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## The Random Integral Representation Conjecture: a quarter of a century later

**Abstract.** In Jurek 1985 and 1988 the random integral representations conjecture was stated. It claims that (some) limit laws can be written as probability distributions of random integrals of the form  $\int_{(a,b]} h(t)dY_\nu(r(t))$ , for some deterministic functions  $h, r$  and a Lévy process  $Y_\nu(t), t \geq 0$ . Here we review situations where a such claim holds true. Each theorem is followed by a remark which gives references to other related papers, results as well as some historical comments. Moreover, some open questions are stated.

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*Key words and phrases:* Class  $L$  distributions or selfdecomposable distributions; infinite divisibility; Lévy-Khintchine formula; class  $\mathcal{U}$  distributions or  $s$ -selfdecomposable distributions; Euclidean space; Lévy process; random integral; Banach space .

*Abbreviated title:* Integral Representation Conjecture

In 1985 in *The Annals of Probability*, vol. 13, No. 2, on the page 607 and later on in 1988 in *Probability Theory and Related Fields*, vol. 78, on the page 474, it was conjectured that:

***Each class of limit distributions, derived from sequences of independent random variables, is the image of some subset of ID by some mapping defined as a random integral.***<sup>1</sup>

More formally, one claims that for a class  $K$  of limiting probability distributions on a Banach space  $E$  there exist: a function  $h$  (a space scaling), a function  $r$  (a time change), an interval  $A$  (in a positive half-line) and a subset  $\mathcal{D}$  of  $ID$  (the class of all infinitely divisible distributions) such that

$$K \equiv I_A^{h,r}(\mathcal{D}) := \{I_A^{h,r}(\nu) := \mathcal{L}\left(\int_A h(s) dY_\nu(r(s))\right) : \nu \in \mathcal{D}\}, \quad (1)$$

where  $Y_\nu(s)$ ,  $s \geq 0$ , is an  $E$ -valued Lévy process with cadlag paths such that its probability distribution at time 1,  $\mathcal{L}(Y_\nu(1)) = \nu$  and  $\mathcal{D}$  denotes the domain of existence of the above random integral; cf. [www.math.uni.wroc.pl/~zjjurek](http://www.math.uni.wroc.pl/~zjjurek) (The Conjecture.)

Note that the notation  $I_A^{h,r}$  for the random integral transformation can be simplified as follows

$$I_A^{h,r}(\nu) = \int_0^\infty 1_A(r^*(s))h(r^*(s)) dY_\nu(s) = I_{(0,\infty)}^{\tilde{h}(s),s}(\nu) \equiv I^{\tilde{h}}(\nu), \quad (1')$$

where  $\tilde{h}(s) := 1_A(r^*(s))h(r^*(s))$  and  $r^*$  is the inverse function of  $r$ .

The term *random integral* emphasizes the fact that the integrand  $h$  is a deterministic function. Thus for  $A = (a, b]$  we may define the random integral by the formal integration by parts formula, i. e.,

$$\int_A h(s) dY_\nu(r(s)) := h(b)Y_\nu(r(b)) - h(a)Y_\nu(r(a)) - \int_A Y_\nu(r(s))dh(s), \quad (2)$$

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<sup>1</sup>Chatterji's subsequence principle claiming that: *Given a limit theorem for independent identically distributed random variables under certain moment conditions, there exists an analogous theorem such that an arbitrary-dependent sequence (under the same moment conditions) always contains a subsequence satisfying this analogous theorem*, was proved by David J. Aldous (1977). Although we do not expect that the above Conjecture and Chatterji's subsequence principle are mathematically related, however, one may see a "philosophical" relation between those two.

provided  $h$  is of a bounded variation. Thus approximating the right-hand side integral, by Riemann -Stieltjes sums, we get the formula for the Fourier transform

$$\log(\widehat{I_A^{h,r}}(\nu))(y) = \int_A \log \widehat{\nu}(h(s)y) dr(s), \quad y \in E' \text{ is the dual Banach space.} \quad (3)$$

Random integrals on half-lines  $(a, \infty)$  are defined as weak limits of the integrals (1) for  $(a, b]$  as  $b \rightarrow \infty$ .

Below we review the old results as well as the more recent ones. In remarks after each theorem we point out to other related facts and papers. This survey is divided into three basic parts. While the last one rephrases the conjecture.

### 1. From a class of limit laws to a class of random integrals.

(a) For the Lévy class  $L$  of selfdecomposable probability measures, that coincides with the class of limiting distributions of the following of sequences

$$T_{a_n}(\xi_1 + \xi_2 + \dots + \xi_n) + x_n, \quad T_a(x) := ax, \quad a > 0, \quad x \in E, \quad (4)$$

where  $(\xi_i)$  are independent  $E$ -valued random variables,  $x_n \in E$ ,  $a_n > 0$  and the summands in (4) are uniformly infinitesimal we have:

**THEOREM 1.** (Jurek and Vervaat(1983)). *For the class  $L$  we have that*

$$L = \{ I_{(0,\infty)}^{e^{-s},s}(\nu) : \nu \in ID_{\log} \} = \{ \mathcal{L}(\int_{(0,\infty)} e^{-s} dY_\nu(s)) : \nu \in ID_{\log} \},$$

where  $ID_{\log}$  is the class of all infinitely divisible measures on  $E$  that integrate the function  $\log(1 + ||x||)$ .

**Remark 1.** (i) S. J. Wolfe (1982) and K. Sato with M. Yamazato (1984) had similar characterizations but with proofs valid only in Euclidean spaces. (ii) To the processes  $Y_\nu$  above and more generally to ones in (1) we refer to as *the background driving Lévy processes; in short: BDLP*; cf. Jurek (1996). (iii) A connection between selfdecomposable distributions and the one -dimensional Ising models in statistical physics were shown in Jurek (2001). (iv) Replacing  $T_a$ 's in (4) by arbitrary linear operators we get so called *operator-limit distributions theory*; cf. Jurek and Mason (1993) or Meerschaert and Scheffler (2001). See also Urbanik (1972, 1978).

(b) If one assumes that  $L_1 \equiv L$  and for positive integer  $m \geq 2$  one defines the class  $L_m$  as a class of limits of (4) but such that  $\mathcal{L}(\xi_i) \in L_{m-1}$  then  $L_{m+1} \subset L_m$  and moreover we have :

**THEOREM 2.** (Jurek(1983)). For the class  $L_m, m = 1, 2, \dots$  we have that

$$L_m = \{ I_{(0,\infty)}^{e^{-s}, s^m/m!}(\nu) : \nu \in ID_{\log^m} \} = \{ \mathcal{L}(\int_{(0,\infty)} e^{-s} dY_\nu(\frac{s^m}{m!}) : \nu \in ID_{\log^m} \},$$

where  $ID_{\log^m}$  is the class of all infinitely divisible measures on  $E$  that integrate the function  $\log^m(1 + ||x||)$ .

**Remark 2.** (i) The idea of classes  $L_m$  belongs to Urbanik (1973) with a different scheme of summation; see also Kumar and Schreiber (1979). The iterative approach was proposed in K. Sato (1980) (for Euclidean spaces) and later generalized in Jurek (1983a) in two direction: replacing Euclidean space by an arbitrary separable Banach space  $E$  and more importantly replacing the group  $(T_a, a > 0)$  of dilation by an arbitrary strongly continuous one-parameter group  $\mathbb{U}$  of bounded linear operators on  $E$ . (ii) The particular case of the group  $\mathbb{U} := \{e^{-tQ} : t \in \mathbb{R}\}$ , where  $Q$  is a fixed bounded linear operator on a Banach space  $E$ , was investigated in Jurek (1983b), where it was shown that  $L_m(Q) = \{ I_{(0,\infty)}^{e^{-tQ}, t}(\nu) : \nu \in ID_{\log^m} \}$ , i.e., in Theorem 2 the scalar function  $e^{-t}$  is replaced by the operator-valued function  $e^{-tQ}$ . Here one needs a new norm and the polar coordinates in a Banach space; cf. Jurek (1984). (iii) N. Thu (1986) extended classes  $L_m, m = 1, 2, \dots$ , to  $L_\alpha, \alpha > 0$ , by using the fractional calculus.

(c) Let us replace the linear normalization in (4) by the non-linear shrinking s-operation  $U_r$  ( $r > 0$ ) and consider the class  $\mathcal{U}$  of limiting distributions in the following scheme

$$U_{r_n}(\xi_1) + U_{r_n}(\xi_2) + \dots + U_{r_n}(\xi_n) + x_n, \quad U_r(x) := \max(||x|| - r, 0) \frac{x}{||x||}, \quad x \neq 0, \quad (5)$$

where the summands are uniformly infinitesimal. Limiting distributions of (5) are called *s-selfdecomposable distributions*.

**THEOREM 3.** (Jurek(1985)). For the class  $\mathcal{U}$  of *s-selfdecomposable distributions* we have that

$$\mathcal{U} = \{ I_{(0,1)}^{s,s}(\nu) : \nu \in ID \} = \{ \mathcal{L}(\int_{(0,1)} s dY_\nu(s)) : \nu \in ID \},$$

where  $ID$  is the class of all infinitely divisible measures .

**Remark 3.** (i) Note that  $U_r(U_s(x)) = U_{r+s}(x)$  (semigroup of non-linear transformations) and for positive random variable  $\xi > 0$ , we get  $U_r(\xi) = (\xi - r)^+$  (the positive part), i.e., it coincides with the famous financial derivative *call option*. (ii) Characterizations of the s-selfdecomposable distributions in terms of the Lévy-Khintchine formula were presented during 2<sup>nd</sup> Vilnius Conference; cf. Jurek (1977). Complete proofs were given in Jurek (1981). (iii) The CLT for s-operations  $U_r$  was proved by Housworth and Shao (2000). (iv) Classes  $\mathcal{U}_\beta := I_{(0,1)}^{s, s^\beta}(ID)$ , were investigated in a series of papers: Jurek (1988) for  $\beta > 0$ , Jurek (1989) for  $-1 \leq \beta < 0$  and Jurek and Schreiber (1992) for  $-2 < \beta \leq -1$ . In the last two cases, the stable distributions appeared as convolution factors of the limiting distributions. Measures from  $\mathcal{U}_\beta$  are called *generalized s-selfdecomposable distributions*.

In a similar way as the classes  $L_m$  were introduced in Theorem 2, one may iterate the random integral mapping  $I_{(0,1)}^{s,s}$  from Theorem 3, and get the classes  $\mathcal{U}^{<m>}$  for which we have

**THEOREM 4.** (Jurek(2004)). *For the class  $\mathcal{U}^{<m>}$  (with  $m = 1, 2, \dots$ ) of  $m$ -times s-selfdecomposable distributions we have that*

$$\mathcal{U}^{<m>} = \{ I_{(0,1)}^{s, \tau_m(s)}(\nu) : \nu \in ID \} = \{ \mathcal{L} \left( \int_{(0,1)} s dY_\nu(\tau_m(s)) : \nu \in ID \right), \\ \tau_m(s) := \frac{1}{(m-1)!} \int_0^s (-\log u)^{m-1} du, \quad 0 < s \leq 1, \}$$

where  $ID$  is the class of all infinitely divisible measures .

Although the classes  $L_m$  and  $\mathcal{U}^{<m>}$  originated in two different limiting schemes ( via the linear dilations  $T_a$  and the non-linear s-operations  $U_r$ , respectively) they still admit some unexpected relations.

**COROLLARY 1.** (Jurek (2004)) (a) *We have inclusions*

$$L_{m+1} \subset \mathcal{U}^{<m+1>} \subset \mathcal{U}^{<m>} \subset ID, \quad m = 1, 2, \dots$$

$$(b) \quad L_\infty := \bigcap_{m=1}^{\infty} L_m = \mathcal{U}^{<\infty>} := \bigcap_{m=1}^{\infty} \mathcal{U}^{<m>}$$

*= the smallest closed convolution semigroup that contains all stable measures.*

**Remark 4.** (i) Maejima and Sato (2009) proved that besides the two instances described in Corollary 1 (b) there are another three classes for which infinitely many integral iterations lead to the smallest closed convolution

semigroup that contains all stable measures. (ii) Still an open question is to describe  $L_\infty(\mathbb{U}) := \bigcap_{m=1}^\infty L_m(\mathbb{U})$ , where  $\mathbb{U}$  is an one-parameter group of bounded linear operators on a Banach space  $E$ ; comp. Remark 2(i) and Jurek (1983).

## 2. From a class of random integrals to ...

The original aim (in the 80's of the last century) was to identify *a given class  $K$  of limit distributions* as a collection of probability distributions of some random integrals; comp. the above Section 1. Later on, more often questions were asked whether *given class of distributions* (or Fourier transforms or Lévy spectral measures), can be described in terms of some random integrals. In this section we discuss only two of such examples.

(a) D. Voiculescu and others studying so called *free-probability* introduced new binary operations on probability measures and termed them *free-convolutions*; cf. Bercovici-Voiculescu (1993) and references therein. For the *additive free-convolution*  $\square$  the Voiculescu transform  $V_\nu(z), z \in \mathbb{C}$ , (an analogue of the characteristic function  $\hat{\nu}(t), t \in \mathbb{R}$ ) is additive. Namely,  $V_{\nu_1 \square \nu_2}(z) = V_{\nu_1}(z) + V_{\nu_2}(z)$ . This property allowed to introduce a notion of free-infinite divisibility.

**THEOREM 5.** (Jurek (2007).) *A probability measure  $\nu$  is  $\square$ -infinitely divisible if and only if there exist a unique  $*$ -infinitely divisible probability measure  $\mu$  such that*

$$(it) V_\nu((it)^{-1}) = \log \left( I_{(0,\infty)}^{s, 1-e^{-s}}(\mu) \right)^\wedge(t) = \log \left( \mathcal{L} \left( \int_0^\infty s dY_\mu(1-e^{-s}) \right) \right)^\wedge(t), \quad t \neq 0,$$

where  $(Y_\mu(t), t \geq 0)$  is a Lévy process such that  $\mathcal{L}(Y_\mu(1)) = \mu$ .

**Remark 5.** Using Theorem 5 one can easily see that for we have the integral mapping

$$\mathcal{K}(\mu) := I_{(0,\infty)}^{s, 1-e^{-s}}(\mu) = I_{(0,1)}^{-\log s, s}(\mu), \quad \mu \in ID. \quad (6)$$

Mapping  $\mathcal{K}$  was called the  $\Upsilon$  (upsilon) transform and studied from a different point of view by Barndorff-Nielsen, Maejima and Sato (2006); also by Barndorff-Nielsen, Rosiński and Thorbjorsen (2008), and by Maejima and Sato (2009).

(b) Thorin class  $T(\mathbb{R}^d)$  is an example of the class of infinitely divisible distributions defined by properties of their Lévy spectral measures that, later on, was characterized by some random integrals; for more details cf. Maejima and Sato (2009), p. 121; for related results cf. Grigelionis (2007). For the class  $G$  distributions see Aoyama and Maejima (2007).

**THEOREM 6.** (Maejima and Sato (2009))

$$T(\mathbb{R}^d) = \left\{ \mathcal{L} \left( \int_0^\infty e^*(t) dY_\mu(t) \right) : \mu \in ID_{\log} \right\} = \left\{ I_{(0,\infty)}^{s,e(s)}(\mu) : \mu \in ID_{\log} \right\},$$

where  $e(s) := \int_s^\infty u^{-1} e^{-u} du$ ,  $s > 0$  and  $e^*(t)$ ,  $t > 0$  is its inverse function.

**Remark 6.** In Jurek (2007), Proposition 4, the class  $\mathcal{TS}_\alpha$  of tempered stable distributions with the index  $0 < \alpha < 1$  was identified as the class of random integrals

$$\mathcal{TS}_\alpha = \left\{ I_{(0,\infty)}^{s,\Gamma(-\alpha,s)}(\mu) : \mu \in ID_\alpha \right\}, \text{ and } \Gamma(-\alpha, s) := \int_s^\infty w^{-\alpha-1} e^{-w} dw, \quad s > 0,$$

and  $ID_\alpha$  denotes the set of all infinitely measures whose Lévy spectral measures integrate  $\|x\|^\alpha$  over the space  $E$ .

### 3. A calculus on (Lévy exponents of) ID distributions.

On the random integrals (1), (viewed as mappings defined on (some) infinitely divisible probability measures  $\mu$ ), one can perform transformations such as compositions or the arithmetic operations. Because of the formula (3) all of that operations have natural generalizations to the Lévy exponents, that is, the logarithms  $\Phi := \log \hat{\mu}$  of Fourier transforms of  $ID$  measures  $\mu$ . Such a calculus may lead to new factorization properties. For the simplicity of the notations let us pu

$$\mathcal{I}(\mu) \equiv I_{(0,\infty)}^{e^{-s},s}(\mu), \text{ for } \mu \in ID_{\log} \quad \text{and} \quad \mathcal{J}(\mu) \equiv I_{(0,1)}^{s,s}(\mu), \text{ for } \mu \in ID. \quad (7)$$

Then we have

**THEOREM 7.** (Jurek (1985) and (2008)) (i) For the mappings  $\mathcal{I}$  and  $\mathcal{J}$  and  $\nu \in ID_{\log}$  we have the identity

$$\mathcal{I}(\nu * \mathcal{J}(\nu)) = \mathcal{I}(\nu) * \mathcal{I}(\mathcal{J}(\nu)) = \mathcal{I}(\nu) * \mathcal{J}(\mathcal{I}(\nu)) = \mathcal{J}(\nu);$$

(ii) For each selfdecomposable measure  $\mu \in L$  there exists a unique  $s$ -selfdecomposable measure  $\tilde{\mu} \in \mathcal{U}$  such that

$$\mu = \tilde{\mu} * \mathcal{I}(\tilde{\mu}) \quad \text{and} \quad \mathcal{J}(\mu) = \mathcal{I}(\tilde{\mu}). \quad (8)$$

**Remark 7.** (i) When one considers  $\mathcal{I}$  and  $\mathcal{J}$  as the mappings defined on the Lévy exponents  $\Phi$ 's (via the equation (3)) then one gets  $\mathcal{I}(\mathcal{J}) = \mathcal{I} - \mathcal{J}$  or  $\mathcal{J}(I + \mathcal{I}) = \mathcal{I}$  or  $\mathcal{I}(I - \mathcal{J}) = \mathcal{J}$  or  $(I - \mathcal{J})(I + \mathcal{I}) = I$ . (ii) Iterating the property (8) we get a convolution decomposition of selfdecomposable distributions with  $m$ -times  $s$ -selfdecomposable distributions (from the class  $\mathcal{U}^{<m>}$ ) as the factors; cf. Jurek (2008).

From Theorem 1 and formula (7) we have that  $L = \mathcal{I}(ID_{\log})$ . We say that a selfdecomposable  $\mu = \mathcal{I}(\rho)$  has the *factorization property* if  $\mathcal{I}(\rho) * \rho \in L$  and let  $L^f$  denotes the class of all class  $L$  distributions having the factorization property.

**THEOREM 8.** (Ikasnov, Jurek and Schreiber (2004); Czyżewska-Jankowska and Jurek (2009)) (i)  $L^f = \mathcal{I}(\mathcal{J}(ID_{\log}))$ ;

$$(ii) L^f = I_{(0,\infty)}^{e^{-s}, s+e^{-s}-1}(ID_{\log}) = \left\{ \mathcal{L} \left( \int_0^\infty e^{-s} dY_\nu(s+e^{-s}-1) \right) : \nu \in ID_{\log} \right\}.$$

**Remark 8.** (i) Note that  $I_{(0,\infty)}^{e^{-s}, s+e^{-s}-1} = I_{(0,1)}^{s, s-\log s-1}$ . (ii) One of the most important examples of the class  $L^f$  distribution is the Lévy's stochastic area integral (the hyperbolic sine characteristic function). In Jurek and Yor (2004) BDLP were identified as Bessel squared processes. (iii) Above we have that  $I_{(0,1)}^{s,s}(I_{(0,\infty)}^{e^{-s},s}) = I_{(0,\infty)}^{e^{-s}, s+e^{-s}-1}$ , it means that a composition of two random integrals is again a random integral.

#### 4. Concluding remarks.

Taking into account the above historical survey :

( $\alpha$ ) Can we still hope for a general proof of the RANDOM INTEGRAL REPRESENTATION CONJECTURE ?

Even without settling the previous question:

( $\beta$ ) Can we develop "an abstract theory" of a calculus on random integral mappings  $I_A^{h,r}$  (or on the corresponding Lévy exponents or Lévy spectral measures)?



Since (many) classes, of probability distribution of random integral mappings discussed above, naturally form convolution semigroups:

( $\gamma$ ) Can we find structural descriptions of ALL (closed) convolution subsemigroups of the semigroup  $ID$  ( of all infinitely divisible distributions)?

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