Fixed points of the smoothing transform: Two-sided solutions

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September 14, 2010

Abstract

Given a sequence $(C, T) = (C, T_1, T_2, \ldots)$ of real-valued random variables with $T_j \geq 0$ for all $j \geq 1$ and almost surely finite $N = \sup\{j \geq 1 : T_j > 0\}$, the smoothing transform associated with (C, T), defined on the set $\mathcal{P}(\mathbb{R})$ of probability distributions on the real line, maps an element $P \in \mathcal{P}(\mathbb{R})$ to the law of $C + \sum_{j\geq 1} T_j X_j$, where $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables independent of (C, T) and with distribution P. In this paper, we study the fixed points of the smoothing transform, that is, the solutions to the stochastic fixed-point equation $X_1 \stackrel{\mathrm{d}}{=} C + \sum_{j\geq 1} T_j X_j$. By drawing on recent work by the authors with J. D. Biggins, a full description of the set of solutions is provided under weak assumptions on the sequence (C, T). This solves a problem posed by Janson and Fill [16]. Our results include precise characterizations of the sets of solutions to large classes of stochastic fixed-point equations that appear in the asymptotic analysis of divide-and-conquer algorithms, for instance the `Quicksort` equation. As a by-product of our analysis, we obtain a result on Seneta-Heyde norming of general branching processes which generalises earlier work by Cohn and may be of interest in its own right.

Keywords: Branching random walk; characteristic function; general branching processes; infinite divisibility; multiplicative martingales; smoothing transformation; stable distribution; stochastic fixed-point equation; weighted branching process

2010 Mathematics Subject Classification: Primary: 60E05
Secondary: 39B32, 60E10, 60J80

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1 Introduction

Let $(C, T) = (C, T_1, T_2, \ldots)$ be a given sequence of real-valued random variables such that the T_j are non-negative and

$$\mathbb{P}(N < \infty) \; = \; 1, \tag{1.1}$$

where $N = \sup\{j \geq 1 : T_j > 0\}$. On the set $\mathcal{P}(\mathbb{R})$ of probability distributions on the line, the smoothing transform (associated with (C, T)) is defined as the mapping

$$P \; \mapsto \; \mathcal{L}\left(C + \sum_{j \geq 1} T_j X_j\right),$$

where $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with common distribution P and independent of (C, T) and where $\mathcal{L}(X)$ denotes the law of a random variable X. A fixed point of this smoothing transform is given by any $P \in \mathcal{P}(\mathbb{R})$ such that, if X has distribution P, the equation

$$X \; \stackrel{\mathrm{d}}{=} \; C + \sum_{j \geq 1} T_j X_j \tag{1.2}$$

holds true. We call this equation homogeneous if $C = 0$, that is, if

$$X \; \stackrel{\mathrm{d}}{=} \; \sum_{j \geq 1} T_j X_j. \tag{1.3}$$

On the set of *non-negative* solutions to (1.3), there is a substantial literature, [7, 15, 24, 9, 23, 20, 10, 6, 2], and relatively complete results. Two-sided solutions to the homogeneous equation, with special focus on symmetric ones and those with finite variance, have been studied in [12, 13] which also allow real-valued T_j, $j \geq 1$.

Our approach to Eqs. (1.2) and (1.3) is heavily based on the use of characteristic functions. Indeed, (1.2) has an equivalent reformulation in terms of the characteristic function $\phi(t) := \mathbb{E}\exp(\mathrm{i}\, tX)$ of X ($t \in \mathbb{R}$, i the imaginary unit), *viz.*

$$\phi(t) \; = \; \mathbb{E}\exp(\mathrm{i}\, Ct) \prod_{j \geq 1} \phi(T_j t) \quad (t \in \mathbb{R}). \tag{1.4}$$

In the homogeneous case, this equation takes the form

$$\phi(t) \; = \; \mathbb{E} \prod_{j \geq 1} \phi(T_j t) \quad (t \in \mathbb{R}). \tag{1.5}$$

Without loss of generality, we assume that N satisfies

$$N \; = \; \sum_{j \geq 1} \mathbb{1}_{\{T_j > 0\}} \tag{1.6}$$

and define the function

$$m : [0, \infty) \to [0, \infty], \quad \theta \mapsto \mathbb{E} \sum_{j=1}^{N} T_j^{\theta}. \tag{1.7}$$

m plays a crucial role in the analysis of (1.3) and can be viewed as the Laplace transform of the intensity measure of the point process

$$\mathcal{Z} \;:=\; \sum_{j=1}^{N} \delta_{S(j)}, \tag{1.8}$$

where $S(j) := -\log T_j$. Hence, m is a convex and continuous function on the possibly unbounded interval $\{m < \infty\}$.

Throughout the paper, we will make the following standing assumptions:

$$\mathbb{P}\left(T \in \{0\} \cup r^{\mathbb{Z}}\right) < 1 \quad \text{for all } r \geq 1. \tag{A1}$$

$$m(0) = \mathbb{E}\, N > 1. \tag{A2}$$

$$1 = m(\alpha) < m(\beta) \text{ for some } \alpha > 0 \text{ and all } \beta \in [0, \alpha). \tag{A3}$$

Condition (A1) ensures that the point process $\mathcal{Z}$ is not concentrated on any lattice $\lambda\,\mathbb{Z}$, $\lambda > 0$, which is a natural assumption in view of examples of Eq. (1.2) coming from applications. Further, as explained in Caliebe [12], only simple cases are ruled out when assuming (A2). Moreover, in view of previous studies of (1.3) in more restrictive settings [15, 24, 6, 5], it is natural to make the assumption (A3) on the behaviour of m. We refer to [5, Theorem 6.1, Example 6.4] for the most recent discussion. We refer to α as the *characteristic exponent (of T)*.

2 Main results and applications

Let $\mathfrak{F}$ denote the set of characteristic functions of probability measures on $\mathbb{R}$, and let ϕ_0 be the characteristic function of the Dirac measure at 0, *i.e.*, $\phi_0(t) = 1$ for all $t \in \mathbb{R}$. We then define

$$\mathcal{S}(\mathfrak{F}) \;:=\; \{\phi \in \mathfrak{F} \setminus \{\phi_0\} : \phi \text{ solves } (1.5)\} \tag{2.1}$$

and

$$\mathcal{S}(\mathfrak{F})(C,T) \;:=\; \{\phi \in \mathfrak{F} : \phi \text{ solves } (1.4)\}. \tag{2.2}$$

Our aim is to provide a full description of the sets $\mathcal{S}(\mathfrak{F})$ and $\mathcal{S}(\mathfrak{F})(C,T)$. We begin with the homogeneous case $C = 0$.

2.1 The homogeneous case

In order to determine $\mathcal{S}(\mathfrak{F})$, we need the following additional weak assumption.

$$\text{(A4a) or (A4b) holds,} \tag{A4}$$

where

$$\mathbb{E} \sum_{j=1}^{N} T_j^{\alpha} \log T_j \;\in\; (-\infty, 0) \text{ and } \mathbb{E}\Big(\sum_{j=1}^{N} T_j^{\alpha}\Big) \log^{+}\Big(\sum_{j=1}^{N} T_j^{\alpha}\Big) \;<\; \infty; \tag{A4a}$$

$$\text{There exists some } \theta \in [0,\alpha) \text{ satisfying } m(\theta) \ < \ \infty. \tag{A4b}$$

Indeed, (A4) is enough to determine $\mathcal{S}(\mathfrak{F})$ in the case that $\alpha \neq 1$. Before we proceed with stating our main results in the homogeneous case, we provide some background information on non-negative solutions to

$$X \ \stackrel{\mathrm{d}}{=} \ \sum_{j\geq 1} T_j^\alpha X_j \tag{2.3}$$

where $X_1, X_2, \dots$ are i.i.d. copies of X and independent of T. Solutions to (2.3) play an important role because they appear as mixing distributions in all other cases. Theorem 2.3 in [2] states that there is a non-negative, non-trivial (*i.e.* non-zero) solution to (2.3) and that its distribution is unique up to scaling. In what follows, we fix a particular, non-negative and non-trivial solution to (2.3) and denote it by W. Further information on W will be provided in Subsection 2.4.

Theorem 2.1. *Assume that* (A1)-(A4) *and* $\alpha \in (0,2] \setminus \{1\}$ *hold true. Then* $\mathcal{S}(\mathfrak{F})$ *is given by the family*

$$\phi(t) \ = \ \begin{cases} \mathbb{E}\exp\left(-\sigma^\alpha W|t|^\alpha\left[1 - \mathrm{i}\,\beta\frac{t}{|t|}\tan\left(\frac{\pi\alpha}{2}\right)\right]\right), & \text{if } \alpha \neq 2, \\ \mathbb{E}\exp(-\sigma^2 W t^2), & \text{if } \alpha = 2. \end{cases} \tag{2.4}$$

The range of the parameters is given by $\sigma > 0$, $\beta \in [-1,1]$ *if* $\alpha \neq 2$, *and* $\sigma > 0$ *if* $\alpha = 2$.

The case $\alpha = 1$ is more involved than the case $\alpha \neq 1$ due to a phenomenon called *endogeneity*, a notion coined by Aldous and Bandyopadhyay [1]. In order to determine $\mathcal{S}(\mathfrak{F})$ despite the appearance of additional endogeneity effects, we need one more assumption concerning the sequence T:

$$\mathbb{E}\sum_{j=1}^N T_j^\alpha (\log^- T_j)^2 \ < \ \infty. \tag{A5}$$

Note that (A5) is implied by (A4b) whereas it constitutes a non-void assumption if (A4a) holds but (A4b) fails.

Theorem 2.2. *Suppose that* (A1)-(A5) *and* $\alpha = 1$ *hold true. Then* $\mathcal{S}(\mathfrak{F})$ *is given by the family*

$$\phi(t) \ = \ \mathbb{E}\exp\left(\mathrm{i}\,\mu W t - \sigma W|t|\right), \tag{2.5}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ *and* $(\mu,\sigma) \neq (0,0)$.

2.2 The inhomogeneous case

In order to solve the inhomogeneous equation, we do not only need assumptions on T as before but also on C. Two important assumptions here are:

$$m(1) < \infty, \ \mathbb{E}\,|C| < \infty, \text{ and } W_n^* \text{ is } \mathcal{L}^p\text{-bounded for some } p \geq 1. \tag{C1}$$

$$m(\beta) < 1 \text{ and } \mathbb{E}\,|C|^\beta < \infty \text{ for some } 0 < \beta \leq 1. \tag{C2}$$

Theorem 2.3. *Suppose that* (A1)-(A4) *and one of the conditions* (C1) *and* (C2) *hold true. Additionally assume* (A5) *in the case* $\alpha = 1$. *Then there exists a coupling* (W^*, W) *of random variables such that* W^* *solves* (1.2), W *is a non-negative solution to* (2.3) *and the set of characteristic functions of solutions to* (1.2) *is given by the family*

$$\phi(t) \;=\; \begin{cases} \mathbb{E}\exp\left(\mathrm{i}\, W^* t - \sigma^\alpha W|t|^\alpha \left[1 - \mathrm{i}\,\beta \frac{t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right)\right]\right), & \text{if } \alpha \notin \{1,2\}, \\ \mathbb{E}\exp\left(\mathrm{i}\,(W^* + \mu W)t - \sigma W|t|\right), & \text{if } \alpha = 1, \\ \mathbb{E}\exp(\mathrm{i}\, W^* t - \sigma^2 W t^2), & \text{if } \alpha = 2. \end{cases} \tag{2.6}$$

The range of the parameters is given by $\sigma \geq 0$, $\beta \in [-1, 1]$ *if* $\alpha \notin \{1, 2\}$, $\mu \in \mathbb{R}$, $\sigma \geq 0$ *if* $\alpha = 1$, *and* $\sigma \geq 0$ *if* $\alpha = 2$. *The coupling* (W^*, W) *can be explicitly constructed in terms of the weighted branching model introduced in Subsection 3.1:* W *can be constructed via* (4.23) *and* W^* *by taking the limit in* (5.3).

2.3 Applications

Examples of the stochastic fixed-point equation (1.2) and its homogeneous counterpart (1.3) abound in the asymptotic analysis of random recursive structures, see *e.g.* [26] and [1] and the references therein. For their occurrence in stochastic geometry see [28] and [29]. Here we confine ourselves to an explicit mention of a particularly prominent example of (1.2), *viz.* the `Quicksort` equation:

$$X \;\stackrel{\mathrm{d}}{=}\; UX_1 + (1-U)X_2 + g(U) \tag{2.7}$$

where $U \sim \mathrm{Unif}(0,1)$, X_1, X_2 are i.i.d. copies of X independent of U, and

$$g : (0,1) \to (0,1), \quad u \mapsto 2u \log u + 2(1-u)\log(1-u) + 1.$$

This equation arises in the study of the asymptotic behaviour of the number C_n of key comparisons `Quicksort` requires to sort a list of n distinct reals, see [30]. More presicely, $(C_n - \mathbb{E}\, C_n)/n$ converges weakly as $n \to \infty$ to a distribution P on $\mathbb{R}$ which is a solution to (2.7). It is the unique solution with mean 0 and finite variance. The set of all solutions to (2.7) (without any moment constraints) has been determined by Fill and Janson [16]. Their result is included in Theorem 2.3 and stated next as a corollary.

Corollary 2.4. *The set of characteristic functions* ϕ *of solutions* X *to the* `Quicksort` *equation* (2.7) *is given by the family*

$$\phi(t) \;=\; \psi(t)\, \exp(\mathrm{i}\,\mu t - \sigma|t|), \quad t \in \mathbb{R}$$

with $\mu \in \mathbb{R}$, $\sigma \geq 0$ *and* ψ *denoting the characteristic function of the distributional limit of* $(C_n - \mathbb{E}\, C_n)/n$.

In other words, the set of solutions to (2.7) equals the set

$$\{P * \mathcal{C}(\mu, \sigma) : \mu \in \mathbb{R}, \sigma \geq 0\}$$

where P is the distribution peraining to ψ and $\mathcal{C}(\mu, \sigma)$ denotes the Cauchy distribution with parameters μ and σ. Here we interpret $\mathcal{C}(\mu, 0)$ as the Dirac measure at μ.

2.4 Discussion of the main results

We continue with a definition of stable distributions following [31]. We say that Y has distribution $\mathcal{S}_\alpha(\sigma,\beta,\mu)$ for $\alpha \in (0,2]$, $\sigma \geq 0$, $\beta \in [-1,1]$ and $\mu \in \mathbb{R}$ if Y has characteristic function $\exp(\psi_Y)$, where $\psi_Y(0) = 0$ and, for $t \neq 0$,

$$\psi_Y(t) \; = \; \begin{cases} \mathrm{i}\,\mu t - \sigma^\alpha |t|^\alpha \left(1 - \mathrm{i}\,\beta \frac{t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right)\right) & \text{if } \alpha \neq 1, \\ \mathrm{i}\,\mu t - \sigma |t| \left(1 + \mathrm{i}\,\beta \frac{t}{|t|} \frac{2}{\pi} \log|t|\right) & \text{if } \alpha = 1. \end{cases}$$

Here, α is called index of stability, σ scale parameter, β skewness parameter, and μ shift parameter. Notice that if $\alpha = 2$, β becomes meaningless so that we can assume $\beta = 0$ in this case. Now suppose that Y is a random variable, defined on the same probability space as and independent of the random variable W in Theorem 2.1, with distribution $Y \sim \mathcal{S}_\alpha(\sigma,\beta,0)$ for $\alpha \in (0,2)\setminus\{1\}$, $\sigma > 0$ and $\beta \in [-1,1]$. Then a standard calculation shows that $W^{1/\alpha}Y$ has characteristic function ϕ as in (2.4). Thus Theorem 2.1 implies that any solution X to (1.3) has a representation of the form

$$X \; \stackrel{\mathrm{d}}{=} \; W^{1/\alpha} Y \tag{2.8}$$

for an appropriate stable random variable Y. Similarly, in the inhomogeneous case, any solution X to (1.2) has a representation of the form

$$X \; \stackrel{\mathrm{d}}{=} \; W^* + W^{1/\alpha} Y \tag{2.9}$$

where (W^*, W) is the coupling from Theorem 2.3 and Y is independent of (W^*, W) and has an appropriate stable law. Analogous constructions can be made in the cases $\alpha = 1$ and $\alpha = 2$.

The random variables W^* and W have been studied in the literature. If (A1)-(A3) and (A4a) hold, then W can be chosen as the intrinsic martingale limit of an appropriate branching random walk, see the beginning of Subsection 4.8 for more details. This limit has been well studied in a host of articles concerning existence of moments or its tail behaviour, see *e.g.* [20, 3, 22] to name but a few. The random variable W^* can be constructed from the weighted branching process introduced in Subsection 3.1 and there are also results on the moments of W^* in more restrictive settings, see *e.g.* [22].

We finish this section with an overview of the further organization of this work and an outline of the proof of our main results. The first step will be the formulation of a weighted branching model in Section 3 that allows the iteration of (1.2) and (1.3) on a fixed probability space. In the course of Section 3, we will further introduce an equivalent branching random walk model. The following Section 4 is devoted to the solutions of the homogeneous equation (1.3). The simple inclusion there is to verify that the functions ϕ defined in Theorems 2.1 and 2.2 are actually characteristic functions solving the functional equation (1.5). This will be done in Subsection 4.1. The proof of the reverse inclusion is more involved. The basic tool there is the use of multiplicative martingales derived from the characteristic functions of solutions to (1.3), see Subsection 4.2. The limits of these martingales are one-to-one with the solutions to the functional equation (1.5). Further, they are stochastic processes that satisfy a pathwise counterpart of the functional equation (1.5). Since, on the other hand, it is known from earlier work by Caliebe [12] that the paths of the martingale limits are characteristic functions of infinitely divisible distributions and thus possessing a unique Lévy representation, one can deduce a pathwise equation for the random Lévy exponent involved, see Eq. (4.12). Starting from this equation, which constitutes the heart of our approach, we determine the random Lévy measures of the martingale limits in Subsection 4.5. In Subsection 4.9, we completely solve Eq. (4.12), which in turn immediately leads to a proof of our main Theorems 2.1 and 2.2. But before we can solve (4.12), we have to deal with the already mentioned phenomenon of *endogeny*. Endogeneous fixed points will be introduced in Subsection 4.4. They are special solutions to (1.3) that can entirely be defined in terms of the underlying weighted branching process. Their appearance significantly complicates the analysis of Eq. (4.12). Therefore, we first determine all endogeneous solutions to (1.3) in Subsection 4.8. To accomplish this, we make heavy use of results on the asymptotic behaviour of general branching processes. These results are derived in Subsection 4.7 including a theorem on Seneta-Heyde norming for general branching processes (Theorem 4.11) that may be of interest in its own right. Section 5 is devoted to the study of the inhomogeneous equation (1.2). Using again multiplicative martingales, we show the existence of an explicit one-to-one correspondence between the solutions to the inhomogeneous equation and the corresponding homogeneous one. From this result, it is easy to deduce Theorem 2.3.

3 Iterating the fixed-point equation

Iteration forms a natural tool in the study of a functional equation which, in the case of Eqs (1.2) and (1.3), leads to a weighted branching model associated with the input variable (C,T). This model will be introduced next. It is intimately connected to the branching random walk based on the point process $\mathcal{Z}$ as introduced in Eq. (1.8). We will discuss the connection to this branching random walk and general branching processes in Subsection 3.2.

3.1 The weighted branching model

Let $\mathbb{V} := \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n$ denote the infinite Ulam-Harris tree, where $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$. The elements $v \in \mathbb{V}$ will be called individuals or vertices. We abbreviate $v = (v_1,\ldots,v_n)$ by $v_1\ldots v_n$ and write $v|k$ for the restriction of v to the first k entries, *i.e.*, $v|k := v_1\ldots v_k$ if $k \leq n$ and $v|k := v$ if $|v| > n$. Let vw denote the concatenation of the vertices v and w, that is, the vertex $v_1\ldots v_n w_1 \ldots w_m$ where $w = w_1\ldots w_m$. In this case, we say that v is an ancestor of vw. The length of a node v is denoted by $|v|$, thus $|v| = n$ iff $v \in \mathbb{N}^n$. Now let $\mathbf{C}\otimes\mathbf{T} := ((C(v),T(v)))_{v\in\mathbb{V}}$ be a family of i.i.d. copies of (C,T), where $(C(\varnothing),T(\varnothing)) = (C,T)$. We refer to $(C,T) = (C,T_1,T_2,\ldots)$ as the *basic sequence (of the weighted branching model)* and interpret $C(v)$ as a weight attached to the vertex v and $T_i(v)$ as a weight attached to the edge (v, vi) in the infinite tree $\mathbb{V}$. Then define $L(\varnothing) := 1$ and, recursively,

$$L(vi) \ := \ L(v)T_i(v) \tag{3.1}$$

for $v \in \mathbb{V}$ and $i \in \mathbb{N}$. For $n \in \mathbb{N}_0$, let $\mathcal{A}_n$ denote the σ-algebra generated by the $(C(v), T(v))$, $|v| < n$. Put also $\mathcal{A}_\infty := \sigma(\mathcal{A}_n : n \geq 0) = \sigma(\mathbf{C} \otimes \mathbf{T})$.

Further, we assume the existence of a family $\mathbf{X} = (X(v))_{v \in \mathbb{V}}$ of i.i.d. copies of X which is independent of $\mathbf{C} \otimes \mathbf{T}$. Then $n$fold iteration of (1.2) can be expressed in terms of the weighted branching model:

$$X \ \stackrel{\mathrm{d}}{=} \ \sum_{|u|<n} L(u)C(u) + \sum_{|v|=n} L(v)X(v). \tag{3.2}$$

In the homogeneous case, the first sum on the right-hand side vanishes and (3.2) simplifies to

$$X \ \stackrel{\mathrm{d}}{=} \ \sum_{|v|=n} L(v)X(v). \tag{3.3}$$

The functional equations (1.4) and (1.5) after n iterations become

$$\phi(t) \ = \ \mathbb{E}\left[\exp\left(\mathrm{i}\, t \sum_{|u|<n} L(u)C(u)\right) \prod_{|v|=n} \phi(L(v)t)\right] \qquad (t \in \mathbb{R}) \tag{3.4}$$

and

$$\phi(t) \ = \ \mathbb{E} \prod_{|v|=n} \phi(L(v)t) \qquad (t \in \mathbb{R}), \tag{3.5}$$

respectively.

We close this subsection with the definition of the shift operators $[\cdot]_u$, $u \in \mathbb{V}$. Given any function $\Psi = \psi(\mathbf{C} \otimes \mathbf{T})$ of the weight family $\mathbf{C} \otimes \mathbf{T}$ pertaining to $\mathbb{V}$, let

$$[\Psi]_u \ := \ \psi(((C(uv), T(uv)))_{v \in \mathbb{V}})$$

be the very same function but for the weight ensemble pertaining to the subtree rooted at $u \in \mathbb{V}$. Any branch weight $L(v)$ can be viewed as such a function, and we thus have $[L(v)]_u = T_{v_1}(u) \cdot ... \cdot T_{v_n}(uv_1...v_{n-1})$ if $v = v_1...v_n$. In the particular case that $L(u) > 0$, this can be rephrased as $[L(v)]_u = L(uv)/L(u)$.

3.2 The corresponding branching random walk

The weighted branching model introduced in Subsection 3.1 turns into a classical *branching random walk (BRW)* model after logarithmic scaling. For $v \in \mathbb{V}$, define

$$S(v) \ := \ -\log L(v) \tag{3.6}$$

where $-\log 0 := \infty$ is stipulated. Further, let

$$\mathcal{Z} \ := \ \sum_{j=1}^{N} \delta_{S(j)} \tag{3.7}$$

and

$$\mathcal{Z}_n \;:=\; \sum_{\substack{|v|=n:\\ S(v)<\infty}} \delta_{S(v)}, \quad n \in \mathbb{N}_0\,. \tag{3.8}$$

Then the sequence of point processes $(\mathcal{Z}_n)_{n\geq 0}$ forms a classical BRW based on the family $([\mathcal{Z}]_v)_{v\in\mathbb{V}}$ of point processes. BRWs have been studied in many articles, see *e.g.* [7, 9, 8, 19] and the references therein.

We continue with a collection of some known facts about BRWs which will be useful in the course of the proof of our main results.

Let $\mathcal{S}$ denote the set of survival of the branching process:

$$\mathcal{S} \;:=\; \{\sup_{|v|=n} L(v) > 0 \text{ for all } n \geq 0\}. \tag{3.9}$$

The supercriticality assumption (A2) guarantees that $\mathbb{P}(\mathcal{S}) > 0$. More precisely, $1 - \mathbb{P}(\mathcal{S})$ is the unique fixed point of the generating function of the reproduction counting variable N in the interval $(0, 1)$.

Our approach to understanding Eq. (1.3) is heavily based on the analysis of its iterated version (3.3). To understand the latter equation, we need input on the asymptotic behaviour of the weights $L(v)$ or, equivalently, the corresponding positions $S(v)$ in the branching random walk model.

Lemma 3.1. *Under* (A1)-(A3), $B_n := \inf_{|v|=n} S(v) \to \infty$ *almost surely on* $\mathcal{S}$. *In particular,*

$$\sup_{|v|=n} L(v) \;=\; e^{-B_n} \;\to\; 0 \quad \textit{almost surely as } n \to \infty. \tag{3.10}$$

Source. The result follows from [8, Theorem 3]. □

3.3 The embedded BRW with positive steps only

In some arguments involving the BRW $(\mathcal{Z}_n)_{n\geq 0}$, it will be convenient to consider an embedded BRW with positive steps only. This idea is not new, see *e.g.* [10, Section 3]. Therefore, we will keep the construction of the embedded BRW short.

Let $\mathcal{G}_n := \{v \in \mathbb{N}^n : S(v) < \infty\} = \{v \in \mathbb{N}^n : L(v) > 0\}$. The $\mathcal{G}_n$ are the generations of the original BRW. $\mathcal{G} := \bigcup_{n\geq 0} \mathcal{G}_n$ is the set of all population members of the original BRW. Now let $\mathcal{G}_0^{>} := \{\varnothing\}$,

$$\mathcal{G}_1^{>} \;:=\; \{v \in \mathcal{G} : S(v) > 0, S(v|k) \leq 0 \text{ for all } 0 \leq k < |v|\} \tag{3.11}$$

and, recursively, for $n > 1$,

$$\mathcal{G}_n^{>} \;:=\; \{vw : v \in \mathcal{G}_{n-1}^{>}, w \in [\mathcal{G}_1^{>}]_v\}. \tag{3.12}$$

The sequence $(\mathcal{G}_n^{>})_{n\geq 0}$ is an embedded generation sequence that contains exactly those individuals v the positions of which are strict records in the random walk $S(\varnothing), S(v|1), \ldots, S(v)$. Using the $\mathcal{G}_n^>$, we can define the nth generation point process of the embedded BRW of strictly increasing ladder heights by

$$\mathcal{Z}_n^> \;:=\; \sum_{v \in \mathcal{G}_n^>} \delta_{S(v)}. \tag{3.13}$$

$(\mathcal{Z}_n^>)_{n\geq 0}$ is again a BRW but with positive steps only. The following result states that the assumptions (A1)-(A5), which can be interpreted as assumptions on the point process $\mathcal{Z}$, are passed on to the point process $\mathcal{Z}^> := \mathcal{Z}_1^>$:

Proposition 3.2. *Assuming* (A1)-(A3)*, the following assertions hold:*

(a) $\mathbb{P}(|\mathcal{G}_1^>| < \infty) = 1$.

(b) $\mathcal{Z}^>$ *satisfies* (A1)-(A3).

(c) If $\mathcal{Z}$ *further satisfies* (A4a) *or* (A4b)*, then the same holds true for* $\mathcal{Z}^>$*, respectively.*

(d) If $\mathcal{Z}$ *satisfies* (A4) *and* (A5)*, then so does* $\mathcal{Z}^>$*.*

Proof. Assertion (a) follows from [10, Theorem 10(d)]. Assertions (b) and (c) follow from [2, Lemma 8.1]. It remains to prove that, given (A1)-(A5), (A5) holds for $\mathcal{Z}^>$ as well, in other words, that

$$\mathbb{E}\sum_{j=1}^{N} e^{-\alpha S(j)}(S(j)^+)^2 \;<\; \infty \quad \text{implies} \quad \mathbb{E}\sum_{v \in \mathcal{G}_1^>} e^{-\alpha S(v)}(S(v)^+)^2 \;<\; \infty.$$

By what has already been shown, if $\mathcal{Z}$ satisfies (A4b), then so does $\mathcal{Z}^>$. Then, since (A4b) implies (A5), (A5) also holds for $\mathcal{Z}^>$. Therefore, we may assume that $\mathcal{Z}$ satisfies (A4a) but not necessarily (A4b). Let $(S_n)_{n\geq 0}$ be a standard random walk with increment distribution $\mu_\alpha = \mathbb{E}\sum_{j=1}^{N} e^{-\alpha S(j)}\delta_{S(j)}$ (notice that (A3) renders μ_α a probability distribution on $\mathbb{R}$). From (6.2) in [2], we infer that $\mu_\alpha^> := \mathbb{E}\sum_{v \in \mathcal{G}_1^>} e^{-\alpha S(v)}\delta_{S(v)}$ is the distribution of the first ladder height of the random walk $(S_n)_{n\geq 0}$. To be more precise,

$$\mu_\alpha^>(\cdot) \;=\; \mathbb{P}(S_\sigma \in \cdot)$$

where $\sigma := \inf\{n \geq 0 : S_n > 0\}$. Now (A5) for $\mathcal{Z}$ can be restated as $\mathbb{E}(S_1^+)^2 < \infty$, whereas (A5) for $\mathcal{Z}^>$ means that $\mathbb{E}\,S_\sigma^2 < \infty$. But $\mathbb{E}(S_1^+)^2 < \infty$ and $\mathbb{E}\,S_\sigma^2 < \infty$ are actually equivalent, for $\mathbb{E}\,S_1 = -m'(\alpha) \in (0, \infty)$, see Theorem 3.1 in [18]. □

4 Solving the homogeneous equation

4.1 The simple inclusions

We begin our analysis of the homogeneous equation by verifying the simple inclusions in our main results. To be more precise, we prove in this subsection that any ϕ as defined in (2.4) or (2.5) is an element of $\mathcal{S}(\mathfrak{F})$.

Proof of Theorems 2.1 and 2.2: The simple inclusions.
Recall that W denotes a fixed non-trivial non-negative random variable satisfying (2.3). Assume that $\alpha \in (0,2) \setminus \{1\}$. Choose any $\sigma \geq 0$ and $\beta \in [-1,1]$ and assume that ϕ is given as in Eq. (2.4). As explained in Subsection 2.4 it follows that ϕ is the characteristic function of $W^{1/\alpha}Y$ for some random variable $Y \sim \mathcal{S}_\alpha(\sigma,\beta,0)$ which is independent of W. In particular, $\phi \in \mathfrak{F}$. Thus, it remains to show that ϕ solves (1.5). To this end, denote by Ψ the moment generating function of W, that is, $\Psi(z) = \mathbb{E}\exp(zW)$ for all $z \in \mathbb{C}$ with $\mathbb{E}\,|\exp(zW)| = \mathbb{E}\exp(\mathrm{Re}(z)W) < \infty$. Note that the latter condition holds true on the halfplane $\{\mathrm{Re}(z) \leq 0\}$, for W is non-negative. Denote by $W_1, W_2, \ldots$ a sequence of i.i.d. copies of W independent of T. Then

$$\phi(t) \;=\; \mathbb{E}\,\Psi\left(-\sigma^\alpha|t|^\alpha\left[1-\mathrm{i}\,\beta\frac{t}{|t|}\tan\left(\frac{\pi\alpha}{2}\right)\right]\right).$$

On the other hand,

$$\begin{aligned}
\phi(t) &= \mathbb{E}\exp\left(-\sigma^\alpha W|t|^\alpha\left[1-\mathrm{i}\,\beta\frac{t}{|t|}\tan\left(\frac{\pi\alpha}{2}\right)\right]\right)\\
&= \mathbb{E}\exp\left(-\sum_{j\geq 1}\sigma^\alpha(T_j|t|)^\alpha W_j\left[1-\mathrm{i}\,\beta\frac{t}{|t|}\tan\left(\frac{\pi\alpha}{2}\right)\right]\right)\\
&= \mathbb{E}\prod_{j=1}^{N}\Psi\left(-\sigma^\alpha(|T_jt|)^\alpha\left[1-\mathrm{i}\,\beta\frac{T_jt}{|T_jt|}\tan\left(\frac{\pi\alpha}{2}\right)\right]\right)\\
&= \mathbb{E}\prod_{j\geq 1}\phi(tT_j).
\end{aligned}$$

Similar arguments apply when $\alpha \in \{1,2\}$. □

4.2 Disintegration

Our analysis of Eq. (1.3) is based on a disintegrated pathwise counterpart of Eq. (1.5). For $\phi \in \mathcal{S}(\mathfrak{F})$, define

$$\Phi_n(t) \;:=\; \Phi_n(t,\mathbf{L}) \;:=\; \prod_{|v|=n}\phi(L(v)t), \quad n \geq 0. \tag{4.1}$$

Caliebe [12] proved that, as $n \to \infty$, almost all paths of Φ_n tend to characteristic functions of infinitely divisible distributions. Since this result is of major importance for our further analysis, we will state it here in a form adapted to our notation. Recall that a measure ν on $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$ is called a *Lévy measure* if

$$\int \frac{x^2}{1+x^2}\,\nu(\mathrm{d}x) \;<\; \infty. \tag{4.2}$$

Proposition 4.1. *Let $\phi \in \mathcal{S}(\mathfrak{F})$. Then, almost surely as $n \to \infty$, $(\Phi_n)_{n\geq 0}$ converges pointwise to a random characteristic function Φ of the form $\Phi = \exp(\Psi)$ with*

$$\Psi(t) \;=\; \mathrm{i}\,W_1 t - \frac{W_2t^2}{2} + \int\left(e^{\mathrm{i}\,tx} - 1 - \frac{\mathrm{i}\,tx}{1+x^2}\right)\nu(\mathrm{d}x), \quad t \in \mathbb{R}, \tag{4.3}$$

where W_1 and W_2 are $\mathbb{R}$- and $[0,\infty)$-valued $\mathbf{L}$-measurable random variables, respectively, and ν is a (random) Lévy measure such that, for any $t > 0$, $\nu([t,\infty))$ and $\nu((-\infty,-t])$ are $\mathbf{L}$-measurable. Moreover,

$$\mathbb{E}\,\Phi(t) \;=\; \phi(t) \quad \textit{for all } t \in \mathbb{R}\,. \tag{4.4}$$

Since the proposition appears in a slightly different form than Theorem 1 in [12], we show how it can be concluded from Caliebe's result.

Proof. Clearly, given $\mathbf{L} = \mathbf{l} = (l(v))_{v\in\mathbb{V}} \in [0,\infty)^{\mathbb{V}}$, $\Phi_n = \Phi_n(\cdot,\mathbf{l})$ is the characteristic function of $Y_n(\mathbf{l}) := \sum_{|v|=n} l(v)X(v)$ where $(X(v))_{v\in\mathbb{V}}$ denotes a sequence of i.i.d. random variables with characteristic function ϕ. Due to Lemma 3.1, we may apply Theorem 1 in [12], which entails that $\sum_{|v|=n} l(v)X(v)$ converges weakly to some limit $Y(\mathbf{l})$ as $n \to \infty$ for $\mathbb{P}(\mathbf{L} \in \cdot)$-almost all $\mathbf{l}$. In particular, the corresponding characteristic functions $\Phi_n(\cdot,\mathbf{l})$ tend to the characteristic function $\Phi(\cdot,\mathbf{l})$ of $Y(\mathbf{l})$. Again by Theorem 1 in [12], $\Phi(\cdot,\mathbf{l})$ is of the form $\exp(\Psi(\cdot,\mathbf{l}))$ with

$$\Psi(t,\mathbf{l}) \;=\; \mathrm{i}\,\gamma(\mathbf{l})t - \frac{\sigma^2(\mathbf{l})t^2}{2} + \int \left(e^{\mathrm{i}\,tx} - 1 - \frac{\mathrm{i}\,tx}{1+x^2}\right) \nu(\mathbf{l})(\mathrm{d}x), \tag{4.5}$$

where $\gamma(\mathbf{l}) \in \mathbb{R}$, $\sigma^2(\mathbf{l}) \geq 0$ and $\nu(\mathbf{l})$ is a Lévy measure. In other words, $(\gamma(\mathbf{l}),\sigma^2(\mathbf{l}),\nu(\mathbf{l}))$ is a Lévy triple depending on $\mathbf{l}$. Then (4.3) follows when defining $W_1 := \gamma(\mathbf{L})$, $W_2 := \sigma^2(\mathbf{L})$ and $\nu := \nu(\mathbf{L})$. It remains to show that the claimed measurability statements hold. To this end, we consider the explicit form of $(\gamma(\mathbf{l}),\sigma^2(\mathbf{l}),\nu(\mathbf{l}))$ provided by Caliebe in [12, Lemma 6]. For ease of reference here and later, this lemma is stated next.

Lemma 4.2 (Lemma 6 in [12]). *In the given situation, let X be a solution to* (1.3) *and denote by F its distribution function, i.e., $F(t) = \mathbb{P}(X \leq t)$. Let (W_1,W_2,ν) be as in Proposition 4.1. Then, almost surely and for any continuity point of ν,*

$$\nu((-\infty,t]) \;=\; \lim_{n\to\infty} \sum_{|v|=n} F(t/L(v)), \qquad \textit{if} \quad t<0 \tag{4.6}$$

$$\textit{and} \quad \nu([t,\infty)) \;=\; \lim_{n\to\infty} \sum_{|v|=n} (1-F(t/L(v))), \qquad \textit{if} \quad t>0. \tag{4.7}$$

Furthermore,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\liminf_{n\to\infty} \sum_{|v|=n} L(v)^2 \left(\int_{\{|x|<\varepsilon/L(v)\}} x^2\,F(\mathrm{d}x) - \left[\int_{\{|x|<\varepsilon/L(v)\}} x\,F(\mathrm{d}x)\right]^2\right) \\
&\;=\; \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \sum_{|v|=n} L(v)^2 \left(\int_{\{|x|<\varepsilon/L(v)\}} x^2\,F(\mathrm{d}x) - \left[\int_{\{|x|<\varepsilon/L(v)\}} x\,F(\mathrm{d}x)\right]^2\right) \\
&\;=\; W_2.
\end{aligned} \tag{4.8}$$

Finally, if

$$W_1(\tau) \;:=\; \lim_{n\to\infty} \sum_{|v|=n} L(v) \int_{\{|x|<\tau/L(v)\}} x\,F(\mathrm{d}x), \tag{4.9}$$

for $\tau > 0$, then

$$W_1 \;=\; W_1(\tau) - \int_{\{|x|<\tau\}} \frac{x^3}{1+x^2}\,\nu(\mathrm{d}x) + \int_{\{|x|\geq\tau\}} \frac{x}{1+x^2}\,\nu(\mathrm{d}x) \qquad (4.10)$$

whenever τ and $-\tau$ are continuity points of ν.

Returning to the proof of Proposition 4.1, we particularly infer from Eqs. (4.6) and (4.7) that $\nu((-\infty,-t])$ and $\nu([t,\infty))$ are **L**-measurable for all $t > 0$. From (4.8) we obtain the **L**-measurability of W_2. Since we do not know the (random) points of incontinuity of ν at this point, a proof of the **L**-measurability of W_1 using the explicit representation of W_1 provided by Eq. (4.10) could easily become messy. For this reason, we postpone the proof of the **L**-measurability of W_1 until the end of Subsection 4.5 when we have derived the explicit form of ν. □

The next result is a key to our further analysis and provides us with a functional equation for the disintegrated characteristic functions.

Lemma 4.3 (see Lemma 5.2 in [4])**.** *Let $\phi \in \mathcal{S}(\mathfrak{F})$ and denote by Φ the disintegration of ϕ. Then, for all $n \in \mathbb{N}_0$,*

$$\Phi(t) \;=\; \prod_{|v|=n} [\Phi]_v(L(v)t) \quad \textit{for all } t \in \mathbb{R} \textit{ almost surely.} \qquad (4.11)$$

In particular, the characteristic exponent Ψ of the corresponding disintegration Φ satisfies

$$\Psi(t) \;=\; \sum_{|v|=n} [\Psi]_v(L(v)t) \quad \textit{for all } t \in \mathbb{R} \textit{ almost surely.} \qquad (4.12)$$

Proof. For $t \geq 0$

$$\begin{aligned}\Phi(t) &= \lim_{k\to\infty} \prod_{|v|=n+k} \phi(L(v)t)\\ &= \lim_{k\to\infty} \prod_{|v|=n} \prod_{|w|=k} \phi([L(w)]_v L(v)t)\\ &= \prod_{|v|=n} [\Phi]_v(L(v)t) \qquad \text{almost surely}\end{aligned}$$

where we made use of the fact that $\mathcal{G}_n = \{v \in \mathbb{N}^n : L(v) > 0\}$ is finite by assumption (1.1). Since Φ and all the $[\Phi]_v$ are continuous functions in t almost surely, this equation actually holds almost surely for all $t \geq 0$ simultaneously. As to (4.12), notice that by (4.11), $\exp(\Psi(t)) = \exp(\sum_{|v|=n}[\Psi]_v(L(v)t))$ for all $t \in \mathbb{R}$ almost surely, that is, $\Psi(t)$ and $\sum_{|v|=n}[\Psi]_v(L(v)t))$ are both continuous logarithms of Φ. Since both functions assume the value 0 at 0, and since continuous logarithms of continuous curves in $\mathbb{C}\setminus\{0\}$ can differ only by constant multiples of $2\pi\mathrm{i}$, (4.12) must hold. □

4.3 Disintegration along ladder lines

As in [2], we use the concept of ladder lines when studying disintegrations. We are particularly interested in approximating a disintegration Φ not only via the corresponding multiplicative martingale $(\Phi_n)_{n\geq 0}$, but also via terms of the form

$$\Phi_{\mathcal{T}_u}(t) \ := \ \prod_{v\in\mathcal{T}_u} \phi(L(v)t), \quad t \in \mathbb{R},\ u \geq 0, \tag{4.13}$$

where $\mathcal{T}_u$ is the first exit line of the interval $(-\infty, u]$, that is,

$$\mathcal{T}_u \ := \ \{v \in \mathcal{G} : S(v) > u \text{ and } S(v|k) \leq u \text{ for } k = 0, \ldots, |v| - 1\}. \tag{4.14}$$

Lemma 4.4. *Given $\phi \in \mathcal{S}(\mathfrak{F})$ with disintegration Φ,*

$$\lim_{u\to\infty} \Phi_{\mathcal{T}_u}(t) \ = \ \lim_{u\to\infty} \prod_{v\in\mathcal{T}_u} \phi(L(v)t) \ = \ \Phi(t) \quad \textit{almost surely.} \tag{4.15}$$

for any $t \in \mathbb{R}$, and outside a $\mathbb{P}$-null set the convergence holds for all t simultaneously.

Proof. That (4.15) holds along a fixed sequence $u_n \to \infty$ can be derived from the arguments in the proofs of Lemma 7.4 and Lemma 7.5(b) in [2]. Further, since by assumption (1.1) and Lemma 3.1 the points $S(v)$, $v \in \mathcal{G}$ do not accumulate on finite intervals in $\mathbb{R}$, $(\Phi_{\mathcal{T}_u}(t))_{u\geq 0}$ constitutes a right-continuous martingale and convergence holds outside a $\mathbb{P}$-null set for any sequence $u \to \infty$. Now let

$$N^c \ := \ \left\{\lim_{u\to\infty} \Phi_{\mathcal{T}_u}(t) = \Phi(t) \text{ for all } t \in \mathbb{Q}\right\}.$$

Then $\mathbb{P}(N) = 0$. Next, consider $\Phi_{\mathcal{T}_u}(\cdot) = \Phi_{\mathcal{T}_u}(\cdot, \mathbf{L})$ as a function of the family $\mathbf{L} = (L(v))_{v\in\mathbb{V}}$. Note that $\mathcal{T}_u$ also depends on $\mathbf{L}$, thus $\mathcal{T}_u = \mathcal{T}_u(\mathbf{L})$. Given a realisation $\mathbf{l} = (l(v))_{v\in\mathbb{V}}$ of $\mathbf{L}$, $\Phi_{\mathcal{T}_u(\mathbf{l})}(\cdot, \mathbf{l})$ is the characteristic function of the sum $\sum_{v\in\mathcal{T}_u(\mathbf{l})} l(v)X(v)$, where the family $(X(v))_{v\in\mathbb{V}}$ is a family of i.i.d. copies of a random variable X having characteristic function ϕ. Since $\mathbb{P}(N^c) = 1$, $\Phi_{\mathcal{T}_u(\mathbf{l})}(t, \mathbf{l})$ converges to $\Phi(t, \mathbf{l})$ as $u \to \infty$ for all $t \in \mathbb{Q}$ and $\mathbb{P}(\mathbf{L} \in \cdot)$-almost all $\mathbf{l}$. Now fix any $\mathbf{l}$ for which this convergence at the rationals hold. Further, fix an arbitrary sequence $(u_n)_{n\geq 0}$ such that $0 \leq u_n \to \infty$. Then every vague limit of a vaguely convergent subsequence of the distributions of $\sum_{v\in\mathcal{T}_{u_n}(\mathbf{l})} l(v)X(v)$, $n \geq 0$ has a characteristic function coinciding with $\Phi(t, \mathbf{l})$ at each rational point $t \neq 0$. Since characteristic functions of probability distributions are continuous on $\mathbb{R}$, the limit thus has characteristic function $\Phi(\cdot, \mathbf{l})$. By the direct half of Lévy's continuity theorem, we then get $\Phi_{\mathcal{T}_{u_n}(\mathbf{l})}(t, \mathbf{l}) \to \Phi(t, \mathbf{l})$ as $n \to \infty$ for all $t \in \mathbb{R}$. $\square$

Lemma 4.4 allows us to prove a useful extension of Lemma 4.2.

Lemma 4.5. *Let X be a solution to* (1.3) *with distribution function F, i.e., $F(t) = \mathbb{P}(X \leq t)$. Let further (W_1, W_2, ν) be the random Lévy triple of the* *disintegration Φ of the characteristic function ϕ of X, see Proposition 4.1. Then, for any continuity point t of ν, almost surely,*

$$\nu((-\infty,t]) \;=\; \lim_{u\to\infty}\sum_{v\in\mathcal{T}_u} F(t/L(v)), \qquad \text{if} \quad t<0 \tag{4.16}$$

$$\text{and} \quad \nu([t,\infty)) \;=\; \lim_{u\to\infty}\sum_{v\in\mathcal{T}_u} (1-F(t/L(v))), \qquad \text{if} \quad t>0. \tag{4.17}$$

Finally, if

$$W_1(\tau) \;:=\; \lim_{u\to\infty}\sum_{v\in\mathcal{T}_u} L(v)\int_{\{|x|<\tau/L(v)\}} x\,F(\mathrm{d}x), \tag{4.18}$$

for $\tau>0$, then

$$W_1 \;=\; W_1(\tau) - \int_{\{|x|<\tau\}} \frac{x^3}{1+x^2}\,\nu(\mathrm{d}x) + \int_{\{|x|\geq\tau\}} \frac{x}{1+x^2}\,\nu(\mathrm{d}x) \tag{4.19}$$

whenever τ and $-\tau$ are continuity points of ν.

Proof. For any fixed sequence $u_n\uparrow\infty$ and $\mathbb{P}(\mathbf{L}\in\cdot)$-almost all $\mathbf{l}=(l(v))_{v\in\mathbb{V}}\in[0,\infty)^{\mathbb{V}}$, $((l(v)X(v))_{v\in\mathcal{T}_{u_n}(\mathbf{l})})_{n\geq0}$ is an infinitesimal triangular array, see [17, Eq. (2) on p. 95]. We further infer from Lemma 4.4 that $\sum_{v\in\mathcal{T}_{u_n}(\mathbf{l})} l(v)X(v)$ converges weakly to the distribution with characteristic function $\Phi(\cdot,\mathbf{l})$ as $n\to\infty$. Eqs. (4.16), (4.17), and (4.19) can therefore be derived from [17, Theorem 1 on p. 116], see [17, Eq. (9) on p. 84] concerning (4.19). Finally note that according to Lemma 4.4, the exceptional $\mathbb{P}(\mathbf{L}\in\cdot)$-null set can be chosen independently of the particular sequence $(u_n)_{n\geq0}$. □

4.4 Endogeneous fixed points

This subsection is devoted to the introduction of *endogeneous fixed points*. These are special solutions to (1.3) with the property that all their randomness can be expressed in terms of the weights $L(v)$, $v\in\mathbb{V}$ with no further randomness needed. Note that the definition given here is more general than [2, Definition 5.1] since we allow an endogeneous fixed point to be both positive and negative with positive probabilities.

Definition 4.6. Let $\beta>0$ and define $T^{(\beta)}:=(T_j^\beta)_{j\geq1}$. A random variable W_β (or its distribution) is called an *endogeneous fixed point of the smoothing transform with respect to (w.r.t.) $T^{(\beta)}$* if there exists a Borel measurable function $g:[0,\infty)^{\mathbb{V}}\to\mathbb{R}$ such that $W_\beta:=g(\mathbf{L})$ and

$$W_\beta \;=\; \sum_{|v|=n} L(v)^\beta\,[W_\beta]_v \quad \text{almost surely} \tag{4.20}$$

for all $n\geq0$. W_β is called non-trivial if $\mathbb{P}(W_\beta\neq0)>0$.

Notice that by definition there is always the trivial endogeneous fixed point 0. From [2, Proposition 5.3 and Theorem 7.2] we infer that under (A1)-(A4) there exists a unique (up to a positive scaling constant) non-trivial non-negative endogeneous fixed point w.r.t. $T^{(\alpha)}$. For the rest of this article, we fix one particular such fixed point and denote it by W, which complies with and thus only further specifies our previous choice of W being a fixed random variable solving (2.3). For ease of reference, we state the uniqueness result for non-negative endogeneous fixed points from [2]:

Proposition 4.7. *Suppose that* (A1)-(A4) *hold true. Let* W_α *be a non-negative endogeneous fixed point w.r.t.* $T^{(\alpha)}$. *Then* $W_\alpha = cW$ *a.s. for some* $c > 0$.

Source. This is Proposition 5.3 in [2]. □

Henceforth, let

$$\varphi(t) \;:=\; \mathbb{E}\, e^{-tW}, \quad t \geq 0 \tag{4.21}$$

denote the Laplace transform of W. From Theorem 2.3 in [2] it follows that $1 - \varphi(t)$ is regularly varying of index 1 at the origin. Equivalently,

$$D_1(t) \;:=\; \frac{1-\varphi(t)}{t}, \quad t > 0 \tag{4.22}$$

is slowly varying at the origin. W can be explicitly constructed from φ via

$$W \;=\; \lim_{n\to\infty} \sum_{|v|=n} 1 - \varphi(L(v)^\alpha) \;=\; \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} 1 - \varphi(L(v)^\alpha) \tag{4.23}$$

$$\;=\; \lim_{t\to\infty} D_1(e^{-\alpha t}) \sum_{v\in\mathcal{T}_t} L(v)^\alpha \;=\; \lim_{t\to\infty} D_1(e^{-\alpha t}) W_n^{(\alpha)} \text{ a.s.}, \tag{4.24}$$

where $W_n^{(\alpha)} := \sum_{|v|=n} L(v)^\alpha = \sum_{|v|=n} e^{-\alpha S(v)}$. (4.23) follows from Proposition 4.7 here and Theorem 7.2 and Lemma 7.5 in [2]. (4.24) follows from Theorem 10.2 in the same reference. $(W_n^{(\alpha)})_{n\geq 0}$ is a non-negative martingale sometimes called Biggings' martingale. Given (A1)-(A3) and (A4a), the distinguished endogeneous fixed point W equals a positive constant times the limit of Biggins' martingale since φ possesses a finite derivative at 0 in this case. In this situation, for convenience, we assume $W = W^{(\alpha)}$.

We will return to endogeneous fixed points in Subsection 4.8 and show there that under (A1)-(A5) endogeneous fixed points exist only w.r.t. to $T^{(\alpha)}$ and that they are always non-negative or non-positive.

4.5 Identifying the random Lévy measure

Lemma 4.8. *Suppose that* (A1)-(A4) *hold. Let* $\phi \in \mathcal{S}(\mathfrak{F})$ *with disintegration* $\Phi = \exp(\Psi)$, *where*

$$\Psi(t) \;=\; \mathrm{i}\, W_1 t - \frac{W_2 t^2}{2} + \int \left(e^{\mathrm{i}\, tx} - 1 - \frac{\mathrm{i}\, tx}{1+x^2} \right) \nu(\mathrm{d}x), \quad t \in \mathbb{R}$$

as (4.3) *in Proposition 4.1. Then, for* $t > 0$,

$$\nu([t,\infty)) = W c_1 t^{-\alpha} \quad \textit{and} \quad \nu((-\infty,-t]) = W c_2 t^{-\alpha} \tag{4.25}$$

for the fixed non-negative endogeneous fixed point W *w.r.t.* $T^{(\alpha)}$ *and constants* $c_1, c_2 \geq 0$. *Moreover, if* $\alpha \geq 2$, *then* $\nu = 0$ *almost surely.*

Proof. By (4.3) and (4.12),

$$
\begin{aligned}
\mathrm{i}\, W_1 t - \frac{W_2 t^2}{2} + \int \left(e^{\mathrm{i}\, tx} - 1 - \frac{\mathrm{i}\, tx}{1+x^2} \right) \nu(\mathrm{d}x) \\
= \ \mathrm{i} \sum_{|v|=n} L(v)[W_1]_v t \ - \ \frac{\sum_{|v|=n} L(v)^2 [W_2]_v t^2}{2} \\
+ \sum_{|v|=n} \int \left(e^{\mathrm{i}\, L(v)tx} - 1 - \frac{\mathrm{i}\, L(v)tx}{1+x^2} \right) [\nu]_v(\mathrm{d}x) \\
= \ \mathrm{i}\, t \sum_{|v|=n} L(v) \left([W_1]_v + \int \left[\frac{x}{1+(L(v)x)^2} - \frac{x}{1+x^2} \right] [\nu]_v(\mathrm{d}x) \right) \\
- \frac{\sum_{|v|=n} L(v)^2 [W_2]_v t^2}{2} \\
+ \sum_{|v|=n} \int \left(e^{\mathrm{i}\, L(v)tx} - 1 - \frac{\mathrm{i}\, L(v)tx}{1+(L(v)x)^2} \right) [\nu]_v(\mathrm{d}x), \quad t \in \mathbb{R}\,.
\end{aligned}
$$

From the uniqueness of the Lévy triple we particularly infer that

$$
W_2 \ = \ \sum_{|v|=n} L(v)^2 [W_2]_v, \tag{4.26}
$$

$$
\int g(x)\, \nu(\mathrm{d}x) \ = \ \sum_{|v|=n} \int g(L(v)x)\, [\nu]_v(\mathrm{d}x) \tag{4.27}
$$

almost surely for all $n \geq 0$ and all non-negative Borel-measurable functions g on $\mathbb{R}$. Now consider the function $g(x) = \mathbb{1}_{[t,\infty)}(x)$ for any fixed $t > 0$. Then (4.27) turns into

$$
\nu([t,\infty)) \ = \ \sum_{|v|=n} [\nu]_v([L(v)^{-1}t, \infty))
$$

almost surely for all $n \geq 0$. Defining $f : [0,\infty) \to [0,1]$ by

$$
f(t) \ = \ \begin{cases} 1 & \text{if } t = 0, \\ \mathbb{E} \exp(-\nu([t^{-1},\infty))) & \text{if } t > 0, \end{cases}
$$

gives a decreasing function with $\lim_{t\to 0} f(t) = 1 = f(0)$. If $f(t) = 1$ for all $t \geq 0$, then ν assigns no mass to $(0,\infty)$ and we can choose $c_1 = 0$ in (4.25). If $f(t) < 1$ for some $t > 0$ or, equivalently, if ν assigns positive mass to the positive halfline, then f is a monotone, non-trivial solution to the functional equation (1.5). Let $M_n(t) = \prod_{|v|=n} f(L(v)t)$ denote the corresponding multiplicative martingale with almost sure limit $M(t)$, see [2, Lemma 7.1]. From [2, Theorem 7.2], we infer that $M(t)$ has the form

$$
M(t) \ = \ \exp(-W c_1 t^\alpha) \quad \text{almost surely}
$$

for some $c_1 > 0$ and each $t \geq 0$. Using that $\exp(-\nu([t^{-1},\infty)))$ is bounded and $\mathbf{L}$-measurable and the fact that $f(t) = \mathbb{E}\, M(t)$, see [2, Lemma 7.1], we infer that

$$\begin{aligned}
\exp(-\nu([t^{-1},\infty))) &= \lim_{n\to\infty} \mathbb{E}[\exp(-\nu([t^{-1},\infty))) \mid \mathcal{A}_n] \\
&= \lim_{n\to\infty} \mathbb{E}\Big[\prod_{|v|=n} \exp(-[\nu]_v([(L(v)^{-1}t^{-1},\infty))) \,\Big|\, \mathcal{A}_n \Big] \\
&= \lim_{n\to\infty} \prod_{|v|=n} f(L(v)t) \\
&= M(t) \;=\; \exp(-Wc_1 t^{\alpha}) \quad \text{almost surely.}
\end{aligned}$$

In particular, $\nu([t,\infty)) = Wc_1 t^{-\alpha}$ almost surely for all $t > 0$. An analogous argument yields that $\nu((-\infty,t]) = Wc_2|t|^{-\alpha}$, for some $c_2 \geq 0$ and all $t < 0$. Since ν is a random Lévy measure, it particularly almost surely satisfies (4.2) necessitating that $\alpha \in (0,2)$ or $\nu = 0$ almost surely. □

Now we can close the gap in the proof of Proposition 4.1 and show that W_1 is $\mathbf{L}$ measurable:

Completion of the proof of Proposition 4.1. Since the random Lévy measure is almost surely continuous w.r.t. to Lebesgue measure, we can choose an arbitrary $\tau > 0$ to calculate W_1 from (4.9) and (4.10). It is easy to check that the right-hand side of (4.10) is $\mathbf{L}$-measurable. □

4.6 Tail bounds for the fixed points

In this subsection, we fix a solution X to (1.3) with distribution function $F(t) = \mathbb{P}(X \leq t)$ and Fourier transform ϕ. From Proposition 4.1, we know that ϕ has a disintegration of the form $\Phi = \exp(\Psi)$ with Ψ being the (random) characteristic exponent of an infinitely divisible distribution with Lévy triple (W_1, W_2, ν). From Lemma 4.8, we infer that ν is of the form

$$\nu([t,\infty)) \;=\; c_1 W t^{-\alpha} \quad \text{almost surely} \quad (t > 0). \tag{4.28}$$

where $c_1 \geq 0$, and W is the unique non-trivial non-negative endogeneous fixed point w.r.t. $T^{(\alpha)}$. Analogously,

$$\nu((-\infty,t]) \;=\; c_2 W |t|^{-\alpha} \quad \text{almost surely} \quad (t < 0). \tag{4.29}$$

where $c_2 \geq 0$. Using these results as a starting point, we derive tail bounds for X by comparing the tail probabilities of X with the behaviour of the Laplace transform φ of W at 0. To this end, recall the definition of D_1 from (4.22). By Theorem 2.3 in [2], $D_1(t)$ is slowly varying as $t \downarrow 0$. In what follows, we are interested in

$$\mathrm{K_u} \;:=\; \limsup_{t\to\infty} \frac{\mathbb{P}(|X| > t)}{1-\varphi(t^{-\alpha})}.$$

Lemma 4.9. *Suppose that* (A1)-(A4) *hold. Then, in the given situation, the following assertions hold:*

(a) $0 \leq \mathrm{K_u} < \infty$.

(b) *If* $c_1 + c_2 = 0$*, then* $\mathrm{K_u} = 0$.

Proof. From Lemma 4.5, (4.28), and (4.29), we infer that

$$c_1 W t^{-\alpha} = \lim_{u\to\infty} \sum_{v\in\mathcal{T}_u} (1 - F(t/L(v))), \quad \text{and} \tag{4.30}$$

$$c_2 W t^{-\alpha} = \lim_{u\to\infty} \sum_{v\in\mathcal{T}_u} F(-t/L(v)) \tag{4.31}$$

almost surely for any $t > 0$. Then, with $D(x) := x^{-\alpha}\,\mathbb{P}(|X| > x^{-1})$ and for $t = 1$, we infer

$$\lim_{u\to\infty} \sum_{v\in\mathcal{T}_u} e^{-\alpha S(v)} D(e^{-S(v)}) = (c_1 + c_2)W \quad \text{almost surely.} \tag{4.32}$$

(4.32) is the analogue to formula (11.9) in [2]. Since, furthermore, $\mathbb{P}(|X| > x^{-1})$ is decreasing in x, Lemma 11.4 in [2] also holds for D (instead of D_α there). As pointed out right before Lemma 11.4 in [2], these two properties, namely, that D satisfies (4.32) and the assertion of Lemma 11.4 in [2], are the only properties needed in the proof of Lemma 11.5 in [2]. Therefore, arguing as in the proof of Lemma 11.5 in [2], we can conclude the analogue of Eq. (11.11) there, namely,

$$(c_1 + c_2)W \ \geq\ e^{-\delta}\,\mathrm{K_u}(1-\varepsilon)W$$

almost surely for some $\delta > 0$ and $\varepsilon \in (0, 1)$. From this, assertions (a) and (b) immediately follow since W is almost surely finite and positive with positive probability. ☐

4.7 Asymptotic results for general branching processes

Similar to [2, Section 9], we will make use of results concerning the asymptotic behaviour of certain general branching processes derived from the BRW $(\mathcal{Z}_n)_{n\geq 0}$. The first result in this subsection provides a Seneta-Heyde norming for general branching processes. Though we do not need this theorem in full generality, we provide it in a general form since the result may be of interest in its own right.

To this end, we need to make the construction of the underlying probability space in Subsection 3.1 more explicit. Suppose that C, T and X are defined on a probability space $(\mathcal{X}, \mathcal{B}, \mathbf{P})$. For each $v \in \mathbb{V}$, let $(\mathcal{X}_v, \mathcal{B}_v, \mathbf{P}_v)$ be a copy of $(\mathcal{X}, \mathcal{B}, \mathbf{P})$. We choose the underlying probability space in Subsection 3.1 as the product space $(\Omega, \mathfrak{A}, \mathbb{P}) = (\prod_{v\in\mathbb{V}} \mathcal{X}_v, \bigotimes_{v\in\mathbb{V}} \mathcal{B}_v, \bigotimes_{v\in\mathbb{V}} \mathbf{P}_v)$. Then, of course, on the product space, $C(v)$ is defined to be $C \circ p_v$, where p_v denotes the projection from Ω to $\mathcal{X}_v$, $v \in \mathbb{V}$. $T(v)$ and $X(v)$ are to be defined analogously. Now let $\phi : \mathbb{R} \times \Omega \to \mathbb{R} \cup \{\infty\}$ denote a product-measurable, separable stochastic process. ϕ can be interpreted as a general characteristic of the process, see [27]. We suppress the dependence of ϕ on ω hereafter, *i.e.*, we write $\phi(t)$ and think of it as the random variable $\omega \mapsto \phi(t, \omega)$. Moreover, the shift operator

$[\cdot]_u$ can be defined as the mapping $[\cdot]_u : \Omega \to \Omega$, $\omega = (\omega_v)_{v\in\mathbb{V}} \mapsto (\omega_{uv})_{v\in\mathbb{V}}$. Then (*cf.* [21, p. 167] and [27, Eq. (1.11)]) we call

$$Z_t^\phi \;:=\; \sum_v [\phi]_v(t - S(v)) \tag{4.33}$$

a general branching process counted with characteristic ϕ. For our theorem on Seneta-Heyde norming of Z_t^ϕ, we need the following condition on ϕ:

Condition 4.10. There exists some $\beta < \alpha$ such that

$$M \;:=\; \sup_{t\geq 0} e^{-\beta t}\phi(t)$$

has finite expectation.

Before presenting our theorem, the reader is reminded that W denotes the distinguished non-trivial non-negative endogeneous fixed point w.r.t. $T^{(\alpha)}$ and that φ denotes the corresponding Laplace transform.

Theorem 4.11. *Assume that* (A1)-(A3) *and* (A4b) *hold and that* $\mathcal{Z} = \sum_j \delta_{S(j)}$ *is concentrated on* $[0,\infty)$ *almost surely. Further, assume that* ϕ *satisfies Condition 4.10 and has càdlàg paths. Then, almost surely as* $t \to \infty$,

$$(1 - \varphi(e^{-\alpha t}))Z_t^\phi \;\to\; \frac{W}{-m'(\alpha)} \int_0^\infty e^{-\alpha x}\,\mathbb{E}\,\phi(x)\;\mathrm{d}x\,. \tag{4.34}$$

This result is a generalization of Theorem 6.1 in [14] because, besides a number of identical assumptions, we only assume the existence of some $\theta < \alpha$ satisfying $m(\theta) < \infty$, whereas Theorem 6.1 in [14] requires $m(0) < \infty$. We also believe that, by using the approach in [25], Theorem 4.11 could be extended to BRWs on the real line, which allows for $\mathbb{P}(\mathcal{Z}((-\infty,0)) > 0) > 0$.

Proof. Let $\psi(t) := \sum_{j=1}^N e^{\alpha(t-S(j))}\,\mathbb{1}_{[0,S(j))}(t)$. We show that all assumptions of Theorem 6.3 in [27] are fulfilled: (A1) ensures that $\mathcal{Z}$ is non-lattice, (A2) the supercriticality. The existence of a Malthusian parameter follows from (A3), while (A4b) and the fact that $\mathcal{Z}$ is concentrated on the positive halfline imply that $m'(\alpha) \in (-\infty, 0)$, which is Nerman's assumption (1.5). Moreover, (A4b) implies Nerman's condition 6.1. Then, with θ as in (A4b), we have

$$\sup_{t\geq 0} e^{-\theta t}\psi(t) \;=\; \sup_{t\geq 0} e^{-\theta t}\sum_{j=1}^N e^{\alpha(t-S(j))}\,\mathbb{1}_{[0,S(j))}(t) \;\leq\; \sum_{j=1}^N e^{-\theta S(j)}, \tag{4.35}$$

where in the last step we have used that $e^{\alpha(t-S(j))} \leq e^{\theta(t-S(j))}$ for $t < S(j)$. Condition (A4b) ensures the finiteness of the expectation of the sum on the right-hand side of (4.35). Consequently, ψ satisfies Condition 6.2 in [27]. Clearly, ψ has càdlàg paths. Furthermore, by assumption, ϕ also satisfies these conditions. Again (A4b) implies that the point process $\mathcal{Z}$ satisfies Condition 6.1 in [27]. Therefore (and by the remarks in Section 7 of [27]), we can apply Nerman's convergence-of-ratios theorem [27, Theorem 6.3] and infer that

$$\frac{Z_t^\phi}{Z_t^\psi} \;\to\; \frac{\int_0^\infty e^{-\alpha x}\,\mathbb{E}\,\phi(x)\;\mathrm{d}x}{\int_0^\infty e^{-\alpha x}\,\mathbb{E}\,\psi(x)\;\mathrm{d}x} \;=\; \frac{\int_0^\infty e^{-\alpha x}\,\mathbb{E}\,\phi(x)\;\mathrm{d}x}{-m'(\alpha)} \tag{4.36}$$

almost surely on $\mathcal{S}$ as $t \to \infty$. Taking into account that

$$e^{-\alpha t} Z_t^{\psi} \;=\; \sum_{v \in \mathcal{T}_t} e^{-\alpha S(v)} \;=:\; W_{\mathcal{T}_t}^{(\alpha)}, \tag{4.37}$$

we infer that

$$(1-\varphi(e^{-\alpha t}))Z_t^{\phi} \;=\; D_1(e^{-\alpha t}) W_{\mathcal{T}_t}^{(\alpha)} \frac{Z_t^{\phi}}{Z_t^{\psi}} \;\to\; W \, \frac{\int_0^\infty e^{-\alpha x}\, \mathbb{E}\, \phi(x)\, \mathrm{d}x}{-m'(\alpha)}$$

almost surely on $\mathcal{S}$ by (4.36) and (4.24). Since (4.34) trivially holds on $\mathcal{S}^c$, the proof is herewith complete. □

Our next result is on ratio convergence on $\mathcal{S}$ of certain general branching processes.

Proposition 4.12. *Assume that* (A1)-(A3) *and* (A4b) *hold and let* $\varepsilon > 0$. *Then for any* $\beta > \theta$ *and all sufficiently large* c *(which may depend on* β*),*

$$\frac{\sum_{v \in \mathcal{T}_t} e^{-\beta(S(v)-t)}(S(v)-t))\, \mathbb{1}_{\{S(v)>t+c\}}}{\sum_{v \in \mathcal{T}_t} e^{-\alpha(S(v)-t)}(S(v)-t)} \;\to\; \varepsilon(c) \le \varepsilon \text{ on } \mathcal{S} \tag{4.38}$$

almost surely as $t \to \infty$.

Proof. Since the sum in the numerator is decreasing in β, we can w.l.o.g. assume that $\theta < \beta < \alpha$. Further, notice that the sums over $v \in \mathcal{T}_t$ in (4.38) remain unaffected, when replacing the underlying BRW $(\mathcal{Z}_n)_{n\ge0}$ by the embedded BRW $(\mathcal{Z}_n^{>})_{n\ge0}$ the construction of which has been described in Subsection 3.3. This is due to the fact that the first crossing of the level t necessarily takes place at a vertex v such that $S(v)$ is a strict record in the finite sequence $0, S(v|1), \ldots, S(v)$. Therefore and in view of Proposition 3.2, it constitutes no loss of generality to assume that beyond (A1)-(A3) and (A4b), the additional assumption that $\mathbb{P}(\mathcal{Z}((-\infty,0)) > 0) = 0$ is fulfilled. One can apply the reasoning at the beginning of the proof of Theorem 4.11 to show that the assumptions of Theorem 6.3 in [27] are fulfilled. What remains to show is that numerator and denominator in (4.38) derive from characteristics that satisfy Condition 6.2 in [27]. To this end, note that, following Nerman's notation, the numerator is derived from

$$\phi(t) \;=\; \mathbb{1}_{[0,\infty)}(t) \sum_{j=1}^{N} e^{-\beta(S(j)-t)}(S(j)-t)\, \mathbb{1}_{\{S(j)>t+c\}},$$

while the denominator is derived from

$$\psi(t) \;=\; \mathbb{1}_{[0,\infty)}(t) \sum_{j=1}^{N} e^{-\alpha(S(j)-t)}(S(j)-t)\, \mathbb{1}_{\{S(j)>t\}}\,.$$

Plainly, ϕ and ψ have càdlàg paths and $\mathbb{E}\,\phi(t)$ and $\mathbb{E}\,\psi(t)$ are continuous almost everywhere w.r.t. Lebesgue measure. Furthermore,

$$e^{-\beta t}\phi(t) \;=\; \mathbb{1}_{[0,\infty)}(t) \sum_{j=1}^{N} e^{-\beta S(j)}(S(j)-t)\, \mathbb{1}_{\{S(j)>t+c\}} \;\le\; \sum_{j=1}^{N} e^{-\beta S(j)} S(j)$$

which is integrable by assumption (A4b), for $\beta > \theta$. Thus, ϕ satisfies Nerman's Condition 6.2. Analogously, one can deduce that ψ satisfies the same condition. Now applying Theorem 6.3 in [27], we infer that the ratio in (4.38) tends to

$$\frac{\int_0^\infty e^{-\alpha x}\,\mathbb{E}\,\phi(x)\;\mathrm{d}x}{\int_0^\infty e^{-\alpha x}\,\mathbb{E}\,\psi(x)\;\mathrm{d}x} \;\le\; \frac{\int_0^\infty \mathbb{E}\sum_{j=1}^N e^{-\beta S(j)}(S(j)-x)\,\mathbb{1}_{\{S(j)>x+c\}}\;\mathrm{d}x}{\int_0^\infty \mathbb{E}\sum_{j=1}^N e^{-\alpha S(j)}(S(j)-x)\,\mathbb{1}_{\{S(j)>x\}}\;\mathrm{d}x} \;=:\; \varepsilon(c)$$

almost surely on $\mathcal{S}$ as $t \to \infty$. (Notice that we have used that $\beta < \alpha$ to derive the inequality for the numerator.) This completes the proof since $\varepsilon(c) \to 0$ as $c \to \infty$. ☐

Later, we will need the following result that may be viewed a kind of converse of Theorem 9.3 in [2]:

Theorem 4.13. *Suppose that* (A1)-(A3) *and* (A4b) *hold. Assume also that the following conditions hold:*

(i) There are a non-negative function H and a random variable $\widetilde{W}$ such that

$$H(t)\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}(S(v)-t) \;\to\; \widetilde{W}$$

almost surely as $t \to \infty$.

(ii) For some $h \in (0,\infty)$

$$\varepsilon_t(a) \;=\; \frac{H(a+t)}{H(t)} - h \;\to\; 0$$

as $t \to \infty$ uniformly on compact subsets of $[0,\infty)$.

(iii) For a finite K, some $\theta < \beta < \alpha$, all $a \ge 0$, and all sufficiently large $t > 0$,

$$\frac{H(a+t)}{H(t)} \;\le\; Ke^{(\alpha-\beta)a}.$$

Then

$$\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} H(S(v))(S(v)-t) \;\to\; h^{-1}\widetilde{W} \tag{4.39}$$

almost surely as $t \to \infty$.

Proof. In this proof, $\sum$ denotes the sum over $v \in \mathcal{T}_t$. It is clear that the assertion holds on $\mathcal{S}^c$ so that we can focus on what happens on the survival set $\mathcal{S}$ in what follows. By increasing K if necessary, we can assume that for all sufficiently large t we have

$$\varepsilon_t(a) \;\le\; Ke^{(\alpha-\beta)a}$$

for all $a \geq 0$. Similar as in the proof of Theorem 9.3 in [2], we consider the ratio

$$\begin{aligned}\frac{\sum e^{-\alpha S(v)} H(S(v))(S(v)-t)}{H(t)\sum e^{-\alpha S(v)}(S(v)-t)} &= \frac{\sum e^{-\alpha S(v)} H(S(v))/H(t)(S(v)-t)}{\sum e^{-\alpha S(v)}(S(v)-t)} \\ &= \frac{\sum e^{-\alpha S(v)}(h+\varepsilon_t(S(v)-t))(S(v)-t)}{\sum e^{-\alpha S(v)}(S(v)-t)} \\ &= h + \frac{\sum e^{-\alpha S(v)}\varepsilon_t(S(v)-t)(S(v)-t)}{\sum e^{-\alpha S(v)}(S(v)-t)}.\end{aligned}$$

The fraction on the right-hand side can be estimated as follows:

$$\begin{aligned}&\left|\frac{\sum e^{-\alpha S(v)}\varepsilon_t(S(v)-t)(S(v)-t)}{\sum e^{-\alpha S(v)}(S(v)-t)}\right| \\ &\quad \leq \sup_{0\leq a\leq c}|\varepsilon_t(a)| + \frac{\sum e^{-\alpha S(v)}\varepsilon_t(S(v)-t)(S(v)-t)\,\mathbb{1}_{\{S(v)>t+c\}}}{\sum e^{-\alpha S(v)}(S(v)-t)}.\end{aligned}$$

Here, the first term on the right-hand side tends to 0 by Condition (ii) whereas the second term converges to 0 as t and then c tend to infinity by Proposition 4.12. □

4.8 Endogeny and disintegration

The following two theorems are the main results on endogeneous fixed points in this subsection.

Theorem 4.14. *Assume that* (A1)-(A4) *hold and that* $\widetilde{W}$ *is an endogeneous fixed point w.r.t.* $T^{(\beta)}$ *for some* $\beta > 0$, $\beta \neq \alpha$. *Then* $\widetilde{W} = 0$ *almost surely.*

For our next theorem, recall that W denotes a fixed non-trivial non-negative endogeneous fixed point w.r.t. $T^{(\alpha)}$.

Theorem 4.15. *If* (A1)-(A5) *hold, then any endogeneous fixed point* W_α *w.r.t.* $T^{(\alpha)}$ *satisfies* $W_\alpha = cW$ *almost surely for some* $c \in \mathbb{R}$.

Before we can prove Theorems 4.14 and 4.15, we need to do some preparatory work.

Lemma 4.16. *Assume that* (A1)-(A4) *hold.*

(a) If $\alpha < 1$, *then* $\sum_{|v|=n} L(v) \to 0$ *almost surely.*

(b) If $\alpha > 1$, *then, almost surely as* $n \to \infty$,

$$\sum_{|v|=n} L(v) \;\to\; \begin{cases}\infty & \text{on } \mathcal{S} \\ 0 & \text{on } \mathcal{S}^c.\end{cases} \tag{4.40}$$

Proof. (a) If $\alpha < 1$, then

$$\sum_{|v|=n} L(v) \;\leq\; W_n^{(\alpha)} \sup_{|v|=n} L(v)^{1-\alpha} \;\to\; 0$$

almost surely because $(W_n^{(\alpha)})_{n\geq 0} = (\sum_{|v|=n} L(v)^\alpha)_{n\geq 0}$ is a non-negative martingale and $\sup_{|v|=n} L(v) \to 0$ by Lemma 3.1.
(b) Clearly, $\sum_{|v|=n} L(v) \to 0$ almost surely as $n \to \infty$ on $\mathcal{S}^c$ whence it remains to prove that this sum tends to ∞ almost surely on $\mathcal{S}$. To this end, recall from (4.24) that

$$\sum_{|v|=n} 1 - \varphi(L(v)^\alpha) \;=\; \sum_{|v|=n} L(v)^\alpha D_1(L(v)^\alpha) \;\to\; W \tag{4.41}$$

almost surely as $n \to \infty$, where φ denotes the Laplace transform of W and $D_1(t) = t^{-1}(1-\varphi(t))$, $t > 0$. $D_1(t)$ is slowly varying as $t \to 0$. Now fix any $\delta > 0$ such that $1 + \delta < \alpha$. By Potter's Theorem [11, Theorem 1.5.6], we infer that $D_1(t^\alpha) \leq 2t^{-\delta}$ for all sufficiently small t. Thus, using Lemma 3.1, we infer that

$$\begin{aligned}\sum_{|v|=n} L(v) \;&=\; \sum_{|v|=n} L(v)^\alpha L(v)^{-\delta} L(v)^{1+\delta-\alpha}\\ &\geq\; \frac{1}{2}\left(\sup_{|v|=n} L(v)\right)^{1+\delta-\alpha} \left(\sum_{|v|=n} L(v)^\alpha D_1(L(v)^\alpha)\right)\end{aligned}$$

for sufficiently large n. Now $\sup_{|v|=n} L(v)$ tends to 0 almost surely as $n \to \infty$ by Lemma 3.1, while the last factor tends to W almost surely by (4.41). Therefore, the desired conclusion follows from the fact that $W > 0$ almost surely on $\mathcal{S}$, which is a commonplace in the study of Eq. (1.3) on the positive halfline (and is an immediate consequence of the fact that $\mathbb{P}(W = 0)$ is a fixed point of the function $f(s) = \mathbb{E}\, s^N$ in $[0, 1)$). ☐

Lemma 4.17. *Assume that $\alpha > 1$ and that W_1 is an endogeneous fixed point w.r.t. T with finite mean. Then $W_1 = 0$ almost surely.*

Proof. We use the same idea as in the proof of Proposition 5.3 in [2]. Since W_1 is endogeneous, it is in particular $\mathcal{A}_\infty$-measurable. Thus, by the integrability of W_1, we have

$$\begin{aligned}W_1 \;&=\; \lim_{n\to\infty} \mathbb{E}[W_1 |\, \mathcal{A}_n]\\ &=\; \lim_{n\to\infty} \mathbb{E}\left[\sum_{|v|=n} L(v)[W_1]_v \,\middle|\, \mathcal{A}_n\right] \;=\; (\mathbb{E}\, W_1) \sum_{|v|=n} L(v)\end{aligned} \tag{4.42}$$

almost surely. By Lemma 4.16, $\sum_{|v|=n} L(v) \to \infty$ almost surely on $\mathcal{S}$ whereas $|W_1| < \infty$ almost surely. Therefore, $\mathbb{E}\, W_1 = 0$ must hold which, in combination with (4.42), implies $W_1 = 0$ almost surely. ☐

Lemma 4.18. *Suppose that* (A1)-(A4) *hold and that* W_α *is an endogeneous fixed point w.r.t.* $T^{(\alpha)}$. *If* $aW_\alpha + bW \geq 0$ *almost surely for some constants* $0 \neq a \in \mathbb{R}$, $b \in \mathbb{R}$, *then* $W_\alpha = cW$ *almost surely for some* $c \in \mathbb{R}$.

Proof. If $aW_\alpha + bW \geq 0$, then $aW_\alpha + bW$ is a non-negative endogeneous fixed point and, therefore, by Proposition 4.7, equals cW for some $c \geq 0$, from which the desired conclusion follows. ☐

Our next result connects the concepts of disintegration and endogeny. In what follows, we call a disintegration Φ *almost surely degenerate* iff there exists an $\mathbf{L}$-measurable random variable W_1 such that $\Phi(t) = \exp(\mathrm{i}\, W_1 t)$ for all $t \in \mathbb{R}$ almost surely.

Proposition 4.19. *Let* $P \in \mathcal{S}(\mathfrak{F})$ *with disintegration* Φ. *Then there exists an endogeneous fixed point* W_1 *w.r.t.* T *with distribution* P *iff* Φ *is almost surely degenerate. Further, in this case,* $\Phi(t) = \exp(\mathrm{i}\, W_1 t)$ *almost surely for all* $t \in \mathbb{R}$.

Proof. First assume that there exists $W_1 \stackrel{\mathrm{d}}{=} P$ which is endogeneous w.r.t. T, thus

$$W_1 \;=\; \sum_{|v|=n} L(v)[W_1]_v \quad \text{almost surely}$$

for all $n \geq 0$. Let ϕ be its characteristic function. Then, with $(\Phi_n)_{n\geq 0}$ denoting the corresponding multiplicative martingale, we have

$$\begin{aligned}
\mathbb{E}[\exp(\mathrm{i}\, W_1 t) \mid \mathcal{A}_n] \;&=\; \mathbb{E}\left[\exp\Big(\mathrm{i}\, t \sum_{|v|=n} L(v)[W_1]_v\Big) \,\Big|\, \mathcal{A}_n\right] \\
&=\; \mathbb{E}\left[\prod_{|v|=n} \exp(\mathrm{i}\, [W_1]_v L(v) t) \,\Big|\, \mathcal{A}_n\right] \\
&=\; \prod_{|v|=n} \phi(L(v)t) \;=\; \Phi_n(t) \underset{n\to\infty}{\to} \Phi(t) \quad \text{almost surely.}
\end{aligned}$$

On the other hand, by the boundedness of $\exp(\mathrm{i}\, W_1 t)$ and the martingale convergence theorem,

$$\mathbb{E}[\exp(\mathrm{i}\, W_1 t) \mid \mathcal{A}_n] \;\to\; \exp(\mathrm{i}\, W_1 t) \quad \text{almost surely as } n \to \infty.$$

This proves that $\Phi(t) = \exp(\mathrm{i}\, W_1 t)$ almost surely, and thus that the disintegration is the characteristic function of a Dirac measure almost surely.

Conversely, if $\phi \in \mathcal{S}(\mathfrak{F})$ has a disintegration Φ of the form $\Phi(t) = \exp(\mathrm{i}\, W_1 t)$ for all $t \in \mathbb{R}$ almost surely for some $\mathbf{L}$ measurable random variable W_1, then, by (4.11),

$$\begin{aligned}
e^{\mathrm{i}\, W_1 t} \;&=\; \Phi(t) \;=\; \prod_{|v|=n} [\Phi]_v(L(v)t) \;=\; \prod_{|v|=n} e^{\mathrm{i}\, [W_1]_v L(v) t} \\
&=\; e^{\mathrm{i} \sum_{|v|=n} L(v)[W_1]_v t} \quad \text{almost surely.}
\end{aligned}$$

Thus, by the uniqueness theorem for characteristic functions,

$$W_1 \;=\; \sum_{|v|=n} L(v)[W_1]_v \quad \text{almost surely}$$

providing the endogeny of W_1. $\square$

Now we are ready to prove the main results in this subsection, Theorems 4.14 and 4.15.

Proof of Theorem 4.14. Let $\beta > 0$, $\beta \neq \alpha$. Since (A1)-(A4) carry over from the sequence T to the sequence $T^{(\beta)} = (T_j^\beta)_{j\geq 1}$ (with the new α being α/β), we can w.l.o.g. assume that $\beta = 1$ and $\alpha \neq 1$. Let $\widetilde{W}$ be an endogeneous fixed point w.r.t. T. We have to show that $\widetilde{W} = 0$ almost surely. By Proposition 4.19, we infer that the disintegration Φ of $\widetilde{W}$ equals $\exp(\mathrm{i}\widetilde{W}t)$. In particular, the random Lévy triple of Φ is of the form $(\widetilde{W}, 0, 0)$. Thus, using (4.9) and (4.10), we infer that

$$\widetilde{W} \;=\; \lim_{n\to\infty} \sum_{|v|=n} L(v) \int_{\{|x|<\tau/L(v)\}} x\, F(\mathrm{d}x) \tag{4.43}$$

for arbitrary $\tau > 0$, where F denotes the distribution function of $\widetilde{W}$. Recall from Lemma 4.9 that $\limsup_{t\to\infty} \mathbb{P}(|\widetilde{W}| > t)/(1-\varphi(t^{-\alpha})) = 0$ with φ denoting the Laplace transform of W, the fixed non-trivial non-negative endogeneous fixed point. Integration by parts further yields

$$\int_{\{|x|<t\}} |x|\, F(\mathrm{d}x) \;=\; \int_0^t \mathbb{P}(|\widetilde{W}| > x)\, \mathrm{d}x - t\, \mathbb{P}(|\widetilde{W}| > t). \tag{4.44}$$

Now suppose first that $\alpha < 1$ and recall that $1 - \varphi(t)$ is regularly varying of index 1 at the origin. Choose an arbitrary $\varepsilon > 0$ and then $t > 0$ large enough such that $\mathbb{P}(|\widetilde{W}| > x) \leq \varepsilon(1 - \varphi(x^{-\alpha}))$ for all $x \geq t$. Then, putting things together, we obtain

$$\begin{aligned}
|\widetilde{W}| \;&\leq\; \limsup_{n\to\infty} \sum_{|v|=n} L(v) \int_{\{|x|<\tau/L(v)\}} |x|\, F(\mathrm{d}x) \\
&\leq\; \limsup_{n\to\infty} \sum_{|v|=n} L(v) \int_0^{\tau/L(v)} \mathbb{P}(|\widetilde{W}| > x)\; \mathrm{d}x \\
&\leq\; \limsup_{n\to\infty} \sum_{|v|=n} L(v) \left(t + \varepsilon \int_t^{\tau/L(v)} 1 - \varphi(x^{-\alpha})\; \mathrm{d}x \right).
\end{aligned}$$

Here, $\sum_{|v|=n} L(v) \to 0$ almost surely as $n \to \infty$ by Lemma 4.16(a). Hence, using Proposition 1.5.8 in [11] and the fact that $1-\varphi(x^{-\alpha})$ is regularly varying of index $-\alpha$ at ∞ by [2, Theorem 2.3], we arrive at

$$\begin{aligned} |\widetilde{W}| &\leq \varepsilon \limsup_{n\to\infty} \sum_{|v|=n} L(v) \int_t^{\tau/L(v)} 1-\varphi(x^{-\alpha}) \, \mathrm{d}x \\ &= \varepsilon \limsup_{n\to\infty} \sum_{|v|=n} L(v) \frac{\tau/L(v)}{1-\alpha}(1-\varphi((\tau/L(v))^{-\alpha})) \, \mathrm{d}x \\ &= \frac{\varepsilon\tau^{1-\alpha}}{1-\alpha} \limsup_{n\to\infty} \sum_{|v|=n} 1-\varphi(L(v)^{\alpha}) \, \mathrm{d}x \\ &= \frac{\varepsilon\tau^{1-\alpha}}{1-\alpha} W \quad \text{almost surely,} \end{aligned}$$

where the last equality follows from (4.23). Letting $\varepsilon \to 0$ yields $|\widetilde{W}| = 0$ almost surely.

If $\alpha > 1$, then Lemma 4.9 provides us with $\mathbb{P}(|\widetilde{W}| > t) = o(1-\varphi(t^{-\alpha}))$ as $t \to \infty$. Since $(1-\varphi(t^{-\alpha}))$ is regularly varying of index $-\alpha$ at infinity, we infer that $\mathbb{E}\,|\widetilde{W}| < \infty$ and thus $\widetilde{W} = 0$ almost surely by Lemma 4.17. □

Proof of Theorem 4.15. Without loss of generality let $\alpha = 1$. Then suppose that W_1 is an endogeneous fixed point w.r.t. T. By Proposition 4.19, its disintegration Φ is of the form $\Phi(t) = \exp(\mathrm{i}\, W_1 t)$ $(t \in \mathbb{R})$ almost surely. In particular, the random Lévy triple corresponding to Φ is of the form $(W_1, 0, 0)$. Therefore, we infer from Eqs. (4.18) and (4.19) that

$$\begin{aligned} W_1 &= \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{\{|x|<L(v)^{-1}\}} x \, F(\mathrm{d}x) \\ &= \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) I(L(v)^{-1}) \quad \text{almost surely} \end{aligned} \tag{4.45}$$

where $I(c) := \int_{\{|x|<c\}} x \, F(\mathrm{d}x)$ $(c \geq 0)$ and F denotes the distribution function of W_1. Now suppose that

$$K \;:=\; \limsup_{t\to\infty} I(t)/D_1(t^{-1}) \;<\; \infty.$$

Then

$$W_1 \;\leq\; K \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) D_1(L(v)) \;=\; KW$$

almost surely by (4.23). Lemma 4.18(b) then implies that $W_1 = aW$ for some $a \in \mathbb{R}$. Using an analogous argument, we arrive at the same conclusion if

$$\liminf_{t\to\infty} I(t)/D_1(t^{-1}) \;>\; -\infty.$$

Therefore, it remains to consider the case

$$-\infty \;=\; \liminf_{t\to\infty} I(t)/D_1(t^{-1}) \;<\; \limsup_{t\to\infty} I(t)/D_1(t^{-1}) \;=\; \infty \tag{4.46}$$

in which there are arbitrarily large t such that $I(e^t) \le 0$. For any such t, we have

$$\begin{aligned}
\sum_{v\in\mathcal{T}_t} L(v)I(L(v)^{-1}) &\le \sum_{v\in\mathcal{T}_t} L(v)(I(L(v)^{-1}) - I(e^t)) \\
&\le \sum_{v\in\mathcal{T}_t} L(v) \int_{\{e^t\le|x|<L(v)^{-1}\}} |x|\, F(\mathrm{d}x) \\
&= \sum_{v\in\mathcal{T}_t} L(v) \Bigg[\int_{e^t}^{L(v)^{-1}} \mathbb{P}(|W_1| > x)\ \mathrm{d}x \\
&\quad -L(v)^{-1}\,\mathbb{P}(|W_1| > L(v)^{-1}) + e^t\,\mathbb{P}(|W_1| > e^t) \Bigg] \\
&\le \sum_{v\in\mathcal{T}_t} L(v) \int_{e^t}^{L(v)^{-1}} \mathbb{P}(|W_1| > x)\ \mathrm{d}x \\
&\quad + e^t\,\mathbb{P}(|W_1| > e^t) \sum_{v\in\mathcal{T}_t} L(v)
\end{aligned}$$

where we have used integration by parts to arrive at the penultimate line. Since W_1 is endogeneous, and therefore, $\nu = 0$ almost surely by Proposition 4.19 (with ν denoting the random Lévy measure of the disintegration Φ of X), Lemma 4.9(b) implies that $\mathbb{P}(|W_1| > x) = o(1 - \varphi(x^{-1}))$ as $x \to \infty$ (with φ denoting the Laplace transform of the distinguished non-negative endogeneous fixed point W). Therefore,

$$e^t\,\mathbb{P}(|W_1| > e^t) \sum_{v\in\mathcal{T}_t} L(v) \;=\; o(D_1(e^{-t})) \sum_{v\in\mathcal{T}_t} L(v) \;=\; o(W)$$

almost surely as $t \to \infty$ by (4.24). Consequently, for any $\varepsilon > 0$, we have that

$$\begin{aligned}
W_1 &= \lim_{t\to\infty,\, I(e^t)\le 0} \sum_{v\in\mathcal{T}_t} L(v)I(L(v)^{-1}) \\
&\le \limsup_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{e^t}^{L(v)^{-1}} \mathbb{P}(|W_1| > x)\ \mathrm{d}x \\
&\le \varepsilon \limsup_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{e^t}^{L(v)^{-1}} x^{-1} D_1(x^{-1})\ \mathrm{d}x\,.
\end{aligned}$$

Now we have to distinguish two cases.

Suppose first that (A4a) and (A5) hold. Then $D_1(x^{-1})$ increases to $\mathbb{E}\,W$

as $x \to \infty$. By convention, $\mathbb{E}\, W = 1$. Therefore,

$$\begin{aligned} W_1 &\le \varepsilon \limsup_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{e^t}^{L(v)^{-1}} x^{-1}\, \mathrm{d}x \\ &\le \varepsilon \limsup_{t\to\infty} \sum_{v\in\mathcal{T}_t} e^{-S(v)}(S(v)-t) \\ &= \varepsilon \limsup_{t\to\infty} e^{-t} \sum_{v\in\mathcal{T}_t} e^{t-S(v)}(S(v)-t) \\ &= \varepsilon \limsup_{t\to\infty} e^{-t} \sum_{v\in\mathbb{V}} [\phi]_v(t-S(v)), \end{aligned} \tag{4.47}$$

where

$$\phi(t) := \sum_{j\ge1} e^{t-S(j)}\, \mathbb{1}_{[0,S(j))}(t)(S(j)-t). \tag{4.48}$$

We further have that

$$\begin{aligned} \int_0^\infty e^{-t}\, \mathbb{E}\, \phi(t)\, \mathrm{d}t &= \int_0^\infty \mathbb{E} \sum_{j\ge1} e^{-S(j)}\, \mathbb{1}_{[0,S(j))}(t)(S(j)-t)\, \mathrm{d}t \\ &= \mathbb{E} \sum_{j\ge1} e^{-S(j)} S(j)^2/2\, \mathrm{d}t < \infty \end{aligned}$$

by (A5). Therefore, [27, Theorem 3.1] yields that the random series in (4.47) tends to cW in probability as $t \to \infty$ for a suitable constant $c \in [0,\infty)$. Choosing an appropriate subsequence, we can assume that the convergence holds almost surely. Consequently, we have $W_1 \le \varepsilon c W$. Letting $\varepsilon \to 0$, we obtain that $W_1 \le 0$ and thus $I \le 0$ which obviously contradicts (4.46).

Now suppose that (A4b) holds true. Then, using that $D_1(x^{-1})$ is increasing in x, we infer

$$\begin{aligned} W_1 &\le \varepsilon \limsup_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{e^t}^{L(v)^{-1}} x^{-1} D_1(x^{-1})\, \mathrm{d}x \\ &\le \varepsilon \limsup_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) D_1(L(v)) \int_{e^t}^{L(v)^{-1}} x^{-1}\, \mathrm{d}x \\ &\le \varepsilon \limsup_{t\to\infty} \sum_{v\in\mathcal{T}_t} e^{-S(v)} D_1(e^{-S(v)})(S(v)-t). \end{aligned} \tag{4.49}$$

By Theorem 4.11,

$$D_1(e^{-t}) \sum_{v\in\mathcal{T}_t} e^{-S(v)}(S(v)-t) = (1-\varphi(e^{-t})) \sum_{v\in\mathcal{T}_t} e^{-(S(v)-t)}(S(v)-t) \to cW$$

almost surely as $t \to \infty$ for some $c \in [0,\infty)$. An application of Theorem 4.13 with $H(t) := D_1(e^{-t})$ (validity of conditions (ii) and (iii) in Theorem 4.13 follow from the fact that D_1 is slowly varying at the origin and Theorems 1.2.1 and 1.5.6 in [11]) yields

$$\sum_{v\in\mathcal{T}_t} e^{-S(v)} D_1(e^{-S(v)})(S(v)-t) \to cW$$

almost surely as $t \to \infty$. Using this in (4.49), we arrive again at $W_1 \leq \varepsilon c W$ almost surely and thus at a contradiction as in the previous case. The proof is herewith complete. □

4.9 The proof of Theorem 2.1

We will derive Theorem 2.1 from the following result that provides a complete description of the disintegrations of solutions to (1.3).

Theorem 4.20. *Suppose that* (A1)-(A4) *hold true and, furthermore,* (A5) *if* $\alpha = 1$. *Then the disintegration* Φ *of any* $\phi \in \mathcal{S}(\mathfrak{F})$ *has a representation of the form*

$$\Phi(t) \;=\; \begin{cases} \exp\left(-\sigma^\alpha W|t|^\alpha \left[1 - \mathrm{i}\,\beta \frac{t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right)\right]\right), & \textit{if } \alpha \notin \{1,2\}, \\ \exp(\mathrm{i}\,\mu W t - \sigma W|t|), & \textit{if } \alpha = 1, \\ \exp(-\sigma^2 W t^2), & \textit{if } \alpha = 2, \end{cases} \tag{4.50}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ *and* $\beta \in [-1,1]$.

Proof of Theorems 2.1 and 2.2 by Theorem 4.20. Both theorems follow immediately from Theorem 4.20 in combination with Eq. (4.4). □

In order to prove Theorem 4.20 we need to evaluate the random integral

$$\int \left(e^{\mathrm{i}\,tx} - 1 - \frac{\mathrm{i}\,tx}{1+x^2}\right) \nu(\mathrm{d}x).$$

We know from Lemma 4.8 that $\nu|_{(0,\infty)}$ and $\nu|_{(-\infty,0)}$, the restrictions of ν to the positive and negative halfline, respectively, are random multiples of $x^{-(\alpha+1)}\,\mathrm{d}x$. The value of the integral can therefore be concluded from existing literature:

Lemma 4.21. *For any* $t > 0$,

$$\begin{aligned} I_1(t) \;&:=\; \int_0^\infty \left(e^{\mathrm{i}\,tx} - 1 - \frac{\mathrm{i}\,tx}{1+x^2}\right) \frac{\mathrm{d}x}{x^{\alpha+1}} \\ &=\; \begin{cases} \mathrm{i}\,ct - t^\alpha e^{-\frac{\pi \mathrm{i}}{2}\alpha} \frac{1}{\alpha}\Gamma(1-\alpha) & \textit{if } 0 < \alpha < 1, \\ \mathrm{i}\,ct - (\pi/2)t - \mathrm{i}\,t\log t & \textit{if } \alpha = 1, \\ \mathrm{i}\,ct - \mathrm{i}\,t^\alpha e^{-\frac{\pi \mathrm{i}}{2}(\alpha-1)} \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} & \textit{if } 1 < \alpha < 2, \end{cases} \end{aligned}$$

where Γ *denotes Euler's Gamma function and* c *is a real constant depending on the value of* α.

Source. See *e.g.* [17, pp. 168]. □

Proof of Theorem 4.20. Let $\phi \in \mathcal{S}(\mathfrak{F})$ with disintegration Φ. From Lemma 4.8 we infer that $\Phi = \exp(\Psi)$ with

$$\Psi(t) \;=\; \mathrm{i}\,W_1 t - \frac{W_2 t^2}{2} + \int \left(e^{\mathrm{i}\,tx} - 1 - \frac{\mathrm{i}\,tx}{1+x^2}\right) \nu(\mathrm{d}x), \quad t \in \mathbb{R} \tag{4.51}$$

Suppose first that $\alpha = 2$. It then follows from Lemma 4.8 that $\nu = 0$ almost surely whence Eq. (4.51) simplifies to

$$\Psi(t) \;=\; \mathrm{i}\, W_1 t - \frac{W_2 t^2}{2} \quad \text{almost surely,}$$

where W_1, W_2 are $\mathbf{L}$ measurable. From (4.12), we infer that for all $t \geq 0$

$$\mathrm{i}\, W_1 t - \frac{W_2 t^2}{2} \;=\; \mathrm{i} \sum_{|v|=n} L(v)[W_1]_v t - \sum_{|v|=n} L(v)^2 [W_2]_v \frac{t^2}{2} \quad \text{almost surely.} \tag{4.52}$$

By linear independence of i and 1, this yields that W_1 and W_2 are endogeneous fixed points w.r.t. $T^{(1)}$ and $T^{(2)}$, respectively. Thus, $W_1 = 0$ almost surely by Theorem 4.14. Since, furthermore, we know that $W_2 \geq 0$ almost surely, we obtain that $W_2 = 2\sigma^2 W$ for some $\sigma \geq 0$ by Proposition 4.7.

Now assume that $0 < \alpha < 2$. Then W_2 is still an endogeneous fixed point w.r.t. $T^{(2)}$ by (4.8). On the other hand, $\alpha < 2$ implies that $W_2 = 0$ almost surely by Theorem 4.14. We proceed with the evaluation of the integral in (4.51). Recall that, by Lemma 4.8, ν can be written as

$$\nu(\mathrm{d}x) \;=\; W(c_1 x^{-(\alpha+1)}\, \mathbb{1}_{(0,\infty)}(x)\, \mathrm{d}x + c_2 |x|^{-(\alpha+1)}\, \mathbb{1}_{(-\infty,0)}(x)\, \mathrm{d}x)$$

for constants $c_1, c_2 \geq 0$ and the non-negative endogeneous fixed point W w.r.t. $T^{(\alpha)}$. Thus, for any $t > 0$,

$$\begin{aligned} \Psi(t) \;&=\; \mathrm{i}\, W_1 t + \int \left(e^{\mathrm{i}\, tx} - 1 - \frac{\mathrm{i}\, tx}{1+x^2} \right) \nu(\mathrm{d}x) \\ &=\; \mathrm{i}\, W_1 t + W(c_1 I_1(t) + c_2 \overline{I_1(t)}), \end{aligned}$$

where $\overline{I_1(t)}$ denotes the complex conjugate of $I_1(t)$. If $c_1 = c_2 = 0$, then $\Psi(t) = \exp(\mathrm{i}\, W_1 t)$ almost surely and by Proposition 4.19, W_1 is an endogeneous fixed point w.r.t. T. Thus, $W_1 = 0$ almost surely, if $\alpha \neq 1$ by Theorem 4.14 and $W_1 = \mu W$ almost surely for some $\mu \in \mathbb{R}$ if $\alpha = 1$ by Theorem 4.15. Therefore, let $c_1 + c_2 > 0$ for the rest of the proof. We will now apply Lemma 4.21:

In the case $0 < \alpha < 1$, this yields

$$\begin{aligned} \Psi(t) \;&=\; \mathrm{i}\, W_1 t + W(c_1(\mathrm{i}\, ct - t^\alpha e^{-\frac{\pi \mathrm{i}}{2}\alpha} \Gamma(1-\alpha)/\alpha) \\ &\qquad\qquad + c_2(-\mathrm{i}\, ct - t^\alpha e^{\frac{\pi \mathrm{i}}{2}\alpha} \Gamma(1-\alpha)/\alpha)) \\ &=\; \mathrm{i}\, (W_1 + c(c_1 - c_2)W)t - W t^\alpha (c_1 e^{-\frac{\pi \mathrm{i}}{2}\alpha} + c_2 e^{\frac{\pi \mathrm{i}}{2}\alpha}) \Gamma(1-\alpha)/\alpha) \\ &=\; \mathrm{i}\, (W_1 + c(c_1 - c_2)W)t \\ &\qquad - \frac{\Gamma(1-\alpha)}{\alpha} W t^\alpha ((c_1 + c_2)\cos(\pi\alpha/2) - \mathrm{i}\, (c_1 - c_2)\sin(\pi\alpha/2)) \\ &=\; \mathrm{i}\, \widetilde{W} t - \sigma^\alpha W |t|^\alpha \left[1 - \mathrm{i}\, \beta \frac{t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right) \right], \end{aligned}$$

where $\widetilde{W} := W_1 + c(c_1 - c_2)W$, $\sigma^\alpha := \frac{\Gamma(1-\alpha)}{\alpha}(c_1 + c_2)\cos(\pi\alpha/2) \geq 0$, and $\beta = (c_1 - c_2)/(c_1 + c_2) \in [-1, 1]$. As in the case $\alpha = 2$, one can show via Eq.

(4.12) that $\widetilde{W}$ is an endogeneous fixed point w.r.t. T. More precisely, (4.12) implies that, almost surely for each $n \geq 0$,

$$\begin{aligned}
&\mathrm{i}\,\widetilde{W}t - \sigma^\alpha W|t|^\alpha \left[1 - \mathrm{i}\,\beta\frac{t}{|t|}\tan\left(\frac{\pi\alpha}{2}\right)\right] \\
&\quad = \mathrm{i}\sum_{|v|=n} L(v)[\widetilde{W}]_v t - \sigma^\alpha \sum_{|v|=n} L(v)^\alpha [W]_v |t|^\alpha \left[1 - \mathrm{i}\,\beta\frac{t}{|t|}\tan\left(\frac{\pi\alpha}{2}\right)\right].
\end{aligned}$$

Dividing by t and letting $t \to \infty$, we see that $\widetilde{W}$ is an endogeneous fixed point w.r.t. T. Since $\alpha < 1$, $\widetilde{W} = 0$ almost surely by Theorem 4.14.

If $1 < \alpha < 2$, a similar argument as before leads to the desired conclusion.

Finally, assume $\alpha = 1$. Then an application of Lemma 4.21 leads to

$$\begin{aligned}
\Psi(t) &= \mathrm{i}\,W_1 t + W(c_1(\mathrm{i}\,ct - \frac{\pi}{2}t - \mathrm{i}\,t\log t) + c_2(-\mathrm{i}\,ct - \frac{\pi}{2}t + \mathrm{i}\,t\log t)) \\
&= \mathrm{i}\,(W_1 + Wc(c_1 - c_2))t + W(c_1(-\frac{\pi}{2}t - \mathrm{i}\,t\log t) + c_2(-\frac{\pi}{2}t + \mathrm{i}\,t\log t)) \\
&= \mathrm{i}\,\widetilde{W}t - \sigma W t\left(1 + \mathrm{i}\,\beta\frac{2}{\pi}\log t\right),
\end{aligned}$$

where $\widetilde{W} = W_1 + Wc(c_1 - c_2)$, $\sigma = (c_1 + c_2)\pi/2 > 0$, and $\beta = (c_1 - c_2)/(c_1 + c_2) \in [-1, 1]$. No use Eq. (4.12) for $t = 1$ to obtain that almost surely for any $n \geq 0$,

$$\mathrm{i}\,\widetilde{W} - \sigma W = \sum_{|v|=n} L(v)[\widetilde{W}]_v - \sigma \sum_{|v|=n} L(v)[W]_v.$$

By linear independence of i and 1, $\widetilde{W}$ is an endogeneous fixed point w.r.t. T. Therefore, $\widetilde{W} = \mu W$ almost surely for some $\mu \in \mathbb{R}$ by Theorem 4.15. Again from (4.12) but for $t > 1$, we infer after some minor manipulations that

$$\begin{aligned}
W\beta\log t &= \sum_{|v|=n} L(v)[W]_v\beta\log(L(v)t) \\
&= \sum_{|v|=n} L(v)[W]_v\beta\log t + \sum_{|v|=n} L(v)\log(L(v))[W]_v\beta \\
&= W\beta\log t + \sum_{|v|=n} L(v)\log(L(v))[W]_v\beta
\end{aligned}$$

almost surely for each $n \geq 0$. Since $\sup_{|v|=n} L(v) \to 0$ almost surely by Lemma 3.1, we infer that $\sum_{|v|=n} L(v)\log(L(v))[W]_v$ is ultimately strictly negative almost surely on $\mathcal{S}$, the set of survival of the supercritical weighted branching process. Thus, $\beta = 0$. □

5 The inhomogeneous equation

We will solve the inhomogeneous equation by another use of disintegration. The strategy is to show that any disintegration of a $\phi \in \mathcal{S}(\mathfrak{F})(C, T)$ can be decomposed into the product of the disintegration of some fixed solution W^* to the inhomogeneous equation and the disintegration of a solution to the homogeneous equation. This approach is taken from [5].

5.1 Disintegration

For $\phi \in \mathcal{S}(\mathfrak{F})(C)$, we define the corresponding multiplicative martingale by

$$\Phi_n(t) \; := \; \phi_n(t, \mathbf{C} \otimes \mathbf{T}) \; := \; \exp\left(\mathrm{i} \sum_{|v|<n} L(v)C(v)t\right) \cdot \prod_{|v|=n} \phi(tL(v)), \quad n \geq 0. \tag{5.1}$$

As in the homogeneous case, $(\Phi_n(t))_{n\geq 0}$ forms a martingale:

Lemma 5.1. *Let $\phi \in \mathcal{S}(\mathfrak{F})(C)$ and $t \in \mathbb{R}$. Then $(\Phi_n(t))_{n\geq 0}$ forms a complex-valued bounded martingale with respect to $(\mathcal{A}_n)_{n\geq 0}$ and thus converges almost surely and in mean to a random variable $\Phi(t) = \Phi(t, \mathbf{C} \otimes \mathbf{T})$ satisfying*

$$\mathbb{E}\,\Phi(t) \; = \; \phi(t). \tag{5.2}$$

Proof. This can be proved analogously to the corresponding result in the homogeneous case. We therefore omit supplying further details. □

In the inhomogeneous case, we define the random variables W_n^*, $n \geq 0$ by

$$W_n^* \; := \; \sum_{|v|\leq n} L(v)C(v). \tag{5.3}$$

By (1.1), we have that W_n^* is the sum of only finitely many non-zero terms almost surely and is therefore well-defined. It is natural to try to construct a fixed point of (1.2) by considering the limit of W_n^* as $n \to \infty$. However, this limit need not exist. In what follows, we make the assumption that W_n^* converges almost surely as $n \to \infty$:

$$W_n^* \underset{n\to\infty}{\to} W^* \text{ almost surely for some finite random variable } W^*. \tag{A6}$$

A number of sufficient conditions for (A6) to hold will be provided in Subsection 5.2.

Lemma 5.2. *If* (A6) *holds, then W^* defines a solution to* (1.2)

Proof. Under (A6),

$$\begin{aligned} W^* \; &= \; \lim_{n\to\infty} \sum_{|v|\leq n} L(v)C(v) \; = \; C(\varnothing) + \lim_{n\to\infty} \sum_{j=1}^{N} T_j \sum_{|v|\leq n-1} [L(v)]_j C(jv) \\ &= \; C(\varnothing) + \sum_{j=1}^{N} T_j \lim_{n\to\infty} [W_{n-1}^*]_j \; = \; C(\varnothing) + \sum_{j=1}^{N} T_j [W^*]_j \quad \text{almost surely,} \end{aligned}$$

where we have utilized that $N < \infty$ almost surely by (1.1). □

Proposition 5.3. *Assume that* (A6) *holds and let $\phi \in \mathcal{S}(\mathfrak{F})(C, T)$ with disintegration Φ. Then*

$$\Phi(t) \; = \; \exp(\mathrm{i}\, W^* t)\, \Phi_{\mathrm{hom}}(t) \quad \textit{almost surely} \quad (t \in \mathbb{R}) \tag{5.4}$$

where Φ_{hom} denotes the disintegration of a (not necessarily non-trivial) solution to (1.3). *Conversely, any characteristic function ϕ obtained by taking the expectation of a process Φ as in* (5.4) *defines a solution to* (1.4).

The following proof is similar to the proof of Theorem 4.4 in [5] but uses characteristic functions instead of Laplace transforms.

Proof. We first prove the converse part of the proposition. To this end, let Φ_{hom} denote the disintegration of some homogeneous solution ϕ_{hom}. Then, with $\Phi(t)$ as defined in (5.4), we have

$$\begin{aligned}\Phi(t) &= \exp(\mathrm{i}\, W^* t)\Phi_{\mathrm{hom}}(t) \\ &= \exp(\mathrm{i}\, C(\varnothing)t) \prod_{j=1}^{N} \exp(\mathrm{i}\, T_j [W^*]_j\, t)\, [\Phi_{\mathrm{hom}}]_j(T_j t),\end{aligned}$$

having used that Φ_{hom} solves the disintegrated version of the homogeneous functional equation by Lemma 4.3. By first taking the conditional expectation given $\mathcal{A}_1$ and then the unconditional expectation yields that $\phi(t) := \mathbb{E}\,\Phi(t)$ satisfies (1.5). Being a mixture of characteristic functions of probability distributions on $\mathbb{R}$, ϕ is itself the characteristic function of a probability measure on $\mathbb{R}$, that is, ϕ defines a solution to (1.4).

Now suppose that ϕ is a characteristic function solving (1.4). Let $(\Phi_n(t))_{n\geq 0}$ denote the corresponding multiplicative martingale with almost sure limit Φ. Then

$$\Phi_n(t) = \prod_{|v|<n} e^{\mathrm{i}\, L(v)C(v)t} \prod_{|v|=n} \phi(L(v)t) = \exp(\mathrm{i}\, W^*_{n-1} t) \prod_{|v|=n} \phi(L(v)t).$$

By (A6), $\exp(\mathrm{i}\, W^*_{n-1}t)$ converges almost surely to $\exp(\mathrm{i}\, W^* t)$. On the other hand, $\Phi_n(t)$ tends to $\Phi(t)$ as $t \to \infty$. Consequently, $\Psi_n(t) := \prod_{|v|=n} \phi(L(v)t)$ tends to $\Psi(t) = \Phi(t)/\exp(\mathrm{i}\, W^* t)$ for all $t \in \mathbb{R}$. From Lemma 4.3, we infer that Φ satisfies (4.11). Using this and the definition of W^*, we obtain

$$\begin{aligned}\Psi(t) &= \frac{\Phi(t)}{\exp(\mathrm{i}\, W^* t)} = \frac{\prod_{|v|<n} e^{\mathrm{i}\, L(v)C(v)t} \cdot \prod_{|v|=n} [\Phi]_v(L(v)t)}{\exp\left(\mathrm{i} \sum_{|v|<n} L(v)C(v)t + \mathrm{i} \sum_{|v|=n} L(v)[W^*]_v t\right)} \\ &= \frac{\prod_{|v|=n} [\Phi]_v(L(v)t)}{\prod_{|v|=n} \exp(\mathrm{i}\, L(v)[W^*]_v t)} = \prod_{|v|=n} [\Psi]_v(L(v)t) \quad \text{almost surely.}\end{aligned}$$

Taking expectations yields that $\psi(t) = \mathbb{E}\,\Psi(t)$ is a solution to (1.4). Moreover, since almost every path of Ψ is the characteristic function of a probability distribution on $\mathbb{R}$, ψ is the characteristic function of a probability distribution on $\mathbb{R}$. Denote by Φ_{hom} the disintegration of ψ. It remains to prove that Φ_{hom} is a version of the process Ψ. But this is immediate from the following calculation based on an application of the martingale convergence theorem:

$$\begin{aligned}\Phi_{\mathrm{hom}}(t) &= \lim_{n\to\infty} \prod_{|v|=n} \psi(L(v)t) = \lim_{n\to\infty} \mathbb{E}\left[\prod_{|v|=n} [\Psi]_v(L(v)t) \,\middle|\, \mathcal{A}_n \right] \\ &= \lim_{n\to\infty} \mathbb{E}[\Psi(t) \mid \mathcal{A}_n] = \Psi(t) \quad \text{almost surely,}\end{aligned}$$

where we have utilized that $\Psi(t)$ is $\mathcal{A}_\infty$-measurable. □

Proof of Theorem 2.3. The Theorem follows immediately from Proposition 5.3 in combination with the main results in the homogeneous case. □

5.2 Sufficient conditions for W^* to be well-defined

Proposition 5.4. *Assume that* (A1)-(A4) *hold true. Then each of the following conditions is sufficient for* (A6) *to hold:*

(i) $m(1) < \infty$, $\mathbb{E}\,|C| < \infty$, *and* W_n^* *is* $\mathcal{L}^p$*-bounded for some* $p \geq 1$.

(ii) $m(\beta) < 1$ *and* $\mathbb{E}\,|C|^\beta < \infty$ *for some* $0 < \beta \leq 1$.

Before we present the proof of Proposition 5.4, we give an auxiliary result.

Lemma 5.5. *Suppose that* (A1)-(A4) *hold. Further, assume that* $m(1) < \infty$ *and* $\mathbb{E}\,|C| < \infty$. *Then*

$$(W_n^*)_{n\geq 0} \text{ is a } \begin{cases} \text{supermartingale} & \\ \text{martingale} & \text{w.r.t. } (\mathcal{A}_{n+1})_{n\geq 0} \text{ iff} \\ \text{submartingale} & \end{cases} \begin{matrix} \mathbb{E}\,C < 0; \\ \mathbb{E}\,C = 0; \\ \mathbb{E}\,C > 0. \end{matrix}$$

Proof. We have $W_n^* - W_{n-1}^* = \sum_{|v|=n} L(v)C(v)$ for each $n \geq 1$. Thus, taking into account that the $L(v)$, $|v| = n$ are $\mathcal{A}_n$-measurable whereas the $C(v)$, $|v| = n$ are independent of $\mathcal{A}_n$,

$$\mathbb{E}[W_n^* - W_{n-1}^* \mid \mathcal{A}_n] \;=\; (\mathbb{E}\,C) \sum_{|v|=n} L(v) \quad \text{almost surely.}$$

□

Proof of Proposition 5.4. If (i) holds, we infer from Lemma 5.5 that $(W_n^*)_{n\geq 0}$ is a (super-,sub-) martingale. Since it is $\mathcal{L}^p$-bounded by (i), an application of the martingale convergence theorem yields the almost sure convergence of $(W_n^*)_{n\geq 0}$.
(ii) follows from the estimate

$$\mathbb{E}\,|W_n^*|^\beta \;\leq\; \mathbb{E} \sum_{|v|\leq n} L(v)|C(v)|^\beta \;\underset{n\to\infty}{\to}\; \frac{\mathbb{E}\,|C|^\beta}{1-m(\beta)}.$$

□

5.3 The fixed points of the `Quicksort` mapping

Recall the `Quicksort` equation (2.7) from Subsection 2.2:

$$X \;\overset{\mathrm{d}}{=}\; UX_1 + (1-U)X_2 + g(U)$$

where $U \sim \mathrm{Unif}(0,1)$, X_1, X_2 are i.i.d. copies of X independent of U, and

$$g : (0,1) \to (0,1), \quad u \mapsto 2u\log u + 2(1-u)\log(1-u) + 1.$$

We will now derive Corollary 2.4 from Theorem 2.3. To this end, notice that in the given context $N = 2$ and

$$m(\theta) \;=\; \mathbb{E}(U^\theta + (1-U)^\theta) \;=\; 2\,\mathbb{E}\,U^\theta \;=\; \frac{2}{1+\theta}, \quad \theta \geq 0.$$

Thus, assumptions (A1)-(A5) are fulfilled with $\alpha = 1$. Moreover, by induction, $\sum_{|v|=n} L(v) = 1$ for all $n \geq 0$, so that $W = 1$ is (up to scaling) the unique positive endogeneous fixed point of (1.3). Furthermore, a standard calculation yields that

$$\mathbb{E}\,|W_n^* - W_{n-1}^*|^2 \;=\; \mathbb{E}\left(\sum_{|v|=n} L(v)C(v)\right)^2 \;=\; (2/3)^2\,\mathbb{E}\,C^2$$

for all $n \geq 0$. Thus, $(W_n^*)_{n\geq 0}$ is an $\mathcal{L}^2$-bounded martingale. From Theorem 2.3, we conclude that any characteristic function ϕ of a solution to (2.7) is of the form

$$\phi(t) \;=\; \mathbb{E}\exp\left(\mathrm{i}\,W^* t + \mathrm{i}\,\mu W t - \sigma W|t|\right) \;=\; \phi^*(t)\,e^{\mathrm{i}\,\mu t - \sigma|t|}$$

where ϕ^* denotes the characteristic function of $W^* = \lim_{n\to\infty}\sum_{|v|\leq n} L(v)C(v)$. As the limit of a mean-zero $\mathcal{L}^2$-bounded martingale, W^* has zero mean and finite variance. On the other hand, if C_n denotes the number of key comparisons `Quicksort` requires to sort a list of n distinct numbers, then P, the distributional limit of $(C_n - n)/n$ as $n \to \infty$, is known [30] to be the unique solution to (2.7) with zero mean and finite variance. Thus, W^* has distribution P and the proof of Corollary 2.4 is complete.

Acknowledgements

The authors sincerely thank Svante Janson for valuable discussions in the course of the project.

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