}}}{\partial z^3} e_3 = 2g^{\mathbb{C}} \begin{pmatrix} \frac{\partial}{\partial z_{22}} & -\frac{\partial}{\partial z_{21}} \\ -\frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{11}} \end{pmatrix}, \\ \nabla f^{\mathbb{C}} &= e_0 \frac{\partial f^{\mathbb{C}}}{\partial z^0} - e_1 \frac{\partial f^{\mathbb{C}}}{\partial z^1} - e_2 \frac{\partial f^{\mathbb{C}}}{\partial z^2} - e_3 \frac{\partial f^{\mathbb{C}}}{\partial z^3} = 2 \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \frac{\partial}{\partial z_{21}} \\ \frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{22}} \end{pmatrix} f^{\mathbb{C}}, \\ g^{\mathbb{C}} \nabla &= \frac{\partial g^{\mathbb{C}}}{\partial z^0} e_0 - \frac{\partial g^{\mathbb{C}}}{\partial z^1} e_1 - \frac{\partial g^{\mathbb{C}}}{\partial z^2} e_2 - \frac{\partial g^{\mathbb{C}}}{\partial z^3} e_3 = 2g^{\mathbb{C}} \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \frac{\partial}{\partial z_{21}} \\ \frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{22}} \end{pmatrix}. \end{aligned}$$

Since we are interested in holomorphic functions only, we will abuse the notation and denote by ∇ and ∇^+ the holomorphic differential operators $\frac{\partial}{\partial z^0} e_0 - \frac{\partial}{\partial z^1} e_1 - \frac{\partial}{\partial z^2} e_2 - \frac{\partial}{\partial z^3} e_3$ and $e_0 \frac{\partial}{\partial z^0} + e_1 \frac{\partial}{\partial z^1} + e_2 \frac{\partial}{\partial z^2} + e_3 \frac{\partial}{\partial z^3}$ respectively.

Proposition 5 For any holomorphic functions $f^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}$ and $g^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}'$,

$$\nabla^+ f^{\mathbb{C}} = 0 \iff Dz \wedge df^{\mathbb{C}} = 0, \quad g^{\mathbb{C}} \nabla^+ = 0 \iff dg^{\mathbb{C}} \wedge Dz = 0.$$

Lemma 6 We have:

1. $\square \frac{1}{N(Z)} = 0$;
2. $\nabla \frac{1}{N(Z)} = \frac{1}{N(Z)} \nabla = -2 \frac{Z^-}{N(Z)} = -2 \frac{Z^+}{N(Z)^2}$;
3. $\frac{Z^-}{N(Z)} = \frac{Z^+}{N(Z)^2}$ is a holomorphic left- and right-regular function defined wherever $N(Z) \neq 0$;
4. The form $\frac{Z^-}{N(Z)} \cdot Dz = \frac{Z^+}{N(Z)^2} \cdot Dz$ is a closed holomorphic $\mathbb{H}_{\mathbb{C}}$ -valued 3-form defined wherever $N(Z) \neq 0$.

Lemma 7 For any differentiable function $\varphi : U^{\mathbb{C}} \rightarrow \mathbb{C}$, we have:

$$\begin{aligned}\nabla(\varphi(Z^+)) &= (\nabla^+ \varphi)(Z^+), & \nabla^+(\varphi(Z^+)) &= (\nabla \varphi)(Z^+), \\ \nabla(\varphi(Z^{-1})) &= -Z^{-1} \cdot (\nabla \varphi)(Z^{-1}) \cdot Z^{-1}.\end{aligned}$$

We will often use the shifted degree operator

$$\widetilde{\text{deg}} = 1 + z_0 \frac{\partial}{\partial z^0} + z_1 \frac{\partial}{\partial z^1} + z_2 \frac{\partial}{\partial z^2} + z_3 \frac{\partial}{\partial z^3}$$

(the degree operator plus identity). By direct computation we obtain

Lemma 8

$$2(\widetilde{\text{deg}} + 1) = Z^+ \nabla^+ + \nabla Z = \nabla^+ Z^+ + Z \nabla. \quad (11)$$

Next we describe actions of the group $GL(2, \mathbb{H}_{\mathbb{C}})$ on the spaces of left-, right-regular and harmonic functions on $\mathbb{H}_{\mathbb{C}}$ with isolated singularities by conformal (fractional linear) transformations. Let

$$h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}}) \quad \text{and write} \quad h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

On the space of left-regular \mathbb{S} -valued functions $GL(2, \mathbb{H}_{\mathbb{C}})$ acts by

$$\pi_l(h) : f(Z) \mapsto (\pi_l(h)f)(Z) = \frac{(cZ + d)^{-1}}{N(cZ + d)} \cdot f((aZ + b)(cZ + d)^{-1}).$$

On the space of right-regular \mathbb{S}' -valued functions $GL(2, \mathbb{H}_{\mathbb{C}})$ acts by

$$\pi_r(h) : g(Z) \mapsto (\pi_r(h)g)(Z) = g((a' - Zc')^{-1}(-b' + Zd')) \cdot \frac{(a' - Zc')^{-1}}{N(a' - Zc')}.$$

On the space of \mathbb{C} -valued harmonic functions we have two different actions:

$$\begin{aligned}\pi_l^0(h) : \varphi(Z) &\mapsto (\pi_l^0(h)\varphi)(Z) = \frac{1}{N(cZ + d)} \cdot \varphi((aZ + b)(cZ + d)^{-1}), \\ \pi_r^0(h) : \varphi(Z) &\mapsto (\pi_r^0(h)\varphi)(Z) = \frac{1}{N(a' - Zc')} \cdot \varphi((a' - Zc')^{-1}(-b' + Zd')).\end{aligned}$$

These two actions coincide on $SL(2, \mathbb{H}_{\mathbb{C}})$, which is defined as the connected Lie subgroup of $GL(2, \mathbb{H}_{\mathbb{C}})$ with Lie algebra

$$\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}}) = \{x \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}); \text{Re}(\text{Tr } x) = 0\}.$$

2.3 Regular Functions on $\mathbb{H}_{\mathbb{R}}$

We introduce linear differential operators on $\mathbb{H}_{\mathbb{R}}$

$$\begin{aligned}\nabla_{\mathbb{R}}^+ &= e_0 \frac{\partial}{\partial x^0} - \tilde{e}_1 \frac{\partial}{\partial x^1} - \tilde{e}_2 \frac{\partial}{\partial x^2} + e_3 \frac{\partial}{\partial x^3} \\ \text{and} \quad \nabla_{\mathbb{R}} &= e_0 \frac{\partial}{\partial x^0} + \tilde{e}_1 \frac{\partial}{\partial x^1} + \tilde{e}_2 \frac{\partial}{\partial x^2} - e_3 \frac{\partial}{\partial x^3}\end{aligned}$$

which may be applied to functions on the left and on the right.

Definition 9 Fix an open subset $U \subset \mathbb{H}_{\mathbb{R}}$. A differentiable function $f : \mathbb{H}_{\mathbb{R}} \rightarrow \mathbb{S}$ is left-regular if it satisfies

$$(\nabla_{\mathbb{R}}^+ f)(X) = e_0 \frac{\partial f}{\partial x^0}(X) - \tilde{e}_1 \frac{\partial f}{\partial x^1}(X) - \tilde{e}_2 \frac{\partial f}{\partial x^2}(X) + e_3 \frac{\partial f}{\partial x^3}(X) = 0, \quad \forall X \in U.$$

Similarly, a differentiable function $g : \mathbb{H}_{\mathbb{R}} \rightarrow \mathbb{S}'$ is right-regular if

$$(g\nabla_{\mathbb{R}}^+)(X) = \frac{\partial g}{\partial x^0}(X)e_0 - \frac{\partial g}{\partial x^1}(X)\tilde{e}_1 - \frac{\partial g}{\partial x^2}(X)\tilde{e}_2 + \frac{\partial g}{\partial x^3}(X)e_3 = 0, \quad \forall X \in U.$$

We denote by $\square_{2,2}$ the ultrahyperbolic wave operator on $\mathbb{H}_{\mathbb{R}}$ which can be factored as follows:

$$\square_{2,2} = \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^1)^2} - \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2} = \nabla_{\mathbb{R}} \nabla_{\mathbb{R}}^+ = \nabla_{\mathbb{R}}^+ \nabla_{\mathbb{R}}.$$

Abusing terminology, we call a smooth function $\varphi : U \rightarrow \mathbb{C}$ harmonic if $\square_{2,2}\varphi = 0$.

Proposition 10 For any \mathcal{C}^1 -functions $f : U \rightarrow \mathbb{S}$ and $g : U \rightarrow \mathbb{S}'$,

$$d(Dx \cdot f) = -Dx \wedge df = (\nabla_{\mathbb{R}}^+ f) dV, \quad d(g \cdot Dx) = dg \wedge Dx = (g\nabla_{\mathbb{R}}^+) dV,$$

In particular,

$$\nabla_{\mathbb{R}}^+ f = 0 \iff Dx \wedge df = 0, \quad g\nabla_{\mathbb{R}}^+ = 0 \iff dg \wedge Dx = 0.$$

Let $U^{\mathbb{C}} \subset \mathbb{H}_{\mathbb{C}}$ be an open set. The restriction relations

$$Dz|_{\mathbb{H}_{\mathbb{R}}} = Dx, \quad Dz|_{\mathbb{H}} = D\tilde{x}$$

imply that the restriction of a holomorphic left- or right-regular function to $U^{\mathbb{R}} = U^{\mathbb{C}} \cap \mathbb{H}_{\mathbb{R}}$ produces a left- or right-regular function on $U^{\mathbb{R}}$ respectively. And the restriction of a holomorphic left- or right-regular function to $U_{\mathbb{H}} = U^{\mathbb{C}} \cap \mathbb{H}$ also produces a left- or right-regular function on $U_{\mathbb{H}}$ respectively. Conversely, if one starts with, say, a left-regular function on $\mathbb{H}_{\mathbb{R}}$, extends it holomorphically to a left-regular function on $\mathbb{H}_{\mathbb{C}}$ and then restricts the extension to \mathbb{H} , the resulting function is left-regular on \mathbb{H} .

Proposition 11 Let $f^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}$ and $g^{\mathbb{C}} : U^{\mathbb{C}} \rightarrow \mathbb{S}'$ be holomorphic functions. Then

$$\nabla_{\mathbb{R}}^+ f^{\mathbb{C}} = \nabla^+ f^{\mathbb{C}}, \quad \nabla_{\mathbb{R}} f^{\mathbb{C}} = \nabla f^{\mathbb{C}}, \quad g^{\mathbb{C}} \nabla_{\mathbb{R}}^+ = g^{\mathbb{C}} \nabla^+, \quad g^{\mathbb{C}} \nabla_{\mathbb{R}} = g^{\mathbb{C}} \nabla.$$

The actions of the group $GL(2, \mathbb{H}_{\mathbb{R}})$ on the spaces of left-, right-regular and harmonic functions on $\mathbb{H}_{\mathbb{R}}$ with isolated singularities are given by the same formulas as before.

2.4 Fueter Formula for Holomorphic Regular Functions on $\mathbb{H}_{\mathbb{R}}$

We are interested in extensions of the Cauchy-Fueter formula to functions on $\mathbb{H}_{\mathbb{R}}$. First we recall the classical version of the integral formula due to Fueter:

Theorem 12 (Cauchy-Fueter Formula [F1, F2]) Let $U_{\mathbb{H}} \subset \mathbb{H}$ be an open bounded subset with piecewise \mathcal{C}^1 boundary $\partial U_{\mathbb{H}}$. Suppose that $f(\tilde{X})$ is left-regular on a neighborhood of the closure $\overline{U_{\mathbb{H}}}$, then

$$\frac{1}{2\pi^2} \int_{\partial U_{\mathbb{H}}} \frac{(\tilde{X} - \tilde{X}_0)^{-1}}{N(\tilde{X} - \tilde{X}_0)} \cdot D\tilde{x} \cdot f(\tilde{X}) = \begin{cases} f(\tilde{X}_0) & \text{if } \tilde{X}_0 \in U_{\mathbb{H}}; \\ 0 & \text{if } \tilde{X}_0 \notin \overline{U_{\mathbb{H}}}. \end{cases}$$

If $g(\tilde{X})$ is right-regular on a neighborhood of the closure $\overline{U_{\mathbb{H}}}$, then

$$\frac{1}{2\pi^2} \int_{\partial U_{\mathbb{H}}} g(\tilde{X}) \cdot D\tilde{x} \cdot \frac{(\tilde{X} - \tilde{X}_0)^{-1}}{N(\tilde{X} - \tilde{X}_0)} = \begin{cases} g(\tilde{X}_0) & \text{if } \tilde{X}_0 \in U_{\mathbb{H}}; \\ 0 & \text{if } \tilde{X}_0 \notin \overline{U_{\mathbb{H}}}. \end{cases}$$

Let $U \subset \mathbb{H}_{\mathbb{R}}$ be an open subset, and let f be a \mathcal{C}^1 -function defined on a neighborhood of \overline{U} such that $\nabla_{\mathbb{R}}^+ f = 0$. In this subsection we extend the Cauchy-Fueter integral formula to left-regular functions which can be extended holomorphically to a neighborhood of \overline{U} in $\mathbb{H}_{\mathbb{C}}$. Observe that the expression in the integral formula $\frac{(\tilde{X}-\tilde{X}_0)^{-1}}{N(\tilde{X}-\tilde{X}_0)} \cdot D\tilde{x}$ is nothing else but the restriction to \mathbb{H} of the holomorphic 3-form $\frac{(Z-\tilde{X}_0)^{-1}}{N(Z-\tilde{X}_0)} \cdot Dz$, which is the form from Lemma 6 translated by \tilde{X}_0 . For this reason we expect an integral formula of the kind

$$f(X_0) = \frac{1}{2\pi^2} \int_{\partial U} \left(\frac{(Z-X_0)^{-1}}{N(Z-X_0)} \cdot Dz \right) \Big|_{\mathbb{H}_{\mathbb{R}}} \cdot f(X), \quad \forall X_0 \in U.$$

However, the integrand is singular wherever $N(Z-X_0) = 0$. We resolve this difficulty by deforming the contour of integration ∂U in such a way that the integral is no longer singular. Fix an $\varepsilon \in \mathbb{R}$ and define an ε -deformation $h_{\varepsilon} : \mathbb{H}_{\mathbb{C}} \rightarrow \mathbb{H}_{\mathbb{C}}$, $Z \mapsto Z_{\varepsilon}$, by:

$$\begin{array}{llll} z_{11} & \mapsto & z_{11} + i\varepsilon z_{11} & z_{12} & \mapsto & z_{12} - i\varepsilon z_{12} \\ z_{21} & \mapsto & z_{21} - i\varepsilon z_{21} & z_{22} & \mapsto & z_{22} + i\varepsilon z_{22}. \end{array}$$

In other words, $Z_{\varepsilon} = Z + i\varepsilon Z^-$. For $Z_0 \in \mathbb{H}_{\mathbb{C}}$ fixed, we use a notation

$$h_{\varepsilon, Z_0}(Z) = Z_0 + h_{\varepsilon}(Z - Z_0) = Z + i\varepsilon(Z - Z_0)^-.$$

Lemma 13 *Define a quadratic form $S(Z) = z_{11}z_{22} + z_{12}z_{21}$ on $\mathbb{H}_{\mathbb{C}}$. We have:*

$$\begin{aligned} N(Z_{\varepsilon}) &= (1 - \varepsilon^2)N(Z) + i2\varepsilon S(Z), \\ S(X) &= \|X\|^2, \quad \forall X \in \mathbb{H}_{\mathbb{R}}. \end{aligned}$$

Theorem 14 *Let $U \subset \mathbb{H}_{\mathbb{R}}$ be an open bounded subset with piecewise \mathcal{C}^1 boundary ∂U , and let $f(X)$ be a function defined on a neighborhood of the closure \overline{U} such that $\nabla_{\mathbb{R}}^+ f = 0$. Suppose that f extends to a holomorphic left-regular function on an open subset $V^{\mathbb{C}} \subset \mathbb{H}_{\mathbb{C}}$ containing \overline{U} , then*

$$-\frac{1}{2\pi^2} \int_{(h_{\varepsilon, X_0})_*(\partial U)} \frac{(Z-X_0)^{-1}}{N(Z-X_0)} \cdot Dz \cdot f^{\mathbb{C}}(Z) = \begin{cases} f(X_0) & \text{if } X_0 \in U; \\ 0 & \text{if } X_0 \notin \overline{U}. \end{cases}$$

for all $\varepsilon \neq 0$ sufficiently close to 0.

Remark 15 *For all $\varepsilon \neq 0$ sufficiently close to 0 the contour of integration $(h_{\varepsilon, X_0})_*(\partial U)$ lies inside $V^{\mathbb{C}}$ and the integrand is non-singular, thus the integrals are well-defined. Moreover, the value of the integral becomes constant when the parameter ε is sufficiently close to 0. Of course, there is a similar formula for right-regular functions on $\mathbb{H}_{\mathbb{R}}$.*

The proof is similar to the proof of Theorem 51 in [FL1]; for this reason we just give an outline. Since the integrand is a closed form, by Stokes' we are free to deform the cycle of integration as long as we stay inside the set

$$\{Z \in V^{\mathbb{C}}; N(X_0 - Z) \neq 0\}. \quad (12)$$

Let $S_r = \{\tilde{X} \in \mathbb{H} + X_0; \|\tilde{X} - X_0\|^2 = r^2\}$ be the sphere of radius r centered at X_0 and lying in the subspace of $\mathbb{H}_{\mathbb{C}}$ parallel to \mathbb{H} . We orient S_r as the boundary of the open ball. One can show that, for $r > 0$ sufficiently small, the cycle of integration $(h_{\varepsilon, X_0})_*(\partial U)$ is homologous to $-S_r$ if $X_0 \in U$ and 0 if $X_0 \notin \overline{U}$ as homology 3-cycles in (12). Then the result follows from the Fueter formula for the regular quaternions (Theorem 12). Alternatively, one can let $r \rightarrow 0^+$ and show

directly that the integral remains unchanged and at the same time approaches $-2\pi^2 f(X_0)$ in the same way the Cauchy and Cauchy-Fueter formulas are proved.

One can drop the assumption that $f(X)$ extends holomorphically to an open neighborhood of \bar{U} in $\mathbb{H}_{\mathbb{C}}$ and prove the following version involving generalized functions:

Theorem 16 (Integral Formula) *Let $U \subset \mathbb{H}_{\mathbb{R}}$ be a bounded open region with smooth boundary ∂U . Let $f : U \rightarrow \mathbb{H}_{\mathbb{C}}$ be a function which extends to a real-differentiable function on an open neighborhood $V \subset \mathbb{H}_{\mathbb{R}}$ of the closure \bar{U} such that $\nabla_{\mathbb{R}}^+ f = 0$. Then, for any point $X_0 \in \mathbb{H}_{\mathbb{R}}$ such that ∂U intersects the cone $\{X \in \mathbb{H}_{\mathbb{R}}; N(X - X_0) = 0\}$ transversally, we have:*

$$\lim_{\varepsilon \rightarrow 0} \frac{-1}{2\pi^2} \int_{\partial U} \frac{(X - X_0)^+}{(N(X - X_0) + i\varepsilon \|X - X_0\|^2)^2} \cdot Dz \cdot f(X) = \begin{cases} f(X_0) & \text{if } X_0 \in U; \\ 0 & \text{if } X_0 \notin \bar{U}. \end{cases}$$

The proof of this theorem is given in [L].

2.5 The Matrix Coefficients of $SU(1, 1)$

The matrix coefficients of the (generalized) principal series representations are functions of $Z \in SU(1, 1)$ given by

$$t_{n\underline{m}}^l(Z) = \frac{1}{2\pi i} \oint (sz_{11} + z_{21})^{l-m} (sz_{12} + z_{22})^{l+m} s^{-l+n} \frac{ds}{s}, \quad Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \quad (13)$$

where the integral is taken over the unit circle $\{s \in \mathbb{C}; |s| = 1\}$ traversed once in the counter-clockwise direction [Vil]. The parameters m and n are either both integers or half-integers:

$$m, n \in \mathbb{Z} \quad \text{or} \quad m, n \in \mathbb{Z} + \frac{1}{2}, \quad \text{and} \quad l \in \mathbb{C}.$$

When the parameters l, m, n range over

$$l = -\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \geq -l$$

we get the matrix coefficients of the holomorphic discrete series representations ($l \leq -1$) together with its limit ($l = -1/2$). When the parameters l, m, n range over

$$l = -\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \leq l$$

we get the matrix coefficients of the antiholomorphic discrete series representations ($l \leq -1$) together with its limit ($l = -1/2$). When the parameter l ranges over

$$l = -\frac{1}{2} + i\lambda, \quad \lambda \in \mathbb{R},$$

we get the matrix coefficients of the continuous series representations and the two limits of the discrete series ($\lambda = 0$ and $m, n \in \mathbb{Z} + \frac{1}{2}$). Formula (13) involves complex numbers raised to complex powers, and we need to clarify it so there is no ambiguity. In the discrete series situation the powers $l \pm m$ are actually integers, so there is no ambiguity at all. In the general case $l \in \mathbb{C}$ we write

$$\begin{aligned} (sz_{11} + z_{21})^{l-m} (sz_{12} + z_{22})^{l+m} s^{-l+n} &= (sz_{11} + z_{21})^{l-m} (z_{12} + s^{-1}z_{22})^{l+m} s^{m+n} \\ &= |sz_{11} + z_{21}|^{2(l-m)} (z_{12} + s^{-1}z_{22})^{2m} s^{m+n} \end{aligned}$$

(recall that $\overline{z_{22}} = z_{11}$, $\overline{z_{12}} = z_{21}$, $s^{-1} = \bar{s}$). The expression $|sz_{11} + z_{21}|^{2(l-m)}$ is well-defined because $|sz_{11} + z_{21}|$ is a positive real number, and the expressions $(z_{12} + s^{-1}z_{22})^{2m}$, s^{m+n} are well-defined as well since the powers $2m$, $m+n$ are integers. For certain values of l , m , n , the matrix coefficients vanish:

$$t_{n\underline{m}}^l(Z) = 0 \quad \text{when } l = 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots \quad \text{and} \quad \begin{array}{l} m \geq -l, n < -l \\ \text{or} \\ m \leq l, n > l. \end{array} \quad (14)$$

We give another expression for the matrix coefficients (13) also due to Vilenkin [Vil]. Fix a parameterization of the group $SU(1, 1)$ which is essentially the KAK decomposition:

$$\begin{aligned} X(\varphi, \tau, \psi) &= \begin{pmatrix} x^0 - ix^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 + ix^3 \end{pmatrix} = \begin{pmatrix} \cosh \frac{\tau}{2} \cdot e^{i\frac{\varphi+\psi}{2}} & \sinh \frac{\tau}{2} \cdot e^{i\frac{\varphi-\psi}{2}} \\ \sinh \frac{\tau}{2} \cdot e^{i\frac{\psi-\varphi}{2}} & \cosh \frac{\tau}{2} \cdot e^{-i\frac{\varphi+\psi}{2}} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cosh \frac{\tau}{2} & \sinh \frac{\tau}{2} \\ \sinh \frac{\tau}{2} & \cosh \frac{\tau}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \end{aligned} \quad (15)$$

with $0 \leq \varphi < 2\pi$, $0 < \tau < \infty$, $-2\pi \leq \psi < 2\pi$. Then we have the following expressions for the $SO(2, 2)$ -invariant measure on $SU(1, 1)$:

$$\frac{dx^1 \wedge dx^2 \wedge dx^3}{x^0} = \frac{\sinh \tau}{8} d\varphi d\tau d\psi = Dx \cdot X^{-1} = \frac{dS}{\|X\|}.$$

In terms of the parameterization (15), the coefficients $t_{n\underline{m}}^l(X)$, $X \in SU(1, 1)$, can be written as

$$t_{n\underline{m}}^l(\varphi, \tau, \psi) = e^{-i(n\varphi+m\psi)} \cdot \mathfrak{P}_{nm}^l(\cosh \tau), \quad (16)$$

where, by definition,

$$\mathfrak{P}_{nm}^l(\cosh \tau) = \frac{1}{2\pi i} \oint \left(s \cosh \frac{\tau}{2} + \sinh \frac{\tau}{2} \right)^{l-m} \left(s \sinh \frac{\tau}{2} + \cosh \frac{\tau}{2} \right)^{l+m} s^{-l+n} \frac{ds}{s}.$$

The functions $\mathfrak{P}_{nm}^l(\cosh \tau)$ are real-valued when $l \in \mathbb{R}$. Moreover,

$$\overline{\mathfrak{P}_{nm}^l(\cosh \tau)} = \mathfrak{P}_{nm}^{\bar{l}}(\cosh \tau),$$

$$\begin{aligned} \mathfrak{P}_{mn}^l(\cosh \tau) &= \mathfrak{P}_{-m-n}^l(\cosh \tau) = (-1)^{m-n} \mathfrak{P}_{nm}^{-l-1}(\cosh \tau) \\ &= \frac{\Gamma(l+n+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l-m+1)} \cdot \mathfrak{P}_{nm}^l(\cosh \tau). \end{aligned}$$

(When $l \in \frac{1}{2}\mathbb{Z}$ two of the Γ -factors become infinite and must be transformed by the formula $\Gamma(x) \cdot \Gamma(1-x) = \pi / \sin(\pi x)$.) They also satisfy the orthogonality relations:

$$\int_1^\infty \mathfrak{P}_{mn}^l(t) \cdot \overline{\mathfrak{P}_{m'n'}^{l'}(t)} dt = 0 \quad \text{if } l \neq l', \quad l, l', m, n \in \frac{1}{2}\mathbb{Z}, \quad (17)$$

$$\int_1^\infty |\mathfrak{P}_{nm}^l(t)|^2 dt = (-1)^{m-n} \frac{-2}{2l+1} \frac{\Gamma(l+m+1)\Gamma(l-m+1)}{\Gamma(l+n+1)\Gamma(l-n+1)}, \quad l = -1, -\frac{3}{2}, -2, \dots \quad (18)$$

The functions $\mathfrak{P}_{nm}^l(\cosh t)$ can be expressed in terms of the hypergeometric function. We will use

$$\begin{aligned} &\mathfrak{P}_{nm}^l(\cosh \tau) \\ &= \frac{\Gamma(l-m+1)(\cosh \frac{\tau}{2})^{m+n} (\sinh \frac{\tau}{2})^{n-m}}{\Gamma(l-n+1)(n-m)!} \cdot {}_2F_1\left(l+n+1, -l+n; n-m+1; -\sinh^2 \frac{\tau}{2}\right) \end{aligned} \quad (19)$$

(valid for $n \geq m$) found in Subsection 6.5.3 of [VilK].

Let $\mathbb{H}_{\mathbb{R}}^{\dagger} = \{X \in \mathbb{H}_{\mathbb{R}}; N(X) > 0\}$. Clearly, the matrix coefficient functions $t_{n\underline{m}}^l$ extend from $SU(1, 1)$ to $\mathbb{H}_{\mathbb{R}}^{\dagger}$. Differentiating (13) under the integral sign we obtain:

Lemma 17

$$\begin{aligned} \frac{\partial}{\partial z_{11}} t_{n\underline{m}}^l(Z) &= (l - m) t_{n+\frac{1}{2}\underline{m}+\frac{1}{2}}^{l-\frac{1}{2}}(Z), & \frac{\partial}{\partial z_{12}} t_{n\underline{m}}^l(Z) &= (l + m) t_{n+\frac{1}{2}\underline{m}-\frac{1}{2}}^{l-\frac{1}{2}}(Z), \\ \frac{\partial}{\partial z_{21}} t_{n\underline{m}}^l(Z) &= (l - m) t_{n-\frac{1}{2}\underline{m}+\frac{1}{2}}^{l-\frac{1}{2}}(Z), & \frac{\partial}{\partial z_{22}} t_{n\underline{m}}^l(Z) &= (l + m) t_{n-\frac{1}{2}\underline{m}-\frac{1}{2}}^{l-\frac{1}{2}}(Z). \end{aligned}$$

We also have the following multiplication identities for matrix coefficients.

Lemma 18

$$\left(t_{m+\frac{1}{2}\underline{n}}^{l-\frac{1}{2}}(Z), t_{m-\frac{1}{2}\underline{n}}^{l-\frac{1}{2}}(Z) \right) \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \left(t_{m\underline{n-\frac{1}{2}}}^l(Z), t_{m\underline{n+\frac{1}{2}}}^l(Z) \right)$$

and

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} (l - m + \frac{1}{2}) t_{n\underline{m+\frac{1}{2}}}^l(Z) \\ (l + m + \frac{1}{2}) t_{n\underline{m-\frac{1}{2}}}^l(Z) \end{pmatrix} = \begin{pmatrix} (l - n + 1) t_{n-\frac{1}{2}\underline{m}}^{l+\frac{1}{2}}(Z) \\ (l + n + 1) t_{n+\frac{1}{2}\underline{m}}^{l+\frac{1}{2}}(Z) \end{pmatrix}.$$

Applying $\square_{2,2} = \frac{\partial^2}{\partial z_{11} \partial z_{22}} - \frac{\partial^2}{\partial z_{12} \partial z_{21}}$ and using Lemma 17 we obtain:

Proposition 19 On $\mathbb{H}_{\mathbb{R}}^{\dagger}$ we have:

$$\square_{2,2} t_{n\underline{m}}^l(Z) = 0 \quad \text{and} \quad \square_{2,2} N(Z)^{-1} \cdot t_{n\underline{m}}^l(Z^{-1}) = 0.$$

Applying $\nabla^+ = 2 \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \frac{\partial}{\partial z_{21}} \\ \frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{22}} \end{pmatrix}$ and using Lemma 17 we obtain:

Proposition 20 The following \mathbb{S} -valued functions on $\mathbb{H}_{\mathbb{R}}^{\dagger}$

$$\begin{pmatrix} (l - m + \frac{1}{2}) t_{n\underline{m+\frac{1}{2}}}^l(Z) \\ (l + m + \frac{1}{2}) t_{n\underline{m-\frac{1}{2}}}^l(Z) \end{pmatrix}, \quad \frac{1}{N(Z)} \begin{pmatrix} (l - n + \frac{1}{2}) t_{n-\frac{1}{2}\underline{m}}^l(Z^{-1}) \\ (l + n + \frac{1}{2}) t_{n+\frac{1}{2}\underline{m}}^l(Z^{-1}) \end{pmatrix}$$

are left-regular. Dually, the following \mathbb{S}' -valued functions on $\mathbb{H}_{\mathbb{R}}^{\dagger}$

$$\frac{1}{N(Z)} \left(t_{m\underline{n-\frac{1}{2}}}^{l+\frac{1}{2}}(Z^{-1}), t_{m\underline{n+\frac{1}{2}}}^{l+\frac{1}{2}}(Z^{-1}) \right), \quad \left(t_{m+\frac{1}{2}\underline{n}}^{l-\frac{1}{2}}(Z), t_{m-\frac{1}{2}\underline{n}}^{l-\frac{1}{2}}(Z) \right)$$

are right-regular.

Next we would like to extend the matrix coefficients (13) holomorphically from $\mathbb{H}_{\mathbb{R}}^{\dagger}$ to an open region in $\mathbb{H}_{\mathbb{C}}$. Define an open region in $\mathbb{H}_{\mathbb{C}}$:

$$U = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}; |z_{21}| < |z_{11}|, |z_{12}| < |z_{22}|, \operatorname{Re}(z_{11} z_{22}) > 0 \text{ or } \operatorname{Im}(z_{11} z_{22}) \neq 0 \right\}.$$

Lemma 21 The matrix coefficients (13) can be holomorphically extended to U .

Proof. We can rewrite

$$t_{n\underline{m}}^l(Z) = \frac{1}{2\pi i} \oint \left(1 + s^{-1} \frac{z_{21}}{z_{11}}\right)^{l-m} \left(1 + s \frac{z_{12}}{z_{22}}\right)^{l+m} (z_{11}z_{22})^{l-m} (z_{22})^{2m} s^{n-m} \frac{ds}{s}. \quad (20)$$

By assumption, $\operatorname{Re}(z_{11}z_{22}) > 0$ or $\operatorname{Im}(z_{11}z_{22}) \neq 0$, which allows us to choose a single branch of the complex multivalued function $(z_{11}z_{22})^{l-m}$. Since $|z_{21}| < |z_{11}|$ and $|z_{12}| < |z_{22}|$,

$$\operatorname{Re}\left(1 + s^{-1} \frac{z_{21}}{z_{11}}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(1 + s \frac{z_{12}}{z_{22}}\right) > 0.$$

Thus we can choose single branches of the multivalued functions $\left(1 + s^{-1} \frac{z_{21}}{z_{11}}\right)^{l-m}$ and $\left(1 + s \frac{z_{12}}{z_{22}}\right)^{l+m}$ as well. \square

Let U^+ be another open region in $\mathbb{H}_{\mathbb{C}}$:

$$\begin{aligned} U^+ &= \{Z \in \mathbb{H}_{\mathbb{C}}; Z^+ \in U\} \\ &= \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}; |z_{12}| < |z_{11}|, |z_{21}| < |z_{22}|, \operatorname{Re}(z_{11}z_{22}) > 0 \text{ or } \operatorname{Im}(z_{11}z_{22}) \neq 0 \right\}. \end{aligned}$$

Observe that, for $Z \in U \cup U^+$, $N(Z)$ is never a negative real number or zero, hence arbitrary complex powers of $N(Z)$ are well-defined. Letting $\tilde{Z} = Z^{-1}$ and writing

$$t_{n\underline{m}}^l(Z) = t_{n\underline{m}}^l(\tilde{Z}^{-1}) = N(\tilde{Z})^{-2l} \cdot t_{n\underline{m}}^l(\tilde{Z}^+)$$

we see that the matrix coefficients $t_{n\underline{m}}^l(Z)$'s can be holomorphically extended to U^+ as well. Now we define $\mathbb{H}_{\mathbb{C}}^+ = U \cup U^+$.

Lemma 22 *The matrix coefficients (13) can be holomorphically extended to $\mathbb{H}_{\mathbb{C}}^+$.*

Proof. Since the matrix coefficients extend to both U and U^+ all we need to show is that we do not run into problems with multivaluedness on $U \cap U^+$. The matrix coefficients of the discrete series and their limits are rational functions in z_{ij} 's and certainly extend to $\mathbb{H}_{\mathbb{C}}^+$. For the matrix coefficients of the continuous series the result follows from the following general observation:

Let $U^{\mathbb{R}}$ be an open region in \mathbb{R}^n , and let $U_1^{\mathbb{C}}$ and $U_2^{\mathbb{C}}$ be two connected open regions in \mathbb{C}^n such that $U^{\mathbb{R}} \subset U_i^{\mathbb{C}} \cap \mathbb{R}^n$, $i = 1, 2$, and every loop in $U_1^{\mathbb{C}} \cup U_2^{\mathbb{C}}$ is homotopic to a loop in $U^{\mathbb{R}}$. Suppose a real-analytic function f on $U^{\mathbb{R}}$ has holomorphic extensions f_1 to $U_1^{\mathbb{C}}$ and f_2 to $U_2^{\mathbb{C}}$, then $f_1 = f_2$ on $U_1^{\mathbb{C}} \cap U_2^{\mathbb{C}}$ and f has a unique holomorphic extension to $U_1^{\mathbb{C}} \cup U_2^{\mathbb{C}}$. \square

We view $\mathbb{H}_{\mathbb{C}}^+$ as an open neighbourhood of $\mathbb{H}_{\mathbb{R}}^+$ in $\mathbb{H}_{\mathbb{C}}$, and in light of the lemma we can regard the matrix coefficients (13) as functions on $\mathbb{H}_{\mathbb{C}}^+$. Lemma 17 and Proposition 20 formally extend to complex variables and $\mathbb{H}_{\mathbb{R}}^+$ replaced with $\mathbb{H}_{\mathbb{C}}^+$.

We conclude this subsection with a list of relations between the matrix coefficients which follow from the relations between $\mathfrak{P}_{nm}^l(t)$'s. When $\operatorname{Re} l = -1/2$, the functions $t_{n\underline{m}}^{\bar{l}}(Z)$ and $N(Z)^{-1} \cdot t_{n\underline{m}}^l(Z/N(Z))$ on $\mathbb{H}_{\mathbb{C}}^+$ are proportional:

$$t_{n\underline{m}}^{\bar{l}}(Z) = (-1)^{m-n} \frac{\Gamma(l-n+1)\Gamma(l+n+1)}{\Gamma(l-m+1)\Gamma(l+m+1)} \cdot \frac{1}{N(Z)} \cdot t_{n\underline{m}}^l(Z/N(Z)), \quad \operatorname{Re} l = -\frac{1}{2}. \quad (21)$$

We also have

$$t_{n\underline{m}}^l(Z^+) = (-1)^{m-n} \frac{\Gamma(l-m+1)\Gamma(l+m+1)}{\Gamma(l-n+1)\Gamma(l+n+1)} \cdot t_{-m\underline{-n}}^l(Z). \quad (22)$$

Combining this with (21) we obtain

$$N(Z)^{-1} \cdot t_{n\underline{m}}^l(Z^{-1}) = t_{-m\underline{-n}}^{\bar{l}}(Z), \quad \operatorname{Re} l = -1/2. \quad (23)$$

2.6 Schwartz Space $\mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$ and Invariant Pairings

Definition 23 Let f be a function on $\mathbb{H}_{\mathbb{R}}^+$ (with values in \mathbb{C} , $\mathbb{H}_{\mathbb{C}}$, \mathbb{S} or \mathbb{S}'). We say f is quasi-regular at the origin if

$$\lim_{t \rightarrow 0^+} t^{1+\delta} f(tX) = 0 \quad \text{for all } X \in \mathbb{H}_{\mathbb{R}}^+ \text{ and all } \delta > 0.$$

If φ is a solution of the wave equation $\square_{2,2}\varphi = 0$ on $\mathbb{H}_{\mathbb{R}}^+$, we say φ is quasi-regular at infinity if

$$\pi_l^0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \varphi = \pi_r^0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \varphi = \frac{1}{N(X)} \cdot \varphi(X^{-1})$$

is quasi-regular at the origin.

Similarly, we say that a left-regular function f or a right-regular function g on $\mathbb{H}_{\mathbb{R}}^+$ is quasi-regular at infinity if

$$\pi_l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f = \frac{X^{-1}}{N(X)} \cdot f(X^{-1}) \quad \text{or} \quad \pi_r \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = g(X^{-1}) \cdot \frac{X^{-1}}{N(X)}$$

is quasi-regular at the origin.

For example, let f be a homogeneous function on $\mathbb{H}_{\mathbb{R}}^+$ of homogeneity degree $\mu \in \mathbb{C}$. Then f is quasi-regular at the origin if and only if $\operatorname{Re} \mu \geq -1$. If f is a solution of the wave equation, then f is quasi-regular at infinity if and only if $\operatorname{Re} \mu \leq -1$. In particular, all matrix coefficients $t_{n\bar{m}}^l(Z)$ are quasi-regular at infinity; moreover, the matrix coefficients of the continuous series and the limits of the discrete series ($\operatorname{Re} l = -1/2$) are also quasi-regular at the origin. Finally, if f is a left- or right-regular function, then f is quasi-regular at infinity if and only if $\operatorname{Re} \mu \leq -2$.

For the purposes of this article, a suitable reference for the Schwartz functions on Lie groups in the sense of Harish-Chandra is [Va]. Our definition of the Schwartz space on $\mathbb{H}_{\mathbb{R}}^+$, denoted by $\mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$, is motivated by the following needs:

- If $f \in \mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$, then the restrictions of f to all hyperboloids H_R , $R > 0$, should be Schwartz functions in the sense of Harish-Chandra;
- Since we often apply the $\widetilde{\operatorname{deg}}$ operator, if $f \in \mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$, then the restrictions $(\operatorname{deg})^d f|_{H_R}$ should be Schwartz functions as well, for all $R > 0$ and integers $d \geq 0$;
- Since we often perform calculations with matrix coefficients $t_{n\bar{m}}^l(Z)$, the matrix coefficients of the discrete series and wave packets formed out of the matrix coefficients of the continuous series should be in $\mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$.

The group $\mathbb{H}_{\mathbb{R}}^+ \times \mathbb{H}_{\mathbb{R}}^+$ acts on $\mathbb{H}_{\mathbb{R}}^+$ by multiplication:

$$(a, b) : X \mapsto aXb^{-1}, \quad (a, b) \in \mathbb{H}_{\mathbb{R}}^+ \times \mathbb{H}_{\mathbb{R}}^+, X \in \mathbb{H}_{\mathbb{R}}^+.$$

Hence we get an action of $\mathbb{H}_{\mathbb{R}}^+ \times \mathbb{H}_{\mathbb{R}}^+$ on the space of smooth functions on $\mathbb{H}_{\mathbb{R}}^+$:

$$(a, b) : f(X) \mapsto f(a^{-1}Xb), \quad f \in C^\infty(\mathbb{H}_{\mathbb{R}}^+).$$

Differentiating, we obtain an action of the Lie algebra $\mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R})$ which extends to an action of its universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(2, \mathbb{R})) \otimes \mathcal{U}(\mathfrak{gl}(2, \mathbb{R}))$ on $C^\infty(\mathbb{H}_{\mathbb{R}}^+)$.

Definition 24 We define $\mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$ – the Schwartz space on $\mathbb{H}_{\mathbb{R}}^+$ – to be the space of smooth functions f on $\mathbb{H}_{\mathbb{R}}^+$ such that, for each $a \in \mathcal{U}(\mathfrak{gl}(2, \mathbb{R})) \otimes \mathcal{U}(\mathfrak{gl}(2, \mathbb{R}))$, each $n \in \mathbb{N}$ and all $R > 1$, there exists a constant $C(a, n, R) > 0$ such that

$$|a \cdot f(X)| \leq C(a, n, R) \cdot \frac{(1 + \log \|X\|)^{-n}}{\|X\|}$$

for all $X \in \mathbb{H}_{\mathbb{R}}^+$ such that $R^{-2} \leq N(X) \leq R^2$.

We shall regard $\mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$ as a Fréchet space with respect to the seminorms

$$\mu_{a,n,R}(f) = \sup_{\{X \in \mathbb{H}_{\mathbb{R}}^+; R^{-2} \leq N(X) \leq R^2\}} |a \cdot f(X)| \cdot \|X\| \cdot (1 + \log \|X\|)^n.$$

Denoting by $\mathcal{C}_c^\infty(\mathbb{H}_{\mathbb{R}}^+)$ the space of compactly supported smooth functions on $\mathbb{H}_{\mathbb{R}}^+$ we get maps

$$\mathcal{C}_c^\infty(\mathbb{H}_{\mathbb{R}}^+) \hookrightarrow \mathcal{S}(\mathbb{H}_{\mathbb{R}}^+) \rightarrow L^2(SU(1, 1)),$$

where the first map is the inclusion and the second map is the restriction map $f \mapsto f|_{SU(1,1)}$. These maps are continuous with dense images.

Let

$$\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+) = \{\varphi \in \mathcal{S}(\mathbb{H}_{\mathbb{R}}^+); \square_{2,2}\varphi = 0\}$$

be the space of “harmonic” functions in $\mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$. Similarly, we denote by $\mathbb{S}(\mathbb{H}_{\mathbb{R}}^+)$ the space of \mathbb{S} -valued left-regular functions and, respectively, $\mathbb{S}'(\mathbb{H}_{\mathbb{R}}^+)$ the space of \mathbb{S}' -valued right-regular functions on $\mathbb{H}_{\mathbb{R}}^+$ with both components in $\mathcal{S}(\mathbb{H}_{\mathbb{R}}^+)$.

Fix an $R > 0$ and define a bilinear form on $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ by

$$\langle \varphi_1, \varphi_2 \rangle_R = -\frac{1}{2\pi^2} \int_{X \in H_R} (\widetilde{\deg \varphi_1})(X) \cdot \varphi_2(X) \frac{dS}{\|X\|}, \quad \varphi_1, \varphi_2 \in \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+). \quad (24)$$

This form is not symmetric, not $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant and depends on the choice of R . In Subsection 4.5 we will extend the space $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ to $\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)$ and define a symmetric $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant nondegenerate bilinear pairing on it.

Lemma 25 Let $\tilde{G}(H_R) \subset GL(2, \mathbb{H}_{\mathbb{C}})$ be the subgroup consisting of all elements of $GL(2, \mathbb{H}_{\mathbb{C}})$ with entries in $\mathbb{H}_{\mathbb{R}}$ and preserving the hyperboloid $H_R = \{X \in \mathbb{H}_{\mathbb{R}}; N(X) = R^2\}$. Then

$$\text{Lie}(\tilde{G}(H_R)) = \left\{ \begin{pmatrix} A & R^2 C^+ \\ C & D \end{pmatrix}; A, C, D \in \mathbb{H}_{\mathbb{R}}, \text{Re } A = \text{Re } D \right\}.$$

Proof. The Lie algebra of $\tilde{G}(H_R)$ consists of all matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A, B, C, D \in \mathbb{H}_{\mathbb{R}}$, which generate vector fields tangent to H_R . Such a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ generates a vector field

$$\frac{d}{dt}((1 + tA)X + tB)(tCX + 1 + tD)^{-1} \Big|_{t=0} = AX + B - XCX - XD.$$

A vector field is tangent to H_R if and only if it is orthogonal with respect to (8) to the vector field X for $N(X) = R^2$:

$$0 = \text{Re}((AX + B - XCX - XD)X^+) = \text{Re}(R^2A - R^2D + BX^+ - R^2XC) = 0, \quad \forall X \in H_R.$$

It follows that $\text{Re } A = \text{Re } D$ and $B = R^2C^+$. \square

Corollary 26 *Let $G(H_R) \subset GL(2, \mathbb{H}_\mathbb{C})$ be the connected subgroup with Lie algebra*

$$\begin{aligned} \mathfrak{g}(H_R) &= \{x \in \text{Lie}(\tilde{G}(H_R)); \text{Re}(\text{Tr } x) = 0\} \\ &= \left\{ \begin{pmatrix} A & R^2 C^+ \\ C & D \end{pmatrix}; A, C, D \in \mathbb{H}_\mathbb{R}, \text{Re } A = \text{Re } D = 0 \right\}. \end{aligned}$$

Then $G(H_R)$ preserves the hyperboloid $H_R = \{X \in \mathbb{H}_\mathbb{R}; N(X) = R^2\}$ and the open sets $\{X \in \mathbb{H}_\mathbb{R}; N(X) > R^2\}$, $\{X \in \mathbb{H}_\mathbb{R}; N(X) < R^2\}$.

The Lie algebra $\mathfrak{g}(H_R)$ and the Lie group $G(H_R)$ are isomorphic to $\mathfrak{so}(3, 2) = \mathfrak{sp}(2, \mathbb{R})$ and $SO^+(3, 2)$ respectively (see, for example, [H]).

Proposition 27 *The bilinear pairing $\langle \varphi_1, \varphi_2 \rangle_R$ is invariant under the π_l^0 action of $G(H_R)$.*

Proof. First we find a convenient pair of subgroups generating $G(H_R)$:

Lemma 28 *The group $G(H_R)$ is generated by $SU(1, 1) \times SU(1, 1)$ realized as the subgroup of diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a, d \in SU(1, 1) \subset \mathbb{H}_\mathbb{R}$, and the one-parameter group*

$$G(H_R)' = \left\{ \begin{pmatrix} \cosh t & R \sinh t \\ R^{-1} \sinh t & \cosh t \end{pmatrix}; t \in \mathbb{R} \right\}.$$

Clearly, the bilinear pairing is invariant under the π_l^0 action of $SU(1, 1) \times SU(1, 1)$. Thus it is sufficient to show it is $G(H_R)'$ -invariant.

Lemma 29 *Fix an element $g = \begin{pmatrix} \cosh t & R \sinh t \\ R^{-1} \sinh t & \cosh t \end{pmatrix} \in G(H_R)'$ and consider its conformal action on H_R :*

$$\pi_l(g) : X \mapsto \tilde{X} = (\cosh t X - R \sinh t)(-R^{-1} \sinh t X + \cosh t)^{-1}.$$

Then the Jacobian J of this map is

$$\pi_l(g)^* \left(\frac{dS}{\|X\|} \right) = J \frac{dS}{\|X\|} = \frac{1}{N(-R^{-1} \sinh t X + \cosh t)} \cdot \frac{R^2 - (\text{Re } \tilde{X})^2}{R^2 - (\text{Re } X)^2} \frac{dS}{\|X\|}.$$

Since $G(H_R)'$ is connected, it is sufficient to verify the $G(H_R)'$ -equivariance of the bilinear pairing on the Lie algebra level. For $t \rightarrow 0$ and modulo terms of order t^2 , we have:

$$\tilde{X} = X + t(X^2 - R^2)/R, \tag{25}$$

$$N(-R^{-1} \sinh t X + \cosh t) = 1 - 2t \text{Re } X/R, \tag{26}$$

$$R^2 - (\text{Re } \tilde{X})^2 = (R^2 - (\text{Re } X)^2)(1 + 4t \text{Re } X/R). \tag{27}$$

Lemma 30 *Modulo terms of order t^2 we have*

$$\widetilde{\text{deg}}_X(\pi_l^0(g)\varphi) = \widetilde{\text{deg}}_X \left(\frac{\varphi(\tilde{X})}{N(-R^{-1} \sinh t X + \cosh t)} \right) = (1 + 4t \text{Re } X/R) \cdot (\widetilde{\text{deg}}_X \varphi)(\tilde{X}).$$

Letting $\tilde{\varphi}_i = \pi_l^0(g)\varphi_i$, $i = 1, 2$, and continuing to work modulo terms of order t^2 , we get

$$\begin{aligned} -2\pi^2 \cdot \langle \tilde{\varphi}_1, \tilde{\varphi}_2 \rangle_R &= \int_{X \in H_R} \widetilde{\text{deg}}_X(\tilde{\varphi}_1)(X) \cdot \tilde{\varphi}_2(X) \frac{dS}{\|X\|} \\ &= \int_{X \in H_R} (\widetilde{\text{deg}}_X \varphi_1)(\tilde{X}) \cdot \varphi_2(\tilde{X}) \cdot \frac{1 + 4t \text{Re } X/R}{N(-R^{-1} \sinh tX + \cosh t)} \frac{dS}{\|X\|} \\ &= \int_{\tilde{X} \in H_R} (\widetilde{\text{deg}}_X \varphi_1)(\tilde{X}) \cdot \varphi_2(\tilde{X}) \frac{dS}{\|\tilde{X}\|} = -2\pi^2 \cdot \langle \varphi_1, \varphi_2 \rangle_R. \end{aligned}$$

This proves that the bilinear pairing $\langle \varphi_1, \varphi_2 \rangle_R$ is $G(H_R)$ -invariant. \square

Next we calculate the pairings between the matrix coefficient functions.

Proposition 31 *The matrix coefficients (13) satisfy the following orthogonality relationships:*

$$\begin{aligned} \left\langle t_{n' \underline{m}'}^{l'}(X), \frac{1}{N(X)} \cdot t_{m \underline{n}}^l(X^{-1}) \right\rangle_R &= - \left\langle \frac{1}{N(X)} \cdot t_{m \underline{n}}^l(X^{-1}), t_{n' \underline{m}'}^{l'}(X) \right\rangle_R \\ &= \begin{cases} \delta_{l'l'} \delta_{mm'} \delta_{nn'} & \text{if } l, l' \neq -1/2; \\ 0 & \text{if } l \text{ or } l' = -1/2, \end{cases} \end{aligned}$$

$$\begin{aligned} R^{-2(2l+1)} \cdot \langle t_{n \underline{m}}^l(X), t_{-n' \underline{-m}'}^{l'}(X) \rangle_R &= -R^{2(2l+1)} \cdot \left\langle \frac{1}{N(X)} \cdot t_{n \underline{m}}^l(X^{-1}), \frac{1}{N(X)} \cdot t_{-n' \underline{-m}'}^{l'}(X^{-1}) \right\rangle_R \\ &= (-1)^{m-n} \frac{\Gamma(l+m+1)\Gamma(l-m+1)}{\Gamma(l+n+1)\Gamma(l-n+1)} \cdot \begin{cases} \delta_{l'l'} \delta_{mm'} \delta_{nn'} & \text{if } l, l' \neq -1/2; \\ 0 & \text{if } l \text{ or } l' = -1/2, \end{cases} \end{aligned}$$

for all $R > 0$, provided that l or l' lies in $\{-\frac{1}{2}, -1, -\frac{3}{2}, \dots\}$. In particular, we get a non-degenerate pairing between the spaces of harmonic functions spanned by $t_{n \underline{m}}^l(X)$'s and by $N(X)^{-1} \cdot t_{m \underline{n}}^l(X^{-1})$'s with $l = -1, -\frac{3}{2}, -2, \dots$, which is independent of the choice of $R > 0$.

Proof. Since the functions $N(X)^{-1} \cdot t_{m \underline{n}}^l(X^{-1})$ and $t_{n' \underline{m}'}^{l'}(X)$ are homogeneous of degree $-2(l+1)$ and $2l'$ respectively, it is sufficient to prove the result for $R = 1$ only. This is done by integrating in coordinates (15) using identities (16) and (22) and the the orthogonality relations for \mathfrak{P}_{nm}^l 's (17)-(18). \square

The $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant pairing between $\mathbb{S}(\mathbb{H}_{\mathbb{R}}^+)$ and $\mathbb{S}'(\mathbb{H}_{\mathbb{R}}^+)$ is given by the formula

$$\langle g, f \rangle = -\frac{1}{2\pi^2} \int_{X \in SU(1,1)} g(X) \cdot Dx \cdot f(X).$$

However, it is only defined when one of the functions has both components in the discrete series (otherwise the integral would not converge).

Proposition 32 *We have the following orthogonality relationships:*

$$\left\langle \frac{1}{N(X)} \left(t_{m \underline{n-\frac{1}{2}}}^l(X^{-1}), t_{m \underline{n+\frac{1}{2}}}^l(X^{-1}) \right), \left(t_{n' \underline{m'+\frac{1}{2}}}^{l'-\frac{1}{2}}(X), t_{n' \underline{m'-\frac{1}{2}}}^{l'-\frac{1}{2}}(X) \right) \right\rangle = \delta_{l'l'} \delta_{mm'} \delta_{nn'},$$

$$\left\langle \left(t_{m'+\frac{1}{2}\underline{n}'}^{l'-\frac{1}{2}}(X), t_{m'-\frac{1}{2}\underline{n}'}^{l'-\frac{1}{2}}(X) \right), \frac{1}{N(X)} \left((l-n+\frac{1}{2})t_{n-\frac{1}{2}\underline{m}}^l(X^{-1}), (l+n+\frac{1}{2})t_{n+\frac{1}{2}\underline{m}}^l(X^{-1}) \right) \right\rangle = \delta_{ll'}\delta_{mm'}\delta_{nn'},$$

$$\left\langle \left(t_{n+\frac{1}{2}\underline{m}}^l(X), t_{n-\frac{1}{2}\underline{m}}^l(X) \right), \left((l'-m')t_{n'm'+\frac{1}{2}}^{l'-\frac{1}{2}}(X), (l'+m')t_{n'm'-\frac{1}{2}}^{l'-\frac{1}{2}}(X) \right) \right\rangle = 0,$$

$$\left\langle \frac{1}{N(X)} \left(t_{n\underline{m}-\frac{1}{2}}^l(X^{-1}), t_{n\underline{m}+\frac{1}{2}}^l(X^{-1}) \right), \frac{1}{N(X)} \left((l'-n')t_{n'-\frac{1}{2}\underline{m}'}^{l'-\frac{1}{2}}(X^{-1}), (l'+n')t_{n'+\frac{1}{2}\underline{m}'}^{l'-\frac{1}{2}}(X^{-1}) \right) \right\rangle = 0,$$

provided that l or $l' - \frac{1}{2}$ lies in the discrete series set $\{-1, -\frac{3}{2}, -2, \dots\}$.

Proof. Using Lemmas 3 and 18, we obtain:

$$\begin{aligned} & \left(t_{m\underline{n}-\frac{1}{2}}^l(X^{-1}), t_{m\underline{n}+\frac{1}{2}}^l(X^{-1}) \right) \cdot X \cdot \left((l'-m')t_{n'm'+\frac{1}{2}}^{l'-\frac{1}{2}}(X), (l'+m')t_{n'm'-\frac{1}{2}}^{l'-\frac{1}{2}}(X) \right) \\ &= (l'-n'+1/2) \cdot t_{n'-\frac{1}{2}\underline{m}'}^{l'}(X) \cdot t_{m\underline{n}-\frac{1}{2}}^l(X^{-1}) + (l'+n'+1/2) \cdot t_{n'+\frac{1}{2}\underline{m}'}^{l'}(X) \cdot t_{m\underline{n}+\frac{1}{2}}^l(X^{-1}), \end{aligned}$$

$$\begin{aligned} & \left(t_{m'+\frac{1}{2}\underline{n}'}^{l'-\frac{1}{2}}(X), t_{m'-\frac{1}{2}\underline{n}'}^{l'-\frac{1}{2}}(X) \right) \cdot X \cdot \left((l-n+\frac{1}{2})t_{n-\frac{1}{2}\underline{m}}^l(X^{-1}), (l+n+\frac{1}{2})t_{n+\frac{1}{2}\underline{m}}^l(X^{-1}) \right) \\ &= (l-n+1/2) \cdot t_{m'\underline{n}'-\frac{1}{2}}^{l'}(X) \cdot t_{n-\frac{1}{2}\underline{m}}^l(X^{-1}) + (l+n+1/2) \cdot t_{m'\underline{n}'+\frac{1}{2}}^{l'}(X) \cdot t_{n+\frac{1}{2}\underline{m}}^l(X^{-1}), \end{aligned}$$

then the result follows from Proposition 31. \square

2.7 Fueter Formula for Hyperboloids

We fix $0 < R' < R$ and let $U = \{X \in \mathbb{H}_{\mathbb{R}}; R'^2 < N(X) < R^2\}$ be the open region in $\mathbb{H}_{\mathbb{R}}^+$ bounded by two hyperboloids. In this section we essentially substitute this unbounded set U and bounded functions f into the integral formula from Theorem 14 and prove that the resulting identity still holds.

Recall the deformation $h_{\varepsilon, Z_0} : \mathbb{H}_{\mathbb{C}} \rightarrow \mathbb{H}_{\mathbb{C}}, Z \mapsto Z + i\varepsilon(Z - Z_0)^-$. First we deal with a technical issue arising from the fact that the image of U under h_{ε, Z_0} does not lie inside $\mathbb{H}_{\mathbb{C}}^+$. Thus we modify the deformation h_{ε, Z_0} as

$$\tilde{h}_{\varepsilon, Z_0} : \mathbb{H}_{\mathbb{C}} \rightarrow \mathbb{H}_{\mathbb{C}}, \quad Z \mapsto Z + \varepsilon(1+i)(Z - Z_0)^-.$$

Note that

$$\tilde{h}_{\varepsilon, Z_0}^*(N(Z - Z_0)) = (N(Z - Z_0) + 2\varepsilon S(Z - Z_0)) + 2i\varepsilon(S(Z - Z_0) + \varepsilon N(Z - Z_0)),$$

so, for $|\varepsilon|$ sufficiently small, the denominator of $\frac{(Z-X_0)^{-1}}{N(Z-X_0)}$ is bounded away from zero on $(\tilde{h}_{\varepsilon, X_0})_*(\partial U)$.

Lemma 33 *Fix any $Z_0 \in \mathbb{H}_{\mathbb{C}}$, then, for $\varepsilon > 0$ sufficiently small, $\tilde{h}_{\varepsilon, Z_0}(\bar{U}) \subset \mathbb{H}_{\mathbb{C}}^+$.*

Proof. Let $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$, $Z_0 = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ and $\tilde{h}_{\varepsilon, Z_0}(Z) = \begin{pmatrix} z'_{11} & z'_{12} \\ z'_{21} & z'_{22} \end{pmatrix}$. We can write

$$\begin{pmatrix} z'_{11} & z'_{12} \\ z'_{21} & z'_{22} \end{pmatrix} = \begin{pmatrix} (1 + \varepsilon(1 + i))z_{11} & (1 - \varepsilon(1 + i))z_{12} \\ (1 - \varepsilon(1 + i))z_{21} & (1 + \varepsilon(1 + i))z_{22} \end{pmatrix} - \varepsilon(1 + i) \begin{pmatrix} y_{11} & -y_{12} \\ -y_{21} & y_{22} \end{pmatrix}.$$

Since $|z_{11}| > |z_{21}|$ and $|z_{22}| > |z_{12}|$ for all $Z \in \bar{U}$, by taking $\varepsilon > 0$ sufficiently small we can arrange that

$$\begin{aligned} |(1 + \varepsilon(1 + i))z_{11}| - |(1 - \varepsilon(1 + i))z_{21}| &> 2\sqrt{2}\varepsilon|y_{ij}|, \\ |(1 + \varepsilon(1 + i))z_{22}| - |(1 - \varepsilon(1 + i))z_{12}| &> 2\sqrt{2}\varepsilon|y_{ij}|. \end{aligned}$$

Then $\tilde{h}_{\varepsilon, Z_0}(Z)$ satisfies the inequalities $|z'_{21}| < |z'_{11}|$ and $|z'_{12}| < |z'_{22}|$. Finally,

$$\begin{aligned} z'_{11}z'_{22} &= ((1 + \varepsilon(1 + i))z_{11} - \varepsilon(1 + i)y_{11})((1 + \varepsilon(1 + i))z_{22} - \varepsilon(1 + i)y_{22}) \\ &= (1 + 2\varepsilon)z_{11}z_{22} + 2i\varepsilon(1 + \varepsilon)z_{11}z_{22} + 2i\varepsilon^2y_{11}y_{22} - \varepsilon(1 + i)(1 + \varepsilon(1 + i))(z_{11}y_{22} + z_{22}y_{11}). \end{aligned}$$

The first term $(1 + 2\varepsilon)z_{11}z_{22}$ is a positive real number, the second and the third terms $2i\varepsilon(1 + \varepsilon)z_{11}z_{22}$ and $2i\varepsilon^2y_{11}y_{22}$ are purely imaginary, finally the fourth term $\varepsilon(1 + i)(1 + \varepsilon(1 + i))(z_{11}y_{22} + z_{22}y_{11})$ becomes smaller in magnitude than $(1 + 2\varepsilon)z_{11}z_{22}$ when $\varepsilon > 0$ is sufficiently small. This proves $\operatorname{Re}(z'_{11}z'_{22}) > 0$. \square

Theorem 34 *Suppose that $f(Z)$ is a left-regular function on $\mathbb{H}_{\mathbb{C}}^+$ such that its components are bounded on closed sets*

$$\left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}; \quad \begin{array}{l} c \leq |z_{11}/z_{21}| \leq 2c, \quad |\bar{z}_{11}/z_{22} - 1| \leq d, \\ c \leq |z_{22}/z_{12}| \leq 2c, \quad |\bar{z}_{12}/z_{21} - 1| \leq d, \end{array} \quad |z_{11}|, |z_{22}| \geq R \right\} \quad (28)$$

for all $c > 1$ sufficiently close to 1 and some fixed values $d, R > 0$. Let $X_0 \in \mathbb{H}_{\mathbb{R}}$, then for $\varepsilon > 0$ sufficiently small

$$-\frac{1}{2\pi^2} \int_{(\tilde{h}_{\varepsilon, X_0})_*(\partial U)} \frac{(Z - X_0)^{-1}}{N(Z - X_0)} \cdot Dz \cdot f(Z) = \begin{cases} f(X_0) & \text{if } X_0 \in U; \\ 0 & \text{if } X_0 \notin \bar{U}. \end{cases}$$

Proof. For $L \in \mathbb{R}$, let B_L denote the open ball $\{X \in \mathbb{H}_{\mathbb{R}}; \|X\| < L\}$ of radius L , and set $U_L = U \cap B_L$. Clearly, the closure of U_L is compact, and the proof of Theorem 14 shows that for L sufficiently large

$$\int_{(\tilde{h}_{\varepsilon, X_0})_*(\partial U_L)} \frac{(Z - X_0)^{-1}}{N(Z - X_0)} \cdot Dz \cdot f(Z) = \begin{cases} -2\pi^2 f(X_0) & \text{if } X_0 \in U; \\ 0 & \text{if } X_0 \notin \bar{U}. \end{cases}$$

The support of ∂U_L consists of a portion which lies in $H_R \cup H_{R'}$ and its complement. We define 3-chains C_L^1 and C_L^2 by

$$\partial U_L = C_L^1 + C_L^2$$

with supports $|C_L^1| \subset H_R \cup H_{R'}$ and $|C_L^2| \subset B_L$. Then

$$\int_{(\tilde{h}_{\varepsilon, X_0})_*(\partial U)} \frac{(Z - X_0)^{-1}}{N(Z - X_0)} \cdot Dz \cdot f(Z) = \lim_{L \rightarrow \infty} \int_{(\tilde{h}_{\varepsilon, X_0})_*(\partial U_L - C_L^2)} \frac{(Z - X_0)^{-1}}{N(Z - X_0)} \cdot Dz \cdot f(Z).$$

Thus the theorem will follow if we can show

$$\lim_{L \rightarrow \infty} \int_{(\tilde{h}_{\varepsilon, X_0})_*(C_L^2)} \frac{(Z - X_0)^{-1}}{N(Z - X_0)} \cdot Dz \cdot f(Z) = 0.$$

First we choose ε small enough so that

$$\left| \frac{1 + \varepsilon(1 + i)}{1 + \varepsilon(1 - i)} - 1 \right| < \frac{d}{2}.$$

Then we choose a number c such that

$$1 < c < \left| \frac{1 + \varepsilon(1 + i)}{1 - \varepsilon(1 + i)} \right| < 2c.$$

As L tends to infinity, the supports of C_L^2 will lie inside the set (28), hence the components of f are bounded on $|C_L^2|$. The volume of $|C_L^2|$ grows as $O(L)$, the denominator of $\frac{(Z-X_0)^+}{N(Z-X_0)^2}$ grows as $O(L^4)$, and the numerator grows as $O(L)$. This proves that the integral over C_L^2 tends to zero. \square

Remark 35 From (20) one can see that the matrix coefficients (13) are bounded on the closed sets (28) as long as $\text{Re } l \leq 0$.

3 The Discrete Series Component on $\mathbb{H}_{\mathbb{R}}$

3.1 The Polynomial Algebra on $\mathbb{H}_{\mathbb{R}}^+$

Let us consider the matrix coefficients $t_{n\underline{m}}^l(Z)$'s of the discrete series together with the limits of the discrete series. It is easy to see that they are rational functions in $z_{11}, z_{12}, z_{21}, z_{22}$, hence uniquely extend to $\mathbb{H}_{\mathbb{C}}$. In this subsection we identify certain spaces of rational functions spanned by these matrix coefficients.

We think of the matrix coefficients $t_{n\underline{m}}^l(Z)$ and $N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l(Z)$ as meromorphic functions on $\mathbb{H}_{\mathbb{C}}$ and introduce the following spaces:

$$\mathcal{D}_{discr}^- = \mathbb{C}\text{-span of } t_{n\underline{m}}^l(Z) \text{ and } N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l(Z), \quad \begin{array}{l} l = -1, -\frac{3}{2}, -2, \dots, \\ m, n \in \mathbb{Z} + l, \quad m, n \geq -l \end{array}$$

(span of the matrix coefficients of the holomorphic discrete series),

$$\mathcal{D}_{lim}^- = \mathbb{C}\text{-span of } t_{n\underline{m}}^{-\frac{1}{2}}(Z), \quad m, n \in \mathbb{Z} + 1/2, \quad m, n \geq 1/2$$

(span of the matrix coefficients of the limit of the holomorphic discrete series),

$$\mathcal{D}_{discr}^+ = \mathbb{C}\text{-span of } t_{n\underline{m}}^l(Z) \text{ and } N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l(Z), \quad \begin{array}{l} l = -1, -\frac{3}{2}, -2, \dots, \\ m, n \in \mathbb{Z} + l, \quad m, n \leq l \end{array}$$

(span of the matrix coefficients of the antiholomorphic discrete series),

$$\mathcal{D}_{lim}^+ = \mathbb{C}\text{-span of } t_{n\underline{m}}^{-\frac{1}{2}}(Z), \quad m, n \in \mathbb{Z} + 1/2, \quad m, n \leq -1/2$$

(span of the matrix coefficients of the limit of the antiholomorphic discrete series),

$$\mathcal{D}^- = \mathcal{D}_{discr}^- \oplus \mathcal{D}_{lim}^-, \quad \mathcal{D}^+ = \mathcal{D}_{discr}^+ \oplus \mathcal{D}_{lim}^+.$$

Note that the matrix entries z_{11} and z_{22} are invertible on $\mathbb{H}_{\mathbb{R}}^+$. We denote by

$$\mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]^{\leq 0} \subset \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]$$

the subalgebra of polynomials in $\mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]$ spanned by monomials $z_{11}^{d_{11}} \cdot z_{12}^{d_{12}} \cdot z_{21}^{d_{21}} \cdot z_{22}^{d_{22}}$ with $d_{11} + d_{12} + d_{21} + d_{22} \leq 0$. Similarly we can define a subalgebra

$$\mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}^{-1}]^{\leq 0} \subset \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}^{-1}].$$

Let $\mathcal{D}_{\leq 0}^-$ (respectively $\mathcal{D}_{\geq 0}^-$) be the \mathbb{C} -span of $t_{n\underline{m}}^l(Z)$ (respectively $t_{n\underline{m}}^l(Z) \cdot N(Z)^{-2l-1}$) with $l = -\frac{1}{2}, -1, -\frac{3}{2}, \dots, m, n \in \mathbb{Z} + l, m, n \geq -l$. Then $\mathcal{D}^- = \mathcal{D}_{\leq 0}^- + \mathcal{D}_{\geq 0}^-$ and $\mathcal{D}_{\geq 0}^-$ is the image of $\mathcal{D}_{\leq 0}^-$ under the inversion map $\varphi(Z) \mapsto N(Z)^{-1} \cdot \varphi(Z/N(Z))$. Similarly, we can define $\mathcal{D}_{\leq 0}^+$, $\mathcal{D}_{\geq 0}^+$ so that $\mathcal{D}^+ = \mathcal{D}_{\leq 0}^+ + \mathcal{D}_{\geq 0}^+$ and $\mathcal{D}_{\geq 0}^+$ is the image of $\mathcal{D}_{\leq 0}^+$ under the inversion map.

Proposition 36 *We have:*

$$\begin{aligned} \mathcal{D}_{\leq 0}^- &= \{\varphi \in z_{11}^{-1} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]^{\leq 0}; \square_{2,2}\varphi = 0\}, \\ \mathcal{D}_{\leq 0}^+ &= \{\varphi \in z_{22}^{-1} \cdot \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}^{-1}]^{\leq 0}; \square_{2,2}\varphi = 0\}. \end{aligned}$$

Remark 37 *Note that \mathcal{D}^- is a proper subspace of $\{\varphi \in z_{11}^{-1} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]; \square_{2,2}\varphi = 0\}$, since the functions $z_{11}^{d_{11}} \cdot z_{12}^{d_{12}}$ and $z_{11}^{d_{11}} \cdot z_{21}^{d_{21}}$ with $d_{11} < 0, d_{12}, d_{21} \geq -d_{11}$ are not in \mathcal{D}^- . Similarly, \mathcal{D}^+ is a proper subspace of $\{\varphi \in z_{22}^{-1} \cdot \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}^{-1}]; \square_{2,2}\varphi = 0\}$.*

Proof. Recall the integral formula for matrix coefficients (13). In the holomorphic case $m, n \geq -l$, we have $l + m \geq 0$ and

$$(sz_{12} + z_{22})^{l+m} = (z_{22})^{l+m} \cdot \left(1 + s \frac{z_{12}}{z_{22}}\right)^{l+m} = (z_{22})^{l+m} \cdot \sum_{q=0}^{l+m} \binom{l+m}{q} \cdot \left(s \frac{z_{12}}{z_{22}}\right)^q;$$

on the other hand $l - m < 0$ and

$$(sz_{11} + z_{21})^{l-m} = (sz_{11})^{l-m} \cdot \left(1 + \frac{z_{21}}{sz_{11}}\right)^{l-m} = (sz_{11})^{l-m} \cdot \sum_{r=0}^{\infty} \binom{l-m}{r} \cdot \left(\frac{z_{21}}{sz_{11}}\right)^r.$$

Substituting into (13) we see that $t_{n\underline{m}}^l(Z)$ is the coefficient of s^0 in the expression

$$\sum_{\substack{0 \leq q \leq l+m \\ r \geq 0}} \binom{l+m}{q} \cdot \binom{l-m}{r} \cdot z_{11}^{l-m-r} \cdot z_{12}^q \cdot z_{21}^r \cdot z_{22}^{l+m-q} \cdot s^{n-m+q-r}.$$

Therefore, $q = r + m - n$ and

$$t_{n\underline{m}}^l(Z) = \sum_{r=\max\{0, n-m\}}^{l+n} \binom{l+m}{r+m-n} \cdot \binom{l-m}{r} \cdot z_{11}^{l-m-r} \cdot z_{12}^{r+m-n} \cdot z_{21}^r \cdot z_{22}^{l+n-r}. \quad (29)$$

Let d_{ij} denote the degree of z_{ij} . Then all the monomials in $t_{n\underline{m}}^l(Z)$ have $d_{11} + d_{12} + d_{21} + d_{22} = 2l \leq -1$. This proves

$$\mathcal{D}_{\leq 0}^- \subset \{\varphi \in z_{11}^{-1} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]^{\leq 0}; \square_{2,2}\varphi = 0\}.$$

It remains to prove the other inclusion. Observe that each $t_{n\underline{m}}^l(Z)$ is a finite linear combination of monomials $z_{11}^{d_{11}} \cdot z_{12}^{d_{12}} \cdot z_{21}^{d_{21}} \cdot z_{22}^{d_{22}}$ with

$$d_{11} + d_{12} = l - n, \quad d_{21} + d_{22} = l + n, \quad d_{11} + d_{21} = l - m, \quad d_{12} + d_{22} = l + m.$$

These sums of degrees can be conveniently arranged in the following form:

$$\begin{array}{ccc} d_{11} + d_{21} & d_{11} + d_{12} & l - n \\ & d_{12} + d_{22} & l + m. \\ & d_{21} + d_{22} & l + n \end{array} = \begin{array}{ccc} & & \\ l - m & & \\ & & \end{array}$$

Each degree invariant

$$\begin{array}{ccc} d_{11} + d_{21} & d_{11} + d_{12} & D_1 \\ & d_{12} + d_{22} & D_2 \quad D_3 \\ & d_{21} + d_{22} & D_4 \end{array} = \begin{array}{ccc} & & \\ D_2 & & \\ & & \end{array} \quad (30)$$

has an obvious restriction $D_1 + D_4 = D_2 + D_3$. These observations lead to the following lemma:

Lemma 38 *The matrix coefficients $t_{n\bar{m}}^l(Z)$ with $m, n \geq -l$ produce all possible degree invariants (30) satisfying $D_3, D_4 \geq 0$ and $D_1 + D_4 = D_2 + D_3 \leq -1$. Moreover, the functions of the type $t_{n\bar{m}}^l(Z)$ can be uniquely recovered from their degree invariant.*

The degree invariants of $z_{11}^{-1}, z_{12}, z_{21}, z_{22}$ are respectively

$$\begin{array}{cccc} & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 1. \\ & 0 & 0 & 1 & 1 & & & \end{array}$$

Thus the monomials in $z_{11}^{-1} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]$ automatically satisfy $D_3, D_4 \geq 0$. The operator $\square_{2,2}$ is homogeneous with respect to these invariants in the sense that it sends

$$\begin{array}{ccc} D_1 & & D_1 - 1 \\ D_2 & D_3 & \mapsto D_2 - 1 & D_3 - 1. \\ D_4 & & D_4 - 1 \end{array}$$

Thus it is enough to show that the space of harmonic polynomials in $z_{11}^{-1} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]$ with a given degree invariant (30) is at most one-dimensional. Fix a degree invariant (30), then any function $\varphi \in z_{11}^{-1} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]$ with that invariant must be a finite linear combination of

$$z_{12}^{D_1} \cdot z_{21}^{D_2} \cdot z_{22}^{D_3 - D_1} \cdot \left(\frac{z_{12}}{z_{22}} \cdot \frac{z_{21}}{z_{11}} \right)^r = z_{11}^{-r} \cdot z_{12}^{D_1 + r} \cdot z_{21}^{D_2 + r} \cdot z_{22}^{D_3 - D_1 - r}, \quad r = 1, \dots, D_3 - D_1.$$

Thus

$$\varphi = \sum_{r=1}^{D_3 - D_1} a_r \cdot z_{11}^{-r} \cdot z_{12}^{D_1 + r} \cdot z_{21}^{D_2 + r} \cdot z_{22}^{D_3 - D_1 - r}.$$

We spell out the equation $\square_{2,2}\varphi = 0$:

$$\begin{aligned} 0 = & \sum_{r=1}^{D_3 - D_1} a_r \cdot r(D_3 - D_1 - r) \cdot z_{11}^{-r-1} \cdot z_{12}^{D_1 + r} \cdot z_{21}^{D_2 + r} \cdot z_{22}^{D_3 - D_1 - r - 1} \\ & + \sum_{r=1}^{D_3 - D_1} a_r \cdot (D_1 + r)(D_2 + r) \cdot z_{11}^{-r} \cdot z_{12}^{D_1 + r - 1} \cdot z_{21}^{D_2 + r - 1} \cdot z_{22}^{D_3 - D_1 - r} \end{aligned}$$

which gives us recursive equations

$$(r-1)(D_3 - D_1 - r + 1) \cdot a_{r-1} = -(D_1 + r)(D_2 + r) \cdot a_r, \quad r = 1, \dots, D_3 - D_1. \quad (31)$$

Hence the space of solutions in $z_{11}^{-1} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]^{\leq 0}$ is at most one-dimensional.

The case $m, n \leq l$ is completely analogous. \square

3.2 The Action of $\mathfrak{sl}(4, \mathbb{C})$

For convenience we restate Lemma 17 of [FL1] describing the π_l^0 and π_r^0 actions of $\mathfrak{gl}(4, \mathbb{C}) = \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ on the spaces of harmonic functions

$$\begin{aligned}
\pi_l^0 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &: \varphi \mapsto -\operatorname{Tr}(A \cdot X \cdot \partial\varphi) \\
\pi_r^0 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &: \varphi \mapsto -\operatorname{Tr}(A \cdot (X \cdot \partial\varphi + \varphi)) \\
\pi_l^0 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \pi_r^0 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} &: \varphi \mapsto -\operatorname{Tr}(B \cdot \partial\varphi) \\
\pi_l^0 \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = \pi_r^0 \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} &: \varphi \mapsto \operatorname{Tr}(C \cdot (X \cdot (\partial\varphi) \cdot X + X\varphi)) = \operatorname{Tr}(C \cdot (X \cdot \partial(X\varphi)) - X\varphi) \\
\pi_l^0 \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} &: \varphi \mapsto \operatorname{Tr}(D \cdot ((\partial\varphi) \cdot X + \varphi)) = \operatorname{Tr}(D \cdot (\partial(X\varphi) - \varphi)) \\
\pi_r^0 \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} &: \varphi \mapsto \operatorname{Tr}(D \cdot (\partial\varphi) \cdot X) = \operatorname{Tr}(D \cdot (\partial(X\varphi) - 2\varphi)),
\end{aligned}$$

where $\partial = \begin{pmatrix} \partial_{11} & \partial_{21} \\ \partial_{12} & \partial_{22} \end{pmatrix} = \frac{1}{2}\nabla$. Recall the representations \mathcal{H}^+ and \mathcal{H}^- of $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$ realized in the space of harmonic functions on \mathbb{H} (see Subsection 2.5 in [FL1]).

Theorem 39 *We have $\mathcal{D}^- \simeq \mathcal{H}^-$ and $\mathcal{D}^+ \simeq \mathcal{H}^+$ as representations of $\mathfrak{sl}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}}) \subset \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$. Moreover, \mathcal{D}^- and \mathcal{D}^+ are irreducible representations of $\mathfrak{sl}(4, \mathbb{C})$ with highest (or lowest) weight vectors $t_{\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}}(Z) = \frac{1}{z_{11}}$ and $t_{-\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}}(Z) = \frac{1}{z_{22}}$ respectively.*

Proof. Using Lemmas 17 and 18 we can compute the Lie algebra actions π_l^0 and π_r^0 of $\mathfrak{gl}(2, \mathbb{H})$ on \mathcal{D}^- and \mathcal{D}^+ :

$$\begin{aligned}
\pi_l^0 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &: t_{n\underline{m}}^l \mapsto -\operatorname{Tr}\left(A \cdot \begin{pmatrix} (l-n)t_{n\underline{m}}^l & (l-n+1)t_{n-1\underline{m}}^l \\ (l+n+1)t_{n+1\underline{m}}^l & (l+n)t_{n\underline{m}}^l \end{pmatrix}\right) \\
\pi_r^0 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &: t_{n\underline{m}}^l \mapsto -\operatorname{Tr}\left(A \cdot \begin{pmatrix} (l-n+1)t_{n\underline{m}}^l & (l-n+1)t_{n-1\underline{m}}^l \\ (l+n+1)t_{n+1\underline{m}}^l & (l+n+1)t_{n\underline{m}}^l \end{pmatrix}\right) \\
\pi_l^0 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \pi_r^0 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} &: t_{n\underline{m}}^l \mapsto -\operatorname{Tr}\left(B \cdot \begin{pmatrix} (l-m)t_{n+\frac{1}{2}\underline{m}+\frac{1}{2}}^{l-\frac{1}{2}} & (l-m)t_{n-\frac{1}{2}\underline{m}+\frac{1}{2}}^{l-\frac{1}{2}} \\ (l+m)t_{n+\frac{1}{2}\underline{m}-\frac{1}{2}}^{l-\frac{1}{2}} & (l+m)t_{n-\frac{1}{2}\underline{m}-\frac{1}{2}}^{l-\frac{1}{2}} \end{pmatrix}\right) \\
\pi_l^0 \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = \pi_r^0 \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} &: t_{n\underline{m}}^l \mapsto \operatorname{Tr}\left(C \cdot \begin{pmatrix} (l-n+1)t_{n-\frac{1}{2}\underline{m}-\frac{1}{2}}^{l+\frac{1}{2}} & (l-n+1)t_{n-\frac{1}{2}\underline{m}+\frac{1}{2}}^{l+\frac{1}{2}} \\ (l+n+1)t_{n+\frac{1}{2}\underline{m}-\frac{1}{2}}^{l+\frac{1}{2}} & (l+n+1)t_{n+\frac{1}{2}\underline{m}+\frac{1}{2}}^{l+\frac{1}{2}} \end{pmatrix}\right) \\
\pi_l^0 \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} &: t_{n\underline{m}}^l \mapsto \operatorname{Tr}\left(D \cdot \begin{pmatrix} (l-m+1)t_{n\underline{m}}^l & (l-m)t_{n\underline{m}+1}^l \\ (l+m)t_{n\underline{m}-1}^l & (l+m+1)t_{n\underline{m}}^l \end{pmatrix}\right) \\
\pi_r^0 \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} &: t_{n\underline{m}}^l \mapsto \operatorname{Tr}\left(D \cdot \begin{pmatrix} (l-m)t_{n\underline{m}}^l & (l-m)t_{n\underline{m}+1}^l \\ (l+m)t_{n\underline{m}-1}^l & (l+m)t_{n\underline{m}}^l \end{pmatrix}\right)
\end{aligned}$$

and similarly for $t_{n\underline{m}}^l(Z) \cdot N(Z)^{-2l-1}$'s. This together with equation (14) shows that the π_l^0 -action of $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$ preserves \mathcal{D}^{\mp} and that $t_{\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}}(Z)$ and $t_{-\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}}(Z)$ generate \mathcal{D}^- and \mathcal{D}^+ respec-

tively. Let $\tilde{e}_3 = ie_3$. Consider an element

$$\gamma_0 = \frac{1}{2} \begin{pmatrix} -i\tilde{e}_3 - i & \tilde{e}_3 - 1 \\ \tilde{e}_3 - 1 & i\tilde{e}_3 + i \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}}) \quad \text{with} \quad \gamma_0^{-1} = \frac{1}{2} \begin{pmatrix} i\tilde{e}_3 + i & \tilde{e}_3 - 1 \\ \tilde{e}_3 - 1 & -i\tilde{e}_3 - i \end{pmatrix}. \quad (32)$$

Then

$$\begin{aligned} \pi_l^0(\gamma_0) : \quad \mathcal{H}^+ \ni 1 &\mapsto \frac{1}{iz_{22}} = -it^{-\frac{1}{2}}_{-\frac{1}{2}}(Z) \in \mathcal{D}^+, \\ \mathcal{H}^- \ni \frac{1}{N(Z)} &\mapsto \frac{i}{z_{11}} = it^{\frac{1}{2}}_{\frac{1}{2}}(Z) \in \mathcal{D}^-. \end{aligned}$$

(This is essentially the composition of the Cayley transform from [FL1] with another Cayley-type transform which will be introduced in Proposition 58.) This proves $\mathcal{D}^{\mp} \simeq \mathcal{H}^{\mp}$, irreducibility and the statement about the highest (or lowest) weight vectors. \square

Representations \mathcal{D}^- and \mathcal{D}^+ are dual to each other. Define a bilinear pairing between \mathcal{D}^- and \mathcal{D}^+ by declaring

$$\begin{aligned} \left\langle t'_{n' \underline{m}'}(Z), \frac{1}{N(Z)} \cdot t^l_{m \underline{n}}(Z^{-1}) \right\rangle_{\mathcal{D}^- \times \mathcal{D}^+} &= \left\langle \frac{1}{N(Z)} \cdot t^l_{m \underline{n}}(Z^{-1}), t'_{n' \underline{m}'}(Z) \right\rangle_{\mathcal{D}^- \times \mathcal{D}^+} = \delta_{ll'} \delta_{mm'} \delta_{nn'}, \\ \left\langle t^l_{n \underline{m}}(Z), t'_{n' \underline{m}'}(Z) \right\rangle_{\mathcal{D}^- \times \mathcal{D}^+} &= \left\langle \frac{1}{N(Z)} \cdot t^l_{m \underline{n}}(Z^{-1}), \frac{1}{N(Z)} \cdot t'_{m' \underline{n}'}(Z^{-1}) \right\rangle_{\mathcal{D}^- \times \mathcal{D}^+} = 0, \end{aligned}$$

$l = -1, -\frac{3}{2}, -2, \dots$. In the second line we exclude $l = -1/2$ because by (23) we have

$$N(Z)^{-1} \cdot t_{-m \underline{-n}}^{-\frac{1}{2}}(Z^{-1}) = t_{-m \underline{-n}}^{-\frac{1}{2}}(Z).$$

By Proposition 31, this pairing partially agrees with the bilinear form (24) up to a sign.

Proposition 40 *This bilinear pairing on $\mathcal{D}^- \times \mathcal{D}^+$ is $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant:*

$$\left\langle \pi_l^0(Z) \varphi_1, \varphi_2 \right\rangle_{\mathcal{D}^- \times \mathcal{D}^+} + \left\langle \varphi_1, \pi_r^0(Z) \varphi_2 \right\rangle_{\mathcal{D}^- \times \mathcal{D}^+} = 0, \quad \forall Z \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}).$$

Proof. Since the elements $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$, $B \in \mathbb{H}_{\mathbb{C}}$, together with their conjugates by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ generate $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$, to prove $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$ -invariance it is enough to check the invariance under the action of $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. By Lemma 17 from [FL1],

$$\pi_l^0 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \pi_r^0 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : \varphi \mapsto \text{Tr}(B \cdot (-\partial\varphi)).$$

Applying Lemmas 17 and 18 repeatedly we find that

$$\begin{aligned} \begin{pmatrix} \partial_{11} & \partial_{21} \\ \partial_{12} & \partial_{22} \end{pmatrix} t^l_{n \underline{m}}(Z) &= \begin{pmatrix} (l-m)t_{n+\frac{1}{2} \underline{m+\frac{1}{2}}}^{l-\frac{1}{2}}(Z) & (l-m)t_{n-\frac{1}{2} \underline{m+\frac{1}{2}}}^{l-\frac{1}{2}}(Z) \\ (l+m)t_{n+\frac{1}{2} \underline{m-\frac{1}{2}}}^{l-\frac{1}{2}}(Z) & (l+m)t_{n-\frac{1}{2} \underline{m-\frac{1}{2}}}^{l-\frac{1}{2}}(Z) \end{pmatrix}, \\ \begin{pmatrix} \partial_{11} & \partial_{21} \\ \partial_{12} & \partial_{22} \end{pmatrix} \left(\frac{1}{N(Z)} \cdot t^l_{m \underline{n}}(Z^{-1}) \right) &= -\frac{1}{N(Z)} \begin{pmatrix} (l-m+1)t_{m-\frac{1}{2} \underline{n-\frac{1}{2}}}^{l+\frac{1}{2}}(Z^{-1}) & (l-m+1)t_{m-\frac{1}{2} \underline{n+\frac{1}{2}}}^{l+\frac{1}{2}}(Z^{-1}) \\ (l+m+1)t_{m+\frac{1}{2} \underline{n-\frac{1}{2}}}^{l+\frac{1}{2}}(Z^{-1}) & (l+m+1)t_{m+\frac{1}{2} \underline{n+\frac{1}{2}}}^{l+\frac{1}{2}}(Z^{-1}) \end{pmatrix}. \end{aligned}$$

We conclude that the bilinear pairing is invariant under the action of $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. Using Lemma 17 from [FL1] it is easy to see that the pairing is invariant under the scalar matrices as well, and the proposition follows. \square

3.3 Ol'shanskii Semigroups

We introduce complex Ol'shanskii semigroups $\Gamma^\pm \subset GL(2, \mathbb{C}) \subset \mathbb{H}_{\mathbb{C}}$. Following [KouØ] and [HN], consider a Hermitian form H on \mathbb{C}^2 defined by

$$H(\zeta, \eta) = -\zeta_1 \overline{\eta_1} + \zeta_2 \overline{\eta_2}.$$

It is easy to check that if $X \in \mathfrak{iu}(1, 1)$, then $H(X\zeta, \zeta) \in \mathbb{R}$ for all $\zeta \in \mathbb{C}^2$. Consider a cone in $\mathfrak{iu}(1, 1)$ defined by

$$C = \{X \in \mathfrak{iu}(1, 1); H(X\zeta, \zeta) \leq 0, \forall \zeta \in \mathbb{C}^2\}.$$

Lemma 41 *The cone C is closed, convex, pointed (i.e. $C \cap -C = \{0\}$), generating (i.e. $C - C = \mathfrak{iu}(1, 1)$) or, equivalently, has non-empty interior), hyperbolic (i.e. for every $X \in C$, the operator $\text{ad}X$ has real eigenvalues and, for every X in the interior of C , $\text{ad}X$ is diagonalizable) and $\text{Ad}(U(1, 1))$ -invariant.*

The set

$$\overline{\Gamma}^- = U(1, 1) \cdot \exp(C)$$

is called a closed complex Ol'shanskii semigroup contained in $GL(2, \mathbb{C})$. The interior of C is

$$C^0 = \{X \in \mathfrak{iu}(1, 1); H(X\zeta, \zeta) < 0, \forall \zeta \in \mathbb{C}^2 \setminus \{0\}\},$$

and the corresponding open Ol'shanskii semigroup is

$$\Gamma^- = U(1, 1) \cdot \exp(C^0).$$

The semigroup Γ^- is an open subset of $GL(2, \mathbb{C})$ and hence of $\mathbb{H}_{\mathbb{C}}$. In fact, Γ^- is the interior of $\overline{\Gamma}^-$. If we re-define for a moment the complex conjugation on $\mathbb{H}_{\mathbb{C}}$ to be relative to $\mathbb{H}_{\mathbb{R}} = \mathfrak{u}(1, 1)$ (i.e. by identifying $\mathbb{H}_{\mathbb{C}}$ with $\mathfrak{u}(1, 1) \oplus \mathfrak{iu}(1, 1)$), then the conjugate Ol'shanskii semigroup $\overline{\Gamma}^-$ coincides with $\overline{\Gamma}^+ = U(1, 1) \cdot \exp(-C)$. Similarly, $\overline{\Gamma}^- = \Gamma^+ = U(1, 1) \cdot \exp(-C^0)$.

Lemma 42 *We have*

$$C = \left\{ X = \begin{pmatrix} a & \gamma \\ -\overline{\gamma} & -b \end{pmatrix}; a, b \in \mathbb{R}, a, b \geq 0, \gamma \in \mathbb{C}, |\gamma|^2 \leq ab \right\} \subset \mathfrak{iu}(1, 1)$$

and

$$C^0 = \left\{ X = \begin{pmatrix} a & \gamma \\ -\overline{\gamma} & -b \end{pmatrix}; a, b \in \mathbb{R}, a, b > 0, \gamma \in \mathbb{C}, |\gamma|^2 < ab \right\} \subset \mathfrak{iu}(1, 1).$$

In particular, each $X \in C^0$ has two distinct real eigenvalues – one positive and one negative.

Fix a maximal compact subgroup $U(1) \times U(1)$ of $U(1, 1)$; its complexification $K_{\mathbb{C}}$ consists of diagonal matrices, and every element in Γ^- is $SU(1, 1)$ -conjugate to an element in

$$\Gamma^- \cap K_{\mathbb{C}} = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \lambda_1, \lambda_2 \in \mathbb{C}, |\lambda_1| > 1 > |\lambda_2| > 0 \right\}.$$

Similarly, every element in Γ^+ is $SU(1, 1)$ -conjugate to an element in

$$\Gamma^+ \cap K_{\mathbb{C}} = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \lambda_1, \lambda_2 \in \mathbb{C}, |\lambda_2| > 1 > |\lambda_1| > 0 \right\}.$$

Proposition 43 *The Ol'shanskii semigroups $\bar{\Gamma}^-$ and Γ^- have the following crucial property:*

$$\gamma_0\gamma \text{ and } \gamma\gamma_0 \in \Gamma^- \quad \text{whenever } \gamma_0 \in \Gamma^- \text{ and } \gamma \in \bar{\Gamma}^-.$$

In particular, if we let

$$W \in \mathbb{H}_{\mathbb{R}}^+ \quad \text{and} \quad Z \in \sqrt{N(W)} \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot SU(1,1) \subset \mathbb{H}_{\mathbb{C}},$$

then $WZ^{-1} \in \Gamma^-$ for all $1 > \sigma > 0$ and $WZ^{-1} \in \Gamma^+$ for all $\sigma > 1$.

Lemma 44 *For $W \in \Gamma^- \cup \Gamma^+$ and $Z \in U(1,1)$, the function $N(Z-W)$ is never zero. Moreover, if we fix $W \in \Gamma^- \cup \Gamma^+$, there exist $c > 0$ and $\varepsilon > 0$ depending on W such that*

$$\frac{1}{N(Z-W)} \leq \frac{c}{\|Z\|}$$

for all $Z \in \mathbb{H}_{\mathbb{R}}$ with $1 - \varepsilon \leq N(Z) \leq 1 + \varepsilon$.

Proof. For concreteness, let us suppose $W \in \Gamma^-$; the other case is similar. We rewrite $N(Z-W)$ as $N(Z^{-1}W - 1) \cdot N(Z)$. Then $Z^{-1}W \in \Gamma^-$ and hence has eigenvalues different from 1. Therefore, $N(Z^{-1}W - 1) \neq 0$.

If $Z \in \mathbb{H}_{\mathbb{R}}$, then $N(Z-W) = N(Z) \cdot N(Z^{-1}W - 1)$. Every element $W \in \Gamma^-$ is $SU(1,1)$ -conjugate to a diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $|\lambda_1| > 1 > |\lambda_2| > 0$. Since the determinant function is $Ad(SU(1,1))$ -invariant, without loss of generality we may assume that $W = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then

$$\begin{aligned} \det \left[\begin{pmatrix} \bar{z}_{11} & -z_{12} \\ -\bar{z}_{12} & z_{11} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] &= \det \begin{pmatrix} \lambda_1 \bar{z}_{11} - 1 & -\lambda_2 z_{12} \\ -\lambda_1 \bar{z}_{12} & \lambda_2 z_{11} - 1 \end{pmatrix} \\ &= (\lambda_1 \bar{z}_{11} - 1)(\lambda_2 z_{11} - 1) - \lambda_1 \lambda_2 |z_{12}|^2 = \lambda_1 \lambda_2 N(Z) + 1 - (\lambda_1 \bar{z}_{11} + \lambda_2 z_{11}). \end{aligned}$$

Now the term $\lambda_1 \lambda_2 N(Z) + 1$ stays bounded while $\|\lambda_1 \bar{z}_{11} + \lambda_2 z_{11}\|$ grows proportionally to $\|Z\|$ as $Z \rightarrow \infty$. \square

As we saw in Subsection 3.1, the matrix coefficients of the discrete series and their limits are rational functions in z_{ij} 's. More precisely, the matrix coefficients of the holomorphic discrete series and its limit lie in $\mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, z_{11}^{-1}]$. Hence they extend uniquely as holomorphic functions to the open set $\{Z \in \mathbb{H}_{\mathbb{C}}; z_{11} \neq 0\}$. On the other hand, Lemma 2.6 from [KouØ] implies that $\Gamma^- \subset \{Z \in \mathbb{H}_{\mathbb{C}}; z_{11} \neq 0\}$. In particular, the matrix coefficients of the holomorphic discrete series and its limit extend holomorphically to Γ^- . Similarly, the matrix coefficients of the antiholomorphic discrete series and its limit lie in $\mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, z_{22}^{-1}]$, extend holomorphically to the open set $\{Z \in \mathbb{H}_{\mathbb{C}}; z_{22} \neq 0\}$. Since $\Gamma^+ \subset \{Z \in \mathbb{H}_{\mathbb{C}}; z_{22} \neq 0\}$, these coefficients extend to Γ^+ .

3.4 Matrix coefficient expansions for $\frac{1}{N(Z-W)}$ and $\frac{(Z-W)^{-1}}{N(Z-W)}$

In this subsection we derive matrix coefficient expansions for $\frac{1}{N(Z-W)}$ and $\frac{(Z-W)^{-1}}{N(Z-W)}$. Using these expansions we obtain maps onto the discrete series components for the spaces of solutions of $\square_{2,2}\varphi = 0$ and left- and right-regular functions on $\mathbb{H}_{\mathbb{R}}^+$.

Proposition 45 *We have the following matrix coefficient expansions*

$$-\frac{1}{N(Z-W)} = \sum_{\substack{l,m,n \\ m,n \geq -l \geq 1}} t_{nm}^l(W) \cdot \frac{1}{N(Z)} \cdot t_{mn}^l(Z^{-1}) + \sum_{\substack{l,m,n \\ m,n \leq -l \leq -1/2}} \frac{1}{N(W)} \cdot t_{nm}^l(W^{-1}) \cdot t_{mn}^l(Z)$$

which converges pointwise absolutely whenever $WZ^{-1} \in \Gamma^-$ and

$$-\frac{1}{N(Z-W)} = \sum_{\substack{l,m,n \\ m,n \leq -l \leq -1}} t_{nm}^l(W) \cdot \frac{1}{N(Z)} \cdot t_{mn}^l(Z^{-1}) + \sum_{\substack{l,m,n \\ m,n \geq -l \geq 1/2}} \frac{1}{N(W)} \cdot t_{nm}^l(W^{-1}) \cdot t_{mn}^l(Z)$$

which converges pointwise absolutely whenever $WZ^{-1} \in \Gamma^+$.

Remark 46 *From (23) we have*

$$N(W)^{-1} \cdot t_{nm}^{-\frac{1}{2}}(W^{-1}) \cdot t_{mn}^{-\frac{1}{2}}(Z) = t_{-m-n}^{-\frac{1}{2}}(W) \cdot t_{nm}^{-\frac{1}{2}}(Z) = t_{-m-n}^{-\frac{1}{2}}(W) \cdot N(Z)^{-1} \cdot t_{-n-m}^{-\frac{1}{2}}(Z^{-1}).$$

Hence, in each expansion of $\frac{1}{N(Z-W)}$, the terms corresponding to the limits of the discrete series (terms with $l = -1/2$) can be included in either the first or the second sum.

Proof. Since every element in Γ^- is $SU(1,1)$ -conjugate to an element $\Gamma^- \cap K_{\mathbb{C}}$, to prove the first expansion we can assume that $WZ^{-1} \in \Gamma^- \cap K_{\mathbb{C}}$, and so $WZ^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_1| > 1 > |\lambda_2| > 0$. The discrete series characters are:

$$\begin{aligned} \tilde{\Theta}_l^+(WZ^{-1}) &= -\frac{\lambda_2^{2l+1}}{\lambda_1 - \lambda_2}, & \tilde{\Theta}_l^+(ZW^{-1}) &= \frac{\lambda_1 \cdot \lambda_2^{-2l}}{\lambda_1 - \lambda_2}, \\ \tilde{\Theta}_l^-(WZ^{-1}) &= \frac{\lambda_1^{2l+1}}{\lambda_1 - \lambda_2}, & \tilde{\Theta}_l^-(ZW^{-1}) &= -\frac{\lambda_2 \cdot \lambda_1^{-2l}}{\lambda_1 - \lambda_2}. \end{aligned} \quad (33)$$

Using the multiplicativity property of matrix coefficients

$$t_{nm}^l(Z_1 Z_2) = \sum_k t_{nk}^l(Z_1) \cdot t_{km}^l(Z_2), \quad (34)$$

the expansion reduces to a geometric series computation:

$$\begin{aligned} & \sum_{\substack{l,m,n \\ m,n \geq -l \geq 1}} t_{nm}^l(W) \cdot \frac{1}{N(Z)} \cdot t_{mn}^l(Z^{-1}) + \sum_{\substack{l,m,n \\ m,n \leq -l \leq -1/2}} \frac{1}{N(W)} \cdot t_{nm}^l(W^{-1}) \cdot t_{mn}^l(Z) \\ &= \frac{1}{N(Z)} \cdot \left(\sum_{\substack{l,n \\ n \geq -l \geq 1}} t_{nm}^l(WZ^{-1}) + N(ZW^{-1}) \cdot \sum_{\substack{l,n \\ n \leq -l \leq -1/2}} t_{nm}^l(ZW^{-1}) \right) \\ &= \frac{1}{N(Z)} \cdot \left(\sum_{l \leq -1} \tilde{\Theta}_l^-(WZ^{-1}) + N(ZW^{-1}) \cdot \sum_{l \leq -1/2} \tilde{\Theta}_l^+(ZW^{-1}) \right) \\ &= \frac{1}{N(Z)} \cdot \left(\sum_{l \leq -1} \frac{\lambda_1^{2l+1}}{\lambda_1 - \lambda_2} + \sum_{l \leq -1/2} \frac{\lambda_2^{-(2l+1)}}{\lambda_1 - \lambda_2} \right) = \frac{1}{N(Z)} \cdot \frac{1}{\lambda_1 - \lambda_2} \cdot \left(\frac{1}{\lambda_1 - 1} + \frac{1}{1 - \lambda_2} \right) \\ &= -\frac{1}{N(Z)} \cdot \frac{1}{(\lambda_1 - 1)(\lambda_2 - 1)} = -\frac{1}{N(Z)} \cdot \frac{1}{N(WZ^{-1} - 1)} = -\frac{1}{N(Z - W)}. \end{aligned}$$

The other matrix coefficient expansion is proved in the same way. \square

For $R > 0$, define operators on $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ by

$$\begin{aligned} (S_R^- \varphi)(W) &= -\frac{1}{2\pi^2} \int_{X \in H_R} \frac{\widetilde{(\deg \varphi)}(X)}{N(X-W)} \cdot \frac{dS}{\|X\|} = \left\langle \varphi(X), \frac{1}{N(X-W)} \right\rangle_R, & W \in R \cdot \Gamma^-, \\ (S_R^+ \varphi)(W) &= -\frac{1}{2\pi^2} \int_{X \in H_R} \frac{\widetilde{(\deg \varphi)}(X)}{N(X-W)} \cdot \frac{dS}{\|X\|} = \left\langle \varphi(X), \frac{1}{N(X-W)} \right\rangle_R, & W \in R \cdot \Gamma^+. \end{aligned}$$

By Lemma 44, these integrals are well-defined. From the orthogonality relations for matrix coefficients (Proposition 31) we obtain:

Theorem 47 *The operators S_R^- and S_R^+ are continuous linear operators $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+) \rightarrow \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. The operator S_R^- annihilates the continuous series, the antiholomorphic discrete series \mathcal{D}_{disc}^+ and sends*

$$\begin{aligned} t_{n\underline{m}}^l(X) &\mapsto -t_{n\underline{m}}^l(W) - R^{2(2l+1)} \cdot N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & l = -1, -\frac{3}{2}, -2, \dots, \\ & & m, n \in \mathbb{Z} + l, \\ N(X)^{-2l-1} \cdot t_{n\underline{m}}^l(X) &\mapsto R^{-2(2l+1)} \cdot t_{n\underline{m}}^l(W) + N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & m, n \geq -l. \end{aligned}$$

The operator S_R^+ annihilates the continuous series, the holomorphic discrete series \mathcal{D}_{disc}^- and sends

$$\begin{aligned} t_{n\underline{m}}^l(X) &\mapsto -t_{n\underline{m}}^l(W) - R^{2(2l+1)} \cdot N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & l = -1, -\frac{3}{2}, -2, \dots, \\ & & m, n \in \mathbb{Z} + l, \\ N(X)^{-2l-1} \cdot t_{n\underline{m}}^l(X) &\mapsto R^{-2(2l+1)} \cdot t_{n\underline{m}}^l(W) + N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & m, n \leq l. \end{aligned}$$

Note that the closure of $R \cdot \Gamma^-$ is $R \cdot \Gamma$ which contains H_R . Also the functions $t_{n\underline{m}}^l(W)$ and $R^{2(2l+1)} \cdot N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W)$ agree on H_R . Thus, the values of the holomorphic discrete series component of a function $\varphi \in \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ on H_R can be determined by continuity. Similarly, one can recover the antiholomorphic discrete series component of φ on H_R .

Differentiating the matrix coefficient expansions for $\frac{1}{N(Z-W)}$ we get two expansions for $\frac{(Z-W)^{-1}}{N(Z-W)}$:

Proposition 48 *We have the following matrix coefficient expansions:*

$$\begin{aligned} \frac{(Z-W)^{-1}}{N(Z-W)} &= \sum_{\substack{l, m, n \\ m, n \leq l \leq -1/2}} \frac{1}{N(W)} \left(\begin{matrix} (l-n+1)t_{n-1\underline{m}}^l(W^{-1}) \\ (l+n)t_{n\underline{m}}^l(W^{-1}) \end{matrix} \right) \left(t_{m+\frac{1}{2}n-\frac{1}{2}}^{l-\frac{1}{2}}(Z), t_{m-\frac{1}{2}n-\frac{1}{2}}^{l-\frac{1}{2}}(Z) \right) \\ &- \sum_{\substack{l, m, n \\ m, n \geq -l \geq 1}} \left(\begin{matrix} (l-m+1)t_{n\underline{m}}^l(W) \\ (l+m)t_{n\underline{m-1}}^l(W) \end{matrix} \right) \left(N(Z)^{-1} \cdot t_{m-\frac{1}{2}n-\frac{1}{2}}^{l+\frac{1}{2}}(Z^{-1}), N(Z)^{-1} \cdot t_{m-\frac{1}{2}n+\frac{1}{2}}^{l+\frac{1}{2}}(Z^{-1}) \right) \end{aligned}$$

which converges pointwise absolutely whenever $WZ^{-1} \in \Gamma^-$ and

$$\begin{aligned} \frac{(Z-W)^{-1}}{N(Z-W)} &= \sum_{\substack{l, m, n \\ m, n \geq -l \geq 1/2}} \frac{1}{N(W)} \left(\begin{matrix} (l-n)t_{n\underline{m}}^l(W^{-1}) \\ (l+n+1)t_{n+1\underline{m}}^l(W^{-1}) \end{matrix} \right) \left(t_{m+\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z), t_{m-\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z) \right) \\ &- \sum_{\substack{l, m, n \\ m, n \leq l \leq -1}} \left(\begin{matrix} (l-m)t_{n\underline{m+1}}^l(W) \\ (l+m+1)t_{n\underline{m}}^l(W) \end{matrix} \right) \left(N(Z)^{-1} \cdot t_{m+\frac{1}{2}n-\frac{1}{2}}^{l+\frac{1}{2}}(Z^{-1}), N(Z)^{-1} \cdot t_{m+\frac{1}{2}n+\frac{1}{2}}^{l+\frac{1}{2}}(Z^{-1}) \right) \end{aligned}$$

which converges pointwise absolutely whenever $WZ^{-1} \in \Gamma^+$.

Proof. The two expansions are proved by applying ∇_Z to the expansions in Proposition 45, where the subscript Z in ∇_Z indicates that the differentiation is done with respect to this variable. We give a proof of the second formula.

$$\begin{aligned} \frac{(Z-W)^{-1}}{N(Z-W)} &= -\frac{1}{2}\nabla_Z \frac{1}{N(Z-W)} = \frac{1}{N(W)} \sum_{\substack{l,m,n' \\ m,n' \geq -l \geq 1/2}} t_{n'\underline{m}}^l(W^{-1}) \cdot \begin{pmatrix} \partial_{11} & \partial_{21} \\ \partial_{12} & \partial_{22} \end{pmatrix}_Z t_{m\underline{n}'}^l(Z) \\ &\quad - \sum_{\substack{l,m',n \\ m',n \leq l \leq -1}} t_{n\underline{m}'}^l(W) \cdot \begin{pmatrix} \partial_{11} & \partial_{21} \\ \partial_{12} & \partial_{22} \end{pmatrix}_Z \left(\frac{-1}{N(Z)} \cdot t_{m'\underline{n}}^l(Z^{-1}) \right). \end{aligned} \quad (35)$$

Using the derivative formulas from Lemma 17 we can expand the first sum in (35) as:

$$\begin{aligned} \sum_{\substack{l,m,n' \\ m,n' \geq -l \geq 1/2}} t_{n'\underline{m}}^l(W^{-1}) \cdot \begin{pmatrix} (l-n')t_{m+\frac{1}{2}n'+\frac{1}{2}}^{l-\frac{1}{2}}(Z) & (l-n')t_{m-\frac{1}{2}n'+\frac{1}{2}}^{l-\frac{1}{2}}(Z) \\ (l+n')t_{m+\frac{1}{2}n'-\frac{1}{2}}^{l-\frac{1}{2}}(Z) & (l+n')t_{m-\frac{1}{2}n'-\frac{1}{2}}^{l-\frac{1}{2}}(Z) \end{pmatrix} = \\ \sum_{\substack{l,m,n' \\ m,n' \geq -l \geq 1/2}} \begin{pmatrix} (l-n')t_{n'\underline{m}}^l(W^{-1}) \cdot t_{m+\frac{1}{2}n'+\frac{1}{2}}^{l-\frac{1}{2}}(Z) & (l-n')t_{n'\underline{m}}^l(W^{-1}) \cdot t_{m-\frac{1}{2}n'+\frac{1}{2}}^{l-\frac{1}{2}}(Z) \\ (l+n')t_{n'\underline{m}}^l(W^{-1}) \cdot t_{m+\frac{1}{2}n'-\frac{1}{2}}^{l-\frac{1}{2}}(Z) & (l+n')t_{n'\underline{m}}^l(W^{-1}) \cdot t_{m-\frac{1}{2}n'-\frac{1}{2}}^{l-\frac{1}{2}}(Z) \end{pmatrix}. \end{aligned}$$

Replacing n' with $n+1$ in the second row we get:

$$\begin{aligned} \sum_{\substack{l,m,n \\ m,n \geq -l \geq 1/2}} \begin{pmatrix} (l-n)t_{n\underline{m}}^l(W^{-1}) \cdot t_{m+\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z) & (l-n)t_{n\underline{m}}^l(W^{-1}) \cdot t_{m-\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z) \\ (l+n+1)t_{n+1\underline{m}}^l(W^{-1}) \cdot t_{m+\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z) & (l+n+1)t_{n+1\underline{m}}^l(W^{-1}) \cdot t_{m-\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z) \end{pmatrix} \\ = \sum_{\substack{l,m,n \\ m,n \geq -l \geq 1/2}} \begin{pmatrix} (l-n)t_{n\underline{m}}^l(W^{-1}) \\ (l+n+1)t_{n+1\underline{m}}^l(W^{-1}) \end{pmatrix} \begin{pmatrix} t_{m+\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z) & t_{m-\frac{1}{2}n+\frac{1}{2}}^{l-\frac{1}{2}}(Z) \end{pmatrix}. \end{aligned}$$

Similarly, we can expand the second sum in (35) using the derivative formulas from Lemmas 7, 17 and the multiplication identities from Lemma 18. \square

From Lemma 44 and Proposition 32 we obtain:

Theorem 49 *For each left-regular function $f \in \mathbb{S}(\mathbb{H}_{\mathbb{R}}^+)$ and $R > 0$, the integral*

$$-\frac{1}{2\pi^2} \int_{X \in H_R} \frac{(X-W)^{-1}}{N(X-W)} \cdot Dx \cdot f(X), \quad W \in R \cdot (\Gamma^- \cup \Gamma^+),$$

converges.

Let P_0^- and P_∞^- be the projections onto the holomorphic discrete components quasi-regular at the origin and infinity respectively. Similarly, let P_0^+ and P_∞^+ be the projections onto the antiholomorphic discrete components quasi-regular at the origin and infinity respectively. This integral gives

$$\begin{aligned} & (P_0^-(f) - P_\infty^-(f))(W) && \text{if } W \in R \cdot \Gamma^- \\ \text{and} & (P_0^+(f) - P_\infty^+(f))(W) && \text{if } W \in R \cdot \Gamma^+. \end{aligned}$$

Similar statement holds for right-regular functions $g \in \mathbb{S}'(\mathbb{H}_{\mathbb{R}}^+)$.

3.5 The Discrete Series Projector

In this subsection we obtain projectors onto the discrete series for solutions of $\square_{2,2}\varphi = 0$, left-regular and right-regular functions on $\mathbb{H}_{\mathbb{R}}^+$.

Observe that, for any $\sigma > 0$, the sets

$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot \mathbb{H}_{\mathbb{R}}^+ \quad \text{and} \quad \mathbb{H}_{\mathbb{R}}^+ \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \quad \text{lie inside } \mathbb{H}_{\mathbb{C}}^+.$$

We introduce cycles in $\mathbb{H}_{\mathbb{C}}^+$

$$C_{R,\sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot H_R \quad \text{and} \quad C'_{R,\sigma} = H_R \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}.$$

If $\sigma > 1$ these cycles lie in $R \cdot \Gamma^-$, and if $0 < \sigma < 1$ these cycles lie in $R \cdot \Gamma^+$. In other words, the sets

$$R \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot \Gamma^- \quad \text{and} \quad R \cdot \Gamma^- \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \quad \text{contain } H_R \text{ when } 0 < \sigma < 1,$$

$$R \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot \Gamma^+ \quad \text{and} \quad R \cdot \Gamma^+ \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \quad \text{contain } H_R \text{ when } \sigma > 1.$$

Proposition 50 *For each $R, \sigma > 0$, we have well-defined continuous linear operators $S_{R,\sigma}^{\mp}$ and $S'_{R,\sigma}{}^{\mp} : \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+) \rightarrow \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$*

$$\begin{aligned} (S_{R,\sigma}^{\mp} \varphi)(W) &= -\frac{1}{2\pi^2} \int_{Z \in C_{R,\sigma}} \frac{Z^{-1} \cdot Dz}{N(Z-W)} \cdot (\widetilde{\text{deg}}\varphi)(Z), & W \in R \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot \Gamma^{\mp}, \\ (S'_{R,\sigma}{}^{\mp} \varphi)(W) &= -\frac{1}{2\pi^2} \int_{Z \in C'_{R,\sigma}} \frac{Dz \cdot Z^{-1}}{N(Z-W)} \cdot (\widetilde{\text{deg}}\varphi)(Z), & W \in R \cdot \Gamma^{\mp} \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}. \end{aligned}$$

Operators $S_{R,\sigma}^-$ and $S'_{R,\sigma}{}^-$ annihilate the continuous series, the antiholomorphic discrete series \mathcal{D}_{disc}^+ and send

$$\begin{aligned} t_{n\underline{m}}^l(Z) &\mapsto -t_{n\underline{m}}^l(W) - R^{2(2l+1)} \cdot N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & l = -1, -\frac{3}{2}, -2, \dots, \\ & & m, n \in \mathbb{Z} + l, \\ N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l(Z) &\mapsto R^{-2(2l+1)} \cdot t_{n\underline{m}}^l(W) + N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & m, n \geq -l. \end{aligned}$$

Operators $S_{R,\sigma}^+$ and $S'_{R,\sigma}{}^+$ annihilates the continuous series, the holomorphic discrete series \mathcal{D}_{disc}^- and send

$$\begin{aligned} t_{n\underline{m}}^l(Z) &\mapsto -t_{n\underline{m}}^l(W) - R^{2(2l+1)} \cdot N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & l = -1, -\frac{3}{2}, -2, \dots, \\ & & m, n \in \mathbb{Z} + l, \\ N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l(Z) &\mapsto R^{-2(2l+1)} \cdot t_{n\underline{m}}^l(W) + N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W), & m, n \leq l. \end{aligned}$$

Proof. From Proposition 11 of [FL1] we see that the differential form $Z^{-1} \cdot Dz$ is invariant under the map $Z \mapsto \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} Z$. Similarly, $Dz \cdot Z^{-1}$ is invariant under the map $Z \mapsto Z \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$.

Let $\tilde{Z} = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix} Z$, $\tilde{W} = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix} W$, and define $\tilde{\varphi}$ by $\tilde{\varphi}(\tilde{Z}) = \varphi(Z)$, Using (13) and (22) one can show

$$t_{n\underline{m}}^l(\tilde{Z}) = \sigma^{-2n} \cdot t_{n\underline{m}}^l(Z),$$

which implies that the map $\varphi \mapsto \tilde{\varphi}$ preserves the holomorphic discrete series, antiholomorphic discrete series and continuous components of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^{\pm})$. We have:

$$(S_{R,\sigma}^{\mp} \varphi)(W) = \frac{-1}{2\pi^2} \int_{Z \in C_{R,\sigma}} \frac{Z^{-1} \cdot Dz \cdot (\widetilde{\deg \varphi})(Z)}{N(Z-W)} = \frac{-1}{2\pi^2} \int_{\tilde{Z} \in H_R} \frac{(\widetilde{\deg \tilde{\varphi}})(\tilde{Z})}{N(\tilde{Z}-\tilde{W})} \frac{dS}{\|\tilde{Z}\|} = (S_{\tilde{R}}^{\mp} \tilde{\varphi})(\tilde{W}),$$

and similarly for $S_{R,\sigma}^{\mp}$. Then the result follows from Theorem 47. \square

Now we redo this for left- and right-regular functions. The same argument that was used to prove Proposition 50 also proves:

Proposition 51 *For each left-regular function $f \in \mathbb{S}(\mathbb{H}_{\mathbb{R}}^{\pm})$ we have:*

$$-\frac{1}{2\pi^2} \int_{Z \in C_{R,\sigma}} \frac{(Z-W)^{-1}}{N(Z-W)} \cdot Dz \cdot f(Z) = \begin{cases} (P_0^-(f) - P_{\infty}^-(f))(W) & \text{if } W \in R \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot \Gamma^-; \\ (P_0^+(f) - P_{\infty}^+(f))(W) & \text{if } W \in R \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot \Gamma^+. \end{cases}$$

Similarly, for each right-regular function $g \in \mathbb{S}'(\mathbb{H}_{\mathbb{R}}^{\pm})$ we have:

$$-\frac{1}{2\pi^2} \int_{Z \in C'_{R,\sigma}} g(Z) \cdot Dz \cdot \frac{(Z-W)^{-1}}{N(Z-W)} = \begin{cases} (P_0^-(g) - P_{\infty}^-(g))(W) & \text{if } W \in R \cdot \Gamma^- \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}; \\ (P_0^+(g) - P_{\infty}^+(g))(W) & \text{if } W \in R \cdot \Gamma^+ \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}. \end{cases}$$

3.6 Second Order Pole in $\mathbb{H}_{\mathbb{R}}$

We consider the linear span of the functions $t_{n\ m}^l(Z) \cdot N(Z)^k$ over

$$l = -1, -3/2, -2, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \geq -l, \quad 0 \leq k \leq -2l - 2$$

and over

$$l = -1, -3/2, -2, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \leq l, \quad 0 \leq k \leq -2l - 2.$$

We denote the first span by \mathcal{D}^{--} and the second one by \mathcal{D}^{++} . It is easy to check that

$$\mathcal{D}^{--} \subsetneq z_{11}^{-2} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]^{\leq 0} \subset \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]$$

and

$$\mathcal{D}^{++} \subsetneq z_{22}^{-2} \cdot \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}^{-1}]^{\leq 0} \subset \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}^{-1}].$$

We have the following analogue of Proposition 19 in [FL1]:

Proposition 52 *Let $\mathcal{D}_{\leq 0}^{--}$ (respectively $\mathcal{D}_{\geq 0}^{--}$) denote the span of $t_{n\ m}^l(Z) \cdot N(Z)^k$ with $l = -1, -3/2, \dots, m, n \in \mathbb{Z} + l, m, n \geq -l, 0 \leq k < -l$ (respectively $-l \leq k \leq -2l - 2$). Then $\mathcal{D}^{--} = \mathcal{D}_{\leq 0}^{--} + \mathcal{D}_{\geq 0}^{--}$, $\mathcal{D}_{\geq 0}^{--}$ is the image of $\mathcal{D}_{\leq 0}^{--}$ under the inversion map $F(Z) \mapsto N(Z)^{-2} \cdot F(Z/N(Z))$, and*

$$\mathcal{D}_{\leq 0}^{--} = z_{11}^{-2} \cdot \mathbb{C}[z_{11}^{-1}, z_{12}, z_{21}, z_{22}]^{\leq 0}.$$

Similar statement holds for \mathcal{D}^{++} .

Recall an action ρ_1 of $GL(2, \mathbb{H}_\mathbb{C})$ on the space of functions on $\mathbb{H}_\mathbb{C}$ with isolated singularities defined in [FL1] by

$$\rho_1(h) : F(Z) \mapsto (\rho_1(h)F)(Z) = \frac{F((aZ + b)(cZ + d)^{-1})}{N(cZ + d) \cdot N(a' - Zc')},$$

$$h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \quad h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C}).$$

Differentiating, we obtain an action of $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \simeq \mathfrak{gl}(4, \mathbb{C})$ also denoted by ρ_1 ; this action was described in Lemma 68 in [FL1]. Recall the representations \mathcal{K}^+ and \mathcal{K}^- of $\mathfrak{sl}(2, \mathbb{H}_\mathbb{C})$ introduced in [FL1].

Theorem 53 *We have $\mathcal{D}^{--} \simeq \mathcal{K}^-$ and $\mathcal{D}^{++} \simeq \mathcal{K}^+$ as representations of $\mathfrak{gl}(4, \mathbb{C})$. Moreover, \mathcal{D}^{--} and \mathcal{D}^{++} are irreducible representations of $\mathfrak{sl}(4, \mathbb{C})$ with highest (or lowest) weight vectors $t_{1\bar{1}}^{-1}(Z) = z_{1\bar{1}}^{-2}$ and $t_{-1\bar{-1}}^{-1}(Z) = z_{2\bar{2}}^{-2}$ respectively.*

Proof. The proof proceeds in the same way as that of Theorem 39. First, we check using Lemma 68 in [FL1] and Lemmas 17, 18 that the spaces \mathcal{D}^{--} and \mathcal{D}^{++} are invariant under the ρ_1 action and that $t_{1\bar{1}}^{-1}(Z)$ and $t_{-1\bar{-1}}^{-1}(Z)$ generate \mathcal{D}^{--} and \mathcal{D}^{++} respectively. Recall the element γ_0 from (32). Then

$$\begin{aligned} \rho_l(\gamma_0) : \quad \mathcal{K}^+ \ni 1 &\mapsto z_{2\bar{2}}^{-2} = t_{-1\bar{-1}}^{-1}(Z) \in \mathcal{D}^{++}, \\ \mathcal{K}^- \ni \frac{1}{N(Z)} &\mapsto z_{1\bar{1}}^{-2} = t_{1\bar{1}}^{-1}(Z) \in \mathcal{D}^{--}. \end{aligned}$$

This proves $\mathcal{D}^{\mp\mp} \simeq \mathcal{K}^\mp$, irreducibility and the statement about the highest (or lowest) weight vectors. \square

Next we establish two expansions for $\frac{1}{N(Z-W)^2}$.

Proposition 54 *We have the following matrix coefficient expansions*

$$\frac{1}{N(Z-W)^2} = \sum_{\substack{k, l, m, n \\ m, n \geq -l \geq 1 \\ 0 \leq k \leq -2l-2}} -(2l+1)t_{n\bar{m}}^l(W) \cdot N(W)^k \cdot t_{m\bar{n}}^l(Z^{-1}) \cdot N(Z)^{-k-2}$$

which converges pointwise absolutely whenever $WZ^{-1} \in \Gamma^-$ and

$$\frac{1}{N(Z-W)^2} = \sum_{\substack{k, l, m, n \\ m, n \leq l \leq -1 \\ 0 \leq k \leq -2l-2}} -(2l+1)t_{n\bar{m}}^l(W) \cdot N(W)^k \cdot t_{m\bar{n}}^l(Z^{-1}) \cdot N(Z)^{-k-2}$$

which converges pointwise absolutely whenever $WZ^{-1} \in \Gamma^+$.

Remark 55 *Note that, unlike the expansions of $\frac{1}{N(Z-W)}$ given in Proposition 45, the matrix coefficients of the limits of the discrete series do not enter the expansions of $\frac{1}{N(Z-W)^2}$.*

Proof. The proof is similar to that of Proposition 45. Thus we can assume $WZ^{-1} \in \Gamma^- \cap K_\mathbb{C}$, and so $WZ^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_1| > 1 > |\lambda_2| > 0$. Using (33) and (34) and letting indices k, l, m, n run over

$$l \leq -1, \quad m, n \geq -l, \quad 0 \leq k \leq -2l-2,$$

we obtain:

$$\begin{aligned}
& \sum_{k,l,m,n} (2l+1)t_{nm}^l(W) \cdot N(W)^k \cdot t_{mn}^l(Z^{-1}) \cdot N(Z)^{-k-2} \\
&= \frac{1}{N(Z)^2} \sum_{k,l,n} (2l+1)t_{nn}^l(WZ^{-1}) \cdot N(WZ^{-1})^k = \frac{1}{N(Z)^2} \sum_{k,l} (2l+1)\tilde{\Theta}_l^-(WZ^{-1}) \cdot N(WZ^{-1})^k \\
&= \frac{1}{N(Z)^2} \sum_{k,l} (2l+1) \frac{\lambda_1^{2l+1}}{\lambda_1 - \lambda_2} \cdot (\lambda_1 \lambda_2)^k = \frac{1}{N(Z)^2} \sum_{l \leq -1} (2l+1) \frac{\lambda_1^{2l+1} - \lambda_2^{-2l-1}}{(\lambda_1 - \lambda_2)(1 - \lambda_1 \lambda_2)} \\
&= -\frac{1}{N(Z)^2} \frac{1}{(1 - \lambda_1)^2 (1 - \lambda_2)^2} = -\frac{1}{N(Z)^2 \cdot N(1 - WZ^{-1})^2} = -\frac{1}{N(Z - W)^2}.
\end{aligned}$$

The other matrix coefficient expansion is proved in the same way. \square

Let $\tilde{e}_3 = ie_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}$. We realize the group $U(2, 2)$ as the subgroup of elements of $GL(2, \mathbb{H}_{\mathbb{C}})$ preserving the Hermitian form on \mathbb{C}^4 given by the 4×4 matrix $\begin{pmatrix} \tilde{e}_3 & 0 \\ 0 & -\tilde{e}_3 \end{pmatrix}$. Explicitly,

$$\begin{aligned}
U(2, 2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{H}_{\mathbb{C}}, \begin{array}{l} a^* \tilde{e}_3 a = \tilde{e}_3 + c^* \tilde{e}_3 c \\ d^* \tilde{e}_3 d = \tilde{e}_3 + b^* \tilde{e}_3 b \\ a^* \tilde{e}_3 b = c^* \tilde{e}_3 d \end{array} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{H}_{\mathbb{C}}, \begin{array}{l} a^* \tilde{e}_3 a = \tilde{e}_3 + b^* \tilde{e}_3 b \\ d^* \tilde{e}_3 d = \tilde{e}_3 + c^* \tilde{e}_3 c \\ a \tilde{e}_3 c^* = b \tilde{e}_3 d^* \end{array} \right\}.
\end{aligned}$$

The Lie algebra of $U(2, 2)$ is

$$\mathfrak{u}(2, 2) = \left\{ \begin{pmatrix} A & B \\ \tilde{e}_3 B^* \tilde{e}_3 & D \end{pmatrix}; A, B, D \in \mathbb{H}_{\mathbb{C}}, \tilde{e}_3 A = -(\tilde{e}_3 A)^*, \tilde{e}_3 D = -(\tilde{e}_3 D)^* \right\}.$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2, 2)$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \tilde{e}_3 \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix} \tilde{e}_3$. The group $U(2, 2)$ acts on $\mathbb{H}_{\mathbb{C}}$ by conformal transformations preserving $U(1, 1)$, where we identify $U(1, 1)$ with a subset of $\mathbb{H}_{\mathbb{C}}$:

$$U(1, 1) = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^* \tilde{e}_3 Z = \tilde{e}_3\}. \quad (36)$$

We orient $U(1, 1)$ so that $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, e_3\}$ is a positive basis of the tangent space at $1 \in U(1, 1)$.

We have a symmetric bilinear pairing on $\mathcal{D}^{--} \oplus \mathcal{D}^{++}$:

$$\langle F_1, F_2 \rangle_1 = \frac{i}{2\pi^3} \int_{U(1,1)} F_1(Z) \cdot F_2(Z) dV$$

(recall that $dV = dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3$ is a holomorphic 4-form on $\mathbb{H}_{\mathbb{C}}$). This pairing is $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant which follows immediately from Lemma 61 in [FL1] and $dZ^4 = dz_{11} \wedge dz_{12} \wedge dz_{21} \wedge dz_{22} = 4dV$. Related to this bilinear pairing we have a $\mathfrak{u}(2, 2)$ -invariant inner product that is non-degenerate on \mathcal{D}^{--} and \mathcal{D}^{++} :

$$(F_1, F_2)_1 = \frac{i}{2\pi^3} \int_{U(1,1)} F_1(Z) \cdot \overline{F_2(Z)} \frac{dV}{N(Z)^2}.$$

(Checking the $\mathfrak{u}(2, 2)$ -invariance requires some verification.)

Every element in $U(1, 1)$ can be uniquely written as $e^{i\theta} \cdot Z$ with $Z \in SU(1, 1)$ and $\theta \in [0, \pi)$. Thus we can identify $U(1, 1)$ with $[0, \pi) \times SU(1, 1)$ and $dV/N(Z)^2$ restricted to $U(1, 1)$ becomes $id\theta \wedge Dz \cdot Z^{-1}$ (since both sides express a $U(1, 1)$ -invariant volume form and coincide at 1). From Proposition 31 we immediately obtain

Proposition 56 *We have the following orthogonality relations:*

$$\langle t_{n' \underline{m}'}^l(Z) \cdot N(Z)^{k'}, t_{m \underline{n}}^l(Z^{-1}) \cdot N(Z)^{-k-2} \rangle_1 = -\frac{1}{2l+1} \delta_{kk'} \delta_{ll'} \delta_{mm'} \delta_{nn'},$$

where the indices k, l, m, n are $l = -1, -\frac{3}{2}, -2, \dots$, $m, n \in \mathbb{Z} + l$, $m, n \geq -l$ or $m, n \leq l$, $k \in \mathbb{Z}$ and similarly for k', l', m', n' .

From these orthogonality relations and expansions for $\frac{1}{N(Z-W)^2}$ we obtain a formula similar to Proposition 73 in [FL1]:

Theorem 57 *Let P^{--} and P^{++} denote the projections of $\mathcal{D}^{--} \oplus \mathcal{D}^{++}$ onto \mathcal{D}^{--} and \mathcal{D}^{++} respectively. For each function $F \in \mathcal{D}^{--} \oplus \mathcal{D}^{++}$ and $R > 0$,*

$$\frac{i}{2\pi^3} \int_{Z \in R \cdot U(1,1)} \frac{F(Z)}{N(Z-W)^2} dV = \begin{cases} (P^{--} F)(W) & \text{if } W \in R \cdot \Gamma^-; \\ (P^{++} F)(W) & \text{if } W \in R \cdot \Gamma^+. \end{cases}$$

In particular, the integral converges absolutely for $W \in R \cdot (\Gamma^- \cup \Gamma^+)$.

4 Separation of the Series for $SL(2, \mathbb{C})/SU(1, 1)$

4.1 The Cayley Transform between $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M}

Recall that $\tilde{e}_3 = ie_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}$ and that $U(1, 1)$ is identified with a subset of $\mathbb{H}_{\mathbb{C}}$ via (36).

Proposition 58 *Consider an element $\gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{e}_3 & i\tilde{e}_3 \\ 1 & -i \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ with $\gamma^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{e}_3 & 1 \\ -i\tilde{e}_3 & i \end{pmatrix}$.*

The fractional linear map on $\mathbb{H}_{\mathbb{C}}$

$$\pi_l(\gamma) : Z \mapsto i(\tilde{e}_3 Z + 1)(\tilde{e}_3 Z - 1)^{-1}$$

maps $\mathbb{M} \rightarrow U(1, 1) \subset \mathbb{H}_{\mathbb{C}}$ (with singularities) and sends the unit one-sheeted hyperboloid $\tilde{H} = \{Y \in \mathbb{M}; N(Y) = 1\}$ into $SU(1, 1) = \{X \in \mathbb{H}_{\mathbb{R}}; N(X) = 1\}$. The singularities of $\pi_l(\gamma)$ on \mathbb{M} lie along the one-sheeted hyperboloid $\{Y \in \mathbb{M}; N(Y) = 1, \text{Re}(e_3 Y) = 0\}$.

Conversely, the fractional linear map on $\mathbb{H}_{\mathbb{C}}$

$$\pi_l(\gamma^{-1}) : Z \mapsto \tilde{e}_3(Z + i)(Z - i)^{-1}$$

maps $U(1, 1) \rightarrow \mathbb{M}$ (with singularities) and sends $SU(1, 1)$ into the hyperboloid \tilde{H} . The singularities of $\pi_l(\gamma^{-1})$ on $\mathbb{H}_{\mathbb{R}}$ lie along the two-sheeted hyperboloid $\{X \in SU(1, 1); \text{Re } X = 0\}$.

The map $\pi_l(\gamma)$ and its inverse were studied in [KouØ], we think of these maps as quaternionic analogues of Cayley transform. For future reference we spell out that if $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in \mathbb{M}$

and $X = i(\tilde{e}_3 Y + 1)(\tilde{e}_3 Y - 1)^{-1}$, then $N(\tilde{e}_3 Y - 1) = 1 - N(Y) + y_{22} - y_{11}$ and

$$\begin{aligned} X &= \frac{-i}{N(\tilde{e}_3 Y - 1)} \begin{pmatrix} 1 + N(Y) + y_{11} + y_{22} & 2y_{12} \\ -2y_{21} & 1 + N(Y) - y_{11} - y_{22} \end{pmatrix} \\ &= i - \frac{2i}{N(\tilde{e}_3 Y - 1)} \begin{pmatrix} 1 + y_{22} & y_{12} \\ -y_{21} & 1 - y_{11} \end{pmatrix}. \end{aligned} \quad (37)$$

We orient the hyperboloid $\tilde{H} \subset \mathbb{M}$ as the boundary of the open set $\{Y \in \mathbb{M}; N(Y) < 1\}$ and denote by $\text{sign}_3(Y)$ the sign of the y^3 -coordinate of $Y = y^0 \tilde{e}_0 + y^1 e_1 + y^2 e_2 + y^3 e_3 \in \mathbb{M}$.

Lemma 59 *The restriction of $\pi_l(\gamma)$ to $\tilde{H} \rightarrow SU(1, 1)$ preserves the orientations for $\{Y \in \tilde{H}; \text{sign}_3(Y) > 0\}$ and reverses the orientations for $\{Y \in \tilde{H}; \text{sign}_3(Y) < 0\}$.*

Conversely, the restriction of $\pi_l(\gamma^{-1})$ to $SU(1, 1) \rightarrow \tilde{H}$ preserves the orientations for $\{X \in SU(1, 1); \text{Re } X < 0\}$ and reverses the orientations for $\{X \in SU(1, 1); \text{Re } X > 0\}$.

Theorem 60 *For any $\varphi_1(X), \varphi_2(X) \in \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ that extend holomorphically to an open neighborhood of $SU(1, 1)$ in $\mathbb{H}_{\mathbb{C}}$ we have:*

$$\begin{aligned} \langle \varphi_1, \varphi_2 \rangle &= -\frac{1}{2\pi^2} \int_{X \in SU(1, 1)} (\widetilde{\text{deg}}_X \varphi_1)(X) \cdot \varphi_2(X) \frac{dS}{\|X\|} \\ &= \frac{i}{2\pi^2} \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot (\widetilde{\text{deg}}_Y (\pi_l^0(\gamma) \varphi_1))(Y) \cdot (\pi_l^0(\gamma) \varphi_2)(Y) \frac{dS}{\|Y\|}, \end{aligned}$$

where

$$(\pi_l^0(\gamma) \varphi_j)(Y) = \frac{-2}{N(\tilde{e}_3 Y - 1)} \cdot \varphi_j(i(\tilde{e}_3 Y + 1)(\tilde{e}_3 Y - 1)^{-1}), \quad j = 1, 2.$$

Proof. The main ingredient of the proof is the following lemma which is obtained by direct computation.

Lemma 61 *For $Y \in \mathbb{H}_{\mathbb{C}}$ and $X = i(\tilde{e}_3 Y + 1)(\tilde{e}_3 Y - 1)^{-1} \in SU(1, 1)$ we have:*

$$\begin{aligned} \frac{N(Y) + 1}{N(\tilde{e}_3 Y - 1)^2} \cdot (\widetilde{\text{deg}}_X \varphi)(X) + i \frac{1 - N(Y)}{N(\tilde{e}_3 Y - 1)^2} (\partial_{11} \varphi + \partial_{22} \varphi)(X) \\ = \widetilde{\text{deg}}_Y \left(\frac{1}{N(\tilde{e}_3 Y - 1)} \cdot \varphi(i(\tilde{e}_3 Y + 1)(\tilde{e}_3 Y - 1)^{-1}) \right). \end{aligned}$$

Writing $X = i(\tilde{e}_3 Y + 1)(\tilde{e}_3 Y - 1)^{-1}$, $dS/\|X\| = iDy/Y$ on \tilde{H} and using Lemma 10 together with Proposition 11 from [FL1], we can rewrite

$$\begin{aligned} \int_{X \in SU(1, 1)} (\widetilde{\text{deg}}_X \varphi_1)(X) \cdot \varphi_2(X) \frac{dS}{\|X\|} &= \int_{X \in SU(1, 1)} (\widetilde{\text{deg}}_X \varphi_1)(X) \cdot \frac{Dx}{X} \cdot \varphi_2(X) \\ &= -8 \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot (\widetilde{\text{deg}}_X \varphi_1)(X) \cdot \frac{(\tilde{e}_3 - Y)^{-1}}{N(\tilde{e}_3 - Y)} \cdot Dy \cdot \frac{(\tilde{e}_3 Y - 1)^{-1}}{N(\tilde{e}_3 Y - 1)} \cdot \frac{\tilde{e}_3 Y - 1}{\tilde{e}_3 Y + 1} \cdot \varphi_2(X) \\ &= -8i \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot (\widetilde{\text{deg}}_X \varphi_1)(X) \cdot \frac{\varphi_2(X)}{N(\tilde{e}_3 Y - 1)^3} \frac{dS}{\|Y\|} \\ &= -4i \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot \widetilde{\text{deg}}_Y \left(\frac{1}{N(\tilde{e}_3 Y - 1)} \cdot \varphi(i(\tilde{e}_3 Y + 1)(\tilde{e}_3 Y - 1)^{-1}) \right) \cdot \frac{\varphi_2(X)}{N(\tilde{e}_3 Y - 1)} \frac{dS}{\|Y\|} \\ &= -i \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot (\widetilde{\text{deg}}_Y (\pi_l^0(\gamma) \varphi_1))(Y) \cdot (\pi_l^0(\gamma) \varphi_2)(Y) \frac{dS}{\|Y\|}. \end{aligned}$$

□

4.2 The Continuous Series Projector on \mathbb{M}

Recall that \mathcal{D}^- and \mathcal{D}^+ are irreducible representations of $\mathfrak{sl}(4, \mathbb{C})$ with highest (or lowest) weight vectors $t_{\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}}(Z) = \frac{1}{z_{11}}$ and $t_{-\frac{1}{2}\frac{1}{2}}^{-\frac{1}{2}}(Z) = \frac{1}{z_{22}}$ respectively (Theorem 39). From (37) we can see that

$$\left(\pi_l^0(\gamma)\left(\frac{1}{z_{11}}\right)\right)(Y) = \frac{-2i}{N(Y+1)} \quad \text{and} \quad \left(\pi_l^0(\gamma)\left(\frac{1}{z_{22}}\right)\right)(Y) = \frac{-2i}{N(Y-1)}.$$

These functions have singularities along the set $\{Y \in \mathbb{M}; N(Y) = -1, \operatorname{Re} Y = 0\}$ which has codimension 2. It is easy to see that they define a tempered distribution on \mathbb{M} , and so their Fourier transforms are L^2 -functions on the light cone in the space dual to \mathbb{M} . The space of functions on \mathbb{M} that arise as Fourier transforms of the L^2 -functions on the light cone is known as the continuous series component of \mathbb{M} . This proves that the Cayley transform switches the discrete series component on $\mathbb{H}_{\mathbb{R}}$ and the continuous series component on \mathbb{M} .

To describe the images of Γ^- and Γ^+ under the Cayley transform $\pi_l(\gamma^{-1})$ we recall the generalized upper and lower half-planes introduced in Section 3.5 in [FL1]:

$$\begin{aligned} \mathbb{T}^- &= \{Z = W_1 + iW_2 \in \mathbb{H}_{\mathbb{C}}; W_1, W_2 \in \mathbb{M}, iW_2 \text{ is positive definite}\}, \\ \mathbb{T}^+ &= \{Z = W_1 + iW_2 \in \mathbb{H}_{\mathbb{C}}; W_1, W_2 \in \mathbb{M}, iW_2 \text{ is negative definite}\}. \end{aligned}$$

In the context of our paper, Lemma 1.1 in [KouØ] can be restated as follows.

Lemma 62 *The Cayley transform $\pi_l(\gamma^{-1})$ sends Γ^- and Γ^+ biholomorphically into respectively \mathbb{T}_0^- and \mathbb{T}_0^+ , where*

$$\mathbb{T}_0^- = \{Z \in \mathbb{T}^-; N(Z - \tilde{e}_3) \neq 0\} \quad \text{and} \quad \mathbb{T}_0^+ = \{Z \in \mathbb{T}^+; N(Z - \tilde{e}_3) \neq 0\}.$$

Recall that \tilde{H} be the unit hyperboloid of one sheet in \mathbb{M} , and let $\mathcal{H}(\tilde{H})$ denote the space of holomorphic functions φ on defined on some connected open neighborhood U_φ of \tilde{H} in $\mathbb{H}_{\mathbb{C}}$ which can be written as $\varphi(Z) = (\pi_l^0(\gamma)\tilde{\varphi})(Z)$ for some $\tilde{\varphi} \in \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ and $Z \in U_\varphi \cap U(1, 1)$. Clearly, functions in $\mathcal{H}(\tilde{H})$ are harmonic. We define operators \tilde{S}^\mp on $\mathcal{H}(\tilde{H})$ by

$$\begin{aligned} (\tilde{S}^- \varphi)(Z) &= \frac{i}{2\pi^2} \int_{Y \in \tilde{H}} \operatorname{sign}_3(Y) \cdot \frac{(\widetilde{\deg \varphi})(Y)}{N(Y-Z)} \frac{dS}{\|Y\|}, \quad Z \in \mathbb{T}_0^-, \\ (\tilde{S}^+ \varphi)(Z) &= \frac{i}{2\pi^2} \int_{Y \in \tilde{H}} \operatorname{sign}_3(Y) \cdot \frac{(\widetilde{\deg \varphi})(Y)}{N(Y-Z)} \frac{dS}{\|Y\|}, \quad Z \in \mathbb{T}_0^+. \end{aligned}$$

Theorem 63 *Let $\varphi \in \mathcal{H}(\tilde{H})$, then $\tilde{S}^\mp \varphi$ are well-defined functions on \mathbb{T}_0^- and \mathbb{T}_0^+ respectively. The operator \tilde{S}^- annihilates the discrete series on \mathbb{M} , the image of the antiholomorphic discrete series \mathcal{D}_{discr}^+ and sends*

$$\begin{aligned} (\pi_l^0(\gamma)t_{n\ m}^l)(Y) &\mapsto -(\pi_l^0(\gamma)t_{n\ m}^l)(Z) - (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\ m}^l))(Z), \\ (\pi_l^0(\gamma)(N(Y)^{-2l-1} \cdot t_{n\ m}^l))(Y) &\mapsto (\pi_l^0(\gamma)t_{n\ m}^l)(Z) + (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\ m}^l))(Z), \\ l &= -1, -3/2, -2, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \geq -l. \end{aligned}$$

The operator \tilde{S}^+ annihilates the discrete series on \mathbb{M} , the image of the holomorphic discrete series \mathcal{D}_{discr}^- and sends

$$\begin{aligned} (\pi_l^0(\gamma)t_{n\ m}^l)(Y) &\mapsto -(\pi_l^0(\gamma)t_{n\ m}^l)(Z) - (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\ m}^l))(Z), \\ (\pi_l^0(\gamma)(N(Y)^{-2l-1} \cdot t_{n\ m}^l))(Y) &\mapsto (\pi_l^0(\gamma)t_{n\ m}^l)(Z) + (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\ m}^l))(Z), \\ l &= -1, -3/2, -2, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \leq -l. \end{aligned}$$

Proof. Let $X = i(\tilde{e}_3 Y + 1)(\tilde{e}_3 Y - 1)^{-1}$, $Z' = i(\tilde{e}_3 Z + 1)(\tilde{e}_3 Z - 1)^{-1}$, and write $\varphi(Y) = (\pi_l^0(\gamma)\tilde{\varphi})(Y)$ for some analytic $\tilde{\varphi} \in \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. By Lemma 10 from [FL1],

$$\left(\pi_l^0(\gamma)_X \frac{1}{N(X - Z')}\right)(Y) = -\frac{1}{2}N(\tilde{e}_3 Z - 1) \cdot \frac{1}{N(Y - Z)}.$$

By Theorem 60 we have

$$\begin{aligned} (\tilde{S}^{\mp} \varphi)(Z) &= \frac{i}{2\pi^2} \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot \frac{(\widetilde{\text{deg}}\varphi)(Y)}{N(Y - Z)} \frac{dS}{\|Y\|} \\ &= \frac{1}{\pi^2 N(\tilde{e}_3 Z - 1)} \int_{X \in SU(1,1)} \frac{(\widetilde{\text{deg}}\tilde{\varphi})(X)}{N(X - Z')} \frac{dS}{\|X\|} = \frac{-2}{N(\tilde{e}_3 Z - 1)} \cdot (S_1^{\mp} \tilde{\varphi})(Z'). \end{aligned}$$

Then the result follows from Theorem 47 and Lemma 62. \square

Note that, for $Z \in \mathbb{M}$ and $s \in \mathbb{R}$, we have $Z + se_0 \in \mathbb{T}^-$ if $s > 0$ and $Z + se_0 \in \mathbb{T}^+$ if $s < 0$. Moreover, $Z + se_0 \in \mathbb{T}_0^- \sqcup \mathbb{T}_0^+$ if $|s|$ is sufficiently small. Taking limits $s \rightarrow 0^{\pm}$ we obtain:

Corollary 64 *Let $Z \in \mathbb{M}$ and let Y range over \tilde{H} , write $Y = t\tilde{e}_0 + y^1 e_1 + y^2 e_2 + y^3 e_3$, $Z = \tilde{t}\tilde{e}_0 + z^1 e_1 + z^2 e_2 + z^3 e_3$, Then the operator on $\mathcal{H}(\tilde{H})$*

$$\varphi(Y) \mapsto \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi^2} \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot \frac{(\widetilde{\text{deg}}\varphi)(Y)}{N(Y - Z) + i\varepsilon \text{sign}(t - \tilde{t})} \frac{dS}{\|Y\|}$$

annihilates the discrete series on \mathbb{M} , the image of the antiholomorphic discrete series \mathcal{D}_{disc}^+ and sends

$$\begin{aligned} (\pi_l^0(\gamma)t_{n\underline{m}}^l)(Y) &\mapsto -(\pi_l^0(\gamma)t_{n\underline{m}}^l)(Z) - (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l))(Z), \\ (\pi_l^0(\gamma)(N(Y)^{-2l-1} \cdot t_{n\underline{m}}^l))(Y) &\mapsto (\pi_l^0(\gamma)t_{n\underline{m}}^l)(Z) + (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l))(Z), \\ l &= -1, -3/2, -2, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \geq -l. \end{aligned}$$

Similarly, the operator on $\mathcal{H}(\tilde{H})$

$$\varphi(Y) \mapsto \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi^2} \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot \frac{(\widetilde{\text{deg}}\varphi)(Y)}{N(Y - Z) - i\varepsilon \text{sign}(t - \tilde{t})} \frac{dS}{\|Y\|}$$

annihilates the discrete series on \mathbb{M} , the image of the holomorphic discrete series \mathcal{D}_{disc}^- and sends

$$\begin{aligned} (\pi_l^0(\gamma)t_{n\underline{m}}^l)(Y) &\mapsto -(\pi_l^0(\gamma)t_{n\underline{m}}^l)(Z) - (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l))(Z), \\ (\pi_l^0(\gamma)(N(Y)^{-2l-1} \cdot t_{n\underline{m}}^l))(Y) &\mapsto (\pi_l^0(\gamma)t_{n\underline{m}}^l)(Z) + (\pi_l^0(\gamma)(N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l))(Z), \\ l &= -1, -3/2, -2, \dots, \quad m, n \in \mathbb{Z} + l, \quad m, n \leq -l. \end{aligned}$$

Next we re-do this for left-regular functions. We denote by $\mathbb{S}(\tilde{H})$ the space of holomorphic left-regular \mathbb{S} -valued functions f defined on some connected open neighborhood U_f of \tilde{H} in $\mathbb{H}_{\mathbb{C}}$ and can be written as $f(Z) = (\pi_l(\gamma)\tilde{f})(Z)$ for some $\tilde{f} \in \mathbb{S}(\mathbb{H}_{\mathbb{R}}^+)$ and $Z \in U_f \cap U(1, 1)$. Similarly we can define the space of right-regular functions $\mathbb{S}'(\tilde{H})$.

Theorem 65 For each left-regular function $f \in \mathbb{S}(\tilde{H})$ we have

$$\frac{1}{2\pi^2} \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot \frac{(Y-Z)^{-1}}{N(Y-Z)} \cdot Dy \cdot f(Y) = \begin{cases} (\tilde{P}_\infty^-(f) - \tilde{P}_0^-(f))(Z) & \text{if } Z \in \mathbb{T}_0^-; \\ (\tilde{P}_\infty^+(f) - \tilde{P}_0^+(f))(Z) & \text{if } Z \in \mathbb{T}_0^+, \end{cases}$$

where $\tilde{P}_0^-, \tilde{P}_\infty^-$ denote the projections onto the $\pi_l(\gamma)$ -images of the holomorphic discrete components of $\mathbb{S}(\mathbb{H}_\mathbb{R}^+)$ quasi-regular at the origin and infinity respectively, and $\tilde{P}_0^+, \tilde{P}_\infty^+$ are the projections onto the $\pi_l(\gamma)$ -images of the antiholomorphic discrete components of $\mathbb{S}(\mathbb{H}_\mathbb{R}^+)$ quasi-regular at the origin and infinity respectively.

Proof. Changing the variables $Y = \tilde{e}_3(X+i)(X-i)^{-1}$, $Z = \tilde{e}_3(Z'+i)(Z'-i)^{-1}$, and using Lemma 10 with Proposition 11 from [FL1],

$$\begin{aligned} & \int_{Y \in \tilde{H}} \text{sign}_3(Y) \cdot \frac{(Y-Z)^{-1}}{N(Y-Z)} \cdot Dy \cdot f(Y) \\ &= \frac{(Z'-i)}{N(Z'-i)^{-1}} \cdot \int_{X \in SU(1,1)} \frac{(X-Z')^{-1}}{N(X-Z')} \cdot Dx \cdot \frac{(X-i)^{-1}}{N(X-i)} \cdot f(\tilde{e}_3(X+i)(X-i)^{-1}) \\ &= -2\pi^2 (\tilde{P}_0^\mp(f) - \tilde{P}_\infty^\mp(f)) (\tilde{e}_3(Z'+i)(Z'-i)^{-1}) = -2\pi^2 (\tilde{P}_0^\mp(f) - \tilde{P}_\infty^\mp(f))(Z) \end{aligned}$$

by Theorem 49. \square

4.3 Basis of Harmonic Functions on \mathbb{M}

Writing

$$\mathbb{M} = \left\{ Y = y^0 \tilde{e}_0 + y^1 e_1 + y^2 e_2 + y^3 e_3 = \begin{pmatrix} -iy^0 - iy^3 & -iy^1 - y^2 \\ -iy^1 + y^2 & -iy^0 + iy^3 \end{pmatrix}; y^0, y^1, y^2, y^3 \in \mathbb{R} \right\},$$

we can embed \mathbb{R}^3 into \mathbb{M} so that $(x^1, x^2, x^3) \leftrightarrow x^1 e_1 + x^2 e_2 + x^3 e_3$. Note that, for $Y = t\tilde{e}_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 \in \mathbb{M}$, $N(Y) = r^2 - t^2$, where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. We parameterize the unit hyperboloid \tilde{H} as

$$\begin{aligned} x^1 &= \cosh \rho \sin \theta \cos \varphi & -\infty < \rho < \infty \\ x^2 &= \cosh \rho \sin \theta \sin \varphi & 0 \leq \varphi < 2\pi \\ x^3 &= \cosh \rho \cos \theta & 0 \leq \theta < \pi \\ t &= \sinh \rho \end{aligned} \tag{38}$$

Lemma 66 We have the following $SO(3,1)$ -invariant measure on the hyperboloid \tilde{H} :

$$-\frac{dx^1 \wedge dx^2 \wedge dx^3}{t} = \cosh^2 \rho \sin \theta d\rho \wedge d\varphi \wedge d\theta = iDy \cdot Y^{-1} = \frac{dS}{\|Y\|}$$

(\tilde{H} is oriented as the boundary of $\{Y \in \mathbb{M}; N(Y) < 1\}$).

The functions (64) can be regarded as functions on $\mathbb{M}^+ = \{Y \in \mathbb{M}; N(Y) > 0\}$, and they satisfy $\square_{3,1}\varphi = 0$. From (60) and (62) by direct computation we obtain:

Lemma 67 We have the following two families of solutions of $\square_{3,1}\varphi = 0$ on \mathbb{M}^+ :

$$(r^2 - t^2)^l \cdot r^{-l-1} \cdot Y_l^m(\theta, \varphi), \quad l = 0, 1, 2, \dots, \quad -l \leq m \leq l,$$

of homogeneity degree $l - 1$ and

$$(r^2 - t^2)^{-l-1} \cdot r^l \cdot Y_l^m(\theta, \varphi), \quad l = 0, 1, 2, \dots, \quad -l \leq m \leq l,$$

of homogeneity degree $-l - 2$.

To construct a basis of solutions of $\square_{3,1}\varphi = 0$ we differentiate these two families with respect to t :

$$\frac{\partial^k}{\partial t^k}(r^2 - t^2)^l \cdot r^{-l-1} \cdot Y_l^m(\theta, \varphi) \quad \text{and} \quad \frac{\partial^k}{\partial t^k}(r^2 - t^2)^{-l-1} \cdot r^l \cdot Y_l^m(\theta, \varphi), \quad (39)$$

where $l = 0, 1, 2, \dots, -l \leq m \leq l, 0 \leq k \leq 2l$. We will see soon that these are two orthogonal families of functions on \mathbb{M}^+ . Let $T = t/r$, then we can rewrite the first family in (39) as

$$\frac{d^k}{dT^k}(1 - T^2)^l \Big|_{T=\frac{t}{r}} \cdot r^{l-k-1} \cdot Y_l^m(\theta, \varphi).$$

Then we rewrite this expression using the associated Legendre functions:

$$(-2)^l \frac{k! \cdot l!}{(2l - k)!} \cdot r^{-1} \cdot (r^2 - t^2)^{(l-k)/2} \cdot P_l^{(l-k)}(t/r) \cdot Y_l^m(\theta, \varphi).$$

Same can be done with the second family in (39).

Theorem 68 *We have two families of solutions of $\square_{3,1}\varphi = 0$ on \mathbb{M}^+ :*

$$f_{l,m,n}^+(Y) = \sqrt{\frac{(2l+1) \cdot (l-n)!}{(l+n)!}} \cdot r^{-1} \cdot (r^2 - t^2)^{n/2} \cdot P_l^{(n)}(t/r) \cdot Y_l^m(\theta, \varphi) \quad (40)$$

and

$$f_{l,m,n}^-(Y) = \sqrt{\frac{(2l+1) \cdot (l-n)!}{(l+n)!}} \cdot r^{-1} \cdot (r^2 - t^2)^{-n/2} \cdot P_l^{(n)}(t/r) \cdot Y_l^m(\theta, \varphi), \quad (41)$$

where $l = 0, 1, 2, \dots, -l \leq m \leq l, 0 \leq n \leq l$.

These families of solutions satisfy the following orthogonality relations:

$$\langle f_{l,m,n}^+(Y), f_{l',-m',n'}^\pm(Y) \rangle = -\langle f_{l,m,n}^-(Y), f_{l',-m',n'}^\pm(Y) \rangle = (-1)^m \frac{2}{\pi i} \delta_{l,l'} \cdot \delta_{m,m'} \cdot \delta_{n,n'} \cdot (1 - \delta_{n,0})$$

with respect to the bilinear pairing

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{2\pi^2 i} \int_{Y \in \tilde{H}} (\widetilde{\deg \varphi_1})(Y) \cdot \varphi_2(Y) \frac{dS}{\|Y\|}. \quad (42)$$

Also, these two families of solutions are related by the involutions $\varphi(Y) \mapsto N(Y)^{-1} \cdot \varphi(Y/N(Y))$ and $\varphi(Y) \mapsto \frac{(-1)^l}{N(Y)} \cdot \varphi(Y^{-1})$.

Proof. The functions (40) are homogeneous of degree $n - 1$ and the functions (41) are homogeneous of degree $-(n + 1)$; if $n = 0$, then $\widetilde{\deg} f_{l,m,n}^\pm(Y) \equiv 0$. To prove the orthogonality relations for $f_{l,m,n}^\pm(Y)$'s, we integrate in coordinates (38) using orthogonality identities (63), (69) and reduce to the following calculation:

$$\begin{aligned} & \int_{-\infty}^{\infty} (\cosh \rho)^{-1} \cdot P_l^{(n)}(\tanh \rho) \cdot (\cosh \rho)^{-1} \cdot P_{l'}^{(n')}(\tanh \rho) \cdot \cosh^2 \rho \, d\rho \\ &= \int_{-\infty}^{\infty} P_l^{(n)}(\tanh \rho) \cdot P_{l'}^{(n')}(\tanh \rho) \, d\rho = \int_{-1}^1 P_l^{(n)}(x) \cdot P_{l'}^{(n')}(x) \frac{dx}{1-x^2} = \frac{(l+n)!}{n(l-n)!} \cdot \delta_{n,n'} \end{aligned} \quad (43)$$

(unless $n = n' = 0$).

The last statement follows immediately from (72) and (77). \square

Remark 69 *When $n = 0$ (40) and (41) yield the same functions. We call these functions $f_{l,m,0}^\pm(Y)$ the limits of the discrete series in \mathbb{M} . Note that $P_l^{(n)}$ and $P_l^{(-n)}$ are proportional according to (66), and so only $n \geq 0$ appear.*

4.4 Space $\mathcal{D}_{\mathbb{M}}^0$ and $\mathfrak{sl}(4, \mathbb{C})$ -action

Let $\mathcal{D}_{\mathbb{M}}^0$ denote the linear span of the functions (40) and (41). In this subsection we characterize the space $\mathcal{D}_{\mathbb{M}}^0$ as a subspace of harmonic polynomials on \mathbb{M} , show that it is preserved by the $\mathfrak{sl}(4, \mathbb{C})$ -action and identify this representation with the minimal representation of $O(3, 3) \simeq SL(2, \mathbb{H}_{\mathbb{R}})/\{\pm 1\}$.

First we identify the linear span of the basis (40). Then the linear span of the basis (41) is obtained by applying an involution $\varphi(Y) \mapsto N(Y)^{-1} \cdot \varphi(Y/N(Y))$ or $\varphi(Y) \mapsto \frac{(-1)^l}{N(Y)} \cdot \varphi(Y^{-1})$. Recall that $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ and consider algebraic functions $\varphi \in \mathbb{C}[t, x^1, x^2, x^3, r^{-1}]$ satisfying $\square_{3,1}\varphi = 0$. We denote by $\mathbb{C}[t, x^1, x^2, x^3, r^{-2}]^{\leq 0}$ the subalgebra of polynomials in $\mathbb{C}[t, x^1, x^2, x^3, r^{-2}]$ spanned by monomials of non-positive degree of homogeneity.

Proposition 70 *The linear span of the basis (40) can be characterized as*

$$\{\varphi \in r^{-1} \cdot \mathbb{C}[t, x^1, x^2, x^3, r^{-2}]^{\leq 0}; \square_{3,1}\varphi = 0\}. \quad (44)$$

Proof. From the definition of the associate Legendre functions (65), (39) and Lemma 99 we see that the basis functions (40) indeed lie in $r^{-1} \cdot \mathbb{C}[t, x^1, x^2, x^3, r^{-2}]^{\leq 0}$. Also, if we fix l and m , there are exactly $l + 1$ such basis functions which are distinguished by their homogeneity degrees d , $-l - 1 \leq d \leq -1$.

Since the polynomials $r^l \cdot Y_l^m(\theta, \varphi)$'s form a basis of polynomial functions on the sphere S^2 lying in the linear span of x^1, x^2, x^3 , every function in $\mathbb{C}[t, x^1, x^2, x^3, r^{-1}]$ can be uniquely expressed as a finite linear combination of the monomials

$$t^a r^b Y_l^m(\theta, \varphi), \quad a, l = 0, 1, 2, \dots, \quad b \in \mathbb{Z}, \quad -l \leq m \leq l.$$

From (60) and (62) we obtain

$$\square_{3,1}(t^a r^b Y_l^m(\theta, \varphi)) = (b(b+1) - l(l+1))t^a r^{b-2} Y_l^m(\theta, \varphi) - a(a-1)t^{a-2} r^b Y_l^m(\theta, \varphi).$$

Note that the coefficient of $t^a r^{b-2} Y_l^m$ is zero if and only if $b = l$ or $b = -l - 1$, and the coefficient of $t^{a-2} r^b Y_l^m$ is zero if and only if $a = 0$ or $a = 1$. This implies that we can find a function $g \in \mathbb{C}[t, r^{\pm 1}]$ such that $\square_{3,1}(g \cdot Y_l^m) = 0$ by starting with a monomial $t^a r^l Y_l^m$ (or $t^a r^{-l-1} Y_l^m$) and inductively finding the coefficients of $t^{a-2} r^{l+2} Y_l^m, t^{a-4} r^{l+4} Y_l^m, \dots$ (or $t^{a-2} r^{-l+1} Y_l^m, t^{a-4} r^{-l+3} Y_l^m, \dots$) until we zig-zag to $t^0 r^* Y_l^m$ or $t^1 r^* Y_l^m$. We can summarize this observation as follows:

Lemma 71 *Every function in the space $\{\varphi \in \mathbb{C}[t, x^1, x^2, x^3, r^{-1}]; \square_{3,1}\varphi = 0\}$, is a finite linear combination of functions of the type $g \cdot Y_l^m(\theta, \varphi)$, where $g \in \mathbb{C}[t, r^{\pm 1}]$ is such that $\square_{3,1}(g \cdot Y_l^m(\theta, \varphi)) = 0$. Moreover, for a fixed Y_l^m and $d \in \mathbb{Z}$,*

$$\dim \left\{ g \in \mathbb{C}[t, r^{\pm 1}]; \begin{array}{l} g \text{ is homogeneous of degree } d \\ \text{and } \square_{3,1}(g \cdot Y_l^m) = 0 \end{array} \right\} = \begin{cases} 2 & \text{if } d \geq l, \\ 1 & \text{if } -l - 1 \leq d \leq l - 1, \\ 0 & \text{if } d < -l - 1. \end{cases}$$

This proves that functions (40) span all of (44). \square

Proposition 72 *The Lie algebra $\mathfrak{sl}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}}) \subset \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ acts on $\mathcal{D}_{\mathbb{M}}^0$ finitely, i.e. it preserves the space of finite linear combinations of functions from (40) and (41). Moreover, $\mathcal{D}_{\mathbb{M}}^0$ is generated by $f_{0,0,0}^+(Y) = f_{0,0,0}^-(Y) = r^{-1}$.*

Proof. It is sufficient to find the actions of $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$, $B \in \mathbb{H}_{\mathbb{C}}$, since together with their conjugates by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ they generate all of $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$. Thus we need to find the actions of $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$, $\frac{\partial}{\partial x^3}$ on $r^{-1} \cdot (r^2 - t^2)^{\pm n/2} \cdot P_l^{(n)}(t/r) \cdot Y_l^m(\theta, \varphi)$. First we consider $\frac{\partial}{\partial t}$, then from (70) and (71) we have:

$$\begin{aligned} & \frac{\partial}{\partial t} \left(r^{-1} \cdot (r^2 - t^2)^{\pm n/2} \cdot P_l^{(n)}(t/r) \cdot Y_l^m(\theta, \varphi) \right) \\ &= r^{-1} \cdot (r^2 - t^2)^{\pm n/2} \cdot \left(\frac{1}{r} \cdot (P_l^{(n)})'(t/r) \mp \frac{nt}{r^2 - t^2} P_l^{(n)}(t/r) \right) \cdot Y_l^m(\theta, \varphi) \\ &= r^{-1} \cdot (r^2 - t^2)^{(\pm n - 1)/2} \cdot P_l^{(n \mp 1)}(t/r) \cdot Y_l^m(\theta, \varphi) \cdot \begin{cases} (l+n)(l-n+1) & \text{if "+n",} \\ -1 & \text{if "-n".} \end{cases} \end{aligned}$$

Hence

$$\frac{\partial}{\partial t} : f_{l,m,n}^+(Y) \mapsto \sqrt{(l+n)(l-n+1)} \cdot f_{l,m,n-1}^+(Y), \quad (45)$$

$$f_{l,m,n}^-(Y) \mapsto -\sqrt{(l+n+1)(l-n)} \cdot f_{l,m,n+1}^-(Y). \quad (46)$$

Next we consider $\frac{\partial}{\partial x^i}$. It is sufficient to show that $\frac{\partial}{\partial x^i}$ acts finitely on the bases (39). By (77) $Y_l^m(\theta, \varphi)$ is proportional to $P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi}$ by (77). Since $\frac{\partial}{\partial x^i}$ commute with $\frac{\partial}{\partial t}$, it is enough to show that

$$\frac{\partial}{\partial x^i} \left((r^2 - t^2)^l \cdot r^{-l-1} \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi} \right) \quad \text{and} \quad \frac{\partial}{\partial x^i} \left((r^2 - t^2)^{-l-1} \cdot r^l \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi} \right)$$

are finite linear combinations of elements from (39). Using $x^3/r = \cos \theta$, (73) and (74), we get:

$$\begin{aligned} & e^{im\varphi} \cdot \frac{\partial}{\partial x^3} \left((r^2 - t^2)^l \cdot r^{-l-1} \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi} \right) \\ &= \left(2l(r^2 - t^2)^{l-1} \cdot r^{-l} - (l+1)(r^2 - t^2)^l \cdot r^{-l-2} \right) \cdot \cos \theta \cdot P_l^{(m)}(\cos \theta) \\ & \quad + (r^2 - t^2)^l \cdot r^{-l-2} \cdot \left((l+m)P_{l-1}^{(m)}(\cos \theta) - l \cos \theta \cdot P_l^{(m)}(\cos \theta) \right) \\ &= (l+m)(r^2 - t^2)^l \cdot r^{-l-2} \cdot P_{l-1}^{(m)}(\cos \theta) + \\ & r^{-l} \cdot \left(2l(r^2 - t^2)^{l-1} - (2l+1)(r^2 - t^2)^l \cdot r^{-2} \right) \cdot \left(\frac{l-m+1}{2l+1} P_{l+1}^{(m)}(\cos \theta) + \frac{l+m}{2l+1} P_{l-1}^{(m)}(\cos \theta) \right) \\ &= \frac{2l(l+m)}{2l+1} (r^2 - t^2)^{l-1} \cdot r^{-l} \cdot P_{l-1}^{(m)}(\cos \theta) \\ & \quad + \frac{l-m+1}{2l+1} \left(2lr^2(r^2 - t^2)^{l-1} - (2l+1)(r^2 - t^2)^l \right) \cdot r^{-l-2} \cdot P_{l+1}^{(m)}(\cos \theta) \\ &= \frac{2l(l+m)}{2l+1} (r^2 - t^2)^{l-1} \cdot r^{-l} \cdot P_{l-1}^{(m)}(\cos \theta) + \frac{l-m+1}{2(2l+1)(l+1)} \frac{\partial^2}{\partial t^2} \left((r^2 - t^2)^{l+1} \cdot r^{-l-2} \cdot P_{l+1}^{(m)}(\cos \theta) \right). \end{aligned}$$

Similarly, we compute

$$\begin{aligned} & \left(\frac{\partial}{\partial x^1} + i \frac{\partial}{\partial x^2} \right) \left((r^2 - t^2)^l \cdot r^{-l-1} \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi} \right) \\ &= \frac{(l-m+1)(l-m+2)}{2(2l+1)(l+1)} \frac{\partial^2}{\partial t^2} \left((r^2 - t^2)^{l+1} \cdot r^{-l-2} \cdot P_{l+1}^{(m-1)}(\cos \theta) \cdot e^{-i(m-1)\varphi} \right) \\ & \quad - \frac{2l(l+m)(l+m+1)}{2l+1} (r^2 - t^2)^{l-1} \cdot r^{-l} \cdot P_{l-1}^{(m-1)}(\cos \theta) \cdot e^{-i(m-1)\varphi}, \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right) \left((r^2 - t^2)^l \cdot r^{-l-1} \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi} \right) \\ &= \frac{2l}{2l+1} (r^2 - t^2)^{l-1} \cdot r^{-l} \cdot P_{l-1}^{(m+1)}(\cos \theta) \cdot e^{-i(m+1)\varphi} \\ & \quad - \frac{l}{(2l+1)(l+1)} \frac{\partial^2}{\partial t^2} \left((r^2 - t^2)^{l+1} \cdot r^{-l-2} \cdot P_{l+1}^{(m+1)}(\cos \theta) \cdot e^{-i(m+1)\varphi} \right). \end{aligned}$$

The derivatives of $(r^2 - t^2)^{-l-1} \cdot r^l \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi}$ are computed in the same way. Finally, it follows from these expressions for the action of $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$ that $f_{0,0,0}^{\pm}$ generates all of $\mathcal{D}_{\mathbb{M}}^0$. \square

Theorem 73 *As a representation of $\mathfrak{sl}(4, \mathbb{C})$, $\mathcal{D}_{\mathbb{M}}^0$ is the minimal representation $(\varpi_{\mathbb{R}^{2,2}}^{\min}, \mathcal{H})$ in the notations of [KobØ]. In particular, it is irreducible.*

Proof. Consider a function

$$\varphi_0(X) = \left((1 + N(X))^2 + 4x_{12}x_{21} \right)^{-1/2}, \quad X \in \mathbb{H}_{\mathbb{R}}. \quad (47)$$

It is easy to see that φ_0 is a real analytic function on $\mathbb{H}_{\mathbb{R}}$ satisfying $\square_{2,2}\varphi_0 = 0$. In fact, φ_0 is the rescaled generating function f_0 from [KobØ], Part III, equation (5.4.1) for $p = q = 3$. Hence it generates the minimal representation $(\varpi_{\mathbb{R}^{2,2}}^{\min}, \mathcal{H})$ of $O(3, 3) \simeq SL(2, \mathbb{H}_{\mathbb{R}})/\{\pm 1\}$. Note that $-N(\tilde{e}_3 Y - 1)^2$ is a negative real number only when $\text{Re}(e_3 Y) = 0$ and

$$(-N(\tilde{e}_3 Y - 1)^2)^{1/2} = -i \text{sign}_3(Y) \cdot N(\tilde{e}_3 Y - 1). \quad (48)$$

Using (37) we obtain:

$$(\pi_l^0(\gamma)\varphi_0)(Y) = 2i \text{sign}_3(Y) \cdot (-4(y_{22} - y_{11})^2 - 16y_{12}y_{21})^{-1/2} = \text{sign}_3(Y) \cdot \frac{i}{2r},$$

which is proportional to $\text{sign}_3(Y) \cdot f_{0,0,0}^{\pm}(Y)$. As far as representations of Lie algebras are concerned, we can drop the factor $\text{sign}_3(Y)$. By the previous proposition, this function generates the whole span, and the result follows. \square

The K -types of the minimal representation $(\varpi_{\mathbb{R}^{2,2}}^{\min}, \mathcal{H})$ of $\mathfrak{sl}(4, \mathbb{C})$ with respect to $K = SO(3) \times SO(3)$ were identified in [KobØ], Part I, as

$$\bigoplus_l V_l \boxtimes V_l, \quad l = 0, 1, 2, \dots,$$

where V_l denotes the irreducible representation of $SO(3)$ of dimension $l + 1$.

Proposition 74 *The basis functions (40) and (41) are the K -finite vectors of the minimal representation $\mathcal{D}_{\mathbb{M}}^0$ of $\mathfrak{sl}(4, \mathbb{C})$ with respect to $K = SO(3) \times SO(3)$.*

Define a nondegenerate symmetric bilinear pairing on the linear span of (40) and (41) by declaring

$$\begin{aligned} \langle f_{l,m,n}^+(Y), f_{l',-m',n'}^-(Y) \rangle_{\min} &= \langle f_{l,m,n}^-(Y), f_{l',-m',n'}^+(Y) \rangle_{\min} = (-1)^m \frac{2}{\pi i} \delta_{l,l'} \cdot \delta_{m,m'} \cdot \delta_{n,n'}, \\ \langle f_{l,m,n}^+(Y), f_{l',m',n'}^+(Y) \rangle_{\min} &= \langle f_{l,m,n}^-(Y), f_{l',m',n'}^-(Y) \rangle_{\min} = 0. \end{aligned}$$

In the second line we exclude $n' = 0$ because $f_{l',m',0}^+(Y) = f_{l',m',0}^-(Y)$. By Theorem 68, this pairing partially agrees with the bilinear form (42) up to a sign.

Proposition 75 *This bilinear pairing on $\mathcal{D}_{\mathbb{M}}^0$ is $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant:*

$$\langle \pi_l^0(Z)\varphi_1, \varphi_2 \rangle_{min} + \langle \varphi_1, \pi_r^0(Z)\varphi_2 \rangle_{min} = 0, \quad \forall Z \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}).$$

Proof. It is sufficient to show invariance of $\langle \cdot, \cdot \rangle_{min}$ under $\mathfrak{sl}(2, \mathbb{C})$ embedded into $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ as in (9), $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ and scalar matrices. The invariance with respect to the scalar matrices and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is clear, and the invariance with respect to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ follows from (45)-(46).

To show the $\mathfrak{sl}(2, \mathbb{C})$ -invariance, we observe that the action of $\mathfrak{sl}(2, \mathbb{C})$ commutes with $\widetilde{\text{deg}}$. Hence, for each fixed n , $\mathfrak{sl}(2, \mathbb{C})$ preserves the linear spans of $\{f_{l,m,n}^+(Y)\}$ and $\{f_{l,m,n}^-(Y)\}$ $l = n, n+1, n+2, \dots, -l \leq m \leq l$. Then the $\mathfrak{sl}(2, \mathbb{C})$ -invariance follows from Theorem 68 for $n > 0$, and the case $n = 0$ has to be considered separately. Heuristically, the action of $\mathfrak{sl}(2, \mathbb{C})$ on the linear span of $\{f_{l,m,0}^+(Y)\}$ is obtained from the action on $\{f_{l,m,n}^{\pm}(Y)\}$, n fixed, by setting the value of the parameter $n = 0$. Since this action is described by algebraic formulas and preserves $\langle \cdot, \cdot \rangle_{min}$ for all $n > 0$, it preserves $\langle \cdot, \cdot \rangle_{min}$ for $n = 0$ as well. \square

4.5 Extension of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$

Let $\overline{\mathcal{D}^0}$ denote the unitary representation of $SL(2, \mathbb{H}_{\mathbb{R}})$ in the Hilbert space generated by the function $\varphi_0(X)$ defined by (47). This is the minimal representation $(\varpi_{\mathbb{R}^{2,2}}^{\min}, \mathcal{H})$ of $SO(3, 3) \simeq SL(2, \mathbb{H}_{\mathbb{R}})/\{\pm 1\}$ in the notations of [KobØ].

Lemma 76 *The function*

$$\tilde{\varphi}_0(X) = \frac{(N(X) + 1) \cdot N(X - i)}{\left((1 + N(X))^2 + 4x_{12}x_{21} \right)^{3/2}}, \quad X \in \mathbb{H}_{\mathbb{R}}$$

also generates the minimal representation $\overline{\mathcal{D}^0}$.

Proof. Using (37) and (48) we obtain:

$$(\pi_l^0(\gamma)\tilde{\varphi}_0)(Y) = \text{sign}_3(Y) \cdot \frac{16i(y_{22} - y_{11})}{(-4(y_{22} - y_{11})^2 - 16y_{12}y_{21})^{3/2}} = -\text{sign}_3(Y) \cdot \frac{\cos \theta}{2r^2},$$

which is proportional to $\text{sign}_3(Y) \cdot f_{1,0,1}^-(Y)$. By Proposition 72 it follows that $\tilde{\varphi}_0$ generates the same representation as φ_0 does. \square

Proposition 77 *The intersection $\overline{\mathcal{D}^0} \cap \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ is precisely the continuous series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$.*

Proof. Clearly, $\tilde{\varphi}_0(X) \in \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. Then Lemma 76 implies that φ_0 cannot have a discrete series component and $\overline{\mathcal{D}^0} \cap \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ is contained in the continuous series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. To prove the other inclusion, it is sufficient to show that

$$\langle t_{\frac{1}{2}, \frac{1}{2}}^l(X), \tilde{\varphi}_0(X) \rangle_1 \neq 0 \quad \text{and} \quad \langle t_{0,0}^l(X), \varphi_0(X) \rangle_1 \neq 0 \quad \text{for all } l = -\frac{1}{2} + i\lambda \text{ with } \lambda \in \mathbb{R}^{\times}.$$

In coordinates (15) the restrictions of φ_0 and $\tilde{\varphi}_0$ to $SU(1, 1)$ become

$$\varphi_0(\varphi, \tau, \psi) = \left(2 \cosh \frac{\tau}{2} \right)^{-1} \quad \text{and} \quad \tilde{\varphi}_0(\varphi, \tau, \psi) = \frac{1}{4i} \left(e^{i\frac{\varphi+\psi}{2}} + e^{-i\frac{\varphi+\psi}{2}} \right) \cdot \left(\cosh \frac{\tau}{2} \right)^{-2}.$$

Substituting $v = \sinh^2 \frac{\tau}{2}$ and using a special case of an integral formula 7.512(10) from [GR]

$$\int_0^\infty (1+x)^{-r} {}_2F_1(a, b; 1; -x) dx = \frac{\Gamma(a+r-1)\Gamma(b+r-1)}{\Gamma(r)\Gamma(a+b+r-1)}$$

valid when $\operatorname{Re}(a+r-1) > 0$, $\operatorname{Re}(b+r-1) > 0$, we obtain

$$\begin{aligned} \langle t_{\frac{1}{2}\frac{1}{2}}^l(X), \tilde{\varphi}_0(X) \rangle_1 &= -\frac{2l+1}{8i} \int_0^\infty \sinh \tau \cdot \left(\cosh \frac{\tau}{2} \right)^{-2} \cdot \mathfrak{P}_{\frac{1}{2}\frac{1}{2}}^l(\cosh \tau) d\tau \\ &= -\frac{2l+1}{4i} \int_0^\infty \sinh \frac{\tau}{2} \cdot {}_2F_1\left(l+3/2, -l+1/2; 1; -\sinh^2 \frac{\tau}{2}\right) d\tau \\ &= -\frac{2l+1}{4i} \int_0^\infty (1+v)^{-\frac{1}{2}} \cdot {}_2F_1(l+3/2, -l+1/2; 1; -v) dv \\ &= -\frac{2l+1}{4i} \cdot \frac{\Gamma(l+1)\Gamma(-l)}{\Gamma(1/2)\Gamma(3/2)} = \frac{2l+1}{2i \sin(\pi l)} = -\frac{2l+1}{2i \cosh(\pi \operatorname{Im} l)} \neq 0. \end{aligned}$$

In the other case we need to deal with convergence issues, so we observe that

$$\mathfrak{P}_{00}^l(\cosh \tau) = \lim_{s \rightarrow 0^+} \mathfrak{P}_{00}^l(\cosh \tau) \cdot \left(\cosh \frac{\tau}{2} \right)^{-2s}$$

as a distribution in $\operatorname{Im} l$. Substituting $v = \sinh^2 \frac{\tau}{2}$ we obtain

$$\begin{aligned} \langle t_{00}^l(X), \varphi_0(X) \rangle_1 &= \lim_{s \rightarrow 0^+} -\frac{2l+1}{4} \int_0^\infty \sinh \tau \cdot \left(\cosh \frac{\tau}{2} \right)^{-1-2s} \cdot \mathfrak{P}_{00}^l(\cosh \tau) d\tau \\ &= \lim_{s \rightarrow 0^+} -\frac{2l+1}{2} \int_0^\infty \sinh \frac{\tau}{2} \cdot \left(\cosh \frac{\tau}{2} \right)^{-2s} \cdot {}_2F_1\left(l+1, -l; 1; -\sinh^2 \frac{\tau}{2}\right) d\tau \\ &= \lim_{s \rightarrow 0^+} -\frac{2l+1}{2} \int_0^\infty (1+v)^{-\frac{1}{2}-s} \cdot {}_2F_1(l+1, -l; 1; -v) dv \\ &= \lim_{s \rightarrow 0^+} -\frac{2l+1}{2} \cdot \frac{\Gamma(l+s+1/2)\Gamma(s-l-1/2)}{(\Gamma(s+1/2))^2} \\ &= -\frac{2l+1}{2} \cdot \frac{\Gamma(l+1/2)\Gamma(-l-1/2)}{(\Gamma(1/2))^2} = \frac{1}{\cos(\pi l)} = \frac{1}{i \sinh(\pi \operatorname{Im} l)} \neq 0. \end{aligned}$$

□

Remark 78 We can summarize the observation made at the beginning of Subsection 4.2, Theorem 73 and Proposition 77 as follows: The Cayley transform switches the discrete and continuous series components of the spaces of harmonic functions on $\mathbb{H}_{\mathbb{R}}$ and \mathbb{M} .

The Schwartz space $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ “almost” contains the representations $\overline{\mathcal{D}^0}$, \mathcal{D}^- and \mathcal{D}^+ , meaning that $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ is missing the limits of the discrete series and the functions on $\mathbb{H}_{\mathbb{R}}$ corresponding to the limits of the discrete series on \mathbb{M} (see Remark 69). From the point of view of representation theory it is natural to add these missing functions to $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. Then the resulting space is just a direct sum of three irreducible representations of $SL(2, \mathbb{H}_{\mathbb{R}})$. Expansions of $\frac{1}{N(X-W)}$ contain these missing functions, which is another reason for extending $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. We define an extended space

$$\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+) = \overline{\mathcal{D}^0} \oplus \overline{\mathcal{D}^-} \oplus \overline{\mathcal{D}^+} \supseteq \mathcal{H}(\mathbb{H}_{\mathbb{R}}^+),$$

where $\overline{\mathcal{D}^\mp}$ denote the Hilbert space completions of \mathcal{D}^\mp with respect to an $\mathfrak{su}(2, 2)$ -invariant inner product (which is unique up to scaling). Similarly we can define

$$\mathcal{H}(\mathbb{M}^+) = \pi_l^0(\gamma)(\overline{\mathcal{D}^-}) \oplus \pi_l^0(\gamma)(\overline{\mathcal{D}^+}) \oplus \{\operatorname{sign}_3(Y) \cdot (\pi_l^0(\gamma)\varphi)(Y); \varphi \in \overline{\mathcal{D}^0}\}. \quad (49)$$

Next we define a nondegenerate $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant symmetric bilinear pairing $\langle \cdot, \cdot \rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)}$ on $\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)$. We declare

$$\langle \overline{\mathcal{D}^0}, \overline{\mathcal{D}^-} \rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)} = \langle \overline{\mathcal{D}^0}, \overline{\mathcal{D}^+} \rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)} = \langle \overline{\mathcal{D}^-}, \overline{\mathcal{D}^-} \rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)} = \langle \overline{\mathcal{D}^+}, \overline{\mathcal{D}^+} \rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)} = 0.$$

Then we extend the pairing $\langle \cdot, \cdot \rangle_{\mathcal{D}^- \times \mathcal{D}^+}$ described in Proposition 40 by continuity to $\overline{\mathcal{D}^-} \oplus \overline{\mathcal{D}^+}$. Finally, we define the pairing on $\overline{\mathcal{D}^0}$ by

$$\langle \varphi_1(X), \varphi_2(X) \rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)} = \langle \text{sign}_3(Y) \cdot (\pi_l^0(\gamma)\varphi_1)(Y), \text{sign}_3(Y) \cdot (\pi_l^0(\gamma)\varphi_2)(Y) \rangle_{\min},$$

where $\langle \cdot, \cdot \rangle_{\min}$ is the pairing described in Proposition 75 and extended by continuity to the last summand of (49).

For example, in this light the matrix coefficient expansions given in Proposition 45 mean that $-\frac{1}{N(Z-W)}$ is the reproducing kernel for $\overline{\mathcal{D}^-}$ and $\overline{\mathcal{D}^+}$ when, respectively, $W \in \Gamma^-$ and Γ^+ . In other words, for each $W \in \Gamma^-$, the function $\frac{1}{N(Z-W)}$ lies in the extended space $\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)$ and

$$\varphi(Z) \mapsto -\left\langle \varphi(Z), \frac{1}{N(Z-W)} \right\rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)}$$

is the projection of φ onto its holomorphic discrete series component. Similarly, for each $W \in \Gamma^+$, the function $\frac{1}{N(Z-W)} \in \widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)$ and

$$\varphi(Z) \mapsto -\left\langle \varphi(Z), \frac{1}{N(Z-W)} \right\rangle_{\widehat{\mathcal{H}}(\mathbb{H}_{\mathbb{R}}^+)}$$

is the projection of φ onto its antiholomorphic discrete series component.

4.6 Expansion of $\frac{1}{N(Z-W)}$

Lemma 79 *Let $Z, W \in \mathbb{M}^+$, write $Z = t\tilde{e}_0 + x^1e_1 + x^2e_2 + x^3e_3$, $W = \tilde{t}\tilde{e}_0 + \tilde{x}^1e_1 + \tilde{x}^2e_2 + \tilde{x}^3e_3$, then we have the following expansions:*

$$\sum_{m=-l}^l (-1)^m Y_l^{-m}(\theta, \varphi) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}) = P_l\left(\frac{x^1\tilde{x}^1 + x^2\tilde{x}^2 + x^3\tilde{x}^3}{r\tilde{r}}\right), \quad (50)$$

$$\sum_{n=-l}^l \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot P_l^{(n)}(t/r) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) = P_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2-t^2+\tilde{r}^2-\tilde{t}^2}{2r\tilde{r}}\right), \quad (51)$$

$$\sum_{n=-l}^l \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot \tilde{Q}_l^{(n)}(t/r) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) = \tilde{Q}_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2-t^2+\tilde{r}^2-\tilde{t}^2}{2r\tilde{r}}\right). \quad (52)$$

(For the last expansion we assume $(2t\tilde{t} + r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2)/2r\tilde{r}$ lies in the domain of \tilde{Q}_l .)

Proof. Since $\cos \theta = x^3/r$, $\cos \tilde{\theta} = \tilde{x}^3/\tilde{r}$ and

$$\sin \theta \sin \tilde{\theta} \cos(\varphi - \tilde{\varphi}) = \sin \theta \cos \varphi \sin \tilde{\theta} \cos \tilde{\varphi} + \sin \theta \sin \varphi \sin \tilde{\theta} \sin \tilde{\varphi} = \frac{x^1}{r} \cdot \frac{\tilde{x}^1}{\tilde{r}} + \frac{x^2}{r} \cdot \frac{\tilde{x}^2}{\tilde{r}},$$

from (77), (78) and (66) we get (50).

Applying (78) with $s \in \mathbb{C}$ such that $e^{is} = \sqrt{\frac{r^2-t^2}{\tilde{r}^2-\tilde{t}^2}}$, $\cos s = (e^{is} + e^{-is})/2$, $\cos \theta = t/r$ and $\cos \tilde{\theta} = \tilde{t}/\tilde{r}$ we get:

$$\begin{aligned} \sum_{n=-l}^l \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot P_l^{(n)}(t/r) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) &= P_l(\cos \theta \cos \tilde{\theta} + \sin \theta \sin \tilde{\theta} \cos \varphi) \\ &= P_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{1}{2}\sqrt{1-\frac{t^2}{r^2}}\sqrt{1-\frac{\tilde{t}^2}{\tilde{r}^2}}\left(\sqrt{\frac{r^2-t^2}{\tilde{r}^2-\tilde{t}^2}} + \sqrt{\frac{\tilde{r}^2-\tilde{t}^2}{r^2-t^2}}\right)\right) \\ &= P_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{|r^2-t^2| + |\tilde{r}^2-\tilde{t}^2|}{2r\tilde{r}}\right) = P_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2-t^2 + \tilde{r}^2-\tilde{t}^2}{2r\tilde{r}}\right), \end{aligned}$$

which proves (51).

Using (79) instead of (78) we prove the last expansion (52). Strictly speaking, (79) applies only if $\sqrt{\frac{r^2-t^2}{\tilde{r}^2-\tilde{t}^2}} = e^{is}$ with $s \in \mathbb{R}$, but both sides depend on s analytically. \square

Proposition 80 *Let $W \in \mathbb{M}^+$ and $Z \in \{\mathbb{M}^+ + ae_0; a \in \mathbb{R}, a \neq 0\} \subset \mathbb{H}_{\mathbb{C}}$. Write $Z = t\tilde{e}_0 + x^1e_1 + x^2e_2 + x^3e_3$ with $t \in \mathbb{C}$, $W = \tilde{t}\tilde{e}_0 + \tilde{x}^1e_1 + \tilde{x}^2e_2 + \tilde{x}^3e_3$, then we have the following expansions:*

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{2l+1}{2r\tilde{r}} \left(\sum_{m,n=-l}^l (-1)^m (i \cdot \text{sign Im } t)^n \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot Q_l^{(n)}(t/r) \cdot Y_l^{-m}(\theta, \varphi) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}) \right) \\ = \frac{1}{N(Z-W)}, \quad \text{provided that } \tilde{t} > \text{Re } t. \end{aligned}$$

Similarly, if $Z = t\tilde{e}_0 + x^1e_1 + x^2e_2 + x^3e_3 \in \mathbb{M}^+$ and $W = \tilde{t}\tilde{e}_0 + \tilde{x}^1e_1 + \tilde{x}^2e_2 + \tilde{x}^3e_3 \in \{\mathbb{M}^+ + ae_0; a \in \mathbb{R}, a \neq 0\}$ with $\tilde{t} \in \mathbb{C}$, then

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{2l+1}{2r\tilde{r}} \left(\sum_{m,n=-l}^l (-1)^m (i \cdot \text{sign Im } \tilde{t})^n \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot P_l^{(n)}(t/r) \cdot Y_l^{-m}(\theta, \varphi) \cdot Q_l^{(n)}(\tilde{t}/\tilde{r}) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}) \right) \\ = \frac{1}{N(Z-W)}, \quad \text{provided that } t > \text{Re } \tilde{t}. \end{aligned}$$

Proof. Letting $t_0 = \text{Re } t$, $a = \text{Im } t$ and using (51), (52), (67), the first sum reduces to

$$\begin{aligned} \sum_{n=-l}^l (i \cdot \text{sign } a)^n \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot Q_l^{(n)}\left(\frac{t_0+ia}{r}\right) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) \\ = \sum_{n=-l}^l \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot \left(\tilde{Q}_l^{(n)}\left(\frac{t_0+ia}{r}\right) - \text{sign } a \cdot \frac{\pi i}{2} P_l^{(n)}\left(\frac{t_0+ia}{r}\right) \right) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) \\ = \tilde{Q}_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2-t^2 + \tilde{r}^2-\tilde{t}^2}{2r\tilde{r}}\right) - \text{sign } a \cdot \frac{\pi i}{2} P_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2-t^2 + \tilde{r}^2-\tilde{t}^2}{2r\tilde{r}}\right) \\ = Q_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2-t^2 + \tilde{r}^2-\tilde{t}^2}{2r\tilde{r}}\right), \end{aligned}$$

provided that

$$\text{Im}\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2-t^2 + \tilde{r}^2-\tilde{t}^2}{2r\tilde{r}}\right) = a \frac{\tilde{t}-t_0}{r\tilde{r}}$$

has the same sign as a . Using

$$N(Z-W) = 2t\tilde{t} + r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2 - 2(x^1\tilde{x}^1 + x^2\tilde{x}^2 + x^3\tilde{x}^3),$$

(80) and (50) we see that

$$\sum_{l=0}^{\infty} \frac{2l+1}{2r\tilde{r}} P_l\left(\frac{x^1\tilde{x}^1 + x^2\tilde{x}^2 + x^3\tilde{x}^3}{r\tilde{r}}\right) Q_l\left(\frac{\tilde{t}}{r\tilde{r}} + \frac{r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2 + \varepsilon^2}{2r\tilde{r}}\right) = \frac{1}{N(Z-W)}.$$

The other expansion is proved by switching the variables Z and W . \square

Let $Z, W \in \mathbb{M}^+$ and write $Z = t\tilde{e}_0 + x^1e_1 + x^2e_2 + x^3e_3$, $W = \tilde{t}\tilde{e}_0 + \tilde{x}^1e_1 + \tilde{x}^2e_2 + \tilde{x}^3e_3$. Then

$$\frac{1}{N(Z-W \pm i\varepsilon\tilde{e}_0)} = \frac{1}{N(Z-W) + \varepsilon^2 \mp 2i\varepsilon(t-\tilde{t})}$$

Letting $\varepsilon \rightarrow 0^+$, from the above expansions and (76) we obtain

$$\begin{aligned} \pi i \sum_{l=0}^{\infty} \frac{2l+1}{2r\tilde{r}} \left(\sum_{m,n=-l}^l (-1)^m \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot P_l^{(n)}(t/r) \cdot Y_l^{-m}(\theta, \varphi) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}) \right) \\ = \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{N(Z-W) + \varepsilon^2 - 2i\varepsilon|t-\tilde{t}|} - \frac{1}{N(Z-W) + \varepsilon^2 + 2i\varepsilon|t-\tilde{t}|} \right). \end{aligned}$$

As distributions on the hyperboloids $\{N(Z) = \text{const}\}$, the limits of $(N(Z-W) + \varepsilon^2 \pm 2i\varepsilon|t-\tilde{t}|)^{-1}$ as $\varepsilon \rightarrow 0^+$, are equivalent to $(N(Z-W) \pm i0)^{-1}$. Thus we can formally write

$$\begin{aligned} \pi i \sum_{l=0}^{\infty} \frac{2l+1}{2r\tilde{r}} \left(\sum_{m,n=-l}^l (-1)^m \frac{(l-n)!}{(l+n)!} \cdot \frac{(r^2-t^2)^{n/2}}{(\tilde{r}^2-\tilde{t}^2)^{n/2}} \cdot P_l^{(n)}(t/r) \cdot Y_l^{-m}(\theta, \varphi) \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}) \right) \\ = \frac{1}{N(Z-W) - i0} - \frac{1}{N(Z-W) + i0}. \quad (53) \end{aligned}$$

4.7 The Discrete Series Projector on \mathbb{M}

Using expansions of $\frac{1}{N(Z-W) \pm i0}$ obtained in previous subsection, we obtain projectors onto the discrete series component on \mathbb{M} . Recall that the space of harmonic functions $\mathcal{H}(\mathbb{M}^+)$ was defined in (49). Let $\tilde{H}_R = \{Y \in \mathbb{M}; N(Y) = R^2\}$, $R > 0$. and define an operator on $\mathcal{H}(\mathbb{M}^+)$ by

$$\begin{aligned} (S_R^{\mathbb{M}} \varphi)(W) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2 i} \int_{Y \in \tilde{H}_R} \left(\frac{1}{N(Y-W) + i\varepsilon} - \frac{1}{N(Y-W) - i\varepsilon} \right) \cdot \widetilde{\text{deg}} \varphi(Y) \frac{dS}{\|Y\|} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2 i} \int_{Y \in \tilde{H}_R} \left(\frac{\widetilde{\text{deg}} \varphi(Y)}{N(Y-W) + i\varepsilon} - \frac{\widetilde{\text{deg}} \varphi(Y)}{N(Y-W) - i\varepsilon} \right) \cosh^2 \rho \sin \theta \, d\rho d\theta d\varphi, \quad (54) \end{aligned}$$

$W \in \mathbb{M}^+$.

Theorem 81 *The operator $S_R^{\mathbb{M}}$ annihilates the continuous series component of $\mathcal{H}(\mathbb{M}^+)$ and acts on the discrete series component of $\mathcal{H}(\mathbb{M}^+)$ by sending*

$$\begin{aligned} f_{l,m,n}^+(Y) &\mapsto -(f_{l,m,n}^+(W) + R^{2n} \cdot f_{l,m,n}^-(W)), \\ f_{l,m,n}^-(Y) &\mapsto R^{-2n} \cdot f_{l,m,n}^+(W) + f_{l,m,n}^-(W) \end{aligned}$$

if $n \neq 0$ and annihilates $f_{l,m,0}^{\pm}(Y)$.

Remark 82 We can “localize” the difference of the integrals at the set where $N(Y - W) = 0$. Thus the operator (54) depends only on the values of $\deg\varphi(Y)$, where Y ranges over a two-dimensional hyperboloid (or cone if $W \in \tilde{H}_R$)

$$\{Y \in \mathbb{M}; N(Y) = R^2, N(Y - W) = 0\}.$$

Proof. The discrete series on \mathbb{M}^+ is spanned by (40) and (41). Hence it is enough to check the statement for the functions $\varphi(Y) = r^{-1} \cdot (r^2 - t^2)^{\pm n/2} \cdot P_l^{(n)}(t/r) \cdot Y_l^m(\theta, \varphi)$, where $l = 1, 2, 3, \dots$, $-l \leq m \leq l$, $1 \leq n \leq l$. First we expand using (80) and (50):

$$\begin{aligned} \frac{1}{N(Y - W) \pm i\varepsilon} &= \sum_{l=0}^{\infty} \frac{2l+1}{2r\tilde{r}} Q_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2}{2r\tilde{r}} \pm \frac{i\varepsilon}{2r\tilde{r}}\right) P_l\left(\frac{x^1\tilde{x}^1 + x^2\tilde{x}^2 + x^3\tilde{x}^3}{r\tilde{r}}\right) \\ &= \sum_{l=0}^{\infty} \frac{2l+1}{2r\tilde{r}} Q_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2}{2r\tilde{r}} \pm \frac{i\varepsilon}{2r\tilde{r}}\right) \left(\sum_{m=-l}^l (-1)^m Y_l^{-m}(\theta, \varphi) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi})\right). \end{aligned}$$

We have $\widetilde{\deg\varphi}(Y) = \pm n\varphi(Y)$. By the orthogonality relation (63), after we integrate out φ and θ we are left with integrating over ρ

$$\begin{aligned} \pm \frac{2\pi n}{r^2\tilde{r}} \left(Q_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2}{2r\tilde{r}} + \frac{i\varepsilon}{r\tilde{r}}\right) - Q_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2}{2r\tilde{r}} - \frac{i\varepsilon}{r\tilde{r}}\right) \right) \\ \times (r^2 - t^2)^{\pm n/2} \cdot P_l^{(n)}(t/r) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}). \end{aligned}$$

As $\varepsilon \rightarrow 0^+$, by (76) we get

$$\mp \frac{2\pi^2 ni R^{\pm n}}{r^2\tilde{r}} P_l\left(\frac{t\tilde{t}}{r\tilde{r}} + \frac{r^2 - t^2 + \tilde{r}^2 - \tilde{t}^2}{2r\tilde{r}}\right) \cdot P_l^{(n)}(t/r) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}).$$

Then using (66), (51) and our earlier calculation (43) we can integrate over ρ to get

$$\mp \tilde{r}^{-1} \cdot R^{\pm n} \cdot P_l^{(n)}(\tilde{t}/\tilde{r}) \cdot Y_l^m(\tilde{\theta}, \tilde{\varphi}) \cdot (R^{-n}(\tilde{r}^2 - \tilde{t}^2)^{n/2} + R^n(\tilde{r}^2 - \tilde{t}^2)^{-n/2}).$$

Finally, the fact that the continuous series on \mathbb{M} gets annihilated follows from Corollary 64 or, alternatively, since our expansion (53) contains the discrete series only. \square

We can rewrite the expression $\frac{1}{N(Y-W)+i0} - \frac{1}{N(Y-W)-i0}$ geometrically as follows.

Lemma 83 Fix $W \in \mathbb{M}^+$ and let Y range over the hyperboloid \tilde{H}_R . Fix a number $\xi \in \mathbb{C} \setminus \mathbb{R}$ and let $W' = (1 + \varepsilon\xi)W \in \mathbb{H}_{\mathbb{C}}$, $\varepsilon > 0$, then $N(Y - W')$ is never zero for all ε sufficiently small.

Proof. Let $A = \text{Tr}(YW^+)$, then

$$N(Y - W') = N(Y) + (1 + \varepsilon\xi)^2 N(W) - (1 + \varepsilon\xi)A.$$

Write $\xi = a + ib$, $a, b \in \mathbb{R}$, then the imaginary part of $N(Y - W')$ is zero if and only if

$$2b(\varepsilon^2 a + \varepsilon)N(W) - \varepsilon b A = 0 \quad \iff \quad A = 2(\varepsilon a + 1)N(W).$$

Then the real part of $N(Y - W')$ is

$$N(Y) + (1 + 2\varepsilon a + \varepsilon^2(a^2 - b^2))N(W) - 2(1 + \varepsilon a)^2 N(W) = R^2 - (1 + 2\varepsilon a + \varepsilon^2|\xi|^2)N(W) \neq 0$$

for all $\varepsilon \ll 1$. \square

When $N(Y - W) = 0$, $N(Y - W') = \varepsilon\xi(N(W) - N(Y)) + \varepsilon^2\xi^2N(W)$. Choose $\xi_1, \xi_2 \in \mathbb{C}$ with $\text{Im}(\xi_1) > 0$ and $\text{Im}(\xi_2) < 0$, then

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \text{sign}(N(W) - R^2) \cdot \int_{Y \in \tilde{H}_R} \left(\frac{1}{N(Y - W) + i\varepsilon} - \frac{1}{N(Y - W) - i\varepsilon} \right) \cdot \widetilde{\text{deg}}\varphi(Y) \frac{dS}{\|Y\|} \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{Y \in \tilde{H}_R} \left(\frac{1}{N(Y - (1 + \varepsilon\xi_1)W)} - \frac{1}{N(Y - (1 + \varepsilon\xi_2)W)} \right) \cdot \widetilde{\text{deg}}\varphi(Y) \frac{dS}{\|Y\|}. \end{aligned}$$

We finish this subsection with an analogue of Theorem 81 for left-regular functions. Of course, similar result holds for right-regular functions. For convergence reasons we need to exclude the functions with components containing $f_{l,m,0}^\pm(Y)$. Thus we define $\mathbb{S}(\mathbb{M}^+)$ to be the space of \mathbb{S} -valued left-regular functions on \mathbb{M}^+ with both components orthogonal to all $f_{l,m,0}^\pm(Y)$'s with respect to $\langle \cdot, \cdot \rangle_{\min}$. Let $P_0^{\text{discr}(\mathbb{M})}$ and $P_\infty^{\text{discr}(\mathbb{M})}$ be the projections onto the discrete series components of $\mathbb{S}(\mathbb{M}^+)$ quasi-regular at 0 and ∞ respectively.

Theorem 84 *Let $f \in \mathbb{S}(\mathbb{M}^+)$ a left-regular function and $W \in \mathbb{M}^+$, then*

$$\begin{aligned} & (P_0^{\text{discr}(\mathbb{M})}(f) - P_\infty^{\text{discr}(\mathbb{M})}(f))(W) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi^2} \int_{Y \in \tilde{H}_R} \left(\frac{(Y - W)^+}{(N(Y - W) + i\varepsilon)^2} - \frac{(Y - W)^+}{(N(Y - W) - i\varepsilon)^2} \right) \cdot Dy \cdot f(Y). \end{aligned}$$

Proof. Suppose that $f : \mathbb{M}^+ \rightarrow \mathbb{S}$ is left-regular and homogeneous of degree $n - 2$ (i.e. $(\widetilde{\text{deg}} + 1)f = nf$) with $n \neq 0$. Let $\varphi(Y) = Yf(Y) : \mathbb{M}^+ \rightarrow \mathbb{S}$ be an \mathbb{S} -valued function, then $\widetilde{\text{deg}}\varphi = n\varphi$, each coordinate component of φ is a harmonic function and “restricting” (11) to \mathbb{M} we obtain

$$\nabla_{\mathbb{M}}\varphi = (Y^+\nabla_{\mathbb{M}}^+ + \nabla_{\mathbb{M}}Y)f = 2(\widetilde{\text{deg}} + 1)f = 2nf \neq 0. \quad (55)$$

For concreteness, let us assume that $n > 0$. (If f is quasi-regular at infinity, the signs get reversed when we apply Theorem 81.) Then

$$\begin{aligned} & \varphi(W) + R^{2n} \cdot N(W)^{-1} \cdot \varphi(W/N(W)) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi^2} \int_{Y \in \tilde{H}_R} \left(\frac{1}{N(Y - W) + i\varepsilon} - \frac{1}{N(Y - W) - i\varepsilon} \right) \cdot (\widetilde{\text{deg}}_Y\varphi)(Y) \frac{dS}{\|Y\|} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi^2} \int_{Y \in \tilde{H}_R} \left(\frac{1}{N(Y - W) + i\varepsilon} - \frac{1}{N(Y - W) - i\varepsilon} \right) \cdot Dy \cdot (\widetilde{\text{deg}}_Y + 1)(Y^{-1}\varphi)(Y) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{i}{4\pi^2} \int_{Y \in \tilde{H}_R} \left(\frac{1}{N(Y - W) + i\varepsilon} - \frac{1}{N(Y - W) - i\varepsilon} \right) \cdot Dy \cdot \nabla_{\mathbb{M}}\varphi(Y). \end{aligned}$$

Differentiating with respect to W we get

$$\begin{aligned} & \nabla_{\mathbb{M}} \left(\varphi(W) + R^{2n} \cdot N(W)^{-1} \cdot \varphi(W/N(W)) \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi^2} \int_{Y \in \tilde{H}_R} \left(\frac{(Y - W)^+}{(N(Y - W) + i\varepsilon)^2} - \frac{(Y - W)^+}{(N(Y - W) - i\varepsilon)^2} \right) \cdot Dy \cdot \nabla_{\mathbb{M}}\varphi(Y) \end{aligned}$$

Finally, by (55),

$$\begin{aligned}
& \nabla_{\mathbb{M}} \left(N(W)^{-1} \cdot \varphi(W/N(W)) \right) \\
&= \frac{1}{N(W)^2} \cdot (\nabla_{\mathbb{M}} \varphi) \left(\frac{W}{N(W)} \right) - 2 \frac{W^{-1}}{N(W)} \cdot \varphi \left(\frac{W}{N(W)} \right) - 2 \frac{W^{-1}}{N(W)^2} \cdot (\deg \varphi) \left(\frac{W}{N(W)} \right) \\
&= \frac{1}{N(W)^2} \cdot (\nabla_{\mathbb{M}} \varphi) \left(\frac{W}{N(W)} \right) - \frac{2}{N(W)^2} \cdot f \left(\frac{W}{N(W)} \right) - \frac{2}{N(W)^2} \cdot (\widetilde{\deg f}) \left(\frac{W}{N(W)} \right) = 0.
\end{aligned}$$

□

5 Separation of the Series and the Plancherel Measure on $SU(1, 1)$

5.1 Convergence and Equivariant Properties of Poisson Integrals

In this section we regularize the Poisson and Cauchy-Fueter integrals by replacing $N(X - W)$ in the denominator with $N(X - W) \pm i\varepsilon$ and letting $\varepsilon \rightarrow 0^+$. First we prove that the resulting limits converge. Recall that $H_R = \{X \in \mathbb{H}_{\mathbb{R}}; N(X) = R^2\} \subset \mathbb{H}_{\mathbb{R}}^+$, where $R > 0$.

Proposition 85 *Fix an $R > 0$ and a positive integer n , then for any Schwartz function ψ on H_R and any $W \in \mathbb{H}_{\mathbb{R}}^+$ with $N(W) \neq R^2$, the limit*

$$\tilde{\psi}(W) = \lim_{\varepsilon \rightarrow 0^+} \int_{H_R} \frac{\psi(X)}{(N(X - W) + i\varepsilon)^n} \frac{dS}{\|X\|} \quad (56)$$

determines a real-analytic function of W . When $n = 1$, $\square_{2,2}\tilde{\psi} = 0$. Similar statements hold for $\varepsilon \rightarrow 0^-$ as well.

Remark 86 *The function ψ itself need not satisfy $\square_{2,2}\psi = 0$. The limits as $\varepsilon \rightarrow 0^+$ and $\varepsilon \rightarrow 0^-$ can yield different functions. We will see later that the difference between these two limits corresponds to the continuous series component of ψ .*

Proof. This proposition would be trivial if $\text{supp } \psi \cap \{X \in H_R; N(X - W) = 0\} = \emptyset$. On the other hand, we will rewrite the integral so that the only “problematic” points are the critical points of $N(X - W)$ restricted to H_R (as a function of X). It is easy to see that the only such critical points are $X_{crit}^{\pm} = \pm \frac{R}{\sqrt{N(W)}} W$, and $N(X_{crit}^{\pm} - W) \neq 0$. Then we apply a partition of unity argument to break up the function ψ into $\psi_0 + \psi_c$ so that ψ_0 is supported away from $\{X \in H_R; N(X - W) = 0\}$ and ψ_c is supported away from X_{crit}^{\pm} .

Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth compactly supported function such that $\eta(t) = 1$ for t near 0 and $\eta(t) = 0$ for t near $N(X_{crit}^{\pm} - W)$. Then the composition function $\eta(N(X - W))$ is one in a neighborhood of $\{X \in H_R; N(X - W) = 0\}$ and zero in a neighborhood of X_{crit}^{\pm} . Define

$$\psi_0(X) = (1 - \eta(N(X - W))) \cdot \psi(X), \quad \psi_c(X) = \eta(N(X - W)) \cdot \psi(X),$$

so that $\psi = \psi_0 + \psi_c$, and set

$$\psi_{0,\varepsilon}(W) = \int_{H_R} \frac{\psi_0(X)}{(N(X - W) + i\varepsilon)^n} \frac{dS}{\|X\|}, \quad \psi_{c,\varepsilon}(W) = \int_{H_R} \frac{\psi_c(X)}{(N(X - W) + i\varepsilon)^n} \frac{dS}{\|X\|}.$$

Clearly, the limit

$$\tilde{\psi}_0(W) = \lim_{\varepsilon \rightarrow 0^+} \psi_{0,\varepsilon}(W) = \int_{H_R} \frac{\psi_0(X)}{N(X - W)^n} \frac{dS}{\|X\|}$$

is a well-defined real-analytic function of W .

Next we analyze $\psi_{c,\varepsilon}(W)$. Consider the case $n = 1$ first and observe that

$$\frac{1}{N(X - W) + i\varepsilon} = (-i) \int_0^\infty \exp(it(N(X - W) + i\varepsilon)) dt,$$

and the integral converges absolutely since the real part of the exponent is $-\varepsilon t < 0$. Thus we can rewrite $\psi_{c,\varepsilon}(W)$ as

$$\psi_{c,\varepsilon}(W) = (-i) \int_0^\infty \left(\int_{H_R} \exp(it(N(X - W) + i\varepsilon)) \cdot \psi_c(X) \frac{dS}{\|X\|} \right) dt,$$

where we are allowed to switch the order of integration because $|\exp(it(N(X - W) + i\varepsilon))| \leq 1$ and the function ψ_c is absolutely integrable on H_R with respect to the measure $\frac{dS}{\|X\|}$. Then the limit

$$\tilde{\psi}_c(W) = \lim_{\varepsilon \rightarrow 0^+} \psi_{c,\varepsilon}(W) = (-i) \int_0^\infty \left(\int_{H_R} \exp(itN(X - W)) \cdot \psi_c(X) \frac{dS}{\|X\|} \right) dt$$

is a well-defined real-analytic function of W . (The integral converges by a standard integration by parts argument; this is where the property $X_{crit}^\pm \notin \text{supp } \psi_c$ is used.)

If $n > 1$, then we can apply the same argument with

$$\frac{1}{(N(X - W) + i\varepsilon)^n} = (-i)^n \int_0^\infty \dots \int_0^\infty \exp(i(t_1 + \dots + t_n)(N(X - W) + i\varepsilon)) dt_1 \dots dt_n.$$

Differentiating under the integral sign we see that $\square_{2,2}\tilde{\psi} = 0$ when $n = 1$. \square

Lemma 87 *The limits*

$$\lim_{\varepsilon \rightarrow 0^+} \int_{H_R} \frac{\psi(X)}{N(X - W) + i\varepsilon} \frac{dS}{\|X\|} \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \int_{H_R} \frac{\psi(X)}{N(X - W) - i\varepsilon} \frac{dS}{\|X\|} \quad (57)$$

determine continuous functions of W on $\mathbb{H}_\mathbb{R}^+$ (even if $N(W) = R^2$).

Proof. Note that when $N(W) = R^2$, i.e. $W \in H_R$, the above argument fails because $X_{crit}^\pm = \pm W$ and $N(X_{crit}^+ - W) = 0$. However, if we remove a small neighborhood V of W in H_R , the above argument still shows that the limits

$$\lim_{\varepsilon \rightarrow 0^+} \int_{H_R \setminus V} \frac{\psi(X)}{N(X - W) \pm i\varepsilon} \frac{dS}{\|X\|}$$

converge. On the other hand, as explained in [GS], the limits

$$\lim_{\varepsilon \rightarrow 0^+} \int_V \frac{\psi(X)}{N(X - W) \pm i\varepsilon} \frac{dS}{\|X\|}$$

converge as well (which is not hard to see directly). Thus we conclude that for $R > 0$, $W \in H_R$ and a Schwartz function ψ on H_R the limits (57) converge. \square

Recall that $\widetilde{\text{deg}}$ is the degree operator plus identity: $\widetilde{\text{deg}} = 1 + \sum_{i=0}^3 x^i \frac{\partial}{\partial x^i}$. For each $R > 0$, we define operators I_R^+ and I_R^- from $\mathcal{H}(\mathbb{H}_\mathbb{R}^+)$ into functions on $\mathbb{H}_\mathbb{R}^+$

$$\begin{aligned} (I_R^+ \varphi)(W) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{H_R} \frac{(\widetilde{\text{deg}}_X \varphi)(X)}{N(X - W) + i\varepsilon} \frac{dS}{\|X\|}, \\ (I_R^- \varphi)(W) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{H_R} \frac{(\widetilde{\text{deg}}_X \varphi)(X)}{N(X - W) - i\varepsilon} \frac{dS}{\|X\|}, \quad \varphi \in \mathcal{H}(\mathbb{H}_\mathbb{R}^+), \quad W \in \mathbb{H}_\mathbb{R}^+. \end{aligned}$$

The functions $(I_R^\pm \varphi)(W)$ are real-analytic and harmonic for $N(W) \neq R^2$, continuous when $N(W) = R^2$. We finish this subsection by proving that the regularized Poisson integrals are equivariant with respect to the $G(H_R)$ -action.

Proposition 88 *The operators I_R^+ and I_R^- are equivariant with respect to the π_l^0 -action of $G(H_R)$, where $G(H_R) \simeq SO^+(3, 2)$ is the group introduced in Corollary 26.*

Proof. The operators I_R^\pm are equivariant with respect to the π_l^0 action of $SU(1, 1) \times SU(1, 1)$. By Lemma 28 it is sufficient to show that I_R^\pm are equivariant with respect to the one-parameter group $G(H_R)'$ introduced in that lemma. We want to compare $(I_R^\pm \tilde{\varphi})(W)$ with $(\widetilde{I_R^\pm \varphi})(W)$, where $\tilde{\varphi} = \pi_l^0(g)\varphi$ and $(\widetilde{I_R^\pm \varphi}) = \pi_l^0(g)(I_R^\pm(\varphi))$, $g \in G(H_R)'$. Since $G(H_R)'$ is connected, it is sufficient to verify the $G(H_R)'$ -equivariance of I_R^\pm on the Lie algebra level. For $t \rightarrow 0$ and modulo terms of order t^2 , we have (25), (26) and (27). Using Lemma 30, we get

$$\begin{aligned} (I_R^\pm \tilde{\varphi})(W) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{X \in H_R} \frac{1}{N(X - W) \pm i\varepsilon} \cdot \widetilde{\deg}_X \left(\frac{\varphi(\tilde{X})}{N(-R^{-1} \sinh tX + \cosh t)} \right) \frac{dS}{\|X\|} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{X \in H_R} \frac{(1 + 4t \operatorname{Re} X/R) \cdot (\widetilde{\deg}_X \varphi)(\tilde{X})}{N(X - W) \pm i\varepsilon} \frac{dS}{\|X\|}. \end{aligned}$$

On the other hand, using Lemma 10 from [FL1] and Lemma 29,

$$\begin{aligned} (\widetilde{I_R^\pm \varphi})(W) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{N(-R^{-1} \sinh tW + \cosh t)} \cdot \frac{1}{2\pi^2} \int_{\tilde{X} \in H_R} \frac{(\widetilde{\deg}_X \varphi)(\tilde{X})}{N(\tilde{X} - \tilde{W}) \pm i\varepsilon} \frac{dS}{\|\tilde{X}\|} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{X \in H_R} \frac{(1 + 4t \operatorname{Re} X/R) \cdot (\widetilde{\deg}_X \varphi)(\tilde{X})}{N(X - W) \pm i\varepsilon (1 - 2t \operatorname{Re} X/R)(1 - 2t \operatorname{Re} W/R)} \frac{dS}{\|X\|} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{X \in H_R} \frac{(1 + 4t \operatorname{Re} X/R) \cdot (\widetilde{\deg}_X \varphi)(\tilde{X})}{N(X - W) \pm i\varepsilon} \frac{dS}{\|X\|}. \end{aligned}$$

This proves that the maps I_R^\pm are $G(H_R)$ -equivariant. \square

5.2 Regularized Integrals and the Discrete Series Component on $\mathbb{H}_\mathbb{R}$

In this subsection we study the effect of the operators I_R^+ and I_R^- on the discrete series component of $\mathcal{H}(\mathbb{H}_\mathbb{R}^+)$. We prove an analogue of Poisson formula for the discrete series.

Theorem 89 *The operators I_R^+ and I_R^- coincide on the discrete series component of $\mathcal{H}(\mathbb{H}_\mathbb{R}^+)$. Moreover,*

$$\begin{aligned} I_R^\pm : \quad N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l(Z) &\mapsto \begin{cases} -N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W) & \text{if } N(W) \leq R^2; \\ -R^{-2(2l+1)} \cdot t_{n\underline{m}}^l(W) & \text{if } N(W) \geq R^2; \end{cases} \\ t_{n\underline{m}}^l(Z) &\mapsto \begin{cases} R^{2(2l+1)} \cdot N(W)^{-2l-1} \cdot t_{n\underline{m}}^l(W) & \text{if } N(W) \leq R^2; \\ t_{n\underline{m}}^l(W) & \text{if } N(W) \geq R^2. \end{cases} \end{aligned}$$

(Compare with Theorem 47.)

Proof. The proof is based on two lemmas.

Lemma 90 *Let φ be a discrete series matrix coefficient*

$$t_{l\underline{l}}^l(Z), \quad N(Z)^{-2l-1} \cdot t_{l\underline{l}}^l(Z), \quad t_{-l\underline{-l}}^l(Z), \quad N(Z)^{-2l-1} \cdot t_{-l\underline{-l}}^l(Z), \quad l \leq -1, \quad l \in \mathbb{Z}/2,$$

Then, for any $R' > 0$, $R' \neq R$, the functions on $H_{R'}$

$$(I_R^{\pm} \varphi)|_{H_{R'}}, \quad (I_{R'}^{\pm} \varphi)|_{H_{R'}} \quad \text{and} \quad \varphi|_{H_{R'}} \quad \text{are proportional.}$$

Proof. By Proposition 88 the maps $\varphi \mapsto (I_R^{\pm} \varphi)|_{H_{R'}}$ are $(SU(1, 1) \times SU(1, 1))$ -equivariant. Let

$$k_{\varphi} = \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix}, \quad \text{then by (16) we have}$$

$$t_{n\underline{m}}^l(k_{\varphi} \cdot Z \cdot k_{\psi}) = e^{-i(n\varphi+m\psi)} \cdot t_{n\underline{m}}^l(Z), \quad N(k_{\varphi} \cdot Z \cdot k_{\psi}) = N(Z).$$

Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard generators of $\mathfrak{sl}(2, \mathbb{C})$. From (14) and the Lie algebra action formulas given in the proof of Theorem 39 we see that

$$\pi_l^0 \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = \pi_r^0 \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_l^0 \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} = \pi_r^0 \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \quad \text{annihilate} \quad t_{l\underline{l}}^l(Z) \quad \text{and} \\ N(Z)^{-2l-1} \cdot t_{l\underline{l}}^l(Z);$$

$$\pi_l^0 \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} = \pi_r^0 \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_l^0 \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} = \pi_r^0 \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \quad \text{annihilate} \quad t_{-l\underline{-l}}^l(Z) \quad \text{and} \\ N(Z)^{-2l-1} \cdot t_{-l\underline{-l}}^l(Z).$$

At a generic point $g \in SU(1, 1)$, vector fields

$$\pi_l^0 \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_l^0 \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}, \quad \pi_l^0 \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_l^0 \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$$

span the entire complexified tangent space $T_g^{\mathbb{C}} SU(1, 1)$ at g . Since $(I_R^{\pm} \varphi)|_{H_{R'}}$ with $\varphi = t_{l\underline{l}}^l(Z)$ or $N(Z)^{-2l-1} \cdot t_{l\underline{l}}^l(Z)$ are analytic, it follows from the $(SU(1, 1) \times SU(1, 1))$ -equivariance that $(I_R^{\pm} \varphi)|_{H_{R'}}$ are proportional to $\varphi|_{H_{R'}}$. Similarly, $(I_{R'}^{\pm} \varphi)|_{H_{R'}}$ are proportional to $\varphi|_{H_{R'}}$ when $\varphi = t_{-l\underline{-l}}^l(Z)$ or $N(Z)^{-2l-1} \cdot t_{-l\underline{-l}}^l(Z)$. \square

Lemma 91 *Let $\varphi(Z)$ be $t_{l\underline{l}}^l(Z) = (z_{22})^{2l}$ or $t_{-l\underline{-l}}^l(Z) = (z_{11})^{2l}$, and let $W = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ be diagonal with $\lambda > 0$. Then*

$$(I_R^{\pm} t_{l\underline{l}}^l)(W) = (I_R^{\pm} t_{-l\underline{-l}}^l)(W) = \begin{cases} R^{2(2l+1)} \cdot \lambda^{-2(l+1)} & \text{if } \lambda < R; \\ \lambda^{2l} & \text{if } \lambda > R. \end{cases}$$

The proof is done by direct computation, and we postpone it until next subsection. Together Lemmas 90 and 91 imply that the theorem holds for functions

$$t_{l\underline{l}}^l(Z), \quad t_{-l\underline{-l}}^l(Z), \quad N(Z)^{-2l-1} \cdot t_{l\underline{l}}^l(Z), \quad N(Z)^{-2l-1} \cdot t_{-l\underline{-l}}^l(Z)$$

for $l \leq -1$. Then the theorem follows from the $(SU(1, 1) \times SU(1, 1))$ -equivariance of I_R^{\pm} .

\square

5.3 Proof of Lemma 91

In this subsection we give a proof of Lemma 91.

Proof. We use integral formulas 2.117 and 2.662 from [GR]

$$\begin{aligned} \int \frac{dx}{x^n(a+bx)} &= \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + (-1)^n \frac{b^{n-1}}{a^n} \log \frac{a+bx}{x} + C, \\ \int \frac{dx}{x^n(a-bx)} &= -\sum_{k=1}^{n-1} \frac{b^{k-1}}{(n-k)a^k x^{n-k}} - \frac{b^{n-1}}{a^n} \log \frac{bx-a}{x} + C, \\ \int e^{ax} (\cos x)^{2m} dx &= \binom{2m}{m} \frac{e^{ax}}{2^{2m} a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^m \binom{2m}{m-k} \frac{a \cos(2kx) + 2k \sin(2kx)}{a^2 + 4k^2} + C, \\ \int e^{ax} (\cos x)^{2m+1} dx &= \frac{e^{ax}}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{m-k} \frac{a \cos(2k+1)x + (2k+1) \sin(2k+1)x}{a^2 + (2k+1)^2} + C. \end{aligned}$$

We parameterize H_R as in (15) so that

$$X(\varphi, \tau, \psi) = R \begin{pmatrix} \cosh \frac{\tau}{2} \cdot e^{i\frac{\varphi+\psi}{2}} & \sinh \frac{\tau}{2} \cdot e^{i\frac{\varphi-\psi}{2}} \\ \sinh \frac{\tau}{2} \cdot e^{i\frac{\psi-\varphi}{2}} & \cosh \frac{\tau}{2} \cdot e^{-i\frac{\varphi+\psi}{2}} \end{pmatrix}, \quad \begin{aligned} 0 &\leq \varphi < 2\pi, \\ 0 &< \tau < \infty, \\ -2\pi &\leq \psi < 2\pi. \end{aligned}$$

Then $\frac{dS}{\|X\|} = \frac{R^2}{8} \sinh \tau d\varphi d\tau d\psi$ and

$$N(X - W) = R^2 + \lambda^2 - 2R\lambda \cosh \frac{\tau}{2} \cdot \cos \frac{\varphi + \psi}{2}.$$

Changing variables $\theta = \frac{\varphi+\psi}{2}$, $\theta' = \frac{\varphi-\psi}{2}$, $x = \cosh \frac{\tau}{2}$ and letting $a(\varepsilon) = R^2 + \lambda^2 + i\varepsilon$, $b(\theta) = 2R\lambda \cos \theta$, we obtain

$$\begin{aligned} (I_R^\pm t_{l\bar{l}})(W) &= (I_R^\pm t_{-l\underline{-l}})(W) \\ &= \frac{2l+1}{16\pi^2} \lim_{\varepsilon \rightarrow 0^\pm} \int_{\psi=-2\pi}^{2\pi} \int_{\tau=0}^{\infty} \int_{\varphi=0}^{2\pi} \frac{R^{2l+2} \cdot (\cosh \frac{\tau}{2})^{2l} \cdot e^{\pm il(\varphi+\psi)} \cdot \sinh \tau}{N(W-X) + i\varepsilon} d\varphi d\tau d\psi \\ &= \frac{2l+1}{4\pi^2} \lim_{\varepsilon \rightarrow 0^\pm} \int_{\theta=0}^{2\pi} \int_{\tau=0}^{\infty} \int_{\theta'=0}^{2\pi} \frac{R^{2l+2} \cdot (\cosh \frac{\tau}{2})^{2l+1} \cdot e^{\pm i2l\theta} \cdot \sinh \frac{\tau}{2}}{R^2 + \lambda^2 - 2R\lambda \cos \theta \cdot \cosh \frac{\tau}{2} + i\varepsilon} d\theta' d\tau d\theta \\ &= \frac{2l+1}{\pi} \lim_{\varepsilon \rightarrow 0^\pm} \int_{\theta=-\pi/2}^{\pi/2} R^{2l+2} \cdot e^{\pm i2l\theta} \int_{x=1}^{\infty} \left(\frac{x^{2l+1}}{a(\varepsilon) - b(\theta)x} + \frac{(-1)^{2l} x^{2l+1}}{a(\varepsilon) + b(\theta)x} \right) dx d\theta \\ &= \frac{2l+1}{\pi} \lim_{\varepsilon \rightarrow 0^\pm} \int_{\theta=-\pi/2}^{\pi/2} R^{2l+2} \cdot e^{\pm i2l\theta} \left(\frac{a(\varepsilon)^{2l+1}}{b(\theta)^{2l+2}} \left(\log \left(\frac{a(\varepsilon) - b(\theta)}{b(\theta)} \right) + \log \left(\frac{a(\varepsilon) + b(\theta)}{b(\theta)} \right) \mp \pi i \right) \right. \\ &\quad \left. - \sum_{k=1}^{-2l-2} \frac{(1 - (-1)^{2l+k}) \cdot b(\theta)^{k-1}}{(2l+k+1)a(\varepsilon)^k} \right) d\theta \\ &= \frac{2l+1}{4\pi} \cdot \frac{(R^2 + \lambda^2)^{2l+1}}{2^{2l}\lambda^{2l+2}} \lim_{\varepsilon \rightarrow 0^\pm} \int_{\theta=-\pi/2}^{\pi/2} \frac{e^{\pm i2l\theta}}{(\cos \theta)^{2l+2}} \cdot \left(\log \left(\frac{a(\varepsilon) - b(\theta)}{b(\theta)} \right) + \log \left(\frac{a(\varepsilon) + b(\theta)}{b(\theta)} \right) \right) d\theta. \end{aligned}$$

From now on we assume $2l$ is odd; the other case is similar. Integrating by parts we get

$$(I_R^\pm t_{l\bar{l}})(W) = (I_R^\pm t_{-l\underline{-l}})(W) = \frac{2l+1}{2\pi} \cdot \frac{(R^2 + \lambda^2)^{2l+1}}{\lambda^{2l+2}} \sum_{k=0}^{-l-\frac{3}{2}} \binom{-2l-2}{-l-k-\frac{3}{2}} \frac{I(l, k)}{4l^2 - (2k+1)^2},$$

where the integrals $I(l, k)$ can be computed using residues

$$\begin{aligned}
I(l, k) &= \int_{\theta=-\pi/2}^{\pi/2} e^{\pm i2l\theta} ((2k+1) \sin(2k+1)\theta \pm i2l \cos(2k+1)\theta) \cdot \left(\log \left(\frac{(R^2 + \lambda^2)^2 - b^2(\theta)}{b^2(\theta)} \right) \right)' d\theta \\
&= 2(R^2 + \lambda^2)^2 \int_{\theta=-\pi/2}^{\pi/2} e^{\pm i2l\theta} \cdot \tan \theta \cdot \frac{(2k+1) \sin(2k+1)\theta \pm i2l \cos(2k+1)\theta}{(R^2 + \lambda^2)^2 - 4R^2\lambda^2 \cos^2 \theta} d\theta \\
&= (R^2 + \lambda^2)^2 \int_{\theta=-\pi}^{\pi} \tan \theta \cdot \frac{(2k+1) \cos(2l\theta) \cdot \sin(2k+1)\theta - 2l \sin(2l\theta) \cdot \cos(2k+1)\theta}{(R^2 + \lambda^2)^2 - 4R^2\lambda^2 \cos^2 \theta} d\theta = \\
&= \frac{i(R^2 + \lambda^2)^2}{4} \oint \frac{s^2 - 1}{s^2 + 1} \cdot \frac{(2k+1)(s^{2l} + s^{-2l})(s^{2k+1} - s^{-2k-1}) - 2l(s^{2l} - s^{-2l})(s^{2k+1} + s^{-2k-1})}{(R^2 + \lambda^2)^2 - R^2\lambda^2(s + s^{-1})^2} \frac{ds}{s}.
\end{aligned}$$

From the factorization

$$(R^2 + \lambda^2)^2 - R^2\lambda^2(s + s^{-1})^2 = -\frac{R^2\lambda^2}{s^2}(s - R/\lambda)(s + R/\lambda)(s - \lambda/R)(s + \lambda/R)$$

and decomposition

$$\frac{(R^2 + \lambda^2)^2 \cdot (s^2 - 1)}{(R^2 + \lambda^2)^2 - R^2\lambda^2(s + s^{-1})^2} = \frac{R^2}{R^2 - \lambda^2 s^2} + \frac{\lambda^2}{\lambda^2 - R^2 s^2} - \frac{2}{1 + s^2}$$

we see that the residues are

$$\begin{aligned}
&\frac{1}{2i} \left((2k+1-2l) \left((\lambda/R)^{2l+2k+1} - (R/\lambda)^{2l+2k+1} \right) \right. \\
&\quad \left. + (2l+2k+1) \left((\lambda/R)^{2k+1-2l} - (R/\lambda)^{2k+1-2l} \right) \right) \cdot \begin{cases} +1 & \text{at } \pm \lambda/R, \\ -1 & \text{at } \pm R/\lambda \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&i(2k+1-2l) \left((\lambda/R)^{2l+2k+1} + (R/\lambda)^{2l+2k+1} - 2 \cdot i^{2l+2k+1} \right) \\
&\quad - i(2l+2k+1) \left((\lambda/R)^{2l-2k-1} + (R/\lambda)^{2l-2k-1} - 2 \cdot i^{2l-2k-1} \right) \quad \text{at } 0.
\end{aligned}$$

Letting $r = \lambda/R$ if $\lambda < R$ and letting $r = R/\lambda$ if $\lambda > R$ we get:

$$\begin{aligned}
(I_R^\pm t_{l_l}^l)(W) &= (I_R^\pm t_{-l}^l)(W) = \\
&= -2(2l+1) \cdot \frac{(R^2 + \lambda^2)^{2l+1}}{\lambda^{2l+2}} \sum_{k=0}^{-l-\frac{3}{2}} \binom{-2l-2}{-l-k-\frac{3}{2}} \left(\frac{r^{-2l-2k-1} - i^{-2l-2k-1}}{-2l-2k-1} + \frac{r^{-2l+2k+1} - i^{-2l+2k+1}}{-2l+2k+1} \right) \\
&= -2(2l+1) \cdot \frac{(R^2 + \lambda^2)^{2l+1}}{\lambda^{2l+2}} \sum_{k=0}^{-l-\frac{3}{2}} \binom{-2l-2}{-l-k-\frac{3}{2}} \int_i^r (x^{-2l-2k-2} + x^{-2l+2k}) dx \\
&= -\frac{(R^2 + \lambda^2)^{2l+1}}{\lambda^{2l+2}} \int_i^r 2(2l+1)x(1+x^2)^{-2l-2} dx \\
&= \frac{(R^2 + \lambda^2)^{2l+1}}{\lambda^{2l+2}} \cdot (1+r^2)^{-2l-1} = \begin{cases} R^{2(2l+1)} \cdot \lambda^{-2(l+1)} & \text{if } \lambda < R; \\ \lambda^{2l} & \text{if } \lambda > R. \end{cases}
\end{aligned}$$

□

5.4 Regularized Integrals and the Continuous Series Component on $\mathbb{H}_{\mathbb{R}}$

In this subsection we determine the actions of the operators I_R^{\pm} on the continuous series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^{\pm})$. Combining this with Theorem 89 we get a complete description of the actions of I_R^{\pm} on $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^{\pm})$. We state the main result:

Theorem 92 *The operator $I_R^+ - I_R^-$ annihilates the discrete series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. Its action on the continuous series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$ is given by*

$$((I_R^+ - I_R^-)t_{n\underline{m}}^l)(W) = (1 + R^{4\text{Im}l} \cdot N(W)^{-2l-1}) \cdot t_{n\underline{m}}^l(W) \cdot \begin{cases} \coth(\pi \text{Im}l) & \text{if } m, n \in \mathbb{Z}; \\ \tanh(\pi \text{Im}l) & \text{if } m, n \in \mathbb{Z} + \frac{1}{2}, \end{cases}$$

where $\text{Re}l = -\frac{1}{2}$ and $W \in \mathbb{H}_{\mathbb{R}}^+$.

The proof of this theorem will be given in the next subsection.

Remark 93 *Strictly speaking, the matrix coefficient functions of the continuous series representations $t_{n\underline{m}}^l(X)$, $l = -1/2 + i\lambda$ with $\lambda \in \mathbb{R}$, do not belong to $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. So the theorem should be restated as follows: If*

$$\varphi(X) = \int_{\lambda \in \mathbb{R}} t_{n\underline{m}}^{-\frac{1}{2}+i\lambda}(X) \cdot \eta(\lambda) d\lambda,$$

where $\eta(\lambda)$ is a smooth compactly supported function on $\mathbb{R} \setminus \{0\}$, then $((I_R^+ - I_R^-)\varphi)(W)$ is

$$\begin{cases} \int_{\lambda \in \mathbb{R}} \coth(\pi\lambda) \cdot (1 + R^{4\lambda} \cdot N(W)^{-2\lambda}) \cdot t_{n\underline{m}}^{-\frac{1}{2}+i\lambda}(W) \cdot \eta(\lambda) d\lambda & \text{if } m, n \in \mathbb{Z}; \\ \int_{\lambda \in \mathbb{R}} \tanh(\pi\lambda) \cdot (1 + R^{4\lambda} \cdot N(W)^{-2\lambda}) \cdot t_{n\underline{m}}^{-\frac{1}{2}+i\lambda}(W) \cdot \eta(\lambda) d\lambda & \text{if } m, n \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

Note that the expression

$$\begin{cases} \frac{\coth(\pi \text{Im}l)}{\text{Im}l} & \text{if } m, n \in \mathbb{Z}; \\ \frac{\tanh(\pi \text{Im}l)}{\text{Im}l} & \text{if } m, n \in \mathbb{Z} + \frac{1}{2} \end{cases}$$

is precisely the inverse of the Plancherel measure on $SU(1, 1)$! Since $\widetilde{\text{deg}}t_{n\underline{m}}^l = 2 \text{Im}l \cdot t_{n\underline{m}}^l$, we can reformulate Theorem 92 as follows:

Theorem 94 *The operator*

$$\varphi(X) \mapsto (\text{Pl}_R \varphi)(W) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi^2} \int_{X \in H_R} \left(\frac{1}{N(X - W) + i\varepsilon} - \frac{1}{N(X - W) - i\varepsilon} \right) \varphi(X) \frac{dS}{\|X\|}$$

annihilates the discrete series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. If $\text{Re}l = -\frac{1}{2}$ and $W \in \mathbb{H}_{\mathbb{R}}^+$,

$$(\text{Pl}_R t_{n\underline{m}}^l)(W) = (1 + R^{4\text{Im}l} \cdot N(W)^{-2l-1}) \cdot t_{n\underline{m}}^l(W) \cdot \begin{cases} \frac{\coth(\pi \text{Im}l)}{\text{Im}l} & \text{if } m, n \in \mathbb{Z}; \\ \frac{\tanh(\pi \text{Im}l)}{\text{Im}l} & \text{if } m, n \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

Remark 95 *As in the Minkowski case (see Remark 82), for each $W \in \mathbb{H}_{\mathbb{R}}^+$, integrals $((I_R^+ - I_R^-)\varphi)(W)$ and $(\text{Pl}_R \varphi)(W)$ depend only on the values of $(\widetilde{\text{deg}}\varphi)(X)$, where X ranges over a two-dimensional hyperboloid (or cone if $W \in H_R$)*

$$\{X \in \mathbb{H}_{\mathbb{R}}; N(X) = R^2, N(X - W) = 0\}.$$

We can rewrite the expression $\frac{1}{N(X-W)+i0} - \frac{1}{N(X-W)-i0}$ geometrically as follows. Let us consider $W' = (1 + \varepsilon\xi)W \in \mathbb{H}_{\mathbb{C}}$, $\varepsilon > 0$, $\xi \in \mathbb{C} \setminus \mathbb{R}$. Then, for $X \in SU(1, 1)$, $W \in \mathbb{H}_{\mathbb{R}}^+$,

$$N(X - W') = N(W) \cdot N(X \cdot W^{-1} - (1 + \varepsilon\xi)) \neq 0$$

for all $\varepsilon \ll 1$ because $X \cdot W^{-1} \in N(W)^{-1/2} \cdot SU(1, 1)$ cannot have an eigenvalue $1 + \varepsilon\xi$. When $N(X - W) = 0$, $N(X - W') = \varepsilon\xi(N(W) - N(X)) + \varepsilon^2\xi^2N(W)$. Choose $\xi_1, \xi_2 \in \mathbb{C}$ with $\text{Im}(\xi_1) > 0$ and $\text{Im}(\xi_2) < 0$, then

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \text{sign}(N(W) - 1) \cdot \int_{X \in SU(1,1)} \left(\frac{1}{N(X - W) + i\varepsilon} - \frac{1}{N(X - W) - i\varepsilon} \right) \varphi(X) \frac{dS}{\|X\|} \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{X \in SU(1,1)} \left(\frac{1}{N(X - (1 + \varepsilon\xi_1)W)} - \frac{1}{N(X - (1 + \varepsilon\xi_2)W)} \right) \varphi(X) \frac{dS}{\|X\|}. \end{aligned}$$

Proposition 96 *The operator $I_R^- + I_R^+$ annihilates the continuous series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$.*

Proof. Let φ be in the continuous series component of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$. Since $(I_R^- + I_R^+)\varphi$ is analytic, it is enough to show that $\widetilde{\text{deg}}^k((I_R^- + I_R^+)\varphi)$ vanishes on H_R for $k = 0, 1, 2, \dots$. By Proposition 50, $(S_{R, \sigma^{-1}}^- \varphi)(W)$ and $(S_{R, \sigma}^+ \varphi)(W)$ vanish identically for W in the open set

$$R \cdot \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix} \cdot \Gamma^- \cap R \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \cdot \Gamma^+, \quad \forall \sigma > 1.$$

For $W \in H_R$ we have:

$$\begin{aligned} & \lim_{\sigma \rightarrow 1^+} (S_{R, \sigma^{-1}}^- \varphi + S_{R, \sigma}^+ \varphi)(W) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{-1}{2\pi^2} \int_{X \in H_R} \left(\frac{\widetilde{(\text{deg}\varphi)}(X)}{N(X - W) + i\varepsilon} + \frac{\widetilde{(\text{deg}\varphi)}(X)}{N(X - W) - i\varepsilon} \right) \frac{dS}{\|X\|} = -(I_R^- \varphi + I_R^+ \varphi)(W). \end{aligned}$$

Hence $(I_R^- + I_R^+)\varphi$ vanishes on H_R along with all derivatives $\widetilde{\text{deg}}^k((I_R^- + I_R^+)\varphi)$, $k = 0, 1, 2, \dots$. \square

This proposition combined with Theorems 89 and 92 gives us a complete description of the actions of I_R^- and I_R^+ on the discrete and continuous series components of $\mathcal{H}(\mathbb{H}_{\mathbb{R}}^+)$.

We conclude this subsection with a result for left-regular functions. (Of course, similar result holds for right-regular functions.)

Theorem 97 *Let $f \in \mathbb{S}(\mathbb{H}_{\mathbb{R}}^+)$ a left-regular function and $W \in \mathbb{H}_{\mathbb{R}}^+$, then the operator*

$$f(X) \mapsto \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{X \in H_R} \left(\frac{(X - W)^+}{(N(X - W) + i\varepsilon)^2} - \frac{(X - W)^+}{(N(X - W) - i\varepsilon)^2} \right) \cdot Dx \cdot f(X)$$

annihilates the discrete series component. If $\text{Re} l = -\frac{1}{2}$,

$$\begin{pmatrix} (l - m + \frac{1}{2})t_{nm+\frac{1}{2}}^l(X) \\ (l + m + \frac{1}{2})t_{n\underline{m-\frac{1}{2}}}^l(X) \end{pmatrix} \mapsto \begin{pmatrix} (l - m + \frac{1}{2})t_{nm+\frac{1}{2}}^l(W) \\ (l + m + \frac{1}{2})t_{n\underline{m-\frac{1}{2}}}^l(W) \end{pmatrix} \cdot \begin{cases} \coth(\pi \text{Im} l) & \text{if } m, n \in \mathbb{Z}; \\ \tanh(\pi \text{Im} l) & \text{if } m, n \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

This theorem follows from Theorem 92 in exactly the same way Theorem 84 follows from Theorem 81 in the Minkowski case.

5.5 Proof of Theorem 92

In this subsection we give a proof of Theorem 92.

Proof. The operators I_R^\pm are equivariant with respect to the subgroup $SU(1,1) \times SU(1,1)$ of $SL(2, \mathbb{H}_\mathbb{C})$. Also, by Theorem 89 the operators I_R^\pm coincide on the discrete series component of $\mathcal{H}(\mathbb{H}_\mathbb{R}^+)$. Therefore, the operator $I_R^+ - I_R^-$ annihilates the discrete series and sends the continuous series component into itself. To determine the action of $I_R^+ - I_R^-$ on the continuous series component we use a couple of reductions. First, it is sufficient to prove the theorem for $R = 1$ only. Indeed, substituting $X' = R^{-1} \cdot X \in SU(1,1)$ and $W' = R^{-1} \cdot W$ we get

$$(I_R^\pm t_{n\mathbf{m}}^l)(W) = R^{2l} \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi^2} \int_{X' \in H_1} \frac{(\widetilde{\deg}_{X'} t_{n\mathbf{m}}^l)(X')}{N(X' - W') \pm i\varepsilon} \frac{dS}{\|X'\|} = R^{2l} \cdot (I_1^\pm t_{n\mathbf{m}}^l)(R^{-1} \cdot W).$$

Secondly, we can reduce the proof of the theorem to the case $N(W) = 1$:

Proposition 98 *When $N(W) = 1$, and $\text{Re } l = -\frac{1}{2}$*

$$((I_1^+ - I_1^-) t_{n\mathbf{m}}^l)(W) = 2t_{n\mathbf{m}}^l(W) \cdot \begin{cases} \coth(\pi \text{Im } l) & \text{if } m, n \in \mathbb{Z}; \\ \tanh(\pi \text{Im } l) & \text{if } m, n \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

We postpone the proof of this proposition until the end of this subsection. By Lemma 10 in [FL1] and direct computation we have:

$$\pi_l^0(\gamma)_X \circ \pi_l^0(\gamma)_W \left(\frac{1}{N(X - W) \pm i\varepsilon} \right) = \frac{1}{N(X' - Y) \pm \frac{i\varepsilon}{4} N(\tilde{e}_3 X' - 1) N(\tilde{e}_3 Y - 1)},$$

where $X' = \pi_l(\gamma^{-1})(X)$ and $Y = \pi_l(\gamma^{-1})(W)$. When $N(X) = N(W) = 1$,

$$N(\tilde{e}_3 X' - 1) N(\tilde{e}_3 Y - 1) = \text{Tr}(\tilde{e}_3 X') \text{Tr}(\tilde{e}_3 Y) = -4 \text{Re}(e_3 X') \text{Re}(e_3 Y).$$

Therefore, by Theorem 60,

$$\begin{aligned} & \text{sign}_3(Y) \cdot \pi_l^0(\gamma) \left((I_1^+ - I_1^-) \varphi \right) (Y) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi^2} \int_{X' \in \tilde{H}} \left(\frac{1}{N(X' - Y) + i\varepsilon} - \frac{1}{N(X' - Y) - i\varepsilon} \right) \cdot (\widetilde{\deg}_{X'}(\pi_l^0(\gamma)\varphi))(X') \frac{dS}{\|X'\|} \\ &= -S_1^{\mathbb{M}}(\pi_l^0(\gamma)\varphi)(Y). \quad (58) \end{aligned}$$

Then by Theorem 81 and Lemma 61 we have $\widetilde{\deg}((I_1^+ - I_1^-)\varphi)(W) = 0$ whenever $W \in SU(1,1)$. Now Theorem 92 follow from Proposition 98 and the uniqueness part of the theorem given on page 343 in [St]. That theorem essentially states that the Cauchy problem for $\square_{2,2}\varphi = 0$ with boundary data on $SU(1,1)$ in the continuous series component of $L^2(SU(1,1))$ has a unique analytic solution. (Caution: there can be other non-analytic solutions.) \square

The rest of this subsection will be devoted to the proof of Proposition 98.

Proof. We start with (58) and track down carefully which functions in the continuous part of $\mathcal{H}(\mathbb{H}_\mathbb{R}^+)$ get sent where in $\mathcal{H}(\mathbb{M}^+)$. The operator $\varphi \mapsto (I_1^+ - I_1^-)\varphi$ is $SU(1,1) \times SU(1,1)$ -equivariant. This means that if φ is a continuous series function, then $(I_1^+ - I_1^-)\varphi|_{SU(1,1)}$ is proportional to φ with a coefficient of proportionality possibly depending on l and whether m, n are integers or half-integers. We essentially compare the expansions of $\widetilde{\deg}_Y(\pi_l^0(\gamma)\varphi)(Y)$ and $\text{sign}_3(Y) \cdot (\pi_l^0(\gamma)\varphi)(Y)$ in terms of basis functions (40) or (41). And to find this coefficient of proportionality we pick a basis function $f_{l,m,n}^\pm$ and find the ratio between the coefficients in those expansions.

Let $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in \tilde{H} \subset \mathbb{M}^+$ and $W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \pi_l(\gamma)(Y) \in SU(1, 1)$. By (37),

$$W = \frac{-i}{y_{22} - y_{11}} \begin{pmatrix} 2 + y_{11} + y_{22} & 2y_{12} \\ -2y_{21} & 2 - y_{11} - y_{22} \end{pmatrix}.$$

From (16)

$$t_{n\underline{m}}^l(\psi_1, \tau, \psi_2) = e^{-i(n+m)\frac{\psi_1+\psi_2}{2}} \cdot e^{-i(n-m)\frac{\psi_1-\psi_2}{2}} \cdot \mathfrak{P}_{n\underline{m}}^l(\cosh \tau).$$

Since

$$\cosh^2 \frac{\tau}{2} = x_{11}x_{22} = -\frac{4 - (y_{11} + y_{22})^2}{(y_{22} - y_{11})^2} = \frac{1 + t^2}{(x^3)^2} = \frac{1}{\cos^2 \theta}, \quad \cosh \tau = \frac{2}{\cos^2 \theta} - 1,$$

$$e^{i(\psi_1+\psi_2)} = \frac{x_{11}}{x_{22}} = \frac{2 + y_{11} + y_{22}}{2 - (y_{11} + y_{22})} = \frac{1 - it}{1 + it} = \frac{1 - i \sinh \rho}{1 + i \sinh \rho}, \quad e^{-i\frac{\psi_1+\psi_2}{2}} = \pm \sqrt{\frac{1 + i \sinh \rho}{1 - i \sinh \rho}}$$

(which is ambiguous), and

$$e^{i\frac{\psi_1-\psi_2}{2}} = \frac{x_{12}}{\sinh \frac{\tau}{2}} = -\frac{2i}{\sinh \frac{\tau}{2}} \frac{y_{12}}{y_{22} - y_{11}} = i \frac{\tan \theta}{\sinh \frac{\tau}{2}} e^{-i\varphi} = i \operatorname{sign}(\cos \theta) \cdot e^{-i\varphi},$$

we obtain

$$t_{n\underline{m}}^l(Y) = (i \operatorname{sign}(\cos \theta))^{m-n} \cdot \mathfrak{P}_{n\underline{m}}^l\left(\frac{2}{\cos^2 \theta} - 1\right) \cdot \left(\pm \sqrt{\frac{1 + i \sinh \rho}{1 - i \sinh \rho}}\right)^{n+m} \cdot e^{i(n-m)\varphi}.$$

To avoid the sign ambiguity we simply set $m + n = 0$. Then, by Lemma 61,

$$\begin{aligned} \operatorname{sign}_3(Y) \cdot (\pi_l^0(\gamma)t_{-m\underline{m}}^l)(Y) &= i \frac{(i \operatorname{sign}(\cos \theta))^{2m}}{\cosh \rho \cdot |\cos \theta|} \cdot \mathfrak{P}_{-m\underline{m}}^l\left(\frac{2}{\cos^2 \theta} - 1\right) \cdot e^{-2im\varphi}, \\ \widetilde{\operatorname{deg}}_Y(\pi_l^0(\gamma)t_{-m\underline{m}}^l)(Y) &= (2l+1) \frac{(i \operatorname{sign}(\cos \theta))^{2m}}{\cosh^2 \rho \cdot \cos^2 \theta} \cdot \mathfrak{P}_{-m\underline{m}}^l\left(\frac{2}{\cos^2 \theta} - 1\right) \cdot e^{-2im\varphi}. \end{aligned}$$

On the other hand, when $N(Y) = 1$, the functions $f_{\lambda, \mu, \nu}^\pm(Y)$ are proportional to

$$r^{-1} \cdot P_\lambda^{(\nu)}(t/r) \cdot P_\lambda^{(\mu)}(\cos \theta) \cdot e^{-i\mu\varphi} = (\cosh \rho)^{-1} \cdot P_\lambda^{(\nu)}(\tanh \rho) \cdot P_\lambda^{(\mu)}(\cos \theta) \cdot e^{-i\mu\varphi}.$$

Thus $\mu = 2m$ and we need to compare the integrals

$$i \left(\int_0^\pi (\operatorname{sign}(\cos \theta))^{2m} \cdot \mathfrak{P}_{-m\underline{m}}^l\left(\frac{2}{\cos^2 \theta} - 1\right) \cdot P_\lambda^{(2m)}(\cos \theta) \cdot |\tan \theta| d\theta \right) \cdot \left(\int_{-\infty}^\infty P_\lambda^{(\nu)}(\tanh \rho) d\rho \right)$$

and

$$(2l+1) \left(\int_0^\pi (\operatorname{sign}(\cos \theta))^{2m} \cdot \mathfrak{P}_{-m\underline{m}}^l\left(\frac{2}{\cos^2 \theta} - 1\right) \cdot P_\lambda^{(2m)}(\cos \theta) \cdot \frac{\tan \theta}{\cos \theta} d\theta \right) \cdot \left(\int_{-\infty}^\infty \frac{P_\lambda^{(\nu)}(\tanh \rho)}{\cosh \rho} d\rho \right).$$

Substituting $u = \tanh \rho$, we find

$$\int_{-\infty}^\infty P_\lambda^{(\nu)}(\tanh \rho) d\rho = \int_{-1}^1 \frac{P_\lambda^{(\nu)}(u)}{1-u^2} du, \quad \int_{-\infty}^\infty \frac{P_\lambda^{(\nu)}(\tanh \rho)}{\cosh \rho} d\rho = \int_{-1}^1 \frac{P_\lambda^{(\nu)}(u)}{\sqrt{1-u^2}} du.$$

If we take $\nu = \lambda$, $P_\lambda^{(\nu)}(u)$ is proportional to $(1 - u^2)^{\lambda/2}$, and

$$\int_{-1}^1 (1 - u^2)^{a-1} du = \frac{\sqrt{\pi} \Gamma(a)}{\Gamma(a + 1/2)}.$$

Thus,

$$\frac{\int_{-\infty}^{\infty} \frac{P_\lambda^{(\lambda)}(\tanh \rho)}{\cosh \rho} d\rho}{\int_{-\infty}^{\infty} P_\lambda^{(\lambda)}(\tanh \rho) d\rho} = \frac{(\Gamma(\lambda/2 + 1/2))^2}{\Gamma(\lambda/2) \cdot \Gamma(\lambda/2 + 1)}.$$

It remains to compare the integrals over θ . When $m \leq 0$, by (19),

$$\mathfrak{P}_{-m}^l \left(\frac{2}{\cos^2 \theta} - 1 \right) = \frac{\Gamma(l - m + 1)}{\Gamma(l + m + 1)(-2m)!} \cdot |\tan \theta|^{-2m} \cdot {}_2F_1(l - m + 1, -l - m; -2m + 1; -\tan^2 \theta).$$

If m is a half-integer, we need $P_\lambda^{(2m)}(x)$ to be odd, so we can take $\lambda = 1 - 2m$, then $P_\lambda^{(2m)}(\cos \theta)$ is proportional to $\cos \theta \cdot \sin^{-2m} \theta$. Substituting $v = \tan^2 \theta$ and using a special case of an integral formula 7.512(10) from [GR]

$$\int_0^\infty x^{c-1} (1+x)^{-r} {}_2F_1(a, b; c; -x) dx = \frac{\Gamma(c)\Gamma(a+r-c)\Gamma(b+r-c)}{\Gamma(r)\Gamma(a+b+r-c)}$$

valid when $\operatorname{Re} c > 0$, $\operatorname{Re}(a+r-c) > 0$, $\operatorname{Re}(b+r-c) > 0$, we obtain

$$\begin{aligned} & 2 \frac{\Gamma(l+m+1)(-2m)!}{\Gamma(l-m+1)} \int_0^{\pi/2} \mathfrak{P}_{-m}^l \left(\frac{2}{\cos^2 \theta} - 1 \right) \cdot \cos \theta \cdot \sin^{-2m} \theta \cdot \tan \theta d\theta \\ &= \int_0^\infty v^{-2m} \cdot (1+v)^{m-3/2} \cdot {}_2F_1(l-m+1, -l-m; -2m+1; -v) dv \\ &= \frac{\Gamma(1-2m)\Gamma(3/2+l)\Gamma(1/2-l)}{(\Gamma(3/2-m))^2} = \frac{\pi}{2 \cos(\pi l)} \frac{(2l+1)\Gamma(1-2m)}{(\Gamma(3/2-m))^2}, \end{aligned}$$

$$\begin{aligned} & 2 \frac{\Gamma(l+m+1)(-2m)!}{\Gamma(l-m+1)} \int_0^{\pi/2} \mathfrak{P}_{-m}^l \left(\frac{2}{\cos^2 \theta} - 1 \right) \cdot \sin^{-2m} \theta \cdot \tan \theta d\theta \\ &= \int_0^\infty v^{-2m} \cdot (1+v)^{m-1} \cdot {}_2F_1(l-m+1, -l-m; -2m+1; -v) dv \\ &= \frac{\Gamma(1-2m)\Gamma(1+l)\Gamma(-l)}{(\Gamma(1-m))^2} = -\frac{\pi}{\sin(\pi l)} \frac{\Gamma(1-2m)}{(\Gamma(1-m))^2}. \quad (59) \end{aligned}$$

Hence the ratio of the coefficients of $f_{\lambda, 1-\lambda, \lambda}^\pm(Y)$ in the expansions of $\widetilde{\operatorname{deg}}_Y(\pi_l^0(\gamma)t_{-m}^l)(Y)$ and $\operatorname{sign}_3(Y) \cdot (\pi_l^0(\gamma)t_{-m}^l)(Y)$ is

$$2i \cot(\pi l) \cdot \frac{(\Gamma(3/2-m))^2}{(\Gamma(1-m))^2} \cdot \frac{(\Gamma(\lambda/2+1/2))^2}{\Gamma(\lambda/2) \cdot \Gamma(\lambda/2+1)} = \lambda \tanh(\pi \operatorname{Im} l).$$

If m is an integer, we need $P_\lambda^{(2m)}(x)$ to be even, so we can take $\lambda = -2m$, then $P_\lambda^{(2m)}(\cos \theta)$ is proportional to $\sin^{-2m} \theta$. As was already computed in (59),

$$2 \frac{\Gamma(l+m+1)(-2m)!}{\Gamma(l-m+1)} \int_0^{\pi/2} \mathfrak{P}_{-m}^l \left(\frac{2}{\cos^2 \theta} - 1 \right) \cdot \sin^{-2m} \theta \cdot \tan \theta d\theta = \frac{-\pi}{\sin(\pi l)} \frac{\Gamma(1-2m)}{(\Gamma(1-m))^2}.$$

In the other case we need to deal with convergence issues, so we observe that

$$\mathfrak{P}_{-mm}^l\left(\frac{2}{\cos^2\theta} - 1\right) = \lim_{r \rightarrow 0^+} \mathfrak{P}_{-mm}^l\left(\frac{2}{\cos^2\theta} - 1\right) \cdot \cos^{2r}\theta$$

as a distribution in $\text{Im } l$. Then substituting $v = \tan^2\theta$,

$$\begin{aligned} & 2 \frac{\Gamma(l+m+1)(-2m)!}{\Gamma(l-m+1)} \int_0^{\pi/2} \mathfrak{P}_{-mm}^l\left(\frac{2}{\cos^2\theta} - 1\right) \cdot \cos^{2r-1}\theta \cdot \sin^{-2m}\theta \cdot \tan\theta \, d\theta \\ &= \int_0^\infty v^{-2m} \cdot (1+v)^{m-1/2-r} \cdot {}_2F_1(l-m+1, -l-m; -2m+1; -v) \, dv \\ &= \frac{\Gamma(1-2m)\Gamma(1/2+l+r)\Gamma(-1/2-l+r)}{(\Gamma(1/2-m+r))^2} \xrightarrow{\text{as } r \rightarrow 0^+} \frac{-2\pi}{\cos(\pi l)} \frac{\Gamma(1-2m)}{(2l+1)(\Gamma(1/2-m))^2} \end{aligned}$$

as a distribution in $\text{Im } l$. Hence the ratio of the coefficients of $f_{\lambda, -\lambda, \lambda}^\pm(Y)$ in the expansions of $\widetilde{\text{deg}}_Y(\pi_l^0(\gamma)t_{-m}^l)(Y)$ and $\text{sign}_3(Y) \cdot (\pi_l^0(\gamma)t_{-m}^l)(Y)$ is

$$-2i \tan(\pi l) \cdot \frac{(\Gamma(1-m))^2}{(\Gamma(1/2-m))^2} \cdot \frac{(\Gamma(\lambda/2+1/2))^2}{\Gamma(\lambda/2) \cdot \Gamma(\lambda/2+1)} = \lambda \coth(\pi \text{Im } l).$$

□

6 Appendix: Review of Special Functions

In this appendix we review special functions used in this paper: the spherical harmonics $Y_l^m(\theta, \varphi)$, the associated Legendre functions $P_l^{(m)}(x)$, the associated Legendre functions of the second kind $Q_l^{(m)}(x)$ and some of their identities. References: [Er], [GR], [Vil].

6.1 Spherical Harmonics

Let \mathbb{R}^3 be the three-dimensional space with coordinates (x^1, x^2, x^3) . We denote the Laplacian by

$$\Delta_3 = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}.$$

We use the spherical coordinates

$$\begin{aligned} x^1 &= r \sin\theta \cos\varphi & r &\geq 0 \\ x^2 &= r \sin\theta \sin\varphi & 0 &\leq \varphi < 2\pi \\ x^3 &= r \cos\theta & 0 &\leq \theta < \pi. \end{aligned}$$

In these coordinates

$$\Delta_3 f = \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2 f}{\partial \varphi^2}. \quad (60)$$

Let \mathbf{t}_{mn}^l denote the matrix coefficients of $SU(2)$:

$$\begin{aligned} \mathbf{t}_{mn}^l(X) &= \frac{1}{2\pi i} \sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}} \oint (sx_{11} + x_{21})^{l-n} (sx_{12} + x_{22})^{l+n} s^{-l+m} \frac{ds}{s}, \\ X &= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in SU(2), & l &= 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, \\ & & m, n &= -l, -l+1, \dots, l, \end{aligned}$$

where the integral is taken over a loop in \mathbb{C} going once around the origin in the counterclockwise direction (see [Vil]). We define the *spherical harmonics* on \mathbb{R}^3 by setting

$$Y_l^m(\theta, \varphi) = \mathbf{t}_{m0}^l(g), \quad l = 0, 1, 2, \dots, \quad m = -l, -l+1, \dots, l,$$

where

$$g = \begin{pmatrix} \cos \frac{\theta}{2} \cdot e^{i\frac{\varphi+\psi}{2}} & i \sin \frac{\theta}{2} \cdot e^{i\frac{\varphi-\psi}{2}} \\ i \sin \frac{\theta}{2} \cdot e^{i\frac{\psi-\varphi}{2}} & \cos \frac{\theta}{2} \cdot e^{-i\frac{\varphi+\psi}{2}} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \quad (61)$$

with $0 \leq \varphi < 2\pi$, $0 < \theta < \pi$, $-2\pi \leq \psi < 2\pi$. We will see later that $r^l \cdot Y_l^m(\theta, \varphi)$ is a polynomial in x^1, x^2, x^3 of degree l . The functions $Y_l^m(\theta, \varphi)$'s satisfy

$$\Delta_3 Y_l^m(\theta, \varphi) = -\frac{l(l+1)}{r^2} \cdot Y_l^m(\theta, \varphi) \quad (62)$$

and form a complete orthogonal basis of functions on the unit sphere S^2 . They also satisfy an orthogonality relation

$$\frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} Y_l^m(\theta, \varphi) \cdot Y_{l'}^{-m'}(\theta, \varphi) \cdot \sin \theta \, d\varphi d\theta = \frac{(-1)^m}{2l+1} \cdot \delta_{l,l'} \cdot \delta_{m,m'}, \quad (63)$$

which follows from (77), (68) and (66) using substitution $x = \cos \theta$. From (60) and (62) we obtain two classes of solutions of the harmonic equation $\Delta_3 f = 0$:

$$r^l \cdot Y_l^m(\theta, \varphi) \quad \text{and} \quad r^{-l-1} \cdot Y_l^m(\theta, \varphi), \quad l = 0, 1, 2, \dots, \quad -l \leq m \leq l. \quad (64)$$

6.2 Legendre Functions

The associated Legendre functions $P_l^{(m)}(x)$ and the associated Legendre functions of the second kind $Q_l^{(m)}(x)$ are two linearly independent solutions of the second order differential equation

$$(1-x^2) \frac{d^2 f}{dx^2} - 2x \frac{df}{dx} + \left(l(l+1) - \frac{m^2}{1-x^2} \right) f = 0.$$

While the parameters l and m can be arbitrary complex numbers, we will only be interested in $l = 0, 1, 2, \dots$, $m \in \mathbb{Z}$, $-l \leq m \leq l$. The *associated Legendre functions (of the first kind)* are functions defined by the formula

$$P_l^{(m)}(x) = \frac{(-1)^{l+m}}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (1-x^2)^l, \quad (65)$$

If we set $m = 0$ we get *Legendre polynomials*:

$$P_l(x) = P_l^{(0)}(x) = \frac{(-1)^l}{2^l l!} \frac{d^l}{dx^l} (1-x^2)^l, \quad x \in \mathbb{C}.$$

Then, for $m \geq 0$,

$$P_l^{(m)}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x).$$

And $P_l^{(m)}$ and $P_l^{(-m)}$ are related as follows:

$$P_l^{(-m)}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} \cdot P_l^{(m)}(x). \quad (66)$$

When m is even $P_l^{(m)}(x)$ is a polynomial of degree l and defined on the entire complex plane. On the other hand, when m is odd, $P_l^{(m)}(x)$ is defined as long as $\sqrt{1-x^2}$ is single-valued. There are different conventions for choosing the domain of $P_l^{(m)}(x)$, ours is that $P_l^{(m)}(x)$ is defined on the complex plane away from the two cuts along the real line: along $(-\infty, -1]$ and $[1, \infty)$. (But, for example, in [Er] the associated Legendre functions are defined on the complex plane with the interval $[-1, 1]$ removed from the real line.)

The *associated Legendre functions of the second kind* are defined on the complex plane with the interval $[-1, 1]$ removed from the real line. One way to define these functions is by declaring

$$Q_0(z) = Q_0^{(0)}(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \quad Q_1(z) = Q_1^{(0)}(z) = \frac{z}{2} \log\left(\frac{1+z}{1-z}\right) - 1,$$

and then by recursive relations

$$(2l+1)z \cdot Q_l^{(m)}(z) = (l-m+1) \cdot Q_{l+1}^{(m)}(z) + (l+m) \cdot Q_{l-1}^{(m)}(z),$$

which is formally identical to (73),

$$Q_l^{(m)}(z) = (z^2 - 1)^{m/2} \frac{d^m}{dz^m} Q_l(z), \quad m \geq 0,$$

and

$$Q_l^{(-m)}(z) = \frac{(l-m)!}{(l+m)!} \cdot Q_l^{(m)}(z).$$

Finally, we introduce functions $\tilde{Q}_l^{(m)}(z)$ defined on the complex plane away from the two cuts along the real line: along $(-\infty, -1]$ and $[1, \infty)$ (same domain as for $P_l^{(m)}(z)$'s). We set

$$\tilde{Q}_l^{(m)}(x) = \frac{i^m}{2} Q_l^{(m)}(x+i0) + \frac{i^{-m}}{2} Q_l^{(m)}(x-i0), \quad x \in (-1, 1) \subset \mathbb{R}.$$

From the identity

$$Q_l^{(m)}(x \pm i0) = i^{\mp m} \left(\tilde{Q}_l^{(m)}(x) \mp \frac{\pi i}{2} P_l^{(m)}(x) \right), \quad x \in (-1, 1),$$

we see that $\tilde{Q}_l^{(m)}(x)$ extends analytically to the upper and lower half-planes by

$$\tilde{Q}_l^{(m)}(z) = \begin{cases} i^m Q_l^{(m)}(z) + \frac{\pi i}{2} P_l^{(m)}(z) & \text{if } \text{Im } z > 0; \\ i^{-m} Q_l^{(m)}(z) - \frac{\pi i}{2} P_l^{(m)}(z) & \text{if } \text{Im } z < 0. \end{cases} \quad (67)$$

6.3 Identities

The associated Legendre functions satisfy two kinds of orthogonality relations:

$$\int_{-1}^1 P_k^{(m)}(x) \cdot P_l^{(m)}(x) dx = \frac{2(l+m)!}{(2l+1)(l-m)!} \cdot \delta_{k,l}, \quad 0 \leq m \leq l, \quad (68)$$

and

$$\int_{-1}^1 P_l^{(m)}(x) \cdot P_l^{(n)}(x) \frac{dx}{1-x^2} = \begin{cases} 0 & \text{if } m \neq n; \\ \frac{(l+m)!}{m(l-m)!} & \text{if } m = n \neq 0; \\ \infty & \text{if } m = n = 0. \end{cases} \quad (69)$$

There is a recursive relation for the associated Legendre functions:

$$\frac{d}{dx}P_l^{(m)}(x) + \frac{mx}{1-x^2} \cdot P_l^{(m)}(x) = -\frac{1}{\sqrt{1-x^2}} \cdot P_l^{(m+1)}(x). \quad (70)$$

Combining it with (66) we obtain:

$$\frac{d}{dx}P_l^{(m)}(x) - \frac{mx}{1-x^2} \cdot P_l^{(m)}(x) = \frac{(l+m)(l-m+1)}{\sqrt{1-x^2}} \cdot P_l^{(m-1)}(x). \quad (71)$$

We will also use the following identities:

$$P_l^{(m)}(-x) = (-1)^{l+m} \cdot P_l^{(m)}(x), \quad (72)$$

$$(2l+1)x \cdot P_l^{(m)}(x) = (l-m+1) \cdot P_{l+1}^{(m)}(x) + (l+m) \cdot P_{l-1}^{(m)}(x), \quad (73)$$

$$(1-x^2) \cdot \frac{d}{dx}P_l^{(m)}(x) = -lx \cdot P_l^{(m)}(x) + (l+m) \cdot P_{l-1}^{(m)}(x), \quad (74)$$

$$\tilde{Q}_l^{(-m)}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} \cdot \tilde{Q}_l^{(m)}(x), \quad x \in (-1, 1), \quad (75)$$

$$i^m Q_l^{(m)}(x+i0) - i^{-m} Q_l^{(m)}(x-i0) = -\pi i P_l^{(m)}(x), \quad x \in (-1, 1). \quad (76)$$

The associated Legendre functions and the spherical harmonics on \mathbb{R}^3 are related as follows:

$$Y_l^m(\theta, \varphi) = \mathbf{t}_{m,0}^l(g) = (-i)^m \cdot \sqrt{\frac{(l-m)!}{(l+m)!}} \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi}. \quad (77)$$

where g is as in (61). From the multiplicativity property for the matrix coefficients \mathbf{t}_{mn}^l 's and (77) one can obtain

$$\sum_{m=-l}^l \frac{(l-m)!}{(l+m)!} e^{im\varphi} \cdot P_l^{(m)}(\cos \theta) \cdot P_l^{(m)}(\cos \tilde{\theta}) = P_l(\cos \gamma), \quad \varphi \in \mathbb{C}, \quad (78)$$

where $\cos \gamma = \cos \theta \cos \tilde{\theta} + \sin \theta \sin \tilde{\theta} \cos \varphi$. There also is a similar formula involving the associated Legendre functions of the second kind:

$$\sum_{m=-l}^l \frac{(l-m)!}{(l+m)!} e^{im\varphi} \cdot P_l^{(m)}(\cos \theta) \cdot \tilde{Q}_l^{(m)}(\cos \tilde{\theta}) = \tilde{Q}_l(\cos \gamma), \quad \varphi \in \mathbb{R}, \quad (79)$$

which follows from (75) and

$$\tilde{Q}_l(\cos \gamma) = P_l(\cos \theta) \cdot \tilde{Q}_l(\cos \tilde{\theta}) + 2 \sum_{m=1}^l \frac{(l-m)!}{(l+m)!} P_l^{(m)}(\cos \theta) \cdot \tilde{Q}_l^{(m)}(\cos \tilde{\theta}) \cdot \cos(m\varphi).$$

We will also use the following relation:

$$\sum_{l=0}^{\infty} (2l+1) P_l(x) Q_l(y) = \frac{1}{y-x}, \quad (80)$$

the sum converges uniformly when x ranges over compact subsets lying inside the ellipse passing through y and foci at the points ± 1 . We conclude this appendix with a lemma.

Lemma 99 *The functions $r^l \cdot Y_l^m(\theta, \varphi)$ are polynomial in x^1, x^2, x^3 of degree l .*

Proof. By (77) it is enough to show that the functions $r^l \cdot P_l^{(m)}(\cos \theta) \cdot e^{-im\varphi}$ are polynomials. We use induction in l . One can check directly that for $l = 0, 1$ we indeed get polynomials. For $m \leq l$, using the recursive relation (73) we get:

$$\begin{aligned} & (l - m + 1) \cdot r^{l+1} \cdot P_{l+1}^{(m)}(\cos \theta) \cdot e^{-im\varphi} \\ &= \left((2l + 1)x^3 \cdot r^l \cdot P_l^{(m)}(\cos \theta) - (l + m)((x^1)^2 + (x^2)^2 + (x^3)^2) \cdot r^{l-1} \cdot P_{l-1}^{(m)}(\cos \theta) \right) \cdot e^{-im\varphi} \end{aligned}$$

which is a polynomial by induction hypothesis. Finally, for $m = l + 1$, $P_{l+1}^{(l+1)}(x)$ is proportional to $(1 - x^2)^{(l+1)/2}$, so $r^{l+1} \cdot P_{l+1}^{(l+1)}(\cos \theta) \cdot e^{-i(l+1)\varphi}$ is proportional to

$$r^{l+1} \cdot (1 - \cos^2 \theta)^{(l+1)/2} \cdot e^{-i(l+1)\varphi} = r^{l+1} \cdot (\sin \theta)^{l+1} \cdot (\cos \varphi - i \sin \varphi)^{l+1} = (x^1 - ix^2)^{l+1},$$

which is a polynomial as well. \square

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Department of Mathematics, Yale University, P.O. Box 208283, New Haven, CT 06520-8283
Department of Mathematics, Indiana University, Rawles Hall, 831 East 3rd St, Bloomington, IN 47405