PROBABILISTIC REPRESENTATION OF BERNOULLI, EULER, AND CARLITZ HERMITE POLYNOMIALS

C. VIGNAT, UNIVERSITÉ D'ORSAY

Abstract. We revisit in a probabilistic framework the umbral approach of Bernoulli numbers, Euler numbers and Carlitz Hermite polynomials by Gessel [1]. This study allows to explicit equivalents of some famous umbræ.

1. Introduction

In [1], Gessel shows how the umbral calculus, introduced by Blissard and later developed by Rota and Roman [2], allows to derive some elementary but also more elaborate results about Bernoulli numbers and about classical polynomials such as Charlier and Hermite polynomials. In this note, we show that another approach based on probabilistic representations can be substituted to the umbral formalism. This substitution brings interesting results about the relationship between random variables and umbræ.

2. Bernoulli numbers and Bernoulli polynomials

The sequence of Bernoulli numbers $\{B_n\}$ is defined by its generating function

$$\sum_{n=0}^{+\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1},\ |t| < 2\pi$$

and the sequence of Bernoulli polynomials $\{B_n(x)\}$ by their generating function

$$\sum_{n=0}^{+\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{tx}}{e^t - 1},\ |t| < 2\pi,\ x \in \mathbb{R},$$

so that the Bernoulli numbers verify $B_n = B_n(0)$. First values are

$$B_0(x) = 1;\ B_1(x) = x - \frac{1}{2};\ B_2(x) = x^2 - x + \frac{1}{6};\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}$$

and

$$B_0 = 1;\ B_1 = -\frac{1}{2};\ B_2 = \frac{1}{6};\ B_3 = 0;\ B_4 = -\frac{1}{30}.$$

In [3], Sun gives the following probabilistic representation of the Bernoulli polynomials

Theorem. *[Sun] Given a sequence $\{L_n\}$ of independent random variables, each with Laplace distribution $\frac{1}{2}\exp(-|x|)$, $x \in \mathbb{R}$, define the random variable*

$$L = \sum_{k=1}^{+\infty} \frac{L_k}{2\pi k}. \tag{2.1}$$

Then the following probabilistic representations hold [1]

$$B_n(x) = E\left(\imath L + x - \frac{1}{2}\right)^n,\ n \geq 0,\ x \in \mathbb{R} \tag{2.2}$$

and

$$B_n = E\left(\imath L - \frac{1}{2}\right)^n,\ n \geq 0. \tag{2.3}$$

[1] In this paper, the symbol E denotes the expectation operator $Eg(X) = \int f(X)\, g(X)\, dX$ where f is the probability density of the relevant random variable.

The random variable L, being defined as a series of i.i.d. random variables, is not easy to characterize. Thus we propose the equivalent result.

Theorem. *The random variable in* (2.1) *follows a logistic distribution, with density*

$$f_L(x) = \frac{\pi}{2} sech^2(\pi x), \; x \in \mathbb{R}.$$

Proof. The random variable L in (2.1) has characteristic function

$$E\left(e^{\imath t L}\right) = \frac{\frac{t}{2}}{\sinh\left(\frac{t}{2}\right)}.$$

But from [5, 1.9.2]

$$\int_0^{+\infty} \operatorname{sech}^2(ax)\cos(xt)\,dx = \frac{\pi t}{2a^2}\operatorname{csch}\left(\frac{\pi t}{2a}\right)$$

so that, with $a = \pi$, the density of L is

$$f_L(x) = \frac{\pi}{2}\operatorname{sech}^2(\pi x).$$

□

We note from [8, p. 471] that the random variable L can be obtained as

$$L = \frac{1}{2\pi}\log\frac{U}{1-U} = \frac{1}{2\pi}\log\frac{E_1}{E_2}$$

where U is uniformly distributed on $[-1,+1]$, E_1 and E_2 are independent with exponential distribution $f_E(x) = \exp(-x), \; x \in [0,+\infty[$ and equality is in the sense of distributions.

Corollary 1. *The Bernoulli polynomials read*

$$B_n(x) = E\left(\frac{\imath}{2\pi}\log\frac{U}{1-U} + x - \frac{1}{2}\right)^n = E\left(\frac{\imath}{2\pi}\log\frac{E_1}{E_2} + x - \frac{1}{2}\right)^n \tag{2.4}$$

and the Bernoulli numbers

$$B_n = E\left(\frac{\imath}{2\pi}\log\frac{U}{1-U} - \frac{1}{2}\right)^n = E\left(\frac{\imath}{2\pi}\log\frac{E_1}{E_2} - \frac{1}{2}\right)^n, \; n \geq 0.$$

We remark that the result by Sun was already given by Talacko in [4].

We note that L is also [9] a Gaussian scale mixture

$$L = 2K.N$$

where N is Gaussian and K follows the Kolmogorov-Smirnov distribution

$$f_K(x) = 8x\sum_{n=1}^{+\infty}(-1)^{n+1} n^2 \exp\left(-2n^2x^2\right), \; x \geq 0.$$

This result was given first by Barndorff-Nielsen et al. in [10], where the mixing distribution of $2K$ is also characterized by its moment generating function

$$\varphi_K(s) = E^{Ks} = \frac{\pi\sqrt{2s}}{\sin\left(\pi\sqrt{2s}\right)}.$$

Proposition 2. *The Bernoulli numbers satisfy*

$$B_n = E\left(\imath L + \frac{1}{2}\right)^n, \; n \neq 1.$$

Proof. We first compute the generating function

$$\begin{aligned}\sum_{n=0}^{+\infty} B_n \frac{t^n}{n!} &= E\sum_{n=0}^{+\infty}\left(\imath L - \frac{1}{2}\right)^n \frac{t^n}{n!} \\ &= \exp\left(-\frac{t}{2}\right)E\exp(\imath t L) = e^{-\frac{t}{2}}\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{t}{e^t - 1}\end{aligned}$$

as it should be. Now let us consider

$$E\sum_{n=0}^{+\infty}\left(\imath L+\frac{1}{2}\right)^{n}\frac{t^{n}}{n!}=\frac{te^{t}}{e^{t}-1}=\frac{t}{e^{t}-1}+t$$

so that, since $\mathrm{E}\big(\imath L+\frac{1}{2}\big)=\frac{1}{2}$,

$$E\sum_{n=2}^{+\infty}\left(\imath L-\frac{1}{2}\right)^{n}\frac{t^{n}}{n!}=E\sum_{n=2}^{+\infty}\left(\imath L+\frac{1}{2}\right)^{n}\frac{t^{n}}{n!}$$

which shows the result. □

Let us show now that the representation (2.3) allows to recover easily some fundamental results about Bernoulli numbers.

In umbral calculus, as described in [1], a Bernoulli number B_n is represented by an umbra B which should be replaced by B_n each time it appears as B^n.

Proposition 3. *[Gessel, (7.2)] The Bernoulli numbers satisfy, in the umbral notation,*

$$(B+1)^{n}=B^{n},\ \forall n\neq 1.$$

Proof. In a probabilistic setting, this is a direct consequence of the result of Proposition 2 for $n\neq 1$,

$$B_{n}=E\left(\imath L-\frac{1}{2}\right)^{n}=E\left(\imath L+\frac{1}{2}\right)^{n}.$$

□

Proposition 4. *The Bernoulli numbers verify, in the umbral notation,*

$$(B+1)^{n}=(-1)^{n}B^{n},\ n\neq 1.$$

Proof. Since by symmetry

$$\begin{aligned}E\left(\frac{\imath}{2\pi}\log\frac{E_1}{E_2}-\frac{1}{2}\right)^{n} &= E\left(\frac{\imath}{2\pi}\log\frac{E_2}{E_1}-\frac{1}{2}\right)^{n}\\ &= (-1)^{n}E\left(\frac{\imath}{2\pi}\log\frac{E_1}{E_2}+\frac{1}{2}\right)^{n}=(-1)^{n}B_{n}\end{aligned}$$

for $n\geq 2$, we deduce the result. □

Other integral representations of Bernoulli numbers and polynomials exist in the litterature; for example, (2.4) can be considered as a substitute for the Mellin Barnes integral [15, 24.7.11]

$$B_{n}(x)=\frac{1}{2\pi\imath}\int_{-c-\imath\infty}^{-c+\imath\infty}(x+t)^{n}\left(\frac{\pi}{\sin\pi t}\right)^{2}dt;$$

moreover, the integral representation [6, p.39, eq. (27)]

$$B_{2n}=(-1)^{n+1}\pi\int_{0}^{+\infty}t^{2n}\operatorname{csch}^{2}(\pi t)\,dt$$

can be easily deduced from (2.3); however, this identity holds only for even order Bernoulli numbers - the odd order ones being equal to 0 except B_1, as a consequence of propositions 3 and 4. This can be a difficulty in the calculation of some series which involve all Bernoulli numbers. As an example, we provide a short proof of Kaneko's theorem.

Theorem. *[Kaneko] The Bernoulli numbers verify*

$$\sum_{i=0}^{n+1}\binom{n+1}{i}(n+i+1)B_{n+i}=0,\ n\geq 0.$$

Proof. It can be easily checked that $\forall z \in \mathbb{C},\ n \geq 0,$

$$\sum_{i=0}^{n+1} \binom{n+1}{i} (n+i+1)\, z^{n+i} = (n+1)\, z^n (1+z)^n (1+2z)$$

so that

$$\begin{aligned}\sum_{i=0}^{n+1} \binom{n+1}{i} (n+i+1)\, B_{n+i} &= (n+1)\, E\left(\imath L - \frac{1}{2}\right)^n \left(\imath L + \frac{1}{2}\right)^n (2\imath L) \\ &= 2\imath\, (n+1)\, E\left[L\left(-L^2 - \frac{1}{4}\right)^n\right]\end{aligned}$$

The random variable L being symmetric, i.e. $-L$ being distributed as $+L$, the former expectation can be computed as

$$E\left[L\left(-L^2 - \frac{1}{4}\right)^n\right] = E\left[-L\left(-L^2 - \frac{1}{4}\right)^n\right] = 0$$

what proves the result. □

Kaneko's theorem is a special case of Momiyama's identity

$$(-1)^m \sum_{k=0}^{m} \binom{m+1}{k} (n+k+1)\, B_{n+k} = (-1)^{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (m+k+1)\, B_{m+k}$$

which can also be easily proved by remarking that

$$\begin{aligned}\sum_{k=0}^{m} \binom{m+1}{k} (n+k+1)\, B_{n+k} &= E_L\, (n+1) \left(\imath L - \frac{1}{2}\right)^n \left(\imath L + \frac{1}{2}\right)^{m+1} \\ &+ (m+1) \left(\imath L - \frac{1}{2}\right)^{n+1} \left(\imath L + \frac{1}{2}\right)^m \\ &- (m+n+2) \left(\imath L - \frac{1}{2}\right)^{m+n+1}\end{aligned}$$

which, by the symmetry of L, can be shown to coincide with the right-hand side sum.

Another famous identity can be proved easily using the probabilistic approach

Theorem 5. *The following identity holds*

$$\int_x^y B_n(z)\, dz = \frac{B_{n+1}(y) - B_{n+1}(x)}{n+1},\ n \geq 2$$

Proof. Using the probabilistic representation (2.2),

$$\begin{aligned}\int_x^y B_n(z)\, dz &= E \int_x^y \left(\imath L + z - \frac{1}{2}\right)^n \\ &= \frac{1}{n+1} \left(E\left(\imath L + y + \frac{1}{2}\right)^{n+1} - E\left(\imath L + x + \frac{1}{2}\right)^{n+1}\right) \\ &= \frac{B_{n+1}(y) - B_{n+1}(x)}{n+1}.\end{aligned}$$

□

An equivalent version of Theorem 5 is

$$\frac{d}{dz} B_n(z) = n B_{n-1}(z)$$

which shows that Bernoulli polynomials are Appell polynomials.

A related result is the following [15, 24.13.2]

Theorem 6. *The Bernoulli polynomials satisfy*

$$\int_x^{x+1} B_n(z)\, dz = x^n. \tag{2.5}$$

Proof. From Thm 5, this integral is equal to

$$\frac{B_{n+1}(x+1)-B_{n+1}(x)}{n+1}=\frac{E\left(iL+x+\frac{1}{2}\right)^{n+1}-E\left(iL+x-\frac{1}{2}\right)^{n+1}}{n+1}.$$

Expanding each power with the binomial formula yields

$$\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}x^k\left\{E\left(iL+\frac{1}{2}\right)^{n+1-k}-E\left(iL-\frac{1}{2}\right)^{n+1-k}\right\}.$$

By (3), all terms vanish except for $k=n$, which yields the result. □

Another identity, used by Gessel in the proof of Kaneko's formula, and also proved in [7] using the extended Zeilberger's algorithm, reads

Theorem 7. *The Bernoulli numbers satisfy the following identity*

$$\sum_{k=0}^{m}\binom{m}{k}B_{n+k}=(-1)^{m+n}\sum_{k=0}^{n}\binom{n}{k}B_{m+k}.$$

Proof. The proof is straightforward using (2.2) since the left-hand side is

$$\sum_{k=0}^{m}\binom{m}{k}B_{n+k}=E\left(\imath L-\frac{1}{2}\right)^n\sum_{k=0}^{m}\binom{m}{k}\left(\imath L-\frac{1}{2}\right)^k=E\left(\imath L-\frac{1}{2}\right)^n\left(\imath L+\frac{1}{2}\right)^m$$

while the right-hand side is $E\left(\imath L-\frac{1}{2}\right)^m\left(\imath L+\frac{1}{2}\right)^n$. Since L is distributed as $-L$,

$$\begin{aligned}E\left(\imath L-\frac{1}{2}\right)^n\left(\imath L+\frac{1}{2}\right)^m &= E\left(-\imath L-\frac{1}{2}\right)^n\left(-\imath L+\frac{1}{2}\right)^m\\ &= (-1)^{m+n}E\left(\imath L+\frac{1}{2}\right)^n\left(\imath L-\frac{1}{2}\right)^m\end{aligned}$$

which concludes the proof. □

In [7], K.W. Chen considers the sequence of numbers $\{K_n\},\ n\ge 0$ defined as

$$K_n=\sum_{i=0}^{n}\binom{n}{i}B_{n+i+1},$$

and proves the following

Theorem 8. *[Chen] The sequence $\{K_n\}$ satisfies the identities*

$$\sum_{k=0}^{n}\binom{2n-k}{k}\frac{2n}{2n-k}K_k=-B_{2n},\ n\ge 1$$

and

$$\sum_{k=0}^{n-1}\binom{2n-k-1}{k}\frac{2n-1}{2n-k-1}K_k=B_{2n-1},\ n\ge 1.$$

Proof. We give a short proof of the first identity remarking that $K_n=E\left(\left(\imath L-\frac{1}{2}\right)^{n+1}\left(\imath L+\frac{1}{2}\right)^n\right)$ and that

$$\sum_{k=0}^{n}\binom{2n-k}{k}\frac{2n}{2n-k}X^k=\left(\frac{1}{2}+X-\frac{\sqrt{1-4X}}{2}\right)^n+\left(\frac{1}{2}+X+\frac{\sqrt{1-4X}}{2}\right)^n$$

so that

$$\sum_{k=0}^{n}\binom{2n-k}{k}\frac{2n}{2n-k}K_k=EU\left(\left(\frac{1}{2}+UV-\frac{\sqrt{1-4UV}}{2}\right)^n+\left(\frac{1}{2}+UV+\frac{\sqrt{1-4UV}}{2}\right)^n\right)$$

with $U=\imath L-\frac{1}{2}$ and $V=\imath L+\frac{1}{2}$ so that $UV=-L^2-\frac{1}{4}$ and $\sqrt{1-4UV}=\imath L$. Finally,

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n-k}{k}\frac{2n}{2n-k}K_k &= E\left(\imath L-\frac{1}{2}\right)\left(\left(-\imath L+\frac{1}{2}\right)^{2n}+\left(\imath L+\frac{1}{2}\right)^{2n}\right)\\ &= E\left(\imath L-\frac{1}{2}\right)^{2n+1}+E\left(\imath L-\frac{1}{2}\right)E\left(\imath L+\frac{1}{2}\right)^{2n}.\end{aligned}$$

The first term is $B_{2n+1}=0$ while the second reads

$$E\left(\imath L-\frac{1}{2}\right)E\left(\imath L+\frac{1}{2}\right)^{2n}=\left(E\left(\imath L+\frac{1}{2}\right)-1\right)E\left(\imath L+\frac{1}{2}\right)^{2n}$$

which is equal to $B_{2n+1}-B_{2n}=-B_{2n}$ since $n\geq 1$. The proof of the second identity is equally easy. □

A last and less easy identity is the following from [11].

Theorem 9. *[Gessel] Denote the power sum polynomial*

$$S_k(n)=\sum_{i=1}^{n}i^k=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{n^{i+1}}{i+1}B_{k-i}. \tag{2.6}$$

Then, with $a\in\mathbb{N},\ a\geq 1$ *and* $n\in\mathbb{N}$,

$$B_n=\frac{1}{a(1-a^n)}\sum_{k=0}^{n-1}a^k\binom{n}{k}B_kS_{n-k}(a-1).$$

A quick proof can be given using the following

Lemma 10. *The power sum polynomial has integral representation* $\forall k,\forall n\in\mathbb{N}$,

$$S_k(n)=\int_0^n B_k(z+1)\,dz.$$

Proof. This can be verified considering $n\in\mathbb{R}$ in the right-hand side of (2.6) so that

$$\begin{aligned}\frac{d}{dn}S_k(n) &= E_L\sum_{i=0}^{k}\binom{k}{i}n^i(-1)^{k-i}\left(\imath L-\frac{1}{2}\right)^{k-i}=E_L\left(-\imath L+n+\frac{1}{2}\right)^k\\ &= E_L\left(\imath L+n+\frac{1}{2}\right)^k\end{aligned}$$

and, since $S_k(0)=0$,

$$S_k(n)=\int_0^n E_L\left(\imath L+z+\frac{1}{2}\right)^n dz=\int_0^n B_n(z+1)\,dz$$

and the result holds in particular $\forall n\in\mathbb{N}$. □

Another simple proof of this lemma can obviously be deduced from identity (2.5).

Using this integral representation, we compute now

$$\sum_{k=0}^{n-1}a^k\binom{n}{k}B_kS_{n-k}(a-1)=\sum_{k=0}^{n}a^k\binom{n}{k}B_kS_{n-k}(a-1)-a^n(a-1)B_n.$$

The right-hand side sum is

$$E_{L_1}\sum_{k=0}^{n}a^k\binom{n}{k}\left(\imath L_1-\frac{1}{2}\right)^k\int_0^{a-1}B_{n-k}(z+1)\,dz=\int_0^{a-1}E_{L_1,L_2}\left[a\left(\imath L_1-\frac{1}{2}\right)+\left(\imath L_2+z+\frac{1}{2}\right)\right]^n dz$$

which can be integrated as

$$\frac{1}{n+1}E_{L_1,L_2}\left[\left(a\left(\imath L_1-\frac{1}{2}\right)-\left(\imath L_2+a-\frac{1}{2}\right)\right)^{n+1}-\left(a\left(\imath L_1-\frac{1}{2}\right)-\left(\imath L_2+\frac{1}{2}\right)\right)^{n+1}\right].$$

Expanding both $(n+1)$ powers, we obtain

$$\frac{1}{n+1}E_{L_1,L_2}\sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}\left(\left(\imath L_1+\frac{1}{2}\right)^{n+1-k}\left(\imath L_2-\frac{1}{2}\right)^{k}-\left(\imath L_1-\frac{1}{2}\right)^{n+1-k}\left(\imath L_2+\frac{1}{2}\right)^{k}\right).$$

But by (3), all terms cancel in this sum except for $k=1$ and $k=n$ in which cases they add up, so that we obtain

$$\frac{1}{n+1}\left(-\left(n+1\right)a^{n}B_n\right)+\frac{1}{n+1}\left(\left(n+1\right)aB_n\right)=B_n a\left(1-a^{n-1}\right)$$

which, adding the remaining term $-a^n\left(a-1\right)B_n$, yields the result.

3. Euler numbers and Euler polynomials

3.1. **definition and characterisation.** We derive here analogous results for Euler numbers and Euler polynomials. A generating function for the sequence $\{E_n\}$ of Euler numbers is

$$\sum_{n=0}^{+\infty}E_n\frac{t^n}{n!}=\operatorname{sech}\left(t\right)$$

and for the Euler poynomials $\{E_n\left(x\right)\}$

$$\sum_{n=0}^{+\infty}E_n\left(x\right)\frac{t^n}{n!}=\frac{2e^{tx}}{e^t+1}.$$

The Euler numbers are obtained as

$$E_n=\frac{1}{2^n}E_n\left(\frac{1}{2}\right).$$

First values are

$$E_0\left(x\right)=1;\ E_1\left(x\right)=x-\frac{1}{2};\ E_2\left(x\right)=x^2-x;\ E_3\left(x\right)=x^3-\frac{3}{2}x^2+\frac{1}{4}$$

and

$$E_0=1;\ E_1=0;\ E_2=-1;\ E_3=0;\ E_4=5.$$

In [3], the following formula is derived

Theorem 11. *[Sun] If L_0 is defined as*

$$L_0=\sum_{k=1}^{+\infty}\frac{L_k}{\left(2k-1\right)\pi}\tag{3.1}$$

where L_k are independent and Laplace distributed, then the Euler polynomials read

$$E_n\left(x\right)=E\left(\imath L_0+x-\frac{1}{2}\right)^n\tag{3.2}$$

and the Euler numbers

$$E_n=2^nE_n\left(\frac{1}{2}\right)=2^nE\left(\imath L_0\right)^n.$$

We provide a more convenient characterization of the random variable L_0 as follows.

Theorem 12. *The random variable L_0 follows the hyperbolic secant distribution*

$$f_{L_0}\left(x\right)=sech\left(\pi x\right).$$

Proof. The characteristic function of L_0 is

$$Ee^{\imath L_0 t}=\operatorname{sech}\left(\frac{t}{2}\right).$$

From [5, 1.9.1],

$$\int_0^{+\infty}\operatorname{sech}\left(ax\right)\cos\left(xt\right)dx=\frac{\pi}{2a}\operatorname{sech}\left(\frac{\pi}{2a}t\right)$$

so that, with $a=\pi$, the density of L_0 is

$$f_{L_0}\left(x\right)=\operatorname{sech}\left(\pi x\right).$$

Thus πL_0 follows an hyperbolic secant distribution. □

We note from [8] that the random variable L_0 can be obtained as

$$L_0 = \frac{1}{\pi}\log|C| = \frac{1}{\pi}\left(\log|N_1| - \log|N_2|\right) \tag{3.3}$$

where C is Cauchy distributed and N_1 and N_2 are two independent standard Gaussian random variables.

The random variable L_0 is also a scale mixture of Gaussian with mixing distribution given in [10], the moment generating function of which reads

$$\varphi(s) = \frac{1}{\cos\left(\pi\sqrt{2s}\right)}.$$

At last, the random variable $\imath L_0$ has the same moments as the Lévy area - that is the signed area - of a Brownian motion B_t for $0 \le t \le 1$, see [12].

We leave to the reader the proofs of basic identities such as

$$E_n(1-x) = (-1)^n E_n(x), \tag{3.4}$$

$$\sum_{r=0}^{n}\binom{n}{r}E_r(x) = E_n(x+1)$$

and

$$\sum_{r=0}^{n}\binom{2n}{2r}E_{2r} = 0 \tag{3.5}$$

as a consequence of the probabilistic representation (3.2): the first identity is a consequence of the symmetry of the distribution of L_0, the second is obtained using the binomial theorem.

3.2. **links between Bernoulli and Euler polynomials.**

3.2.1. *a summation identity.* An interesting identity that links Bernoulli and Euler polynomials is the following [13, p. xxxiii]

$$B_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}E_k(2x)$$

and its more general version [15, 24.14.5]

$$B_n\left(\frac{x+y}{2}\right) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)E_k(y).$$

The proof reads

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)E_k(y) = \frac{1}{2^n}E\left(\imath L_e + x - \frac{1}{2} + \imath L_0 + y - \frac{1}{2}\right)^n = E\left(\imath\frac{L_0+L_e}{2} + \frac{x+y}{2} - \frac{1}{2}\right)^n$$

and to conclude, we need simply to show that $\frac{L_0+L_e}{2} \sim L_e$: this can be deduced immediately from the identities (3.1) and (2.1); or from

$$\frac{L_0+L_e}{2} = \frac{1}{2\pi}\log|C| + \frac{1}{2\pi}\log\sqrt{\frac{E_1}{E_2}} = \frac{1}{2\pi}\log\frac{|N_1|\sqrt{E_1}}{\sqrt{E_2}|N_2|}$$

where C is Cauchy, E is exponential and since $\frac{N}{\sqrt{2E}}$ is Laplace distributed and $|L|$ is exponentially distributed, we deduce

$$\frac{L_0+L_e}{2} \sim \frac{1}{2\pi}\log\frac{E_1}{E_2} \sim L_e.$$

3.2.2. *an integral identity.* Another similar identity is [15, 24.13.3]

$$\int_x^{x+\frac{1}{2}} B_n(z)\,dz = \frac{E_n(2x)}{2^{n+1}} \tag{3.6}$$

which can be proved as follows

$$\begin{aligned}
\int_x^{x+\frac{1}{2}} B_n(z)\,dz &= \frac{1}{n+1}\left\{E\left(iL_e+x\right)^{n+1} - E\left(iL_e + x - \frac{1}{2}\right)^{n+1}\right\}\\
&= \frac{1}{n+1}\left\{E\left(i\frac{L_e}{2}+i\frac{L_0}{2}+\frac{2x}{2}\right)^{n+1} - E\left(i\frac{L_e}{2}+i\frac{L_0}{2}+\frac{2x-1}{2}\right)^{n+1}\right\}\\
&= \frac{1}{2^{n+1}(n+1)}\left\{E\left(\left(iL_e+\frac{1}{2}\right)+\left(iL_0+2x-\frac{1}{2}\right)\right)^{n+1} - E\left(\left(iL_e-\frac{1}{2}\right)+\left(iL_0+2x-\frac{1}{2}\right)\right)^{n+1}\right\}
\end{aligned}$$

By property (4), all terms cancel in the binomial expansions of the $(n+1)$ powers, except for $k=n$ and the only remaining term is

$$\frac{1}{2^{n+1}(n+1)}\binom{n+1}{n}E\left(iL_0+2x-\frac{1}{2}\right)^n = \frac{E_n(2x)}{2^{n+1}}.$$

3.2.3. *a consequence of* (3.6). As a consequence of the identity (3.6), we deduce the identity

$$E_n(x+1)+E_n(x) = 2x^n$$

as follows:

$$E_n(x+1)+E_n(x) = 2^{n+1}\int_{\frac{x}{2}}^{\frac{x}{2}+1} B_n(z)\,dz = 2^{n+1}\left(\frac{x}{2}\right)^n = 2x^n$$

where we applied the result (2.5).

4. Hermite polynomials

In [1], the study of Hermite polynomials involves the definition of the umbra M such that

$$e^{Mx} = e^{-x^2},\ x\in\mathbb{C}.$$

In the rest of this section, we provide a probabilistic interpretation to this umbra, and using a master identity on Gaussian random variables, derive new simple proofs of the results by Gessel.

Theorem 13. *The umbra M is equivalent to the expectation*

$$f(Mx) = Ef(-\imath Zx) \tag{4.1}$$

where Z is a Gaussian random variable with zero mean and variance $\sigma^2=2$, which we denote as $Z\sim\mathcal{N}(0,2)$, for any admissible function f.

Proof. It suffices to prove

$$Ee^{-iZx} = e^{-x^2}$$

which is nothing but the expression of the characteristic function of the Gaussian random variable Z. □

We need the following master identity.

Proposition 14. *In umbral notation,*

$$e^{Mx+M^2y} = \frac{1}{\sqrt{1+4y^2}}\exp\left(-\frac{x^2}{1+4y^2}\right). \tag{4.2}$$

Proof. Expressing

$$e^{Mx+M^2y} = E_V e^{Mx+MVy} = E_V e^{-(x+Vy)^2}$$

where $V\sim\mathcal{N}(0,2)$, we need to compute the expectation

$$\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\frac{V^2}{4}}e^{-(x+Vy)^2}dV.$$

We recognize here, up to a constant term $\frac{\sqrt{\pi}}{y}$, the convolution, evaluated at the point $\frac{x}{y}$, of a variable $V \sim \mathcal{N}(0,2)$ with a random variable $W \sim \mathcal{N}\left(0, \frac{1}{2y^2}\right)$; it can easily be checked to be the Gaussian with variance $\mathcal{N}\left(0, 1+4y^2\right)$, i.e.

$$\frac{1}{\sqrt{1+4y^2}} \exp\left(-\frac{x^2}{1+4y^2}\right).$$

□

As special cases, we recover

(1) for $y=0$, $e^{Mx} = \exp\left(-x^2\right)$
(2) for $x=0$, $e^{M^2x} = \frac{1}{\sqrt{1+4x^2}}$

Here are now a few results from [1] that can be deduced from the master identity (4.2).

(1) the umbral notation for the Hermite polynomials

$$H_n(u) = (2u+M)^n$$

can be deduced from the identity (4.1) with $f(M) = (2u+M)^n$, so that

$$H_n(u) = E(2u+\imath N)^n$$

where $N \sim \mathcal{N}(0,2)$, which is a well-known result (see for example [14, p.49])

(2) the generating function

$$\sum_{n=0}^{+\infty} H_{2n}(u) \frac{x^n}{n!} = \frac{1}{\sqrt{1+4x}} \exp\left(-\frac{4u^2x}{1+4x}\right)$$

can be derived from

$$\sum_{n=0}^{+\infty} H_{2n}(u) \frac{x^n}{n!} = E_N \exp\left(4x(u+\imath N)^2\right) = E_V \exp\left(-\left(V\sqrt{x} - iu\sqrt{4x}\right)^2\right)$$

and application of the master identity (4.2)

(3) the same approach yields the bivariate generating function

$$\sum_{m,n=0}^{+\infty} H_{2m+n}(u) \frac{x^m}{m!} \frac{y^n}{n!} = \frac{e^{-\frac{y^2}{4x}}}{\sqrt{1+4x}} \exp\left(\frac{\left(2u\sqrt{x} + \frac{y}{2\sqrt{x}}\right)^2}{1+4x}\right).$$

5. Carlitz and Zeilberger's Hermite polynomials

In [1], Gessel proposes an umbral study of Carlitz Hermite polynomials

$$H_{m,n}(u,v) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k} k! u^{m-k} v^{n-k}$$

and of the Zeilberger's Hermite polynomials

$$H_{m,n}(w) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k} k! w^k.$$

with notation $m \wedge n = \min(m,n)$. In this aim, he defines the umbræ A and B by

$$A^m B^n = \delta_{m,n} m!, \ \forall m,n \in \mathbb{N} \tag{5.1}$$

or equivalently as

$$\exp(Ax+By) = \exp(xy) \ \ \forall x,y \in \mathbb{R}. \tag{5.2}$$

The probabilistic counterpart of these umbræ is given by the following result.

Proposition 15. *Let us consider a complex circular normal random variable Z with density*

$$f_Z(z) = \frac{1}{\pi}\exp\left(-|z|^2\right).$$

The umbræ A and B in (5.1) *and* (5.2) *are identified with the expectations with respect to Z and $\bar{Z}$ respectively.*

Proof. It suffices to check that

$$EZ^m\bar{Z}^n = \delta_{m,n}m!$$

which is straightforward.

Equivalently,

$$\begin{aligned} E\exp\left(Zx+\bar{Z}y\right) &= E\exp\left(\left(X+\imath Y\right)x+\left(X-\imath Y\right)y\right) \\ &= E\exp\left(X\left(x+y\right)\right)E\exp\left(\imath Y\left(x-y\right)\right) \\ &= \exp\frac{1}{4}\left(x+y\right)^2\exp-\frac{1}{4}\left(x-y\right)^2 \end{aligned}$$

so that the result holds. □

As a consequence, we deduce a probabilistic representation of Zeilberger's Hermite polynomials as

$$H_{m,n}\left(u\right) = E\left(1+Z\right)^m\left(1+u\bar{Z}\right)^n$$

and of the Carlitz Hermite polynomials as

$$H_{m,n}\left(u,v\right) = u^{m-n}E\left(1+Z\right)^m\left(uv+\bar{Z}\right)^n.$$

Using now classical results about complex Gaussian random variables, we can recover easily some of the results derived by Gessel using umbral calculus. We need the following master identity, the umbral version of which is Lemma 5.3 in [1].

Lemma 16. *If Z is circular normal, then*
(5.3)

$$E\exp\left(Za+\bar{Z}b+Z^2u+Z\bar{Z}v+\bar{Z}^2w\right) = \frac{1}{\sqrt{\left(1-v\right)^2-4uw}}\exp\left(\frac{a^2w+b^2u+ab\left(1-v\right)}{\left(1-v\right)^2-4uw}\right).$$

Proof. The proof is a consequence of the fact that the expectation in (5.3) can be rewritten as

$$E\exp\left(\tilde{Z}a+\bar{\tilde{Z}}b\right)$$

where $\tilde{Z}$ is a complex Gaussian variable with covariance matrix

$$E\tilde{Z}\tilde{Z}^t = \Sigma = \frac{1}{\sqrt{\left(1-v\right)^2-4xu}}\begin{bmatrix} x & \frac{(1-v)}{2} \\ \frac{(1-v)}{2} & u \end{bmatrix}.$$

The latest expectation over $\tilde{Z}$ is nothing but the generating function of $\tilde{Z}$ computed at the point (a,b). □

In order to illustrate the efficiency of the probabilistic approach, we provide some quick derivations of the results obtained by Gessel.

(1) the generating function of Zeilberger's Hermite polynomials reads

$$\begin{aligned} \sum_{m,n=0}^{+\infty} H_{m,n}\left(w\right)\frac{x^m}{m!}\frac{y^n}{n!} &= E_Z\exp\left(x\left(1+Z\right)\right)\exp\left(y\left(1+w\bar{Z}\right)\right) \\ &= \exp\left(x+y\right)E_Z\left(xZ+yw\bar{Z}\right) \\ &= \exp\left(x+y+wxy\right) \end{aligned}$$

where the latest equality is a consequnce of (5.2)

(2) the bilinear generating function of Zeilberger's Hermite polynomials reads

$$\begin{aligned} \sum_{m,n=0}^{+\infty} H_{m,n}\left(u\right)H_{m,n}\left(v\right)\frac{x^m}{m!}\frac{y^n}{n!} &= E\exp x\left(1+Z_1\right)\left(1+Z_2\right)\exp\left(y\left(1+u\bar{Z}_1\right)\left(1+v\bar{Z}_2\right)\right) \\ &= E\exp\left(x\left(1+Z_2\right)+y\left(1+\bar{Z}_2v\right)\right) \\ &\quad\times\exp\left(xZ_1\left(1+Z_2\right)+uy\bar{Z}_1\left(1+v\bar{Z}_2\right)\right) \end{aligned}$$

Taking first expectation on Z_1 and using formula (5.2), then expectation on Z_2 and using formula (5.3) yields the result.

(3) the following general formula holds

$$\sum_{i,j,k,l,m=0}^{+\infty} H_{i+2k+m,j+2l+m}\left(u\right)\frac{v^i}{i!}\frac{w^j}{j!}\frac{x^k}{k!}\frac{y^l}{l!}\frac{t^m}{m!} = \frac{1}{\sqrt{\left(1-ut\right)^2-4u^2xy}} \tag{5.4}$$

$$\times \exp\left(\frac{\left(1+uw\right)^2 x+\left(1+uv\right)^2 y+4uxy+\left(1-ut\right)\left(v+w+t+uvw\right)}{\left(1-ut\right)^2-4u^2xy}\right)$$

The quintuple sum can be computed as

$$E\exp\left[v\left(1+Z\right)+w\left(1+u\bar{Z}\right)+x\left(1+Z\right)^2+y\left(1+u\bar{Z}\right)^2+t\left(1+Z\right)\left(1+u\bar{Z}\right)\right]$$

and using the master lemma (5.3) and simple algebra, we recover (5.4).

6. Conclusion

We have seen that all umbræ used in [1] have a simple probabilistic expectation counterpart. Depending on the complexity of the formula to prove, each approach has its advantages and drawbacks. A future direction of work is to deepen the understanding of the link between the umbral and the probabilistic approach.

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