NON-ISOPARAMETRIC SOLUTIONS OF THE EIKONAL EQUATION

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Abstract. In this paper, we prove that a quartic solution of the eikonal equation $|\nabla_x f|^2 = 16x^6$ in $\mathbb{R}^n$ is either isoparametric or congruent to a polynomial $f = (\sum_{i=1}^n x_i^2)^2 - 8(\sum_{i=1}^k x_i^2)(\sum_{i=k+1}^n x_i^2)$, $k = 0, 1, \ldots, [\frac{n}{2}]$.

1. Introduction

Let $V = \mathbb{R}^n$ be a Euclidean space equipped with the standard scalar product $\langle \cdot, \cdot \rangle$. By $O(V)$ we denote the orthogonal group in V and be $S(V)$ we denote the standard unit sphere in V centered at the origin. Recall that a hypersurface M in $S(V)$ is called *isoparametric* if it has constant principal curvatures [**23**]; see also a recent survey [**5**]. By a celebrated theorem of Münzner [**12**], any isoparametric hypersurface is algebraic and its defining polynomial f is homogeneous of degree $g = 1, 2, 3, 4$ or 6, where g is the number of distinct principal curvatures. Moreover, suitably normalized, f satisfies the system of the so-called Münzner-Cartan differential equations

$$|\nabla_x f|^2 = g^2 x^{2g-2}, \qquad \Delta_x f = \frac{m_2 - m_1}{2} g^2 x^{g-2}, \tag{1}$$

where m_i are the multiplicities of the maximal and minimal principal curvature of M (there holds $m_1 = m_2$ when g is odd). Here and in what follows, if no ambiguity is possible, we omit the norm notation by writing x^k for $|x|^k$.

Isoparametric hypersurfaces of lower degrees $g = 1, 2, 3$ were completely classified by Élie Cartan in the late 1930-s. In the case of three distinct curvatures one has $m_1 = m_2$, so that any isoparametric cubic f is a priori harmonic. In [**4**] Cartan established a remarkable result that there exist exactly four different isoparametric hypersurfaces with three distinct principal curvatures: their dimensions are equal to $3d$, where $d = 1, 2, 4, 8$, and the corresponding defining polynomials f can be naturally expressed in terms of the multiplication in one of four real division algebras $\mathbb{F}_d$ of dimension d, where $\mathbb{F}_1 = \mathbb{R}$ (reals), $\mathbb{F}_2 = \mathbb{C}$ (complexes), $\mathbb{F}_4 = \mathbb{H}$ (quaternions) and $\mathbb{F}_8 = \mathbb{O}$ (octonions). The class isoparametric hypersurfaces with $g = 4$ is very well understood by now, thanks to the works of Ozeki and Takeuchi [**13**], [**14**], Ferus, Karcher, and Münzner [**7**], Cecil, Chi and Jensen [**6**] and several other authors. However, in spite of much recent progress, isoparametric hypersurfaces with $g = 4$ and $g = 6$ distinct principal curvatures are not yet completely classified.

Regarding the Münzner-Cartan differential equations as a system of two differential relations, a very natural question appears: How to characterize polynomial solutions of the first equation in (1) alone? We shall call a homogeneous polynomial f satisfying

$$|\nabla_x f|^2 = g^2 x^{2g-2}, \qquad \deg f = g, \quad x \in \mathbb{R}^n, \tag{2}$$

an *eikonal* polynomial. This problem arises, for example, in geometrical optics and wave propagation [**17**], [**2**], the theory of harmonic morphisms [**1**], entire solutions of the eikonal equation [**10**], transnormal hypersurfaces [**16**], [**3**]. Note also that any eikonal polynomial induces a polynomial map $\mathrm{grad}_x f : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$. Polynomial maps between Euclidean spheres, in particular those with harmonic coordinates (the so-called eigenmaps), has been the subject of considerable recent interest, with applications to the general problem of representing homotopy classes by polynomial maps (see [**25**], [**1**] for the further discussion). Remarkably, for $n = 4$ any nonconstant quadratic eigenmap $F\colon S(\mathbb{R}^4) \to S(\mathbb{R}^4)$ up to isometries of the domain and the range, is the gradient of a Cartan cubic isoparametric polynomial [**8**].

The characterization problem of eikonal polynomials, except for the trivial cases $g = 1, 2$ remains essentially open. Only recently, for $g = 3$ the following complete description of eikonal cubic polynomials was obtained in [**21**]. We have proved that any eikonal polynomial for $g = 3$ is congruent to either one of the four isoparametric Cartan cubic polynomials in dimensions $n = 5, 8, 14$ and 26, or to the polynomial

$$h = x_n^3 - 3x_n(x_1^2 + \ldots + x_{n-1}^2). \tag{3}$$

Recall that two eikonal polynomials f_1 and f_2 are called congruent if there is an orthogonal transformation $U \in O(V)$ such that $f_2(x) = \pm f_1(Ux)$. Observe that $\Delta h = 3(2-n)x_n$, thus h is non-isoparametric for $n \neq 2$. Our proof is heavily based on one result of Yiu [**26**] on quadratic maps between Euclidean spheres and the theory of composition formulas [**18**].

In this paper, we obtain a similar result for the eikonal quartics, that is the solutions of (2) with $g = 4$. Before formulating this result we make some preliminary definitions and observations. The above family (3) has a natural generalization for any degree g. Namely, let us associate with an arbitrary subspace H of V and an integer $g \geq 1$ the following function:

$$h_{g,H} = \mathrm{Re}(|\xi| + |\eta|\sqrt{-1})^g = \sum_{k=0}^{[g/2]} (-1)^k \binom{g}{k} \xi^{g-2k} \eta^{2k}, \tag{4}$$

where ξ and η are the orthogonal projections of x onto H and $H^\perp = V \ominus H$ respectively. Then the cubic h in (3) is exactly $h_{3,H}$ with $H = \mathbb{R}e_n$. In general, one can easily verify that the following is true.

Proposition 1.1. *Suppose that either g is even or g is odd but* $\dim H_1 = 1$. *Then $h_{g,\mathcal{H}}$ is eikonal polynomial of degree g.*

We call a polynomial congruent to (4) a *primitive* eikonal polynomial.

A typical *quartic* primitive polynomial in the above notation reads as follows:

$$h_{4,H} = \xi^4 - 6\xi^2\eta^2 + \eta^4.$$

Proposition 1.2. *Two primitive quartics h_{4,H_1} and h_{4,H_2} in $\mathbb{R}^n$ are congruent if and only if either* $\dim H_1 = \dim H_2$ *or* $\dim H_1 = n - \dim H_2$.

PROOF. It suffices to prove the 'only if' part. Suppose that h_{4,H_1} and h_{4,H_2} are congruent. Then there exists an orthogonal transformation $U \in O(V)$ such that $h_{4,H_2}(x) = \epsilon h_{4,H_1}(Ux)$, $\epsilon^2 = 1$. Since the Laplacian is an invariant operator, the quadratic forms $\Delta_x h_{4,H_1}$ and $\epsilon\Delta_x h_{4,H_2}$ have the same spectrum. We have

$$\Delta_x h_{4,H_i} = (8 + 16p_i - 12n)\xi_i^2 + (4n - 16p_i + 8)\eta_i^2 \equiv \lambda_i \xi_i^2 + \mu_i \eta_i^2,$$

where ξ_i and η_i are the projections on H_i and $H_i^\perp$ respectively, and $p_i = \dim H_i$, $i = 1, 2$. Note that the quadratic forms are diagonal. If $\epsilon = 1$ then a simple analysis implies that either $p_1 = p_2$ or $p_1 = n - p_2$. If $\epsilon = -1$ we have $\lambda_1 + \mu_1 = -\lambda_2 - \mu_2$. On the other hand, $\lambda_i + \mu_i = 16 - 8n$, hence $n = 2$ and in this case the conclusion of the proposition is trivial. □

It follows from Proposition 1.2 that there is exactly $[\frac{n}{2}]$ distinct congruence classes of primitive quartics in $\mathbb{R}^n$. Any such a quartic is congruent to one of the following:

$$f = (\sum_{i=1}^{n} x_i^2)^2 - 8(\sum_{i=1}^{k} x_i^2)(\sum_{i=k+1}^{n} x_i^2), \qquad 0 \le k \le [\frac{n}{2}].$$

The main result of the present paper is the following theorem.

Theorem 1.3. *Any quartic eikonal polynomial is either isoparametric or primitive.*

The proof of Theorem 1.3 will be given in the remaining sections of the paper. We conclude this Introduction by observing that, in view of the remarks made above, the following conjecture seems to be plausible.

Conjecture. An eikonal polynomial of an arbitrary degree $g \ge 2$ is either isoparametric or primitive. In particular, if $g \ne 2, 3, 4, 6$ then f is primitive.

2. Preliminaries

Note that any quartic homogeneous polynomial f can be written in some orthogonal coordinates in the following *normal* form:

$$f = x_n^4 + 2\phi(\bar{x})x_n^2 + 8\psi(\bar{x})x_n + \theta(\bar{x}), \quad \bar{x} = (x_1, \ldots, x_{n-1}), \tag{5}$$

where ϕ, ψ and θ are homogeneous polynomials of degrees 2,3 and 4 respectively. To see this, it suffices to choose an orthonormal basis $e_1, \ldots, e_n$ of V with e_n being a maximum point of the restriction of $f|_{S(V)}$. Note that for an *arbitrary* f, its normal form is by no means unique and the polynomials ϕ, ψ and θ carry no specific intrinsic information on f. As we shall see below, the situation with eikonal polynomials is remarkable and many basic algebraic properties of f can be derived directly from the first component ϕ.

Now we suppose that f written in the normal form (5) is eikonal, that is f satisfies (2) for $g = 4$. Then identifying the coefficients of x_n^j in (2) for $0 \le j \le 4$ yields the following system:

$$8\phi + |\nabla_{\bar{x}}\phi|^2 = 12\bar{x}^2, \tag{6}$$

$$\langle \nabla_{\bar{x}}\phi, \nabla_{\bar{x}}\psi\rangle = -2\psi, \tag{7}$$

$$4\phi^2 + \langle \nabla_{\bar{x}}\phi, \nabla_{\bar{x}}\theta\rangle + 16|\nabla_{\bar{x}}\psi|^2 = 12\bar{x}^4, \tag{8}$$

$$\langle \nabla_{\bar{x}}\psi, \nabla_{\bar{x}}\theta\rangle = -4\phi\psi, \tag{9}$$

$$64\psi^2 + |\nabla_{\bar{x}}\theta|^2 = 16\bar{x}^6. \tag{10}$$

We can assume without loss of generality that the quadratic form ϕ is diagonal, say $\phi = \sum_{i=1}^{n-1}\phi_i x_i^2$, so that (6) implies $\phi_i = 1$ or $\phi_i = -3$. Let us denote by L and M the corresponding eigenspaces of $\bar{V} = V \ominus \mathbb{R}e_n$, and set $\dim L = p$ and $\dim M = q$. By denoting ξ and η the projections of $\bar{x}$ onto L and M respectively, we get

$$\phi = \xi^2 - 3\eta^2, \qquad \bar{x} = (\xi, \eta).$$

Thus, with any eikonal quartic is associated an ordered pair of nonnegative integers (p, q). We shall indicate this by writing $f \in E_{p,q}$. Note that the pair (p, q) is not uniquely determined as the following example shows.

Example 2.1. Let us consider the one-dimensional subspace $H = \mathbb{R}e_1$ spanned on the vector e_1. Then the primitive eikonal polynomial

$$f \equiv h_{4,H} = x_1^4 - 6x_1^2(x_2^2 + \ldots + x_n^2) + (x_2^2 + \ldots + x_n^2)^2$$

is written in the normal form with $\phi = -3(x_2^2 + \ldots + x_n^2)$, thus $f \in E_{0,n-1}$. On the other hand, one can rewrite f as follows:

$$f = x_2^4 + 2(x_3^2 + \ldots x_n^2 - 3x_1^2)x_2^2 + \theta,$$

where θ is a quartic form in $(x_1, x_3, \ldots, x_n)$. This shows that $f \in E_{n-2,1}$.

Thus obtained orthogonal decomposition $\bar{V} = L \oplus M$ induces the corresponding decompositions in the tensor products. We write $h \in \xi^i \otimes \eta^j$ if the polynomial h is homogeneous in ξ and η of degrees i and j respectively. In particular, by decomposing the cubic form ψ into the sum of its homogeneous parts, $\sum_{i=0}^3 \psi_i$, where $\psi_i \in \xi^i \otimes \eta^{3-i}$, we find that

$$\langle \nabla_{\bar{x}}\phi, \nabla_{\bar{x}}\psi\rangle = 2\sum_{i=0}^{3}\langle \xi, \nabla_\xi \psi_i\rangle - 6\sum_{i=0}^{3}\langle \eta, \nabla_\eta \psi_i\rangle = 2\sum_{i=0}^{3}(4i-9)\psi_i,$$

hence by (7), $\sum_{i=0}^3(4i-8)\psi_i = 0$. Taking into account that non-zero ψ_i are linearly independent, we conclude that $\psi_0 = \psi_1 = \psi_3 \equiv 0$. Thus, ψ is completely determined by the component ψ_2 which is a quadratic form in ξ and a linear form in η. In matrix notation this reads as follows:

$$\psi = \xi^{\mathrm{t}} A_\eta \xi, \qquad A_\eta := \sum_{i=1}^{q}\eta_i A_i, \tag{11}$$

where $A_i \in \mathbb{R}^{p\times p}$ are symmetric matrices and ξ^{t} denotes the transpose to ξ.

Now we proceed with (8). We have

$$|\nabla_{\bar{x}}\psi|^2 = |\nabla_\xi\psi|^2 + |\nabla_\eta\psi|^2 = 4\xi^{\mathrm{t}} A_\eta^2\xi + \sum_{i=1}^{q}(\xi^{\mathrm{t}} A_i\xi)^2,$$

and $\langle \nabla_{\bar{x}}\phi, \nabla_{\bar{x}}\theta\rangle = 8\sum_{i=0}^{4}(i-3)\theta_i$, where we decomposed $\theta = \sum_{i=0}^{4}\theta_i$ with $\theta_i \in \xi^i\otimes\eta^{4-i}$. Thus, (8) takes the form

$$16(\xi^2-3\eta^2)^2 + 8\sum_{i=0}^{4}(i-3)\theta_i + 64\xi^{\mathrm{t}} A_\eta^2\xi + 16\sum_{i=1}^{q}(\xi^{\mathrm{t}} A_i\xi)^2 = 12(\xi^2+\eta^2)^2.$$

The latter identity yields

$$\theta_4 - \theta_2 - 2\theta_1 - 3\theta_0 = \xi^4 - 2\sum_{i=1}^{q}(\xi^{\mathrm{t}} A_i\xi)^2 + (6\xi^2\eta^2 - 8\xi^{\mathrm{t}} A_\eta^2\xi) - 3\eta^4,$$

hence by identifying the homogeneous parts, we find that $\theta_1 \equiv 0$ and

$$\begin{aligned}\theta_4 &= \xi^4 - 2\sum_{i=1}^{q}(\xi^{\mathrm{t}} A_i\xi)^2,\\ \theta_2 &= 8\xi^{\mathrm{t}} A_\eta^2\xi - 6\xi^2\eta^2,\\ \theta_0 &= \eta^4.\end{aligned}\tag{12}$$

It follows from (12) and (11) that any solution of (2) f is completely determined by the matrix pencil A_η and the cubic form θ_3. Our next step is to show that the matrix pencil satisfies a generalized Clifford property. To this end, we rewrite (9) in the new notation:

$$\langle \nabla_\xi\psi, \nabla_\xi\theta\rangle + \langle \nabla_\eta\psi, \nabla_\eta\theta\rangle = -4(\xi^2-3\eta^2)\xi^{\mathrm{t}} A_\eta\xi.\tag{13}$$

We have $\nabla_\xi\psi = 2A_\eta\xi$ and $\nabla_\eta\psi = \tau$, where

$$\tau = (\xi^{\mathrm{t}} A_1\xi, \ldots, \xi^{\mathrm{t}} A_q\xi),$$

hence we find from (12) the following gradients:

$$\begin{aligned}\nabla_\xi\theta_2 &= 16A_\eta^2\xi - 12\eta^2\cdot\xi,\\ \partial_{\eta_i}\theta_2 &= 16\xi^{\mathrm{t}} A_iA_\eta\xi - 12\xi^2\cdot\eta_i\\ \nabla_\xi\theta_4 &= 4\xi^2\cdot\xi - 8\sum_{i=1}^{q}(\xi^{\mathrm{t}} A_i\xi)A_i\xi \equiv 4\xi^2\cdot\xi - 8A_\tau\xi,\\ \nabla_\eta\theta_4 &= 4\eta^2\cdot\eta.\end{aligned}\tag{14}$$

Substituting the found relations into (13) yields

$$\begin{aligned}4(3\eta^2-\xi^2)\xi^{\mathrm{t}} A_\eta\xi = \langle\tau, \nabla_\eta\theta_3\rangle + 2\langle A_\eta\xi, \nabla_\xi\theta_3\rangle + 32\xi^{\mathrm{t}} A_\eta^3\xi\\ - 20\eta^2(\xi^{\mathrm{t}} A_\eta\xi) - 4\xi^2(\xi^{\mathrm{t}} A_\eta\xi).\end{aligned}$$

Collecting terms in the latter relation by homogeneity, we obtain additionally the following relations:

$$\langle\tau, \nabla_\eta\theta_3\rangle = 0,\tag{15}$$

$$\langle A_\eta\xi, \nabla_\xi\theta_3\rangle = 0,\tag{16}$$

$$\xi^{\mathrm{t}} A_\eta^3 \xi = \eta^2 (\xi^{\mathrm{t}} A_\eta \xi). \tag{17}$$

Since (17) holds for any ξ, we get the required Clifford type matrix identity:

$$A_\eta^3 = \eta^2 A_\eta. \tag{18}$$

Similarly, collecting the terms in (10) by homogeneity yields

$$|\nabla_\xi \theta_4|^2 + |\nabla_\eta \theta_3|^2 = 16\xi^6, \tag{19}$$

$$\langle \nabla_\xi \theta_4, \nabla_\xi \theta_3\rangle + \langle \nabla_\eta \theta_3, \nabla_\eta \theta_2\rangle = 0, \tag{20}$$

$$64(\xi^{\mathrm{t}} A_\eta \xi)^2 + 2\langle \nabla_\xi \theta_4, \nabla_\xi \theta_2\rangle + |\nabla_\eta \theta_2|^2 + |\nabla_\xi \theta_3|^2 = 48\xi^4\eta^2, \tag{21}$$

$$\langle \nabla_\xi \theta_3, \nabla_\xi \theta_2\rangle + \langle \nabla_\eta \theta_3, \nabla_\eta \theta_0\rangle = 0. \tag{22}$$

Now we are ready to characterize quartic eikonal polynomials with lower dimensions p and q.

Proposition 2.2. *If $f \in E_{0,n-1} \cup E_{n-1,0} \cup E_{n-2,1}$ then f is a primitive quartic.*

Proof. First consider the case $f \in E_{0,n-1} \cup E_{n-1,0}$. Then $qp = 0$, thus $\psi \equiv 0$. If $q = 0$ then $\bar{x} = \xi$ and by (12), $\theta \equiv \theta_4 = \xi^4$. Hence (5) yields the required property, because

$$f = x_n^4 + 2x_n^2\,\bar{x}^2 + \bar{x}^4 = (x_n^2 + \bar{x}^2)^2 = x^4 \equiv h_{4,V}.$$

If $p = 0$ then $\bar{x} = \eta$ and (12) yields $\theta = \theta_0 = \bar{x}^4$. This again shows that f is primitive:

$$f = x_n^4 - 6x_n^2\,\bar{x}^2 + \bar{x}^4 \equiv h_{4,H},$$

where $H = \mathbb{R}e_1$.

It remains to consider the case $q = 1$. Then $\eta \equiv \eta_1$, so that $\theta_3 = g_1(\xi)\eta_1$, where g_1 is a cubic form in ξ. Furthermore, (15) implies $(\xi^{\mathrm{t}} A_1 \xi) g_1(\xi) \equiv 0$. Since there are no zero divisors in the polynomial ring $\mathbb{R}[\xi_1, \ldots, \xi_p]$, either $\xi^{\mathrm{t}} A_1 \xi$ or $g_1(\xi)$ must be identically zero. If $\xi^{\mathrm{t}} A_1 \xi \equiv 0$ then $A_1 = 0$ and by virtue of (12)

$$\theta_4 = \xi^4, \quad \theta_3 = g(\xi)\eta_1, \quad \theta_2 = -6\xi^2\eta_1^2, \quad \theta_0 = \eta_1^4. \tag{23}$$

By using (10) and the last identity one finds $16\xi^6 + g_1^2 = 16\xi^6$, hence $g_1 \equiv 0$. Therefore, by virtue of (23) and (5)

$$\begin{aligned} f &= x_n^4 + 2x_n^2(\xi^2 - 3\eta_1^2) + \xi^4 - 6\xi^2\eta_1^2 + \eta_1^4 \\ &= \eta_1^4 - 6(x_n^2 + \xi^2)\eta_1^2 + (x_n^2 + \xi^2)^2 \equiv h_{4,H}, \end{aligned}$$

where H is a one-dimensional subspace spanned on the coordinate vector corresponding to η_1.

Now suppose that $A_1 \neq 0$. Then $g_1 \equiv 0$ and (18) implies that $A_1^3 = A_1$. Hence A_1 is a symmetric matrix with eigenvalues ± 1 and 0. Denote by $L = L^+ \oplus L^- \oplus L^0$ the corresponding eigen decomposition of L. Since $A_1 \neq 0$, the subspace $L^+ \oplus L^-$ is nontrivial. We claim that $L^0 = \{0\}$. Indeed, let $\xi = u \oplus v \oplus w$ be the decomposition of an arbitrary $\xi \in L$ according to the eigen decomposition of L. Then $\xi^{\mathrm{t}} A_1 \xi =$

$u^2 - v^2$, and by (12), $\theta_4 = (u^2 + v^2 + w^2)^4 - 2(u^2 - v^2)^2$. Since $\theta_3 = g_1\eta_1 \equiv 0$, we obtain by virtue of (19)

$$0 \equiv |\nabla_\xi \theta_4|^2 - 16\xi^6 = -64(u^2 - v^2)^2 w^2,$$

which implies that w the zero vector. Thus L^0 is trivial and it follows that $A_1^2 = \mathbf{1}$. By virtue of (12), we obtain

$$\begin{aligned}\theta &= \xi^4 - 2(\xi^{\mathrm{t}} A_1 \xi)^2 + 8(\xi^{\mathrm{t}} A_1^2 \xi)\eta_1^2 - 6\xi^2\eta^2 + \eta_1^4 \\ &= (u^2 + v^2)^2 - 2(u^2 - v^2)^2 + 8(u^2 + v^2)\eta_1^2 - 6(u^2 + v^2)\eta_1^2 + \eta_1^4 \\ &= (u^2 + v^2 + \eta_1^2)^2 - 2(u^2 - v^2)^2,\end{aligned}$$

hence

$$\begin{aligned}f &= x_n^4 + 2x_n^2(u^2 + v^2 - 3\eta_1^2) + 8x_n\eta_1(u^2 - v^2) + (u^2 + v^2 + \eta_1^2)^2 - 2(u^2 - v^2)^2 \\ &= (x_n^2 + u^2 + v^2 + \eta_1^2)^2 - 2(u^2 - v^2 - 2x_n\eta_1)^2.\end{aligned}$$

By making an orthogonal change of coordinates $\eta_1 = \frac{1}{\sqrt{2}}(t_1 + t_2)$ and $x_n = \frac{1}{\sqrt{2}}(t_2 - t_1)$, we obtain

$$\begin{aligned}f &= (u^2 + t_1^2 + v^2 + t_2^2)^2 - 2(u^2 + t_1^2 - v^2 - t_1^2)^2 \\ &= -(u^2 + t_1^2)^2 + 6(u^2 + t_1^2)(v^2 + t_1^2) - (v^2 + t_2^2)^2 \equiv -h_{4,H},\end{aligned}$$

where H is spanned on u and t_1. It follows that f is primitive. $\square$

3. Harmonicity of θ_3

In what follows we shall assume that $f \in E_{p,q}$ with $p \geq 1$ and $q \geq 2$. We need the following general fact on the matrix solutions of (18).

Lemma 3.1. *Let $A_i \in \mathbb{R}^{p\times p}$, $1 \leq i \leq q$, be symmetric matrices satisfying $A_\eta^3 = \eta^2 A_\eta$, where $A_\eta = \sum_{i=1}^q \eta_i A_i$. If $q \geq 2$ then*

- (i) *there are nonnegative integers ν and μ, $2\nu + \mu = p$, such that for any $\eta \in \mathbb{R}^q$, $\eta \neq 0$ and for any i, $1 \leq i \leq q$, the matrices $\frac{1}{|\eta|}A_\eta$ and A_i have eigenvalues ± 1 of multiplicity ν and the eigenvalue 0 of multiplicity μ;*
- (ii) *all A_i are trace free;*
- (iii) *$A_i^3 = A_i$ for any i.*
- (iv) *for any two vectors $t, s \in \mathbb{R}^q$ such that $\langle s, t\rangle = 0$,*

$$A_s^2 A_t + A_s A_t A_s + A_t A_s^2 = s^2 A_t.$$

Proof. The assumption $A_\eta^3 = \eta^2 A_\eta$ implies that for any $\eta \neq 0$, the eigenvalues of A_η are $\pm|\eta|$ and 0. Denote by $\nu^\pm(\eta)$ and $\mu(\eta)$ the multiplicities of $\pm|\eta|$ and 0 respectively. For $q \geq 2$ the unit sphere $S = \{\eta \in M : |\eta| = 1\}$ is connected and the traces

$$\operatorname{tr} A_\eta = \nu^+(\eta) - \nu^-(\eta), \qquad \operatorname{tr} A_\eta^2 = \nu^+(\eta) + \nu^-(\eta),$$

as functions defined on the unit sphere S are continuous and integer-valued, thus they must be constants. It follows that $\nu^+(\eta)$ and $\nu^-(\eta)$ are also constants. In particular, $\mu(\eta) = q - \nu^+(\eta) - \nu^-(\eta)$ is a constant.

We notice that

$$\operatorname{tr} A_\eta \equiv \sum_{i=1}^{q} \eta_i \operatorname{tr} A_i = |\eta|(\nu^+ - \nu^-). \tag{24}$$

Since $|\eta|$ is non-linear for $q \geq 2$, we conclude that $\nu^+ = \nu^-$. We denote by ν the common value of $\nu^\pm$. Then (24) yields $\operatorname{tr} A_\eta = 0$, hence $\operatorname{tr} A_i = 0$ for all i. Since $A_i = A_{e_i}$ and $|e_i| = 1$, we conclude that A_i has eigenvalues ± 1 and 0 of multiplicities ν and $\mu = q - 2\nu$ respectively. This proves (i)–(iii). In order to prove (iv), we put $\eta = s + \lambda t$ in (18) and identify the coefficient of λ. The proposition is proved. □

Corollary 3.2. *With any solution $f \in E_{p,q}$, $q \geq 2$, one can associate nonnegative integers ν and $\mu = p - 2\nu$ such that the matrix A_η defined by (11) is similar to the diagonal trace free matrix*

$$A_\eta \sim |\eta| \begin{pmatrix} \mathbf{1}_\nu & & \\ & -\mathbf{1}_\nu & \\ & & \mathbf{0}_\mu \end{pmatrix}, \tag{25}$$

where $\mathbf{1}_\nu$ and $\mathbf{0}_\mu$ stands for the unit matrix and the zero matrix of sizes $\nu \times \nu$ and $\mu \times \mu$ respectively and the elements not shown are all zero.

Remark 3.3. For the sake of convenience, we shall indicate the situation in Corollary 3.2 by writing $f \in E^\nu_{p,q}$.

Lemma 3.4. *If $f \in E^\nu_{p,q}$, $q \geq 2$, $p \geq 1$, then*

$$\Delta_x f = 4(p - 3q + 3)(x_n^2 + \xi^2) + 4(8\nu - 1 + q - 3p)\eta^2 + \Delta_\xi \theta_3. \tag{26}$$

Proof. By virtue of (5), $f = x_n^4 + 2(\xi^2 - 3\eta^2)x_n^2 + 8(\xi^{\mathrm{t}} A_\eta \xi)x_n + \theta$. Applying (ii) in Lemma 3.1 we find for the Laplacian

$$\begin{aligned} \Delta_x f &\equiv (\partial^2_{x_n} + \Delta_\xi + \Delta_\eta) f \\ &= 12x_n^2 + 4(\xi^2 - 3\eta^2) + 4x_n^2(p - 3q) + 16x_n \operatorname{tr} A_\eta + \Delta_\xi \theta + \Delta_\eta \theta \\ &= 4(p - 3q + 3)x_n^2 + 4(\xi^2 - \eta^2) + \Delta_\xi \theta + \Delta_\eta \theta. \end{aligned} \tag{27}$$

By (25), $\operatorname{tr} A_\eta^2 = 2\nu\eta^2$, hence we find from (12)

$$\Delta_\xi \theta = 4(p + 2)\xi^2 + (32\nu - 12p)\eta^2 - 16 \sum_{i=1}^{q} \xi^{\mathrm{t}} A_i^2 \xi.$$

Similarly we find $\Delta_\eta \theta = \Delta_\eta \theta_3 + 16 \sum_{i=1}^{q} \xi^{\mathrm{t}} A_i^2 \xi - 12q\xi^2 + 4(q+2)\eta^2$, and combining these formulas with (27) yields (26). □

Now we are ready to proof the main result of this section.

Proposition 3.5. *If $f \in E^\nu_{p,q}$, $q \geq 2$, $p \geq 1$, then $\Delta_\eta \theta_3 = 0$.*

Proof. Write $\theta_3 = 8 \sum_{i=1}^{q} g_i(\xi)\eta_i$, where g_i are cubic forms in ξ. It suffices to show that $\Delta_\xi g_1 \equiv 0$. To this end, let us consider the eigen decomposition

$L = L_1^+ \oplus L_1^- \oplus L_1^0$ associated with A_1 according Lemma 3.1, and decompose the cubic form g_1 into the corresponding homogeneous parts:

$$g_1 = \sum_s G_s, \qquad G_s \in u^{s_1} \otimes v^{s_2} \otimes w^{s_3}, \tag{28}$$

where $s = (s_1, s_2, s_3)$, $s_1 + s_2 + s_3 = 3$, and $\xi = u \oplus v \oplus w$, $u \in L^+$, $v \in L^-$, $w \in L^0$.

By (16), $\langle A_1\xi, \nabla_\xi g_1\rangle = 0$. We have $A_1\xi = u - v$, so that applying (28) and the homogeneity of each G_s, we obtain

$$0 = \sum_s \langle \nabla_\xi G_s, A_1\xi\rangle = \sum_s (\langle \nabla_u G_s, u\rangle - \langle \nabla_v G_s, v\rangle) = \sum_s (s_1 - s_2)G_s.$$

Since non-zero components G_s are linear independent, we have $s_1 = s_2$. This yields

$$s \in \{(1,1,1),\ (0,0,3)\} \tag{29}$$

On the other hand, we infer from (22) by virtue of (14) and (12) that

$$\langle \nabla_\xi \theta_3, A_\eta^2\xi\rangle = 2\eta^2\theta_3.$$

By identifying the coefficients of η_1^3, $\langle \nabla_\xi g_1, A_1^2\xi\rangle = 2g_1$, thus, by virtue of $A_1^2\xi = u + v$,

$$2g_1 = \langle \nabla_\xi g_1, A_1^2\xi\rangle \equiv \sum_s \langle A_1^2\xi, \nabla_\xi G_s\rangle = \sum_s (s_1 + s_2)G_s.$$

Comparing with (28) implies $s_1 + s_2 = 2$, hence by (29) we obtain $s = (1,1,1)$, i.e. $g_1 \in u \otimes v \otimes w$ is a trilinear form. It follows that $\Delta_\xi g_1 = 0$. The theorem is proved. $\square$

Corollary 3.6. *If $f \in E^\nu_{p,q}$, $q \geq 2$, $p \geq 1$, then*

$$\Delta_x f = 4(p - 3q + 3)(x_n^2 + \xi^2) + 4(8\nu - 1 + q - 3p)\eta^2. \tag{30}$$

4. Proof of Theorem 1.3

Suppose that f is a non-primitive eikonal quartic. Then by Proposition 2.2, $f \in E^\nu_{p,q}$ with $p \geq 1$ and $q \geq 2$. Applying (14) to (21), one obtains the following identity:

$$\begin{aligned}\frac{1}{64}|\nabla_\xi\theta_3|^2 &= 4\xi^2(\xi^{\mathrm{t}} A_\eta^2\xi) + 4\xi^{\mathrm{t}} A_\tau A_\eta^2\xi - 3\eta^2(\xi^{\mathrm{t}} A_\tau\xi) - (\xi^{\mathrm{t}} A_\eta\xi)^2\\ &\quad - 4\sum_{i=1}^q (\xi^{\mathrm{t}} A_i A_\eta\xi)^2.\end{aligned} \tag{31}$$

By Proposition 3.5, $\Delta_\xi\theta_3^2 = 2|\nabla_\xi\theta_3|^2$, hence (31) implies

$$\begin{aligned}\frac{1}{128}\Delta_\eta\Delta_\xi\theta_3^2 &= 8\xi^2(\xi^{\mathrm{t}} B\xi) + 8\xi^{\mathrm{t}} A_\tau B\xi - 6q(\xi^{\mathrm{t}} A_\tau\xi)\\ &\quad - \Delta_\eta(\xi^{\mathrm{t}} A_\eta\xi)^2 - 4\Delta_\eta\sum_{i=1}^q (\xi^{\mathrm{t}} A_i A_\eta\xi)^2,\end{aligned} \tag{32}$$

where $B := \sum_{i=1}^q A_i^2$. We have

$$\Delta_\eta(\xi^{\mathrm{t}} A_\eta\xi)^2 = 2\sum_{i=1}^q (\xi^{\mathrm{t}} A_i\xi)^2 = 2\sum_{i=1}^q \tau_i^2 \equiv 2\tau^2,$$

and similarly

$$\Delta_\eta \sum_{i=1}^{q} (\xi^{\mathrm{t}} A_i A_\eta \xi)^2 = 2 \sum_{i,j=1}^{q} (\xi^{\mathrm{t}} A_i A_j \xi)^2.$$

Taking into account that $\xi^{\mathrm{t}} A_\tau \xi = \tau^2$, we get from (32)

$$\frac{1}{128}\Delta_\eta \Delta_\xi \theta_3^2 = 8\xi^2 \cdot \xi^{\mathrm{t}} B\xi + 8\xi^{\mathrm{t}} A_\tau B\xi - 8 \sum_{i,j=1}^{q} (\xi^{\mathrm{t}} A_i A_j \xi)^2 - (6q+2)\tau^2. \tag{33}$$

On the other hand, $\Delta_\eta \theta_3 = 0$ because θ_3 is linear in η, hence

$$\frac{1}{128}\Delta_\eta \Delta_\xi \theta_3^2 = \frac{1}{128}\Delta_\xi \Delta_\eta \theta_3^2 = \frac{1}{64}\Delta_\xi |\nabla_\eta \theta_3|^2. \tag{34}$$

Applying (19) and (14), we find

$$|\nabla_\eta \theta_3|^2 = 16\xi^6 - |\nabla_\xi \theta_4|^2 = 64(\xi^2 \cdot \xi^{\mathrm{t}} A_\tau \xi - \xi^{\mathrm{t}} A_\tau^2 \xi) = 64(\xi^2 \tau^2 - \xi^{\mathrm{t}} A_\tau^2 \xi), \tag{35}$$

where $\tau_i = \xi^{\mathrm{t}} A_i \xi$.

Our next step is the ξ-Laplacian of the right hand side of (35). We have $\nabla_\xi \tau_i = 2A_i\xi$ and by Lemma 3.2, $\Delta_\xi \tau_i = 2\,\mathrm{tr}\, A_i = 0$. Therefore

$$\Delta_\xi \tau^2 = 2\sum_{i=1}^{q} |\nabla_\xi \tau_i|^2 = 8\sum_{i=1}^{q} \xi^{\mathrm{t}} A_i^2 \xi \equiv 8\xi^{\mathrm{t}} B\xi,$$

and $\Delta_\xi(\xi^2\tau^2) = (2p+16)\tau^2 + 8\xi^2(\xi^{\mathrm{t}} B\xi)$. Furthermore,

$$\xi^{\mathrm{t}} A_\tau^2 \xi = \sum_{i,j=1}^{q} \rho_{ij}\tau_i\tau_j, \qquad \rho_{ij} = \xi^{\mathrm{t}} A_i A_j \xi.$$

We find $\nabla_\xi \rho_{ij} = (A_iA_j + A_jA_i)\xi$ and $\Delta_\xi \rho_{ij} = 2\,\mathrm{tr}\, A_iA_j$, hence

$$\begin{aligned}\Delta_\xi(\xi^{\mathrm{t}} A_\tau^2 \xi) &= \sum_{i,j=1}^{q} \tau_i\tau_j \Delta_\xi \rho_{ij} + 4\tau_i \langle \nabla_\xi \rho_{ij}, \nabla_\xi \tau_j\rangle + 2\rho_{ij}\langle \nabla_\xi \tau_i, \nabla_\xi \tau_j\rangle \\ &= 2\,\mathrm{tr}\, A_\tau^2 + 8\sum_{i,j=1}^{q} \xi^{\mathrm{t}} A_j(A_iA_j + A_jA_i)\xi\, \tau_i + 8\sum_{i,j=1}^{q} (\xi^{\mathrm{t}} A_iA_j\xi)^2.\end{aligned}$$

By (iii) and (iv) in Lemma 3.1, $A_j(A_iA_j + A_jA_i) = (1+2\delta_{ij})A_i - A_iA_j^2$, where δ_{ij} is the Kronecker delta, hence

$$\sum_{i,j=1}^{q} \xi^{\mathrm{t}} A_j(A_iA_j + A_jA_i)\xi\, \tau_i = \xi^{\mathrm{t}}(qA_\tau + 2A_\tau - A_\tau B)\xi = (q+2)\tau^2 - \xi^{\mathrm{t}} A_\tau B\xi.$$

Since $\mathrm{tr}\, A_\tau^2 = 2\nu\tau^2$, we obtain

$$\Delta_\xi(\xi^{\mathrm{t}} A_\tau^2 \xi) = (4\nu + 8q + 8)\tau^2 - 8\xi^{\mathrm{t}} A_\tau B\xi + 8\sum_{i,j=1}^{q} (\xi^{\mathrm{t}} A_iA_j\xi)^2.$$

On substituting the found relations into (35), we find

$$\frac{1}{64}\Delta_\xi |\nabla_\eta \theta_3|^2 = 8\xi^2(\xi^{\mathrm{t}} B\xi) + 8\xi^{\mathrm{t}} A_\tau B\xi - 8\sum_{i,j=1}^{q} (\xi^{\mathrm{t}} A_iA_j\xi)^2 + 2(p - 4q - 2\nu)\tau^2,$$

and on combining the last relation with (33) and (34), we arrive at

$$(p + 1 - q - 2\nu)\tau^2 = 0. \tag{36}$$

Suppose that $\tau \equiv 0$. Then $A_i = 0$ for all $i = 1, \dots, q$. It follows from (31) that $\theta_3 \equiv 0$ and on applying (12), $\theta \equiv \theta_4 + \theta_2 + \theta_0 = \xi^4 - 6\xi^2\eta^2 + \eta^4$. This yields

$$f = x_n^4 + 2x_n^2(\xi^2 - 3\eta^2) + \xi^4 - 6\xi^2\eta^2 + \eta^4 = (x_n^2 + \xi^2)^2 - 6\eta^2(x_n^2 + \xi^2) + \eta^4 \equiv h_{4,H},$$

where H is spanned on x_n and ξ, hence f is primitive eikonal quartic, a contradiction. Thus, $\tau \not\equiv 0$ and (36) implies $2\nu = p + 1 - q$. On substituting this into (30), one finds

$$\Delta_x f = 4(p - 3q + 3)(x_n^2 + \xi^2 + \eta^2) = 4(p - 3q + 3)x^2 = 8(\nu - q + 1)x^2.$$

In view of $n = 2\nu + 2q$ one finds by comparing with (1) that $m_1 = q - 1$ and $m_2 = \nu$ are integers. It follows that f is an isoparametric polynomial. The theorem is proved completely.

5. Concluding remarks

After this paper was finished, the author was informed by Professor Tang Zizhou that about the paper [**24**]. In this paper, Q.M. Wang proves that any smooth solution f of the equation $|\overline{\nabla} f|^2 = b(f)$ on $M = \mathbb{R}^n$ and $M = \mathbb{S}^n$ has isoparametric fibration, i.e. must have only the level sets which are isoparametric on M ($\overline{\nabla}$ denotes the covariant derivative on M), see also a recent preprint of R. Miyaoka [**11**]. The Wang theorem clarify, to some content, the appearance and algebraic structure of the primitive eiconal polynomials $h_{g,H}$ in (4) above. Indeed, any solution of the eiconal equation $|\nabla_x f|^2 = k^2 x^{2k-2}$ in $\mathbb{R}^n$ induces (by the restriction) a transnormal function $F := f|_{\mathbb{S}^{n-1}}$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ satisfying the transnormal condition $|\overline{\nabla} F|^2 = k^2(1 - F^2)$. Thus, according to the Wang theorem, the level sets $F = c$ are isoparametric submanifolds in $\mathbb{S}^{n-1}$, which agrees with the conclusion of our Theorem 1.3 because the level set $h_{g,H} = \cos t$, $t \in \mathbb{R}$ splits into standard products of two spheres $\mathbb{S}^p(\cos \frac{t+2\pi m}{g}) \times \mathbb{S}^q(\sin \frac{t+2\pi m}{g})$, $p = \dim H - 1$, $q = n - \dim H - 1$, and $m = 0, 1, \dots, g-1$. The latter tori are well known to be isoparametric hypersurfaces in $\mathbb{S}^{n-1}$ with 2 distinct principal curvatures, see for instance [**5**].

We would like to thank Professor Tang Zizhou for many helpful suggestions, and, especially, for bringing to our attention the Wang paper [**24**] and also the papers [**9**], [**11**], [**15**], [**19**], [**20**] where related questions are treated.

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