On triangles determined by subsets of the Euclidean plane, the associated bilinear operators and applications to discrete geometry

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Abstract. We prove that if the Hausdorff dimension of the set $E \subset \mathbb{R}^2$ is greater than $\frac{7}{4}$, then the three-dimensional Lebesgue measure of the set of triangles determined by E is positive. We establish this fact by showing that the natural measure on the set of triangles determined by E, given by the relation $\int f(a,b,c)d\nu(a,b,c) = \int \int \int f(|x-y|, |x-z|, |y-z|)d\mu(x)d\mu(y)d\mu(z)$, where $d\mu$ is a Frostman measure on E, is in L^{∞} if the Hausdorff dimension of E is greater than $\frac{7}{4}$. Furthermore, we show that the exponent $\frac{7}{4}$ cannot, in general, be improved.

The study of this problem naturally leads to the analysis of the bi-linear convolution operator

$$B(f,g)(x) = \int \int f(x-u)g(x-v)dK(u,v),$$

where dK is the Lebesgue measure on the set $\{(\omega, \omega') \in S^1 \times S^1 : |\omega - \omega'| = 1\}$. We obtain $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to L^1_s(\mathbb{R}^2)$ for the operator B on non-negative functions. It is needed to establish the geometric fact described above. We note that the $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to L^1_s(\mathbb{R}^2)$ estimate holds with $s = \frac{1}{2}$, on characteristic functions, in spite of the fact that the multiplier does not possess the isotropic decay rate at infinity of order $-\frac{1}{2}$. It does decay of this order away from an exception co-dimension 2 plane. The fact that this is enough is a critical feature of the bi-linear setting.

As a consequence of our main result, we show that for a natural class of discrete sets in the plane consisting of n elements, the maximum number of times a given triangle arises, up to congruence, is $O(n^{\frac{9}{7}+\epsilon})$, improving upon the known bound of $O(n^{\frac{4}{3}})$ in this context.

1. Introduction

The classical Falconer distance conjecture says that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of

$$\Delta(E) = \{|x-y| : x, y \in E\}$$

is positive. Here, and throughout, $|\cdot|$ denotes the Euclidean distance. A beautiful example due to Falconer, based on the integer lattice, shows that the exponent $\frac{d}{2}$ is best possible. The best known results, culminating almost three decades of efforts by Falconer ([12]), Mattila ([21]), Bourgain ([3]) and others are due to Wolff ([29]) in two dimensions and to Erdogan ([5]) in higher dimensions. They prove that $\mathcal{L}^1(\Delta(E)) > 0$ if

$$dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}.$$

One may think of the distance set problem as showing that a compact E of sufficiently large Hausdorff dimension determines a positive proportion of all possible two-point configurations. This is because two different two-point configurations are equivalent, up to rigid motions, precisely if the corresponding distances are exactly the same. A natural question to ask is, how large does the Hausdorff dimension of a compact set needs to be to ensure that it determines a positive proportion of finite point configurations with a larger number of points. To this effect, define

$$T_k(E) = E^{k+1}/\sim,$$

where $E^{k+1} = E \times E \times \cdots \times E$, $k+1$ times and

$$(x^1, x^2, \dots, x^{k+1}) \sim (y^1, y^2, \dots, y^{k+1})$$

if there exists an element Λ of the orthogonal group $O(d)$ and a translation $\tau \in \mathbb{R}^d$ such that

$$y^j = \tau + \Lambda(x^j),$$

for all j.

Observe that we may view $T_k(E)$ as a subset of $\mathbb{R}^{\binom{k+1}{2}}$ since rigid motions may be encoded by fixing distances. The problem under consideration was first taken up in [4] where the authors proved that if

$$dim_{\mathcal{H}}(E) > \frac{d+k+1}{2}, \text{ then } \mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0.$$

Unfortunately, these results do not give a non-trivial exponent for what is arguably the most natural case of triangles in the planes, tetrahedra in three space and, more generally, of d-dimensional simplexes in $\mathbb{R}^d$. In this paper we partly fill this gap by establishing a non-trivial exponent for triangles in the plane.

As for counter-examples, it is not difficult to see that $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$ does not hold if the Hausdorff dimension of E is less than or equal to $d-1$. To see this, just take E to be a subset of a $(d-1)$-dimensional plane. We do not currently know if more restrictive examples exist in this context. However, more restrictive counter-examples do exist if we consider the following related question investigated in [4]. Let

$$\mathcal{S}_t^k(E) = \{(x^1, \dots, x^{k+1}) \in E^{k+1} : |x^i - x^j| = t_{ij}\},$$

where $t = \{t_{ij}\}$ is symmetric matrix with zeroes on the diagonal and $E^{k+1} = E \times E \times \cdots \times E$. In [4] the authors investigated conditions under which

$$\dim_{\mathcal{M}}(\mathcal{S}_t^k(E)) \leq (k+1)\dim_{\mathcal{H}}(E) - \binom{k+1}{2}, \tag{1.1}$$

where $\dim_{\mathcal{M}}$ denotes the Minkowski dimension and $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension. See, for example, [13] and [22] for a thorough description of these notions and connections with harmonic analysis.

The estimate (1.1) follows easily if one can show that

$$\mu \times \mu \times \cdots \times \mu\{(x^1, \dots, x^{k+1}) : t_{ij} \leq |x^i - x^j| \leq t_{ij} + \epsilon\} \lesssim \epsilon^{\binom{k+1}{2}}, \tag{1.2}$$

where μ is a Frostman measure (defined in 1.3 below) on E, under the assumption that the Hausdorff dimension of E is greater than some threshold $s_0 < d$. This was shown in [4] under the assumption that the Hausdorff dimension of E is greater than $\frac{k}{k+1}d + \frac{k}{2}$. Observe that this only yields a non-trivial exponent (less than d) if $d > \binom{k}{2}$ and, in particular, does not cover the case $k = d$.

It is also not difficult to show (see Section 2 below) that if the estimate (1.2) holds under the assumption that the Hausdorff dimension of E is greater than s_0, then $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$ for all sets E with Hausdorff dimension greater than s_0. The results in [4] prove a number of estimates of the type (1.2), but, as we note above, they do not cover the case $d = k$, the full simplex in $\mathbb{R}^d$. We establish non-trivial exponents for $d = k = 2$ below by proving the estimate (1.2) in this case for sets E of Hausdorff dimension greater than $\frac{7}{4}$. Furthermore, we use a variant of Mattila's example from [21] to show that our estimate cannot be improved in the sense that for every $s < \frac{7}{4}$ there exists a set E and a Frostman measure μ for which the estimate (1.2) does not hold in the case $d = k = 2$.

1.1. A combinatorial perspective. Finite configuration problems have their roots in geometric combinatorics. For example, the Falconer distance problem is a continuous analog of the celebrated Erdős distance problem. See [24], [20], [1], [28] and the references contained therein.The discrete precursor of the problem discussed in this paper is the following question posed by Erdős and Purdy (see [1], [6] and also [7], [8], [9] [10], [11]):

Q: What is the maximum number of mutually congruent k-simplexes among n points in $\mathbb{R}^d$?

In Section 5 we shall see that the main result of this paper (Theorem 1.1) implies that for a large class of points sets of size n, the maximum number of mutually congruent triangles is $O(n^{\frac{9}{7}+\epsilon})$.

For explicit quantitative connections between discrete and continuous finite configuration problems in other contexts, see, for example, [15], [17] and [18].

1.2. Notation: Throughout the paper, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$ and $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$. We also define $X \lessapprox Y$ as follows. If X and Y are quantities that depend on a large parameter N, then $X \lessapprox Y$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon N^\epsilon Y$. If X and Y depend on a small parameter δ, then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon \delta^{-\epsilon} Y$ as δ tends to 0.

Throughout the paper, we shall work with *Frostman* measures μ. Recall that a probability measure μ on a compact set E is a Frostman measure if for any ball B_δ of radius δ,

$$\mu(B_\delta) \lessapprox \delta^s, \tag{1.3}$$

where s is the Hausdorff dimension of s. For the proof of the existence of such measures for any compact set E see, for example, [22].

Our main analytic result is the following.

Theorem 1.1. *Let $E \subset [0,1]^2$ of Hausdorff dimenion $> \frac{7}{4}$. Then the estimate (1.2) holds in the case $d = k = 2$.*

Consequently, if the Hausdorff dimension of E is greater than $\frac{7}{4}$, then

$$\mathcal{L}^3(T_2(E)) > 0.$$

The proof that the first assertion of Theorem 1.1 implies the second is in Section 2 below.

This result is sharp in the following sense. Define a measure ν on $T_k(E)$ by the relation

(1.4)
$$\int f(t_{12}, t_{13}, t_{23}) d\nu(t_{12}, t_{13}, t_{23}) = \int\int\int f\left(|x^1 - x^2|, |x^1 - x^3|, |x^2 - x^3|\right) d\mu(x^1) d\mu(x^2) d\mu(x^3),$$

where $d\mu$ is a Frostman measure on E. We shall prove that $\nu \in L^\infty$ if the Hausdorff dimension of E is greater than $\frac{7}{4}$, which is just a rephrasing of the statement that (1.2) holds when $d = k = 2$. We will then show in Section 2 that this implies the conclusion of Theorem 1.1. Finally, we will show that if $s < \frac{7}{4}$, then ν need not be, in general, in L^∞ in the sense that for every $s < \frac{7}{4}$ there exists a set E of Hausdorff dimension s and a Frostman measure μ supported on this set, such that ν, defined in 1.4 above, is not in L^∞. This issue is taken up in the Section 4 below.

Theorem 1.1 may be viewed as a local version of the following theorem due to Furstenberg, Katznelson and Weiss ([**14**]). See also [**2**] and [**30**] for subsequent result along these lines.

Theorem 1.2. *Let $E \subset \mathbb{R}^2$, of positive upper Lebesgue density in the sense that*

$$\limsup_{R\to\infty} \frac{\mathcal{L}^d\{E \cap [-R, R]^d\}}{(2R)^d} > 0,$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure.

Let E_δ denote the δ-neighborhood of E. Then, given vectors u, v in $\mathbb{R}^2$, there exists l_0 such that for $l > l_0$ and any $\delta > 0$, there exists $\{x, y, z\} \subset E_\delta$ forming a triangle congruent to $\{\mathbf{0}, lu, lv\}$, where $\mathbf{0}$ denotes the origin in $\mathbb{R}^2$.

We note in passing that it is generally believed that the conclusion of Theorem 1.2 still holds if the δ-neighborhood of E is replaced by E under an additional assumption that the triangles under consideration are non-degenerate. We believe that there is a possibility that techniques of this paper may be applicable to this question and hope to address this issue in a subsequent paper. For degenerate triangles the necessity of considering the δ-neighborhood of E was established by Bourgain. See [**14**] for the description of his argument.

In contrast to Theorem 1.2, we are able in the local version to go beyond subsets of the plane of positive Lebesgue measure, and we do not require dilations. On the other hand, we only obtain a positive Lebesgue measure's worth of all the possible triangles and not all of them.

2. Reduction of the proof of the main result to the estimation of a tri-linear form

Let μ be a Frostman measure on E. Cover $T_2(E)$ by cubes of the form

$$(t_{12}^l - \epsilon_l, t_{12}^l + \epsilon_l) \times (t_{13}^l - \epsilon_l, t_{13}^l + \epsilon_l) \times (t_{23}^l - \epsilon_l, t_{23}^l + \epsilon_l).$$

It follows that

$$1 = \mu \times \mu \times \mu\{E \times E \times E\}$$

$$\leq \sum_l \mu \times \mu \times \mu\{(x^1, x^2, x^3) : t_{ij}^l - \epsilon_l \leq |x^i - x^j| \leq t_{ij}^l + \epsilon_l\}.$$

Suppose that we could show that this expression is

$$\lesssim \sum \epsilon_l^3.$$

It would then follow, by definition of sets of measure 0, that the three dimensional Lebesgue measure of $T_2(E)$ is positive. By standard pigeonholing, we may assume that $t_{ij} \geq c > 0$. Therefore the proof of Theorem 1.1 is reduced to the tri-linear estimate

$$\int\int\int \sigma^{\epsilon}_{t_{12}}(x^1-x^2)\sigma^{\epsilon}_{t_{13}}(x^1-x^3)\sigma^{\epsilon}_{t_{23}}(x^2-x^3)d\mu(x^1)d\mu(x^2)d\mu(x^3) \lesssim 1, \tag{2.1}$$

where σ_r is the Lebesgue measure on the circle of radius.

3. Proof of the estimation of the tri-linear form

Let us consider a slightly modified expression of the form

$$\int\int\int \sigma^{\epsilon}_{t_{12}}(x^1-x^2)\sigma^{\epsilon}_{t_{13}}(x^1-x^3)\sigma^{\epsilon}_{t_{23}}(x^2-x^3)\mu^{\delta}_{\alpha}(x^1)\mu^{\delta}(x^2)\mu^{\delta}_{-\alpha}(x^3)dx^1dx^2dx^3, \tag{3.1}$$

where

$$\mu^{\epsilon}(x) = \frac{1}{\epsilon^2}\rho\left(\frac{\cdot}{\epsilon}\right) * \mu(x),$$

with ρ a smooth approximation to the identity, and μ^{ϵ}_{α} is defined by the relation

$$\mu_{\alpha}(x) = \mu * \lambda_{\alpha}(x),$$

where

$$\lambda_{\alpha}(x) = |x|^{-2+\alpha}(1-\phi(c^{-1}x)), \tag{3.2}$$

with $c > 0$ to be determined, if $x \neq (0,0)$ and 0 if $x = (0,0)$. Here ϕ is a smooth cut-off function identically equal to one in a ball of radius 1 and vanishing identically outside the ball of radius 2.

By symmetry, if we could prove that the expression (3.1) is bounded for some $\alpha > 0$, with constants independent of ϵ and δ, the estimate (2.1) would follow by interpolation, or, more precisely, by the three lines lemma. See, for example, [**26**].

We may replace t_{ij}s by $a, b, 1$ since may take one of the radii to be 1 by scaling. We have

$$\int\int\int \mu^{\delta}_{-\alpha}(x-u)\mu^{\delta}(x-v)K^{\epsilon}(u,v)dudv\ \mu^{\delta}_{\alpha}(x)\psi(x)dx, \tag{3.3}$$

where

$$K^{\epsilon}(u,v) = \sigma^{\epsilon}(u-v)\sigma^{\epsilon}_a(u)\sigma^{\epsilon}_b(v),$$

interpreted in the sense of distributions.

Lemma 3.1. *With the notation above,*

$$||\mu^{\delta}_{\alpha}||_{\infty} \lessapprox 1$$

if $s + \alpha \geq 2$.

To prove the lemma, we observe that

$$\mu_\alpha^\delta(x) \le \int |x-y|^{-2+\alpha} d\mu^\delta(y)$$

$$\approx \sum_m 2^{m(2-\alpha)} \int_{|x-y|\approx 2^{-m}} d\mu^\delta(y)$$

$$\lesssim \sum_m 2^{m(2-\alpha)} 2^{-ms},$$

since μ is a Frostman measure. This quantity is $\lessapprox 1$ if $\alpha + s \ge 2$, as desired.

Using Lemma 3.1 it follows that the expression in (3.3) is bounded by

$$\int\int\int \mu_{-\alpha}^\delta(x-u)\mu^\delta(x-v)K^\epsilon(u,v)\psi(x)dxdudv$$

$$= \int\int F^\delta(u,v)K^\epsilon(u,v)dudv,$$

where

$$F^\delta(u,v) = \int \mu_{-\alpha}^\delta(x-u)\mu^\delta(x-v)\psi(x)dx.$$

We have

$$\widehat{F}^\delta(\xi,\eta) = \widehat{\mu}(\xi)\widehat{\lambda}_{-\alpha}(\xi)\widehat{\mu}(\eta)\widehat{\psi}(\xi+\eta)\widehat{\psi}(\delta\xi)\widehat{\psi}(\delta\eta).$$

So we obtain

$$\int\int \widehat{\mu}(\xi)\widehat{\lambda}_{-\alpha}(\xi)\widehat{\mu}(\eta)\widehat{\psi}(\xi+\eta)\widehat{\psi}(\delta\xi)\widehat{\psi}(\delta\eta)\widehat{K}^\epsilon(\xi,\eta)d\xi d\eta. \tag{3.4}$$

Lemma 3.2. *Let $K(u,v) = K^0(u,v)$, interpreted in the sense of distributions. We have*

$$\widehat{K}(\xi,\eta) = \widehat{\sigma}(U_{a,b}(\xi,\eta)), \tag{3.5}$$

where

$$U_{a,b} : \mathbb{R}^4 \to \mathbb{R}^2$$

and is defined by

$$U_{a,b}(\xi,\eta) = \left(a\xi_1 + b\eta_1\gamma_{a,b} + b\eta_2\sqrt{1-\gamma_{a,b}^2},\, a\xi_2 - b\eta_1\sqrt{1-\gamma_{a,b}^2} + b\gamma_{a,b}\eta_2\right) \tag{3.6}$$

with $\gamma_{a,b} = \frac{a^2+b^2-1}{2ab}$.

Recall that by the standard method of stationary phase (see e.g. [23], [27]),

$$|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{1}{2}}.$$

Consequently,

$$\left|\widehat{K}^\epsilon(\xi,\eta)\widehat{\psi}(\xi+\eta)\right| \lesssim (|\xi|+|\eta|)^{-\frac{1}{2}}. \tag{3.7}$$

To prove the lemma, parameterize the Cartesian product of two circles as

$$\{(a\cos(\theta), a\sin(\theta), b\cos(\phi), b\sin(\phi))\}.$$

The restriction imposed by $\sigma(u-v)$ says that

$$dist((a\cos(\theta), a\sin(\theta)), (b\cos(\phi), b\sin(\phi)) = 1,$$

which implies via standard trigonometric identities that

$$\cos(\theta-\phi) = \frac{a^2+b^2-1}{2ab} \equiv \gamma_{a,b}$$

and thus
$\theta - \phi = \pm\theta_{a,b} = \cos^{-1}(\gamma_{a,b})$. It follows that

$$\widehat{K}(\xi,\eta) = \int_0^{2\pi} e^{2\pi i(a\cos(\theta)\xi_1 + a\sin(\theta)\xi_2 + b\cos(\theta+\theta_{a,b})\eta_1 + b\sin(\theta+\theta_{a,b})\eta_2)} d\theta$$

$$= \widehat{\sigma}(U_{a,b}(\xi,\eta)),$$

as claimed. This proves (3.5). The estimate (3.7) follows in the same way since $\sigma_a^\epsilon(x) = \sigma_a * \rho_\delta(x)$.

Lemma 3.3. *With the notation above,*

$$|\widehat{\lambda}_{-\alpha}(\xi)| \lesssim |\xi|^\alpha \text{ if } |\xi| > \frac{2}{c}.$$

To prove Lemma 3.3, observe that the absolute value of the Fourier transform of the distribution of $|x|^{-2-\alpha}\phi(x)$ is $\lesssim |\xi|^\alpha$. Therefore, it suffices to show that the absolute value of the Fourier transform of $|x|^{-2-\alpha}\phi(c^{-1}x)$ is $\lesssim |\xi|^\alpha$ if $|\xi| > \frac{2}{c}$. The Fourier transform of this function equals a constant times

$$c^2 \int |\xi - z|^\alpha \widehat{\phi}(cz) dz$$

$$\lesssim C_N \cdot c^2 \int |\xi - z|^\alpha (1+|cz|)^{-N} dz$$

$$\lesssim C_N |\xi|^\alpha,$$

provided that $|\xi| > \frac{2}{c}$, as desired.

Set $c = 2$ in Lemma 3.3. In view of Lemma 3.2 and Lemma 3.3, for every $N > 0$ there exists $C_N > 0$ such that the expression (3.4) equals $O(1)$ plus

$$\approx \int_{2\le|\xi|,|\eta|\lesssim\delta^{-1}} \int C_N(1+|\xi+\eta|)^{-N}(1+|U_{a,b}(\xi,\eta)|)^{-\frac{1}{2}} |\xi|^\alpha |\widehat{\mu}(\xi)||\widehat{\mu}(\eta)| d\xi d\eta$$

$$\lesssim \int_{|\xi|\lesssim\delta^{-1}} |\xi|^{-\frac{1}{2}+\alpha} |\widehat{\mu}(\xi)|^2 d\xi.$$

This expression is bounded if $\alpha = \frac{1}{4}$ and $s > \frac{7}{4}$ as the expression above is bounded by a constant multiple of the energy integral

$$\int\int |x-y|^{-\frac{7}{4}} d\mu(x) d\mu(y).$$

See, for example, [22], for the discussion of energy integrals and connections with Hausdorff dimensions. We also include the argument here for the reader's convenience. We have

$$\int\int |x-y|^{-\frac{7}{4}} d\mu(x)d\mu(y) \approx \sum_j 2^{\frac{7}{4}j} \int\int_{2^{-j}\le |x-y|\le 2^{-j+1}} d\mu(x)d\mu(y).$$

By (1.3) this quantity is

$$\lesssim \sum_j 2^{\frac{7}{4}j} 2^{-js}$$

and the geometric series converges if $s > \frac{7}{4}$.

This completes the proof of the estimate (3.1) and thus the proof of Theorem 1.1 is complete.

4. Sharpness of the tri-linear estimate (2.1)

To understand the extent to which this result is sharp, we use a variant of the construction due to Mattila obtained for the case $k=1, d=2$ in [21]. See [19] and [4] where this issue is studied comprehensively. Let C_α denote the standard α-dimensional Cantor set contained in the interval $[0,1]$. Let

$$F_\alpha = (C_\alpha - 1) \cup (C_\alpha + 1)$$

and let μ denote the natural measure on this set. Let $E = F_\alpha \times F_\beta$. Observe that we can a fit a $\sqrt{\epsilon}$ by ϵ rectangle in the annulus $\{x : 1 \le |x| \le 1+\epsilon\}$ near $(0, \pm 1)$ and also near $(\pm 1, 0)$.

Fix x and observe that

$$\mu \times \mu\{(y,z) : 1 \le |x-z| \le 1+\epsilon; 1 \le |x-y| \le 1+\epsilon; \sqrt{2} \le |y-z| \le \sqrt{2}+\epsilon\}$$

$$\approx \epsilon^{\frac{\alpha}{2}+\beta} \cdot \epsilon^{\alpha+\beta} = \epsilon^{\frac{3}{2}\alpha+2\beta}.$$

Integrating in x, we see that

$$\mu \times \mu \times \mu\{(x,y,z) : 1 \le |x-z| \le 1+\epsilon; 1 \le |x-y| \le 1+\epsilon; \sqrt{2} \le |y-z| \le \sqrt{2}+\epsilon\} \gtrsim \epsilon^{\frac{3}{2}\alpha+2\beta}.$$

We need this quantity to be $\lesssim \epsilon^3$, which leads to the equation

$$\frac{3}{2}\alpha + 2\beta \ge 3.$$

Choosing $\alpha = 1$ and $\beta = \frac{3}{4}$ shows that the inequality (2.1) does not in general hold if $s < \frac{7}{4}$. It is important to note that this does not prove that $\mathcal{L}^3(T_2(E)) > 0$ does not in general hold if $s < \frac{7}{4}$.

We stress that the calculation above pertains to the tri-linear expression (2.1). We do not know of any example that shows that $\mathcal{L}^3(T_2(E))$ is not in general positive if the Hausdorff dimension of E is greater than one. The discrepancy here is not particularly surprising because it already takes place in the study of distance sets. For example, as we point out in the introduction, it is known that if the Hausdorff dimension of $E \subset \mathbb{R}^2$ is ≤ 1, then it is not in general true that $\mathcal{L}^1(\Delta(E)) > 0$. A result due to Wolff ([29]) says that if the Hausdorff dimension of E is greater than $\frac{4}{3}$, then $\mathcal{L}^1(\Delta(E)) > 0$. On the other hand, an example due to Mattila ([21]) shows that if the Hausdorff dimension of E is less than $\frac{3}{2}$ and μ is a Frostman measure on E, then

$$\limsup_{\epsilon \to 0} \epsilon^{-1} \mu \times \mu \{(x,y) \in E \times E : 1 \leq |x-y| \leq 1+\epsilon\} = \infty. \tag{4.1}$$

We note that (4.1) is the analog of (1.4). It says that the distance measure, defined by

$$\int f(t) d\nu(t) = \int \int f(|x-y|) d\mu(x) d\mu(y)$$

is not in L^∞.

Thus, in order to prove that $\mathcal{L}^3(T_2(E)) > 0$ for sets of Hausdorff dimension $< \frac{7}{4}$, it is reasonable to try to obtain an L^2, rather an L^∞ bound on the measure ν defined by 1.4 above. We hope to address this issue in a subsequent paper.

5. Application to discrete geometry

Definition 5.1. Let P be a set of n points contained in $[0,1]^2$. Define the measure

$$d\mu_P^s(x) = n^{-1} \cdot n^{\frac{d}{s}} \cdot \sum_{p \in P} \chi_{B_{n^{-\frac{1}{s}}}(p)}(x) dx, \tag{5.1}$$

where $\chi_{B_{n^{-\frac{1}{s}}}(p)}(x)$ is the characteristic function of the ball of radius $n^{-\frac{1}{s}}$ centered at p. We say that P is s-adaptable if

$$I_s(\mu_P) = \int \int |x-y|^{-s} d\mu_P^s(x) d\mu_P^s(y) < \infty.$$

This is equivalent to the statement

$$n^{-2} \sum_{p \neq p' \in P} |p-p'|^{-s} \lesssim 1. \tag{5.2}$$

To understand this condition in more clear geometric terms, suppose that P comes from a 1-separated set A scaled down by its diameter. Then the condition (5.2) takes the form

$$n^{-2} \sum_{a \neq a' \in A} |a-a'|^{-s} \lesssim (diameter(A))^{-s}. \tag{5.3}$$

This says P is s-adaptable if it is a scaled 1-separated set where the expected value of the distance between two points raised to the power $-s$ is comparable to the value of the diameter raised to the power of $-s$. This basically means that for the set to be s-adaptable, clustering is not allowed to be too severe.

To put it in more technical terms, s-adaptability means that a discrete point set P can be thickened into a set which is uniformly s-dimensional in the sense that its energy integral of order s is finite. Unfortunately, it is shown in [**18**] that there exist finite point sets which are not s-adaptable for certain ranges of the parameter s. The point is that the notion of Hausdorff dimension is much more subtle than the simple "size" estimate. However, many natural classes of sets are s-adaptable. For example, homogeneous sets studied by Solymosi, Vu ([**25**]) and others are s adaptable for all

$0 < s < d$. See also [16] where s adaptability of homogeneous sets is used to extract discrete incidence theorems from Fourier type bounds.

Before we state the discrete result that follows from Theorem 1.1, let us briefly review what is known. If P is set of n points in $[0,1]^2$, let $u_{2,2}(n)$ denote the number of times a fixed triangle can arise among points of P. It is not hard to see that

$$u_{2,2}(n) = O(n^{\frac{4}{3}}). \tag{5.4}$$

This follows easily from the fact that a single distance cannot arise more than $O(n^{\frac{4}{3}})$ times, which, in turn, follows from the celebrated Szemeredi-Trotter incidence theorem. See [1] and the references contained therein. By the pigeon-hole principle, one can conclude that

$$\#T_2(P) \gtrsim \frac{n^3}{n^{\frac{4}{3}}} = n^{\frac{5}{3}}. \tag{5.5}$$

However, it is not difficult to see that one can do quite a bit better as far as the lower bound on $\#T_2(P)$ is concerned. It is shown in [1], page 263 that

$$\#T_2(P) \gtrsim n \cdot \#\{|x-y| : x, y \in P\}.$$

A result due to Katz and Tardos, building on a previous breakthrough due to Solymosi and Toth says that

$$\#\{|x-y| : x, y \in P\} \gtrsim n^{\approx .86},$$

and it follows that

$$\#T_2(P) \gtrsim n^{\approx 1.86},$$

which is better than what we can deduce from Theorem 1.1. However, Theorem 1.1 does allow us to obtain an upper bound on $u_{2,2}$ for s-adaptable sets that is better than the one in 5.4. Before we state the main result of this section, we need the following definition.

Definition 5.2. Let P be a subset of $[0,1]^2$ consisting of n points as before. Let $\delta > 0$ and define

$$u^{\delta}_{2,2}(n) = \#\{(x^1, x^2, x^3) \in P \times P \times P : t_{ij} - \delta \le |x^i - x^j| \le t_{ij} + \delta\},$$

where the dependence on $t = \{t_{ij}\}$ is supressed.

Observe that obtaining an upper bound for $u^{\delta}_{2,2}(n)$ with arbitrary t_{ij} immediately implies the same upper bound on $u_{2,2}(n)$ defined above. The main result of this section is the following.

Corollary 5.3. *Let $P \subset [0,1]^2$ be s-adaptable for $s = \frac{7}{4} + a$ for every sufficiently small $a > 0$. Then for every $a > 0$, there exists $C_a > 0$ such that*

$$u^{n^{-\frac{4}{7}-a}}_{2,2}(n) \le C_a n^{\frac{9}{7}+a}. \tag{5.6}$$

The proof is fairly immediately from Theorem 1.1. Let E denote the support of $d\mu^s_P$, defined as in (5.1) above. We know that if $s > \frac{7}{4}$, then

$$\mu^s_P \times \mu^s_P \times \mu^s_P\{(x^1, x^2, x^3) : t_{ij} \le |x^i - x^j| \le t_{ij} + \epsilon\} \lesssim \epsilon^3. \tag{5.7}$$

Taking $\epsilon = n^{-\frac{1}{s}}$, we see that the left hand side is

$$\approx n^{-3} \cdot u^{n^{-\frac{1}{s}}}_{2,2}(n)$$

and we conclude that

$$u^{n^{-\frac{1}{s}}}_{2,2}(n) \lesssim n^{3-\frac{3}{s}},$$

which yields the desired result since $s = \frac{7}{4} + a$.

As we note above, this result is stronger than the previously known $u_{2,2}(n) \lesssim n^{\frac{4}{3}}$. However, our result holds under an additional restriction that P is s-adaptable. We hope to address this issue in a subsequent paper.

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