Splitting Line Patterns in Free Groups

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Abstract. There is a canonical JSJ-decomposition of a free group relative to a line pattern.

This provides a characterization of virtually geometric multiwords. They are the multiwords that are built from geometric pieces.

1. Introduction

Let $F = F_n$ be a free group of finite rank $n > 1$. Let g and w be elements of F with w non-trivial. The w–line through g is the coarse equivalence class of the set $\{gw^m\}_{m\in\mathbb{Z}}$.

If $\mathcal{B} = \{b_1, \dots, b_n\}$ is a free basis for F, the Cayley graph of F with respect to $\mathcal{B}$ is a tree $\mathcal{T}$. There is a unique geodesic coarsely equivalent to the set of vertices $\{gw^m\}_{m\in\mathbb{Z}}$, which we also call the w–line through g. This is simply the axis of the hyperbolic $gw\bar{g}$–action on $\mathcal{T}$, and g is a point on this axis if w is cyclically reduced.

The collection of distinct w–lines is the *line pattern* generated by w. Similarly, given a multiword $\underline{w} = \{w_1, \dots, w_k\}$ with each w_i a non-trivial word, the line pattern $\mathcal{L} = \mathcal{L}_{\underline{w}}$ generated by $\underline{w}$ is the set of distinct lines in the union of the line patterns generated by the w_i.

Another way to realize a line pattern is to consider a rose with n edges labeled $b_1, \dots, b_n$. The fundamental group is F_n. Each w_i corresponds to a loop in this graph. The universal cover of the graph is the tree $\mathcal{T}$, and the collection of lifts of the loops is the line pattern.

Changing a multiword by conjugating an element or replacing an element with a root or a power does not change the associated line pattern. Thus, we will assume that elements of a multiword are indivisible and cyclically reduced, represent distinct conjugacy classes, and are not inverses of one another. With these assumptions, a line pattern determines a generating multiword up to cyclic permutation, inversion of the elements and adding or discarding duplicate elements.

There is a topological space, the *decomposition space*, $\mathbf{D}_{\mathcal{L}}$ associated to a line pattern $\mathcal{L}$ obtained as a quotient of the boundary at infinity of the tree $\mathcal{T}$ by identifying the two endpoints of a line l for each $l \in \mathcal{L}$.

We will say that the multiword (or corresponding line pattern or decomposition space) is *rigid* if the decomposition space is connected without cut points or cut pairs. (Such a line pattern is quasi-isometrically rigid, see Cashen and Macura [4].)

It is an easy consequence of Whitehead's Algorithm that the decomposition space associated to a line pattern $\mathcal{L}$ is disconnected if and only if there exists a

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free splitting $F \cong F' * F''$ such that each element of any multiword generating $\mathcal{L}$ is conjugate into one of the free factors.

Otal observed [10] that if there is a splitting of F as an amalgamated free product or HNN-extension over an infinite cyclic group such that all elements of a multiword are conjugate into the factors, then the corresponding decomposition space contains cut points or cut pairs. He speculates that the converse may be true and proves it in the special case that the decomposition space is planar and each element of the multiword is full width.

We will say F splits (freely or over $\mathbb{Z}$) *relative to* $\mathcal{L}$ if each element of any multiword generating $\mathcal{L}$ is conjugate into one of the factors of the splitting of F.

In Theorem 3.3 we show that the converse to Otal's observation is true in general: If $\mathcal{L}$ is a line pattern in F such that the corresponding decomposition space is connected with cut points or cut pairs then there exists a splitting of F over $\mathbb{Z}$ relative to $\mathcal{L}$.

Moreover, there is a canonical graph of groups decomposition of F relative to $\mathcal{L}$ that encodes all relative cyclic splittings:

Relative JSJ-Decomposition Theorem (Theorem 3.4)**.** *Let $\mathcal{L}$ be a line pattern in F such that the corresponding decomposition space $\boldsymbol{D}$ is connected. There is a canonical graph of groups decomposition of F relative to $\mathcal{L}$ with the following properties:*

(1) *The vertex groups are free (possibly cyclic), and any edge groups are cyclic.*
(2) *In every non-cyclic vertex group G, the decomposition space of the line pattern in G generated by stabilizers of the incident edge groups and generators of $\mathcal{L}$ conjugate into G either is a circle or is rigid.*
(3) *Either $\boldsymbol{D}$ is rigid or a circle or the decomposition is nontrivial and the vertex groups alternate between non-cyclic free groups and cyclic groups.*

If F splits over a cyclic subgroup relative to $\mathcal{L}$ then the cyclic subgroup is contained in either one of the cyclic vertex groups or one of the non-cyclic vertex groups with circle decomposition space.

We will refer to the graph of groups provided by this theorem as *the rJSJ* .

In Section 4 the Relative JSJ-Decomposition Theorem is applied to characterize virtually geometric multiwords.

A multiword in F is said to be *geometric* if it can be represented by an embeded multicurve in the boundary of a (possibly non-orientable) handlebody with fundamental group F. A multiword is *virtually geometric* if it becomes geometric upon passing to a finite index subgroup of F.

Otal's main result in [10], suitable reinterpreted, is that in the case that the rJSJ is a trivial decomposition, a single vertex stabilized by all of F, the multiword is geometric if and only if the decomposition space is planar, and that planarity of the decomposition space can be deduced from the Whitehead graph of the multiword.

The Relative JSJ-Decomposition Theorem provides a reduction of the virtual geometricity question: a multiword is virtually geometric if and only if its decomposition space is planar, and this is true if and only if for each non-cyclic vertex group of the rJSJ, the induced multiword in the vertex group is geometric. Thus, virtually geometric multiwords are exactly those that are built from geometric pieces.

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2. Preliminaries

2.1. **Free Groups, Line Patterns and Decomposition Spaces.**

Let $F = F_n$ be a free group of rank $n > 1$. Let $\bar{g}$ denote g^{-1}.

A nontrivial element $g \in F$ is *indivisible* if is not a proper power of another element. Equivalently, the cyclic subgroup $\langle g \rangle$ is maximal, it is not properly contained in a cyclic subgroup of F.

The *width* of g is the rank of the smallest free factor of F containing g. An element is *full width* if its width is equal to the rank of F. Full width is also known as *diskbusting*, particularly in the context of 3–manifolds.

A *basis* or *free basis* or *free generating set* of F is a generating set consisting of exactly n elements $\mathcal{B} = \{b_1, \dots, b_n\}$.

A element g is *basic* if it is indivisible and width one, or, equivalently, if g belongs to some basis of F. The term *primitive* is usually used for our basic elements, but some authors use primitive to mean indivisible.

The *degree* of a homomorphism from the integers into a free group is index of the image in the maximal cyclic subgroup containing the image.

The Cayley graph of F with respect to a basis $\mathcal{B}$ is a tree $\mathcal{T}$ whose vertices are in bijection with elements of F. There is an edge from vertex g to vertex h if and only if there exists a $b_i \in \mathcal{B}$ such that $gb_i = h$. We make this a metric space by assigning each edge length one; F acts isometrically on $\mathcal{T}$ by left multiplication.

The tree $\mathcal{T}$ has a boundary at infinity $\partial\mathcal{T}$ that is homeomorphic to a Cantor set. This compactifies the tree, $\overline{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$ is a compact topological space whose topology on $\mathcal{T}$ agrees with the metric topology.

Two subsets of $\mathcal{T}$ are *coarsely equivalent*, written $\stackrel{c}{=}$, if they have bounded Hausdorff distance.

For any two points $\xi, \xi' \in \overline{\mathcal{T}}$ there exists a unique geodesic $[\xi, \xi']$ joining them.

For nontrivial $h \in F$, we denote by gh^{∞} the point in $\partial\mathcal{T}$ that is the limit of the sequence of vertices (gh^i) in $\mathcal{T}$. In particular, if l is the w–line through g then l has two endpoints in the boundary: $l^+ = gw^{\infty}$ and $l^- = gw^{-\infty}$.

Let $\mathcal{L} = \mathcal{L}_{\underline{w}}$ be the line pattern generated by a multiword. The decomposition space $\mathbf{D} = \mathbf{D}_{\underline{w}} = \mathbf{D}_{\mathcal{L}}$ is the quotient of $\partial\mathcal{T}$ obtained by identifying the two points l^+ and l^- for each $l \in \mathcal{L}$. Let $q\colon \partial\mathcal{T} \to \mathbf{D}$ be the quotient map.

By construction, distinct lines l and l' in $\mathcal{L}$ never have a common endpoint in $\partial\mathcal{T}$. Thus, the preimage of a point in the decomposition space is either a single point in $\partial\mathcal{T}$ or the pair of endpoints of a line in the pattern.

Any homeomorphism of $\partial\mathcal{T}$ that preserves the collection of pairs of endpoints of lines of $\mathcal{L}$ descends to a homeomorphism of the decomposition space. In particular, any quasi-isometry of $\mathcal{T}$ that, up to coarse equivalence, preserves the line pattern will induce a homeomorphism of the decomposition space.

We will primarily be interested in the case that the decomposition space is connected. A *cut set* is a set whose complement is not connected. The cut set is *minimal* if no proper subset is a cut set. A *cut point* is a point that is a cut set. A two point cut set ought to be called a cut pair, but we will be interested in minimal cut pairs, so the term *cut pair* will be reserved for a two point cut set, neither point of which is a cut point.

If $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are cut pairs, we say $\{x_0, x_1\}$ *crosses* $\{y_0, y_1\}$ if x_0 and x_1 are in different complementary component of $\mathbf{D} \setminus \{y_0, y_1\}$. Since we assume that no point of a cut pair is a cut point, crossing is symmetric: $\{x_0, x_1\}$ crosses $\{y_0, y_1\}$ if and only if $\{y_0, y_1\}$ crosses $\{x_0, x_1\}$, so we just say $\{x_0, x_1\}$ and $\{y_0, y_1\}$ *cross*.

Note that in any connected topological space, a cut point can not be crossed by a cut pair and a cut pair with more than two complementary connected components can not be crossed by a cut pair.

2.2. **A Line Pattern Restricted to a Finite Index Subgroup.** Let G be a finite index subgroup of F. Let $1 = f_1, \dots, f_k$ be right coset representatives, so that $F = \coprod_{i=1\dots k} Gf_i$. The map $\bar{\iota}\colon F \to G$ that sends gf_i to $g \in G$ is a quasi-isometry. It is the coarse inverse to the inclusion $\iota\colon G \to F$.

Let w_j be an element of a multiword $\underline{w}$. Let $\mathcal{L}$ be the line pattern generated by $\underline{w}$. For each i and j there exists a minimal positive integer $a = a(i,j)$ such that $f_i w_j^a \bar{f}_i \in G$. Therefore, for any $h = gf_i$, we have:

$$\{hw_j^m\}_{m\in\mathbb{Z}} \stackrel{c}{=} \{hw_j^{am}\}_{m\in\mathbb{Z}} = \{gf_iw_j^{am}\}_{m\in\mathbb{Z}} \stackrel{c}{=} \{g(f_iw_j^a\bar{f}_i)^m\}_{m\in\mathbb{Z}} \subset G$$

Thus, if $h \in Gf_i$, the map $\bar{\iota}$ sends the w_j–line through h in F to a set of points coarsely equivalent to the $(f_iw_j^a\bar{f}_i)$–line through $\bar{\iota}(h)$ in G.

Consider the set of conjugacy classes in G given by $\{[f_iw_j^{a(i,j)}\bar{f}_i] \mid \text{ for all } i,j\}$. Pick a cyclically reduced representative from each distinct conjugacy class. This gives a multiword $\underline{w}'$ that generates a line pattern in G that is equivalent to the image of $\mathcal{L}$ under $\bar{\iota}$. We call this the line pattern $\mathcal{L}$ *restricted to* G.

The restricted line pattern is independent of the choice of coset representatives and choice of $\underline{w}'$, and $\bar{\iota}$ takes each line of $\mathcal{L}$ to a set coarsely equivalent to a line in the restricted pattern, and similarly for ι. Therefore, $\bar{\iota}$ extends to a G–equivariant homeomorphism $\partial F \to \partial G$ that preserves pairs of endpoints of lines in the pattern. This homeomorphism descends to a G–equivariant homeomorphism of decomposition spaces.

2.3. **Whitehead Graphs.** The primary tool for understanding the topology of the decomposition space associated to a line pattern is the generalized Whitehead graph of a generating multiword. This machinery was developed in [4]. In this section we will recall the relevant definitions and results, but see [4] for details.

The Whitehead graph $\mathrm{Wh}(*)$ of a cyclically reduced word w with respect to a basis $\mathcal{B}$ of F is a graph with $2n$ vertices labeled with the elements of $\mathcal{B}$ and their inverses. One edge joins vertex x to vertex y for each occurrence of $\bar{x}y$ in the word w as a cyclic word.

Similarly, the Whitehead graph of a line pattern is obtained by adding edges for each of the generators of the line pattern.

The number of edges is therefore equal to the sum of the word lengths of the generators with respect to $\mathcal{B}$.

Lemma 2.1. *If for some choice of basis* $\mathrm{Wh}(*)$ *is disconnected, then* $\boldsymbol{D}$ *is disconnected.*

Lemma 2.2. *Suppose there exists a free basis $\mathcal{B}$ of F such that* $\mathrm{Wh}_{\mathcal{B}}(*)$ *is connected without cut vertices. Let $\mathcal{T}$ be the Cayley graph of F corresponding to $\mathcal{B}$. Pick any edge e in $\mathcal{T}$. Let $*$ and v be the endpoints of e. Let $\hat{A}$ be the collection of points $\xi \in \partial\mathcal{T}$ such that v is on $[*, \xi]$. The set $A = q(\hat{A})$ is connected in $\boldsymbol{D}$.*

Let $\{l_1, \dots, l_k\} \subset \mathcal{L}$ be the set of lines that cross e. For each i, the two endpoints l_i^+ and l_i^- are identified in the decomposition space. Thus:

$$q(\hat{A}) \cap q(\hat{A}^c) = \cup_{i=1..k} q(l_i^+)$$

So, if $\mathrm{Wh}(*)$ is connected without cut vertices, then for any edge e in $\mathcal{T}$ the boundaries at infinity of the two connected components of $\mathcal{T} \setminus e$ correspond to connected sets in the decomposition space. Since $\mathrm{Wh}(*)$ is connected there is also at least one line in $\mathcal{L}$ crossing e, so these connected sets have a point in common.

Corollary 2.3. *Suppose* $\mathrm{Wh}(*)$ *has no cut vertices. The decomposition space is connected if and only if* $\mathrm{Wh}(*)$ *is connected. Furthermore, if* $\boldsymbol{D}$ *is connected it is also locally connected.*

Moreover, if none of the $q(l_i^+)$ is a cut point of $\mathbf{D}$ then, in fact,

$$q(\hat{A}) \setminus \cup_{i=1..k} q(l_i^+)$$

is a connected subset of $\mathbf{D}$.

If a Whitehead graph is connected and contains a cut point then there is another choice of basis that will give a Whitehead graph with fewer edges. Thus, by choosing a $\mathcal{B}$ for which the graph has the minimal possible number of edges, we may assume that the Whitehead graph is either disconnected or connected with no cut vertices. For many purposes, however, we do not actually need a minimal Whitehead graph, just a Whitehead graph with no cut vertices.

The decomposition space of a line pattern is disconnected if and only if every Whitehead graph with no cut vertices is disconnected, and this is true if and only if the line pattern splits freely. Thus, we will always assume that we have a line pattern $\mathcal{L}$ and a basis $\mathcal{B}$ such that the Whitehead graph of $\mathcal{L}$ is connected without cut vertices.

The classical Whitehead graph generalizes: if $\mathcal{L}$ is a line pattern in F and $\mathcal{X}$ is a connected subset of $\overline{\mathcal{T}}$ we define the Whitehead graph $\mathrm{Wh}_{\mathcal{B}}(\mathcal{X})\{\mathcal{L}\}$ to be the graph with vertices in bijection with connected components of $\overline{\mathcal{T}} \setminus \mathcal{X}$ and one edge joining v and v' for each line of $\mathcal{L}$ that has one endpoint in the component corresponding to v and the other in the component corresponding to v'. We will omit $\mathcal{B}$ and $\mathcal{L}$ when they are clear and just write $\mathrm{Wh}(\mathcal{X})$. The notation $\mathrm{Wh}(*)$ for the classical Whitehead graph just means the Whitehead graph at a vertex $*$, and because the line pattern is equivariant we get the same graph for any vertex.

Let $\mathcal{X} \subset \mathcal{Y} \subset \overline{\mathcal{T}}$. Let e be an edge of $\mathcal{T}$ incident to exactly one vertex of $\mathcal{X}$.

The edge e corresponds to a vertex in $\mathrm{Wh}(\mathcal{X})$. The graph $\mathrm{Wh}(\mathcal{X}) \setminus e$ is obtained from $\mathrm{Wh}(\mathcal{X})$ by deleting this vertex, but retaining the incident edges as *loose ends at* e.

If v is a vertex of $\mathcal{T}$ that is distance 1 from $\mathcal{X}$, then there is a unique edge e with one endpoint equal to v and the other in $\mathcal{X}$. Define $\mathrm{Wh}(\mathcal{X}) \setminus v = \mathrm{Wh}(\mathcal{X}) \setminus e$.

Similarly, $\mathrm{Wh}(\mathcal{X}) \setminus \mathcal{Y}$ is obtained from $\mathrm{Wh}(\mathcal{X})$ by deleting each vertex of $\mathrm{Wh}(\mathcal{X})$ that corresponds to an edge in $\mathcal{Y}$. Visualizing Whitehead graphs in the tree, $\mathrm{Wh}(\mathcal{X}) \setminus \mathcal{Y}$ is the portion of $\mathrm{Wh}(\mathcal{Y})$ that passes through the set $\mathcal{X}$.

Lemma 2.4. *Let S be a nonempty, finite subset of $\boldsymbol{D}$ whose preimage in $\partial\mathcal{T}$ is more than one point. Let $\mathcal{H}$ be the convex hull of $q^{-1}(S)$. There is a bijection between connected components of* $\mathrm{Wh}(\mathcal{H})$ *and connected components of $\boldsymbol{D} \setminus S$.*

Let S be a finite cut set whose preimage in $\partial\mathcal{T}$ is a pair of points, so that $\mathcal{H}$ is a line. The fact that $\mathrm{Wh}(*)$ is connected without cut vertices implies that every vertex of $\mathrm{Wh}(\mathcal{H})$ belongs to a component that limits to both boundary points of $\mathcal{H}$. It follows that if $\{x, y\}$ is a cut pair in $\mathbf{D}$, then for any small connected neighborhood N of x in $\mathbf{D}$ the number of components of $N\setminus x$ is equal to the number of components of $\mathbf{D} \setminus \{x, y\}$, which is the number of components of $\mathrm{Wh}(\mathcal{H})$.

3. Splittings

3.1. First Splitting Argument.

Proposition 3.1. *Suppose $\mathcal{L}$ is a line pattern in the free group F such that the decomposition space is connected and $q(\{\bar{g}^\infty, g^\infty\})$ is a cut set that is not crossed by any translates of itself. Then F splits over $\langle g\rangle$ relative to $\mathcal{L}$.*

Otal [10] proved a similar statement in the case that $q(\{\bar{g}^\infty, g^\infty\})$ is a cut point. The proof is virtually unchanged. For the reader's convenience we provide a sketch:

Suppose g is an indivisible element of F such that $S = q(\{\bar{g}^\infty, g^\infty\})$ is a cut set that is not crossed by any translates of itself.

Consider translates hS of S by the group action. None of these cross each other, so the orbit of S has a partial ordering relative to S defined by $h'S < hS$ if $h'S$ separates S from hS, that is, if S and hS are in different components of $\mathbf{D} \setminus h'S$.

For any translate hS, there are only finitely many other translates of S that separate hS from S. (To see this, note that for $h'S$ to separate hS from S, it is necessary that the axis of $h'g\bar{h}'$ in $\mathcal{T}$ intersects the finite geodesic segment joining the axis of g to the axis of $hg\bar{h}$.) Therefore, there are *minimal* translates hS that are not separated from S by any translate of S.

Define a simplicial tree on which F acts without inversions as follows. The tree has two classes of vertices. The first class of vertices is in bijection with the orbit of S. Given a vertex in the first class, there is a corresponding cut set hS. This has finitely many complementary connected components. For each of these components, consider the subset of the orbit consisting of hS and all $h'S$ in the component such that $h'S$ is minimal with respect to hS. This subset forms a vertex of class two adjacent to the vertex hS.

The quotient of this tree by the F–action contains a single vertex of class one, and some finite number of adjacent vertices of class two, and the stabilizer of the class one vertex is $\langle g\rangle$. Thus, we have finite graph of groups decomposition of F, and all edge stabilizers are subgroups of $\langle g\rangle$.

The generators of the line pattern must be conjugate into the vertex groups, otherwise we would have a line in the pattern crossing from one component of $\mathrm{Wh}([\bar{g}^\infty, g^\infty])$ to another, which is absurd.

Also note that since the group is free and the element g was assumed to be indivisible, each edge injection into a vertex group other than $\langle g\rangle$ must be degree one. It follows that the vertex groups other than $\langle g\rangle$ are non-cyclic free groups.

3.2. Proof of the Decomposition Theorem.

In Section 3.3 and Section 3.4 we will prove the following proposition:

Proposition 3.2. *If $\mathcal{L}$ is a line pattern in F such that the decomposition space is connected and not rigid, then either:*

1. *the decomposition space is a circle, or*

(2) *there is an indivisible element* $g \in F$ *such that* $q(\{\bar{g}^{\infty}, g^{\infty}\})$ *is a cut set that is not crossed by any cut pair.*

Combining Proposition 3.1 and Proposition 3.2 proves:

Theorem 3.3. *If* $\mathcal{L}$ *is a line pattern in* F *such that the corresponding decomposition space is connected with cut points or cut pairs then there exists a splitting of* F *over* $\mathbb{Z}$ *relative to* $\mathcal{L}$*.*

In case (1) of Proposition 3.2 there are many incompatible splittings, as every pair is a cut pair.

The condition in case (2) is stronger than the hypothesis of Proposition 3.1. The cut set $q(\{\bar{g}^{\infty}, g^{\infty}\})$ is not crossed by *any* other cut pairs. This means that two splittings of this type are compatible, and we may split over all such g, distinct up to inversion and conjugacy, to get a graph of groups decomposition of F relative to the line pattern. By accessibility of splittings over $\mathbb{Z}$, as in Bestvina-Feighn [2], there can be only finitely many compatible splittings, so we get a finite graph of groups decomposition of F.

For every cut point or uncrossed cut pair, the stabilizer is conjugate to one of the cyclic vertex groups of this decomposition. Any remaining cut pair occurs in the image of the boundary of one of the vertex groups. The rigid vertices have no cut pairs in their decomposition spaces, so this must be one of the vertex groups whose decomposition space is a circle. This proves:

Theorem 3.4 (Relative JSJ-Decomposition)**.** *Let* $\mathcal{L}$ *be a line pattern in* F *such that the corresponding decomposition space* $\boldsymbol{D}$ *is connected. There is a canonical graph of groups decomposition of* F *relative to* $\mathcal{L}$ *with the following properties:*

(1) *The vertex groups are free (possibly cyclic), and any edge groups are cyclic.*
(2) *In every non-cyclic vertex group* G*, the decomposition space of the line pattern in* G *generated by stabilizers of the incident edge groups and generators of* $\mathcal{L}$ *conjugate into* G *either is a circle or is rigid.*
(3) *Either* $\boldsymbol{D}$ *is rigid or a circle or the decomposition is nontrivial and the vertex groups alternate between non-cyclic free groups and cyclic groups.*

If F *splits over a cyclic subgroup relative to* $\mathcal{L}$ *then the cyclic subgroup is contained in either one of the cyclic vertex groups or one of the non-cyclic vertex groups with circle decomposition space.*

3.3. **Crossing Pairs and the Circle.** In this subsection we give criteria for the decomposition space to be a circle.

In a circle, the only minimal cut sets are cut pairs, and every cut pair is crossed by a cut pair.

Lemma 3.5. *Suppose* $\{x_0, x_1\}$ *and* $\{y_0, y_1\}$ *are crossing cut pairs in* $\boldsymbol{D}$*. Let* A_0 *and* A_1 *be the two connected components of* $\boldsymbol{D} \setminus \{x_0, x_1\}$*, and assume* $y_0 \in A_0$ *and* $y_1 \in A_1$*. Similarly, let* B_0 *and* B_1 *be the connected components of* $\boldsymbol{D} \setminus \{y_0, y_1\}$*, and assume* $x_0 \in A_0$ *and* $x_1 \in A_1$*. Then* $\{x_0, y_0\}$ *is a cut pair with connected components* $C_0 = A_0 \cap B_0$ *and* $C_1 = A_1 \cup B_1 \cup \{x_1\} \cup \{y_1\}$*.*

Proof. Since $\{x_0, x_1\}$ and $\{y_0, y_1\}$ cross, y_0 is a cut point of the connected set A_0. Any neighborhood of y_0 contains a connected open set N such that $N \setminus y_0$ has exactly two connected components. Therefore, $A_0 \setminus y_0$ has exactly two connected components, which are $A_0 \cap B_0$ and $A_0 \cap B_1$. Similarly, the sets $A_1 \cap B_0$ and $A_1 \cap B_1$ are connected.

It follows that C_1 is a connected set and $\overline{C}_1 \setminus C_1 = \{x_0, y_0\}$.

Now:

$$\{x_0, y_0\} \subset \overline{C}_0 \setminus C_0 \subset \{x_0, x_1, y_0, y_1\}$$

We are done if x_1 and y_1 are not limit points of C_0. Suppose, without loss of generality, that x_1 is a limit point of $C_0 = A_0 \cap B_0$. Since x_1 is also a limit point of $A_0 \cap B_1$ and $A_1 \cap B_1$, this implies that for any small neighborhood N of x_1, the complement $N \setminus x_1$ has at least three connected components, which is a contradiction. □

Proposition 3.6. *The decomposition space* $\boldsymbol{D}$ *is a circle if and only if all of the following conditions are satisfied:*

1. $\boldsymbol{D}$ *is connected.*
2. $\boldsymbol{D}$ *has no cut points.*
3. $\boldsymbol{D}$ *has cut pairs.*
4. *Every cut pair in* $\boldsymbol{D}$ *is crossed by a cut pair.*

Proof. A circle satisfies these conditions, so one direction is clear.

Define a relation on $\mathbf{D}$ by $x \sim y$ if $x = y$ or if $\{x, y\}$ is a cut pair.

Claim 3.6.1. $\sim$ is an equivalence relation.

Proof of Claim. We must show transitivity.

Suppose x, y and z are distinct points with $x \sim y$ and $y \sim z$, so that $\{x, y\}$ and $\{y, z\}$ are cut pairs. By hypothesis, every cut pair is crossed by a cut pair. Therefore, every cut pair has exactly two complementary connected components. Let A_0 and A_1 be the connected components of $\mathbf{D} \setminus \{x, y\}$. Let B_0 and B_1 be the connected components of $\mathbf{D} \setminus \{y, z\}$. Assume that $z \in A_0$ and $x \in B_0$. Arguing as in the proof of Lemma 3.5, $\mathbf{D} \setminus \{x, z\}$ has two connected components, $A_0 \cap B_0$ and $A_1 \cup B_1 \cup \{y\}$. ♢

Claim 3.6.2. Equivalence classes are closed.

Proof of Claim. If an equivalence class consists of a single point it is closed, so suppose $[x]$ is not a single point.

Let $(y_i) \to y$ for $y_i \in [x]$. Choose points $\xi \in q^{-1}(y)$ and $\xi_i \in q^{-1}(y_i)$. After passing to a subsequence, $(\xi_i) \to \xi$ in $\partial\mathcal{T}$.

Pick $\eta \in q^{-1}(x)$. For each i, $\mathrm{Wh}([\eta, \xi_i])$ has two components.

Now consider $[\eta, \xi]$. Number the vertices v_j along this geodesic with consecutive integers, increasing in the ξ direction, where v_0 may be any vertex on $[\eta, \xi]$. Since $(\xi_i) \to \xi$, for every j there is an I_j such that for all $i > I_j$, we have $[\eta, v_{j+1}] \subset [\eta, \xi_i]$. Therefore, $\mathrm{Wh}([\eta, v_j]) \setminus [\eta, \xi] = \mathrm{Wh}([\eta, v_j]) \setminus [\eta, \xi_i]$.

Thus, for all j, the Whitehead graph $\mathrm{Wh}([\eta, v_j]) \setminus [\eta, \xi]$ has two components, which implies $q(\{\eta, \xi\}) = \{x, y\}$ is a cut pair. Hence, $y \in [x]$, and $[x]$ is closed. ♢

Claim 3.6.3. All of $\mathbf{D}$ is in one equivalence class.

Proof of Claim. We have assumed that a cut pair exists, so there is an equivalence class $[x]$ consisting of more than one point. Suppose that $[x]$ is not all of $\mathbf{D}$.

Let U be a connected component of $\mathbf{D} \setminus [x]$. Since $\mathbf{D}$ is locally connected and $[x]$ is closed, U is open in $\mathbf{D}$. Since $\mathbf{D}$ is connected without cut points, U has at least two limit points in $[x]$. Pick distinct points y_0 and y_1 in $\overline{U} \cap [x]$. These points are a cut pair, and $\mathbf{D} \setminus \{y_0, y_1\}$ has exactly two connected components, A_0 and A_1. Assume $U \subset A_0$.

Let $\{z_0, z_1\}$ be a cut pair crossing $\{y_0, y_1\}$ with complementary connected components B_0 and B_1. Assume $z_0 \in A_0$, $z_1 \in A_1$, $y_0 \in B_0$ and $y_1 \in B_1$.

By Lemma 3.5, y_0 and y_1 are in $[x] \subset \mathbf{D} \setminus U$. Thus, U is contained in B_ϵ, where ϵ is either 0 or 1. Since $U \subset A_0$, we have $U \subset A_0 \cap B_\epsilon$.

Now, $\{y_\epsilon, z_0\}$ is a cut pair whose connected components are $C_0 = A_0 \cap B_\epsilon$ and $C_1 = A_1 \cup B_{1+\epsilon} \cup \{y_{1+\epsilon}\} \cup \{z_1\}$ (subscripts modulo 2). However, U, and hence C_0, has $y_{1+\epsilon} \in C_1$ as a limit point, which is a contradiction. Thus, $[x] = \mathbf{D}$. ♢

Since $[x] = \mathbf{D}$, every point of $\mathbf{D}$ is a member of a cut pair, and the only minimal cut sets are of size two. It then follows from [4, Theorem 6.1] that $\mathbf{D}$ is a circle. □

The cut pair equivalence relation in the preceding proof is inspired by a similar construction of Bowditch for boundaries of hyperbolic groups [3].

3.4. **Uncrossed Cut Pairs.** If the decomposition space is connected, not rigid and not a circle, then by Proposition 3.6 there is either a cut point or a cut pair that is not crossed by any other cut pair. To complete the proof of Proposition 3.2, we will show that there is such a cut pair of the form $q(\{\bar{g}^\infty, g^\infty\})$.

First, a lemma.

Lemma 3.7 (cf [4, Lemma 4.12])**.** *Let ξ_0 and ξ_1 be a pair of points in $\partial\mathcal{T}$ such that $q(\{\xi_0, \xi_1\})$ is a cut pair in $\boldsymbol{D}$ and for each i we have $\xi_i = q^{-1}(q(\xi_i))$. There exist elements g, $h \in F$ and $a \in \mathcal{B} \cup \bar{\mathcal{B}}$ such that:*

1. *the oriented edges $[h, ha]$ and $[hg, hga]$ belong to $[\xi_0, \xi_1] \cap [h\bar{g}^\infty, hg^\infty]$,*
2. *components of* $\mathrm{Wh}([ha, hg]) \setminus \{h, hga\}$ *that are in different components of* $\mathrm{Wh}([\xi_0, \xi_1])$ *are in different components of* $\mathrm{Wh}([h\bar{g}^\infty, hg^\infty])$*, and*
3. *for each line $l \in \mathcal{L}$ crossing the edge $[h, ha]$, the lines l and $hg\bar{h}l$ (which crosses $[hg, hga]$) belong to the same component of* $\mathrm{Wh}([\xi_0, \xi_1])$*.*

Proof. Assume $[\xi_0, \xi_1]$ is oriented from ξ_0 to ξ_1.

Pick any $a \in \mathcal{B} \cup \bar{\mathcal{B}}$ such that there are infinitely many directed a–edges in $[\xi_0, \xi_1]$. Fix one of these, $e = [g_0, g_0 a]$.

There are finitely many lines of $\mathcal{L}$ that cross e. Fix a numbering of them $1, \ldots, k$. Partition them into subsets according to the component of $\mathrm{Wh}([\xi_0, \xi_1])$ to which they belong.

Consider an element $g' \in F$ such that the oriented edge $g'e$ is in $[\xi_0, \xi_1]$. There is a bijection $l \mapsto g'l$ between lines of $\mathcal{L}$ crossing e and lines of $\mathcal{L}$ crossing $g'e$, so the numbering of the lines crossing e can be pushed forward to a numbering of the lines crossing $g'e$. We can also partition the lines crossing $g'e$ according to the component of $\mathrm{Wh}([\xi_0, \xi_1])$ to which they belong.

There are infinitely many such g', but only finitely many partitions of k numbers, so for some of these g' the partitions are the same. Thus, there exist g_1 and g_2 such that the oriented edges $g_1 e$ and $g_2 e$ are edges of $[\xi_0, \xi_1]$ (with $g_2 e$ between $g_1 e$ and ξ_1) and for each line $l \in \mathcal{L}$ crossing $g_1 e$, the corresponding line $g_2 \bar{g}_1 l$ crossing $g_2 e$ is in the same component of $\mathrm{Wh}([\xi_0, \xi_1])$.

The desired elements are $h = g_1 g_0$ and $g = \bar{g}_0 \bar{g}_1 g_2 g_0$. □

Corollary 3.8. *With notation as in the previous lemma,* $q(\{h\bar{g}^\infty, hg^\infty\})$ *and* $q(\{\xi_0, hg^\infty\})$ *and* $q(\{h\bar{g}^\infty, \xi_1\})$ *are cut pairs.*

Proposition 3.9. *Suppose the decomposition space is connected, has no cut points, is not a circle, and is not rigid. There is an element* $g \in F$ *such that* $q(\{\bar{g}^\infty, g^\infty\})$ *is a cut pair in* $\boldsymbol{D}$ *that is not crossed by any other cut pair.*

Proof. The remarks at the beginning of this subsection show that the decomposition space has a cut pair $\{x_0, x_1\}$ that is not crossed by any other cut pair.

Claim 3.9.1. For each i, the preimage $q^{-1}(x_i)$ in $\partial\mathcal{T}$ is a single point.

Proof of Claim. Suppose $q^{-1}(x_0) = \{l^-, l^+\}$ for some w–line $l \in \mathcal{L}$.

By an argument similar to the proof of Lemma 3.5, for some i and j, the pair $\{w^{-i}x_1, w^j x_1\}$ is a cut pair crossing $\{x_0, x_1\}$, contrary to hypothesis. ♢

Let $\xi_i = q^{-1}(x_i)$. Let $h, g \in F$ be the elements provided by Lemma 3.7. We may assume h is trivial by replacing x_i with $\bar{h}x_i$.

If $\mathbf{D} \setminus q(\{\bar{g}^\infty, g^\infty\})$ has more than two components then $q(\{\bar{g}^\infty, g^\infty\})$ can not be crossed by a cut pair. Thus, g is the desired element.

Now assume $\mathbf{D} \setminus q(\{\bar{g}^\infty, g^\infty\})$ has exactly two components, and suppose $g^\infty \neq \xi_1$.

Claim 3.9.2. $\{x_0, x_1\}$ has exactly two complementary connected components.

Proof of Claim. Components of $\mathbf{D} \setminus \{x_0, x_1\}$ are in bijection with components of $\mathrm{Wh}([\xi_0, \xi_1])$. Only one of these contains g^∞, which means the rest belong to a common component of $\mathrm{Wh}([\bar{g}^\infty, g^\infty])$. However, separate components of $\mathrm{Wh}([\xi_0, \xi_1])$ remain separate in $\mathrm{Wh}([\bar{g}^\infty, g^\infty])$, so there can be only one component of $\mathrm{Wh}([\xi_0, \xi_1])$ not containing g^∞. Thus, $\{x_0, x_1\}$ has exactly two complementary connected components, the one that contains $q(g^\infty)$ and the one that does not. ♢

Claim 3.9.3. The action of g fixes the two components of $\mathbf{D} \setminus q(\{\bar{g}^\infty, g^\infty\})$.

Proof of Claim. Since $\mathbf{D} \setminus q(\{\bar{g}^\infty, g^\infty\})$ has only two components this follows from property (3) of Lemma 3.7. ♢

Since $g^\infty \neq \xi_1$, we have $\bar{g}\xi_1 \neq \xi_1$.

Claim 3.9.4. $q(\{\bar{g}\xi_1, g^\infty\})$ is a cut pair.

Proof of Claim. $q(\{\bar{g}^\infty, \bar{g}\xi_1\})$ and $q(\{\bar{g}^\infty, g^\infty\})$ are cut pairs with two components each, so $q(\{\bar{g}\xi_1, g^\infty\})$ is a cut pair by the argument of Claim 3.6.1. ♢

Claim 3.9.3 implies that $\bar{g}\xi_1$ and ξ_1 are in the same component of $\mathrm{Wh}([\bar{g}^\infty, g^\infty])$, which implies that $\bar{g}\xi_1$ and g^∞ are in different components of $\mathrm{Wh}([\xi_0, \xi_1])$. However, $q(\{\bar{g}\xi_1, g^\infty\})$ is a cut pair by Claim 3.9.4. Therefore, $q(\{\bar{g}\xi_1, g^\infty\})$ is a cut pair crossing $\{x_0, x_1\}$, which is a contradiction. Thus, $g^\infty = \xi_1$.

A similar argument shows that $\bar{g}^\infty = \xi_0$, so g is the desired element. □

3.5. **Another Splitting Argument.** Suppose $g \in F$ is an element such that $q(\{g^{-\infty}, g^\infty\})$ is a cut set in the decomposition space. In this subsection we show that there is a finite index subgroup G of F such that G splits over $G \cap \langle g \rangle$ relative to the line pattern. The argument is not as elegant as that of Section 3.1, but produces the non-cyclic vertex groups explicitly, which will be useful in Section 4.

Since $q(\{g^{-\infty}, g^{\infty}\})$ is a cut set we know that $\mathrm{Wh}([g^{-\infty}, g^{\infty}])$ has at least two components, and there is some minimal k such that the g^k–action preserves the components. Let G be a finite index subgroup of F such that $u = g^k$ is a basic element of G. Such a group can be constructed by the Schreier Method, see Hall [7, Section 7.2].

Let $\mathcal{B}$ be a free basis for G containing u, and let $\mathcal{L}$ be the induced line pattern in G. It is not hard to see that there is always such a basis such that the Whitehead graph of $\mathcal{L}$ with respect to $\mathcal{B}$ is connected without cut vertices (but it is not true that there is always a basis containing u for which the Whitehead graph is minimal).

Number the finitely many components of $\mathrm{Wh}([\bar{u}^{\infty}, u^{\infty}])$. Let $\mathcal{C}_i$ denote the elements of $\mathcal{B} \cup \bar{\mathcal{B}}$ corresponding to vertices in the i-th component of $\mathrm{Wh}([\bar{u}^{\infty}, u^{\infty}])$. Let

$$\mathcal{B} \cup \bar{\mathcal{B}} = \{\bar{u}, u\} \coprod_i \{a_{i,j}\}_{j=1\dots\alpha_i} \coprod_i \{\bar{a}_{i,j}\}_{j=1\dots\alpha_i} \coprod_i \{b_{i,j}\}_{j=1\dots\beta_i}$$

where for any j, $a_{i,j}$, $\bar{a}_{i,j}$ and $b_{i,j}$ are in $\mathcal{C}_i$. In other words, the a–type generators are those whose inverses belong to the same $\mathcal{C}_i$, whereas $\bar{b}_{i,j} = b_{k,h} \in \mathcal{C}_k$ for some $k \neq i$.

Make a graph with vertices corresponding to the $\mathcal{C}$'s and an edge joining $\mathcal{C}_i$ to $\mathcal{C}_k$ if there is a $b_{i,j} \in \mathcal{C}_i$ such that $\bar{b}_{i,j} = b_{k,h} \in \mathcal{C}_k$. Let $\mathcal{D}_1, \dots, \mathcal{D}_m$ be the subsets of $\mathcal{B} \cup \bar{\mathcal{B}}$ corresponding to connected components of this graph. Each $\mathcal{D}_i$ is closed under inversion, so if $m > 1$ then G splits non-trivially as $G \cong \langle \mathcal{D}_1, u \rangle *_{\langle u \rangle} \langle \mathcal{D}_2, \dots, \mathcal{D}_m, u \rangle$. We will show that this is a splitting relative to the line pattern by showing that any generator of the line pattern that involves generators in $\mathcal{D}_1$ does not involve generators from the other $\mathcal{D}$'s.

The assumption that the u–action preserves components of $\mathrm{Wh}([u^{-\infty}, u^{\infty}])$ means that each component of $\mathrm{Wh}([u^{-\infty}, u^{\infty}])$ consists of vertices in $\cup_{n \in \mathbb{Z}} u^n \mathcal{C}_i$ for some fixed i. For a word w that is a generator of the line pattern, w may or may not involve the b–type generators. If not, then for some i and j we have:

$$w \in \langle a_{i,1}, \dots a_{i,\alpha_i}, u \rangle \subset \mathcal{C}_i \cup \{\bar{u}, u\} \subset \langle \mathcal{D}_j, u \rangle$$

On the other hand, suppose (possibly after inversion or cyclic permutation) that w begins with $\bar{b}_{1,1}$. In the Whitehead graph this gives an edge that begins at $b_{1,1} \in \mathcal{C}_1$ and must end at an element of some $u^n \mathcal{C}_i$. Now, w can not just be $w = \bar{b}_{1,1} u^n$ because the edge would then end at $u^n \bar{b}_{1,1} \notin u^n \mathcal{C}_1$. Suppose the edge ends at an element $u^n a_{1,j}$. There must be more to w, because if $w = \bar{b}_{1,1} u^n a_{1,j}$ we would get an edge from $\bar{a}_{1,j} \in \mathcal{C}_1$ to $\bar{b}_{1,1} \notin \mathcal{C}_1$. So, w may be $\bar{b}_{1,1}$ followed by some element of $\langle u, a_{1,1}, \dots a_{1,\alpha_1} \rangle$, but it must eventually have another b_{1,j_1}. This means w contributes a series of edges in $\mathrm{Wh}([u^{-\infty}, u^{\infty}])$ joining various $u^n \mathcal{C}_i$. The next edge that w contributes begins at $\bar{b}_{1,j_1} \in \mathcal{C}_{\sigma(1,j_1)} \neq \mathcal{C}_1$. Repeating the previous argument, w is of the following form:

$$\begin{aligned} w = \bar{b}_{1,1} \langle u, a_{1,1}, \dots, a_{1,\alpha_1} \rangle b_{1,j_1} & \left\langle u, a_{\sigma(1,j_1),1}, \dots, a_{\sigma(1,j_1),\alpha_{\sigma(1,j_1)}} \right\rangle \\ & \cdot b_{\sigma(1,j_1),j_2} \left\langle u, a_{\sigma(\sigma(1,j_1),j_2),1}, \dots, a_{\sigma(\sigma(1,j_1),j_2),\alpha_{\sigma(\sigma(1,j_1),j_2)}} \right\rangle \\ & \cdots \\ & \cdot b_{x,j_y} \left\langle u, a_{\sigma(x,j_y),1}, \dots, a_{\sigma(x,j_y),\alpha_{\sigma(x,j_y)}} \right\rangle \end{aligned}$$

such that $x = \sigma(\cdots \sigma(\sigma(1, j_1), j_2) \dots j_{y-1}) = \sigma(1, 1)$.

Each subword of the form $\langle u, a_{x,1}, \dots, a_{x,\alpha_x}\rangle\, b_{x,y} \left\langle u, a_{\sigma(x,y),1}, \dots, a_{\sigma(x,y),\alpha_{\sigma(x,y)}}\right\rangle$ indicates that there is a $b_{x,y} \in \mathcal{C}_x$ with $\bar{b}_{x,y} \in \mathcal{C}_{\sigma(x,y)}$, so $\mathcal{C}_x$ and $\mathcal{C}_{\sigma(x,y)}$ belong to a common component $\mathcal{D}_i$ in the graph of $\mathcal{C}$'s, which implies $w \in \langle \mathcal{D}_i, u\rangle$.

Thus, when there is more than one component $\mathcal{D}$ the line pattern splits as an amalgamated product over $\langle u\rangle$.

Of course, there was nothing special about $\mathcal{D}_1$, and if there are more than two $\mathcal{D}$'s then there are further splittings available. G decomposes as a graph of groups with one cyclic vertex group $\langle u\rangle$ and m free vertex groups $\langle \mathcal{D}_i, u\rangle$, all amalgamated over $\langle u\rangle$. For every generator w of the line pattern there is some i such that w is conjugate into the $\langle \mathcal{D}_i, u\rangle$ vertex group.

Now consider a $\mathcal{D}$ that contains multiple $\mathcal{C}$'s, which must be the case, in particular, if there is only one $\mathcal{D}$. Let us assume that $\mathcal{D}_1$ contains $\mathcal{C}_1, \dots, \mathcal{C}_\delta$, and that $\bar{b}_{1,1} \in \mathcal{C}_2$. To simplify notation let $c = b_{1,1}$.

Consider the Whitehead automorphism of G that pushes all the $\mathcal{C}_1$ generators through c:

$$\begin{aligned}
c &\mapsto c \\
b_{1,j} &\mapsto cb_{1,j} \text{ for } j > 1 \\
a_{1,j} &\mapsto ca_{1,j}\bar{c} \text{ for all } j \\
x &\mapsto x \text{ for all generators not determined by the above}
\end{aligned}$$

Pattern generators w that do not involve generators or inverses of generators in $\mathcal{C}_1$ are unchanged by this automorphism. If there is a generator $w \in \langle u, a_{1,1}, \dots, a_{1,\alpha_1}\rangle$, the automorphism changes this to a word in $\langle u, ca_{1,1}\bar{c}, \dots, ca_{1,\alpha_1}\bar{c}\rangle$, which is conjugate to $\langle \bar{c}uc, a_{1,1}, \dots, a_{1,\alpha_1}\rangle$.

Similarly, for the more complicated words involving b's as above, we have:

$$\begin{aligned}
w =& \bar{c}\, \langle u, a_{1,1}, \dots, a_{1,\alpha_1}\rangle\, b_{1,j_1} \dots \\
&\mapsto \bar{c}\, \langle u, ca_{1,1}\bar{c}, \dots, ca_{1,\alpha_1}\bar{c}\rangle\, cb_{1,j_1} \dots \\
&= \langle \bar{c}uc, a_{1,1}, \dots, a_{1,\alpha_1}\rangle\, b_{1,j_1} \dots
\end{aligned}$$

This shows that $\langle u, \mathcal{D}_1\rangle$ splits as an HNN–extension over $\langle u\rangle$ as

$$\langle u, \mathcal{D}_1\rangle \cong \langle a_{i,j}, b_{i,j}, c, u, v \mid 1 \leq i \leq \delta_1, \bar{c}uc = v\rangle \cong \langle a_{i,j}, b_{i,j}, u, v \mid 1 \leq i \leq \delta_1\rangle *_{\mathbb{Z}}$$

with all generators of the line pattern that involve generators of $\mathcal{D}_1$ conjugate into $\langle a_{i,j}, b_{i,j}, u, v \mid 1 \leq i \leq \delta_1\rangle$.

The effect of this in the line pattern in $\langle a_{i,j}, b_{i,j}, u, v \mid 1 \leq i \leq \delta_1\rangle$ is to combine $\mathcal{C}_1$ and $\mathcal{C}_2$. The argument may then be repeated $\delta_1 - 2$ more times, after which $\mathcal{D}_1$ consists of a single component and all the generators are of the a-type.

4. Virtually Geometric Multiwords

A multiword $\underline{w} = \{w_1, \dots, w_k\}$ in $F = F_n$ is *geometric* if there exits a (possibly non-orientable) handlebody H with fundamental group F such that the conjugacy classes of the w_i can be represented by an embedded multicurve in the boundary of H. The multiword is *virtually geometric* if it becomes geometric upon passing to a finite index subgroup of F.

Geometricity is understood to the extent that given a multiword there is an algorithm to determine whether or not it is geometric. If a multiword is geometric then any minimal Whitehead graph must be planar. Furthermore, if the vertices of the Whitehead graph are blown up into discs then there is a way to embed the Whitehead graph in the plane in such a way that the cyclic orderings of edges incident to a vertex and the inverse vertex are preserved by the action of the appropriate generator. This version of the Whitehead graph is an example of a *Heegaard diagram.* These claims follow from work of Zieschang [5], who gives a geometric version of Whitehead's Algorithm [11] (see also Berge's *Documentation for the program Heegaard* [1]).

The Whitehead graph is finite, so it is possible to check every distinct embedding to see if there exists one that is planar and respects cyclic ordering around the vertices.

In contrast, until recently the only method of checking virtual geometricity was to enumerate finite index subgroups of F and check whether the multiword becomes geometric, hoping for success. Gordon and Wilton [6] asked whether every one element multiword is virtually geometric. Manning [8] answered in the negative by constructing an example for which every finite index subgroup has non-planar Whitehead graph.

Otal [10] did not consider the question of virtual geometricity. We will see in Section 4.1 that it follows fairly easily from his work that a rigid multiword is virtually geometric if and only if it is geometric. However, the fact that the existence of cut pairs, hence rigidity, is algorithmically detectable was not known until Cashen and Macura [4].

4.1. **Rigid Multiwords and Geometricity.**

Lemma 4.1 ([10, Proposition 0])**.** *The decomposition space of a geometric multiword is planar.*

Proof. By definition, a geometric multiword can be realized by an embedded multicurve on the surface of a handlebody. The multicurve lifts to a collection of disjoint curves on the boundary surface of the universal cover of the handlebody. This universal cover is a thickened tree, and may be compactified by including the Cantor set boundary of the tree. The resulting space is a 3–ball with a collection of disjoint arcs in the bounding 2–sphere. By Moore's Decomposition Theorem [9], the quotient of the 2–sphere obtained by collapsing each of the curves to a point is still a 2–sphere. The image of the Cantor set in this quotient is exactly the decomposition space. Thus, the decomposition space embedds into S^2. □

Passing to a finite index subgroup induces a homeomorphism of decomposition spaces, so we also have:

Corollary 4.2. *The decomposition space of a virtually geometric multiword is planar.*

Theorem 4.3 (cf [10, Theorem 1])**.** *Let $\underline{w}$ be a multiword that generates a rigid line pattern. The following are equivalent:*

1. *The multiword $\underline{w}$ is geometric.*
2. *The decomposition space $\boldsymbol{D}_{\underline{w}}$ is planar.*
3. *Any minimal Whitehead graph for $\underline{w}$ is planar with consistent cyclic orderings of edges incident to inverse vertices.*

Proof. Lemma 4.1 shows (1) implies (2).

(2) implies (3) is the content of [10, Lemma 4.4]. The hypotheses for this lemma are that every element of the multiword is indecomposable and that the decomposition space embeds into S^2 in such a way that closures of the complementary regions intersect pairwise in at most one point. The first hypothesis is too strong. Indecomposability of each element of the multiword is only used to prove that the decomposition space has no cut points. The second hypothesis is satisfied if the decomposition space has no cut pairs. Therefore, rigidity is a sufficient hypothesis.

Suppose there exists a Whitehead graph that is planar with consistent cyclic orderings of edges incident to inverse vertices. Embed this Whitehead graph on the surface of a 3–ball and attach 1–handles joining inverse vertices. The result is a (possibly non-orientable) handlebody with embedded multicurve representing $\underline{w}$. Thus, (3) implies (1).

□

Corollary 4.4. *A rigid multiword is virtually geometric if and only if it is geometric.*

Manning has shown [8] that the word *bbaaccabc* in $F_3 = \langle a, b, c\rangle$ is not virtually geometric. This is proved by showing that the structure of the Whitehead graph of this word implies that the Whitehead graph in any finite cover is still non-planar.

Alternatively, using the methods of [4] it is possible to show this word generates a rigid line pattern, so non-planarity of the Whitehead graph immediately implies the word is not virtually geometric.

It is also possible to give examples of rigid patterns for which the decomposition space is planar. For instance, the Whitehead graph of the word $a^2b^2a\bar{b}$ in $F_2 = \langle a, b\rangle$ is the complete graph. Such a pattern is always rigid, see [4, Section 6.2], and it may easily be checked that condition (3) of Theorem 4.3 is satisfied.

4.2. **Non-rigid Multiwords and Virtual Geometricity.** The question of virtual geometricity for non-rigid multiwords reduces to the vertex groups of the rJSJ. Since the circle is planar, we are particularly interested in the rigid vertex groups.

As in Section 3.4, if $\mathbf{D}_{\underline{w}}$ is connected and has uncrossed cut pairs then we can find an indivisible element $g \in F$ such that $q(\{\bar{g}^\infty, g^\infty\})$ is an uncrossed cut pair. Up to conjugation and inversion there are only finitely many such g; let the *augmented* multiword $\mathrm{Aug}(\underline{w})$ be the union of $\underline{w}$ with these g's. The decomposition space $\mathbf{D}_{\mathrm{Aug}(\underline{w})}$ is connected with no uncrossed cut pairs (they all become cut points).

Note that the rJSJ is the same for either $\underline{w}$ or $\mathrm{Aug}(\underline{w})$.

Lemma 4.5. *Let $\underline{w}'$ be a multiword in F such that the decomposition space $\boldsymbol{D}_{\underline{w}'}$ is connected. Let $\underline{w} = \mathrm{Aug}(\underline{w}')$. The following are equivalent:*

1. *The multiword $\underline{w}$ is virtually geometric.*
2. *The decomposition space $\boldsymbol{D}_{\underline{w}}$ is planar.*
3. *For every non-cyclic vertex group of the rJSJ, the induced multiword is geometric.*

Proof. Corollary 4.2 shows (1) implies (2).

Consider the rJSJ. Recall that in each non-cyclic vertex group we have an induced multiword consisting of:

- generators of maximal cyclic subgroups containing the images of the edge injections, and

- elements of $\underline{w}$ conjugate into the vertex group that do not generate a cyclic subgroup containing the image of an edge injection.

The decomposition spaces of the vertex groups with induced multiwords must all be planar. This is because they all embed into $\mathbf{D}_{\underline{w}}$, so if one of them is non-planar, $\mathbf{D}_{\underline{w}}$ is non-planar.

For each non-cyclic vertex group of the rJSJ, the induced decomposition space is either rigid or a circle. If it is a circle then the induced multiword is geometric in the vertex group. If it is rigid and the decomposition space is planar then Theorem 4.3 says it is geometric. Thus, (2) implies (3).

Assume (3).

From a graph of groups we may build a corresponding graph of spaces. For each vertex group choose a vertex space with fundamental group isomorphic to the vertex group. For each edge group choose a space with fundamental group isomorphic to the edge group, and let the edge space be the product of that space with the unit interval. Use the edge injections of the graph of groups to define attaching maps of edge spaces to the corresponding vertex spaces. The resulting space will have fundamental group isomorphic to the fundamental group of the graph of groups.

For each non-cyclic vertex group, the induced multiword is geometric, so we can choose the vertex space to be a handlebody with an embedded multicurve in the boundary representing the induced multiword.

For the edge spaces we could use annuli, but later we will want to thicken them to make the resulting graph of spaces a 3–manifold.

For the moment we will also make a geometricity assumption on the cyclic vertex groups. Suppose for a cyclic vertex group $\langle g\rangle$ there are k incident edges and each edge injection is degree one. In this case we choose the vertex space to be a solid torus with $k+1$ disjoint curves on the boundary representing the element g and k attaching curves to which we will glue a boundary curve of an annulus edge space. (In fact, we could also achieve this if the degrees of the edge injections are all d and we replace g by g^d in $\underline{w}$.)

Another possibility is that the degrees of the edge injections are all two except for possibly one of degree one. In this case we choose the vertex space to be a solid Klein bottle, and again we have disjoint curves on the boundary representing g and the attaching curves.

Suppose one of these possibilities is true for every cyclic vertex group.

The resulting graph of spaces has fundamental group F and has an embedded multicurve representing $\underline{w}$ such that the multicurve is disjoint from the edge spaces. It is not yet a 3–manifold with boundary; we need to fatten the annuli. To see if this is possible, consider for each boundary component of each annulus the tubular neighborhood of the attaching curve in the boundary of the corresponding handlebody. If for each annulus the two neighborhoods are either both annuli or both Möbius strips then the annuli may be fattened to make the graph of spaces a 3–manifold, so $\underline{w}$ is geometric.

Thus, assuming (3), there are two possible obstructions to geometricity:

(1) The degrees of the edge injections into some cyclic vertex group are not of one of the two forms described above.
(2) Some annulus can not be fattened because one boundary neighborhood is an annulus and the other is a Möbius strip.

Claim 4.5.1. These obstructions vanish in a finite index subgroup of F, so $\underline{w}$ is virtually geometric.

Proof of Claim. There are finitely many elements $g_i \in \underline{w}$ such that $q(\{\bar{g}_i^\infty, g_i^\infty\})$ is a cut point in $\mathbf{D}_{\underline{w}}$.

From the proof of Proposition 3.1, an edge injection of degree greater than one into a cyclic vertex group $\langle g_i \rangle$ occurs when the g_i–action permutes the components of $\mathbf{D}_{\underline{w}} \setminus q(\{\bar{g}_i^\infty, g_i^\infty\})$. There are only finitely many components, so there exists some minimal power a_i of g_i such that the $g_i^{a_i}$–action fixes them.

Additionally, if obstruction (2) occurs for some conjugate of g_i, and if a_i is odd, then consider $g_i^{2a_i}$.

Let G_i be a finite index subgroup of F in which $g_i^{a_i}$ (or $g_i^{2a_i}$) is basic. Let G be the finite index subgroup $\cap_i G_i$. If we apply the Relative JSJ-Decomposition Theorem to G we get a graph of groups covering the graph of groups decomposition for F. By construction, the smallest power of g_i in G is a multiple of $g_i^{a_i}$, so all edge inclusions are degree one. This takes care of obstruction (1), and we can choose all the cyclic vertex spaces to be solid tori.

Furthermore, we can take the vertex spaces to be handlebodies finitely covering the original handlebodies. If some attaching curve in the original decomposition ran along a Möbius strip then it runs along an even covering of the Möbius strip in the covering handlebodies. Thus, all attaching attaching curves have annulus neighborhoods, which takes care of obstruction (2). $\Diamond$

Thus, (3) $\implies$ (1). $\square$

In the preceding lemma we first augmented the multiword. This is just for technical convenience. The rJSJ is the same in either case. The only difference is that for the original multiword some of the cyclic vertices in the rJSJ may correspond to uncrossed cut pairs in the decomposition space, while for the augmented multiword they all correspond to cut points. As we see in the next theorem, this makes no difference for virtual geometricity.

Theorem 4.6. *Let $\underline{w}$ be a multiword in F. The following are equivalent:*

(1) *The multiword $\underline{w}$ is virtually geometric.*
(2) *The decomposition space $\boldsymbol{D}_{\underline{w}}$ is planar.*
(3) *For every non-cyclic vertex group of the rJSJ, the induced multiword is geometric.*

Proof. If the decomposition space is not connected then F splits freely relative to the line pattern generated by $\underline{w}$ and we may deal with each free factor separately, so assume the decomposition space is connected.

We have already seen that (1) implies (2).

Suppose $\mathbf{D}_{\underline{w}}$ is planar.

Claim 4.6.1. If $\mathbf{D}_{\underline{w}}$ is planar, then $\mathbf{D}_{\mathrm{Aug}(\underline{w})}$ is planar.

By Lemma 4.5, $\mathbf{D}_{\mathrm{Aug}(\underline{w})}$ is planar if and only if the induced multiword in each non-cyclic vertex group of the rJSJ is geometric in the vertex group. By Theorem 4.3, this is true if and only if the decomposition space of each induced multiword in its vertex group is planar. Thus, Claim 4.6.1 reduces to showing the following claim.

Claim 4.6.2. Suppose $\mathbf{D}_{\underline{w}}$ is planar. For each non-cyclic vertex group G of the rJSJ, the induced multiword in G gives a planar decomposition space.

Proof of Claim. Let $\underline{v}$ be the induced multiword in G, and let $\mathbf{E}$ be the decomposition space of ∂G corresponding to $\underline{v}$.

Recall that $\underline{v}$ consists of elements of $\underline{w}$ conjugate into G as well as generators of the images of the edge injections.

If $\mathbf{D}_{\underline{w}}$ has no uncrossed cut pairs then $\underline{v}$ is exactly the elements of $\underline{w}$ conjugate into G, and $\mathbf{E}$ embeds into $\mathbf{D}_{\underline{w}}$ and hence into S^2, and we are done.

Similarly, for each cut point p there is a unique component $\mathcal{C}_p$ of $\mathbf{D}_{\underline{w}} \setminus p$ such that the image of ∂G is contained in $\mathcal{C}_p \cup p$. The intersection over all cut points of the sets $\mathcal{C}_p \cup p$ is a connected subset of $\mathbf{D}_{\underline{w}}$ containing the image of ∂G. Thus, we may assume that there are no cut points.

Now, suppose we have finitely many elements $g_1, \dots, g_k$ in $\underline{v}$ such that $\{\bar{g}_i^\infty, g_i^\infty\}$ gives an uncrossed cut pair in $\mathbf{D}_{\underline{w}}$. This means that $\mathbf{E}$ is a quotient of the image of ∂G in $\mathbf{D}_{\underline{w}}$ obtained by identifying pairs of points $\{h\bar{g}_i^\infty, hg_i^\infty\}$ for each $h \in G$ and $i = 1, \dots, k$.

Embed $\mathbf{D}_{\underline{w}}$ into S^2. We will show that there is a monotone upper semi-continuous decomposition of S^2 whose non-degenerate elements are arcs whose two endpoints are $\{h\bar{g}_i^\infty, hg_i^\infty\}$ for some h and i, and whose interiors are disjoint from the image of ∂G in S^2. Moore's Decomposition Theorem [9] says that the quotient of the sphere obtained by collapsing each of these arcs to a point is again the sphere, and the image of ∂G in this quotient is $\mathbf{E}$. Thus, $\mathbf{E}$ is planar.

A *decomposition* of S^2 is just a way of writing S^2 as a disjoint union of finite unions of compact continua. The *non-degenerate* elements are the non-singletons. The decomposition is monotone if each element is connected. The collection is upper semi-continuous if for each element α of the decomposition, and for each neighborhood U of α, there exists a neighborhood V of α such that any element of the decomposition that meets V is contained in U. For the quotient to be the sphere we also require that the elements of the decomposition are non-separating subsets.

Each uncrossed cut pair $q(\{h\bar{g}_i^\infty, hg_i^\infty\})$ has finitely many complementary components in $\mathbf{D}_{\underline{w}}$, one of which, $\mathcal{C}$, contains the image of ∂G. Choose a connected component of $S^2 \setminus \mathbf{D}_{\underline{w}}$ that limits to $q(h\bar{g}_i^\infty)$ and $q(hg_i^\infty)$. The boundary of this set is a Jordan curve passing through the two points $q(h\bar{g}_i^\infty)$ and $q(hg_i^\infty)$. Choose the arc connecting $q(h\bar{g}_i^\infty)$ and $q(hg_i^\infty)$ in our decomposition to be the sub-arc of this curve that does not go through the component of $\mathbf{D}_{\underline{w}} \setminus \{q(h\bar{g}_i^\infty), q(hg_i^\infty)\}$ containing the image of ∂G.

To satisfy the requirements of Moore's theorem, we must show that for every arc α and every neighborhood U of α there is a neighborhood V of α such that if an arc α' meets V it is contained in U.

By construction, each interior point of each arc has a neighborhood that is not entered by any other arc in the collection, but this is not true for the endpoints of the arc.

Fix an arc α and let β be one of its endpoints. Let $\hat{\beta}$ be the point of $\partial\mathcal{T}$ that is the preimage of this endpoint.

The identity element of F gives a basepoint for the topology on $\overline{\mathcal{T}}$ in the sense that the basic open neighborhoods $N_r(\xi)$ of a point $\xi \in \partial\mathcal{T}$ are the subsets of $\overline{\mathcal{T}}$ consisting of points η such that the geodesic from the basepoint to η coincides with the geodesic from the basepoint to ξ for at least distance r. There is a constant R depending on $\mathrm{Aug}(\underline{w})$ such that if ξ is the endpoint of a line in the line pattern, and if one endpoint of a line in $\mathcal{L}_{\mathrm{Aug}(\underline{w})}$ is in $N_{r+R}(\xi)\setminus\xi$ for some $r>0$, the other is in $N_r\xi$. For this same R, for every $r>0$ it is also true that $q^{-1}(q(N_{r+R}(\xi)))\subset N_r(\xi)$.

Thus, for any $r>0$ there is an open neighborhood U_r of β in $\mathbf{D}_{\underline{w}}$ such that $q(N_{r+R}(\hat{\beta}))\subset U_r\subset q(N_r(\hat{\beta}))$.

Let U be any neighborhood of α. For each interior point γ of α there is a small neighborhood V_γ contained in U that does not contain points of the other arcs. If β is an endpoint of α there is a small open neighborhood of β contained in U of the form U_r for some r, as in the previous paragraph. Let V_β be a neighborhood of β of the form U_{r+2R}. Let V be the union of the U_γ for all $\gamma\in\alpha$. If any arc of the collection enters V then it must enter $V_\beta\subset U_{r+2R}$. This implies that the entire arc is contained in $q(N_{r+R}(\hat{\beta}))\subset U$. $\Diamond$

Thus, $\mathbf{D}_{\mathrm{Aug}(\underline{w})}$ is planar. By Lemma 4.5, this means $\mathrm{Aug}(\underline{w})$ is virtually geometric. Clearly this implies $\underline{w}$ is virtually geometric: just take the graph of spaces from Lemma 4.5 and omit some of the curves running around solid torus vertex spaces to get an embedded multicurve representing $\underline{w}$.

The equivalence of (3) is also a consequence of Lemma 4.5, since the decomposition and induced multiwords are the same for $\underline{w}$ and $\mathrm{Aug}(\underline{w})$. $\square$

Remark. An argument similar to that of Claim 4.6.2 may be used to see the Relative JSJ-Decomposition Theorem from a different viewpoint. Let $\underline{w}$ be a multiword with connected, planar decomposition space. Suppose there is an uncrossed cut pair with k complementary components. It is possible to embed a graph in S^2 with the two points as vertices and k edges such that the interiors of the edges lie in the complement of the decomposition space, and such that complementary components of the decomposition space lie in complementary components of the sphere minus the graph. We can find a similar graph for all uncrossed cut pairs and cut points, and make the collection upper semi-continuous as in Claim 4.6.1.

In general, the quotient of the sphere obtained by collapsing all of these graphs to points is a *cactoid*, but in this case we can say more: it is a tree of spheres whose tree structure mirrors that of the Bass-Serre tree of the rJSJ. The vertices of the Bass-Serre tree with non-cyclic stabilizers correspond to spheres. The cyclic vertices correspond to points of intersection of different spheres. The decomposition space of the augmented multiword embeds into the tree of spheres in such a way that the decomposition space of an induced multiword in a non-cyclic vertex group of the rJSJ embeds into the appropriate sphere in the tree of spheres, and the cut points of the decomposition space are exactly points of intersection of multiple spheres.

4.3. **Examples.**

4.3.1. *Baumslag's Word.* The first example is *Baumslag's word* $w=\bar{a}^2\bar{b}\bar{a}ba\bar{b}ab$ in $F_2=\langle a,b\rangle$. In response to a question of Gordon and Wilton, Manning showed, by enumerating subgroups and checking geometricity, that this word becomes geometric in an orientable handlebody with fundamental group an index four subgroup of F_2.

This word generates a line pattern whose decomposition space is connected with no cut points. From $\mathrm{Wh}(*)\setminus[a^{-\infty},a^{\infty}]$, it is apparent that $a^{-\infty}$ and a^{∞} give a cut pair in the decomposition space. This pair has only two complementary connected components, so it requires more work to see that it is not crossed by any other cut pair. If it were crossed we would be able to find a disconnected Whitehead graph of the form $\mathrm{Wh}([*, a^m]) \setminus \{\bar{b}, a^m b\}$ for some $m \in \mathbb{Z}$. The reader may verify that all such Whitehead graphs are connected, so the cut pair $q(\{a^{-\infty}, a^{\infty}\})$ is not crossed by any other cut pair.

The rJSJ for $F = \langle a, b\rangle \cong \langle a, b, c \mid c = \bar{b}ab\rangle$ is shown in Figure 1. (The arrows at the end of each edge indicate the image of the generator of the cyclic edge group in the vertex group.)

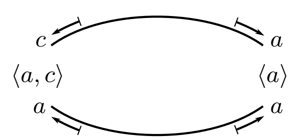

FIGURE 1. rJSJ-Decomposition of $\langle a, b\rangle$ for $\bar{a}^2\bar{b}\bar{a}ba\bar{b}ab$ ($c = \bar{b}ab$)

The induced multiword in the rank two vertex group is $\{\bar{a}^2\bar{c}ac, a, c\}$. This is a geometric multiword that generates a rigid pattern.

In Figure 2 we have a reduced Whitehead graph/Heegaard diagram for this multiword. The cyclic ordering of edges incident to the c and $\bar{c}$ discs is reversed, so we identify these discs by an orientation reversing map to make an orientable c–handle. The cyclic ordering of edges incident to the a and $\bar{a}$ discs is the same, so we identify the discs by an orientation preserving map to make a non-orientable a–handle. For convenience in drawing figures we will leave the a and $\bar{a}$ discs separate in the handlebody figure. Figure 3 shows a (non-orientable) handlebody with embedded multicurve representing $\{\bar{a}^2\bar{c}ac, a, c\}$.

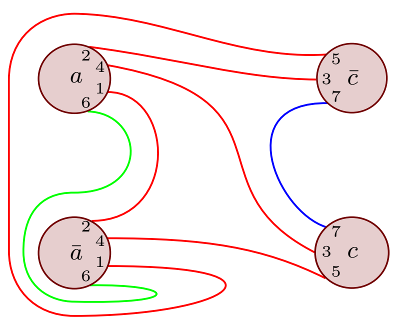

FIGURE 2. Whitehead graph/ Heegaard diagram

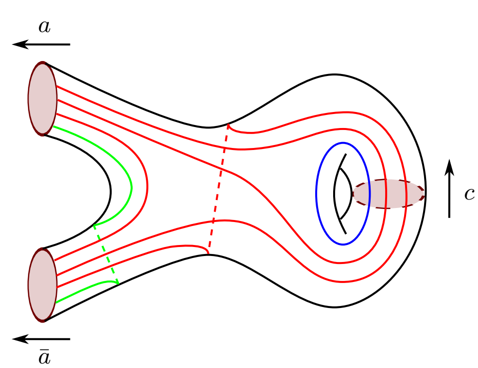

FIGURE 3. Corresponding non-orientable handlebody for $\{\bar{a}^2\bar{c}ac, a, c\}$

The obstruction to geometricity of w is that if we tried to build a graph of spaces with this non-orientable handlebody as the vertex space we would need to conjugate a curve running around an orientable handle to one running around a non-orientable handle (a is conjugate to c).

To correct this problem, pass to the index two subgroup:

$$G = \langle A = a^2, b, B = ab\bar{a}\rangle$$

Baumslag's word is not in this subgroup, but its square is:

$$\begin{aligned} w^2 &= (\bar{a}^2\bar{b}\bar{a}ba\bar{b}ab)^2 \\ &= \bar{a}^2\bar{b}\bar{a}ba\bar{b}ab\bar{a}^2\bar{b}\bar{a}ba\bar{b}ab \\ &= \bar{a}^2 \cdot \bar{b} \cdot \bar{a}^2 \cdot ab\bar{a} \cdot a^2 \cdot \bar{b} \cdot ab\bar{a} \cdot \bar{a}^2 \cdot a\bar{b}\bar{a} \cdot b \cdot a\bar{b}\bar{a} \cdot a^2 \cdot b \\ &= \bar{A}\bar{b}\bar{A}BA\bar{b}B\bar{A}\bar{B}b\bar{B}Ab \end{aligned}$$

We proceed as in Section 3.5. The A–action fixes the two complementary components of $\mathbf{D} \setminus q(\{\bar{A}^\infty, A^\infty\})$. Let $\mathcal{C}_1 = \{b, B\}$ and $\mathcal{C}_2 = \{\bar{b}, \bar{B}\}$. The graph of $\mathcal{C}$'s has a single connected component, since, for example, $b \in \mathcal{C}_1$ and $\bar{b} \in \mathcal{C}_2$. Apply the Whitehead automorphism that pushes B through b. This sends B to bB and $\bar{B}$ to $\bar{B}\bar{b}$, and fixes b and A.

The word becomes $\bar{A}(\bar{b}\bar{A}b)BAB\bar{A}\bar{B}^2(\bar{b}Ab)$.

The splitting over $\langle A\rangle$ is therefore $\langle A, b, B, C \mid C = \bar{b}Ab\rangle$. The induced multiword in the vertex group $\langle A, B, C\rangle$ is $\{A, C, \bar{A}\bar{C}BAB\bar{A}\bar{B}^2C\}$. This multiword is rigid and geometric in a non-orientable handlebody, as seen in Figure 4.

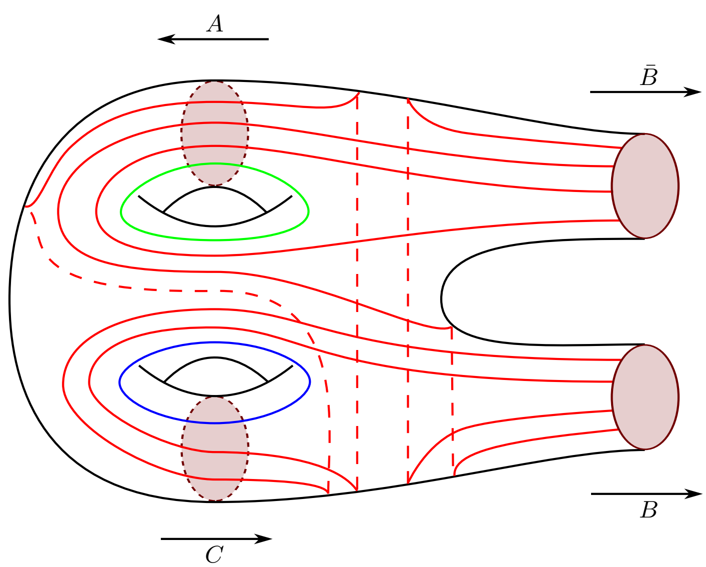

FIGURE 4. A non-orientable handlebody for $\{A, C, \bar{A}\bar{C}BAB\bar{A}\bar{B}^2C\}$

Although the handlebody is non-orientable, this time we can build a 3–manifold graph of spaces because we only need to conjugate an orientable handle to an orientable handle. Gluing on a fattened annulus conjugating A to C gives a non-orientable handlebody with fundamental group isomorphic to G for which the image of w^2 is geometric. One could pass further to a twofold cover of this handlebody to find an index four subgroup of F for which the multiword is geometric in an orientable handlebody, if desired.

4.3.2. *Baumslag-Solitar Words.* Another interesting family of example are given by the Baumslag-Solitar words $w_{p,q} = \bar{a}^q\bar{b}a^pb$ in $F_2 = \langle a, b\rangle$. We will assume that $0 < p \leq q$. Gordon and Wilton [6] have shown that $w_{p,q}$ is virtually geometric when p and q are relatively prime.

The decomposition space associated to this word is connected without cut points. The pair $\{q(a^{-\infty}), q(a^{\infty})\}$ is a cut pair. Figure 5 shows the Whitehead graph:

$$\mathrm{Wh}_{\{a,b\}}([*, a^5])\{\bar{a}^3\bar{b}a^2b\} \setminus [a^{-\infty}, a^{\infty}]$$

This is six copies of Wh$(*)$ spliced together. Note there are $p = 2$ components containing the vertices along the top of the figure, and $q = 3$ components containing the vertices along the bottom. The a–action is a shift that exchanges the two components on the top and cyclically permutes the three components along the bottom, so the a^6–action fixes all five components.

In general there are $p+q$ connected components in the complement of $\{q(a^{-\infty}), q(a^{\infty})\}$. There are two orbits of components under the a–action, one of size p and one of size q.

The case when $p = q = 1$ is special; in this case the Whitehead graph is a circle, which implies the decomposition space is a circle and the word is geometric.

Otherwise, the number of complementary components is $p+q > 2$, so $q(\{\bar{a}^{\infty}, a^{\infty}\})$ is an uncrossed cut pair. The rJSJ is shown in Figure 6.

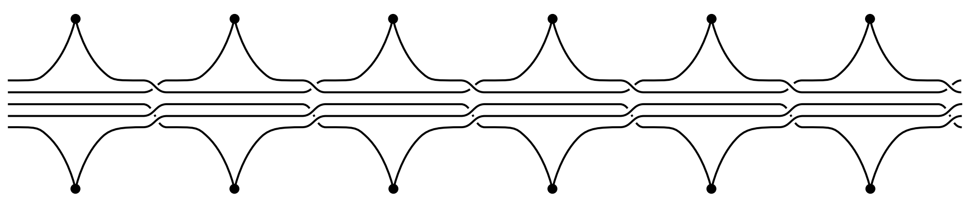

Figure 5. $\mathrm{Wh}_{\{a,b\}}([*, a^5])\{\bar{a}^3\bar{b}a^2b\} \setminus [a^{-\infty}, a^{\infty}]$

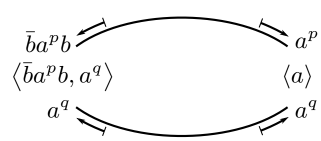

Figure 6. rJSJ-Decomposition of $\langle a, b\rangle$ for $\bar{a}^q\bar{b}a^pb$

The rank two vertex group is $\langle A = a^q, C = \bar{b}a^pb \rangle$, and the induced multiword in this vertex group is $\{A, C, \bar{A}C\}$. The Whitehead graph for this multiword is a circle, which implies the vertex decomposition space is a circle and the induced multiword is geometric. Thus, Theorem 4.6 says $w_{p,q}$ is at least virtually geometric.

The cyclic vertex group has edge inclusions of degrees p and q.

If $p = 1$ and $q = 2$ we can make this geometric by using a solid Klein bottle for the cyclic vertex space. (We saw the non-cyclic vertex space for this example back in Figure 3.)

If $p = q$ the word is also geometric, because two disjoint degree p curves fit into the boundary of a solid torus. However, an additional degree one curve does not fit. This example is notable because the word $w_{p,p}$ is geometric, but the augmented multiword $\{w_{p,p}, a\}$ is only virtually geometric. Augmenting the multiword does not change virtual geometricity, but it may change geometricity. However, $\{w_{p,p}, a^p\}$ is geometric.

Otherwise, the word $w_{p,q}$ is not geometric. Let l be the least common multiple of p and q. It suffices to pass to the index l subgroup:

$$G = \left\langle A, B_0, B_1, \dots, B_{l-1} \mid A = a^l, B_i = a^i b \bar{a}^i \right\rangle$$

Since $A = a^l$ is the smallest power of a in this subgroup, there is still only one orbit of uncrossed cut pair in the decomposition space. The A–action fixes each of the $p+q$ complementary components of $q(\{\bar{A}^\infty, A^\infty\})$. Therefore, the rJSJ has a single cyclic vertex group with all edge inclusions of degree one, and all non-cyclic vertex groups of the rJSJ have a circle for the decomposition space of the induced multiword.

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