THE TAMBARA-YAMAGAMI CATEGORIES AND 3-MANIFOLD INVARIANTS

VLADIMIR TURAEV AND LEONID VAINERMAN

Abstract. We prove that if two Tambara-Yamagami categories $\mathcal{TY}(A,\chi,\nu)$ and $\mathcal{TY}(A',\chi',\nu')$ give rise to the same state sum invariants of 3-manifolds and the order of one of the groups A, A' is odd, then $\nu = \nu'$ and there is a group isomorphism $A \approx A'$ carrying χ to χ'. The proof is based on an explicit computation of the state sum invariants for the lens spaces of type $(k,1)$.

Introduction

It is known that every spherical fusion category $\mathcal{C}$ of non-zero dimension gives rise to a scalar topological invariant $|M|_\mathcal{C}$ of any closed oriented 3-manifold M, see [11], [1]. The definition of $|M|_\mathcal{C}$ proceeds by considering a certain state sum on a triangulation of M and proving that this sum depends only on the topological type of M. The key algebraic ingredients of the state sum are the so-called $6j$-symbols associated with $\mathcal{C}$. This gives a pairing $(M,\mathcal{C}) \mapsto |M|_\mathcal{C}$ between 3-manifolds and spherical fusion categories. A study of this pairing leads to natural questions. Are these invariants sufficient to distinguish 3-manifolds up to homeomorphism? To what extent a spherical fusion category is determined by the associated 3-manifold invariants? We study here the latter question for the class of spherical fusion categories introduced by Tambara and Yamagami [9].

A Tambara-Yamagami category $\mathcal{TY}(A,\chi,\nu)$ is determined by a bicharacter χ on a finite abelian group A and a sign $\nu = \pm 1$. By a *bicharacter* on A we mean a non-degenerate symmetric bilinear pairing $\chi : A \times A \longrightarrow S^1$; the non-degeneracy of χ means that the adjoint homomorphism $A \to \mathrm{Hom}(A, S^1)$ is bijective. The pair (A,χ) will be called a *bicharacter pair*.

Two bicharacter pairs (A,χ) and (A',χ') are said to be isomorphic if there is an isomorphism $A \cong A'$ transforming χ into χ'. It is known that two Tambara-Yamagami categories, $\mathcal{TY}(A,\chi,\nu)$ and $\mathcal{TY}(A',\chi',\nu')$, are monoidally equivalent if and only if the pairs (A,χ) and (A',χ') are isomorphic and $\nu = \nu'$.

Each bicharacter pair (A,χ) splits uniquely as an orthogonal sum

$$(A,\chi) = \bigoplus_p (A^{(p)}, \chi^{(p)}),$$

where p runs over all prime natural numbers, $A^{(p)} \subset A$ is the abelian p-group consisting of the elements of A annihilated by a sufficiently big power of p, and $\chi^{(p)} : A^{(p)} \times A^{(p)} \longrightarrow S^1$ is the restriction of χ to $A^{(p)}$. In the sequel, the order of a group A is denoted $|A|$.

Theorem 0.1. *Let $\mathcal{C} = \mathcal{TY}(A,\chi,\nu)$ and $\mathcal{C}' = \mathcal{TY}(A',\chi',\nu')$ be two Tambara-Yamagami categories such that $|M|_\mathcal{C} = |M|_{\mathcal{C}'}$ for all closed oriented 3-manifolds M.*

(a) We have $|A| = |A'|$. If $|A|$ is not a positive power of 4, then $\nu = \nu'$.

(b) For every odd prime p, the pairs $(A^{(p)},\chi^{(p)})$ and $(A'^{(p)},\chi'^{(p)})$ are isomorphic.

Date: September 10, 2010.

2000 *Mathematics Subject Classification.* 57M27,16W30,18C20.

Combining the claims (a) and (b) we obtain the following corollary.

Corollary 0.2. *Let $\mathcal{C} = \mathcal{TY}(A,\chi,\nu)$ and $\mathcal{C}' = \mathcal{TY}(A',\chi',\nu')$ be two Tambara-Yamagami categories such that $|M|_{\mathcal{C}} = |M|_{\mathcal{C}'}$ for all closed oriented 3-manifolds M. If $|A|$ is odd, then the bicharacter pairs (A,χ) and (A',χ') are isomorphic and $\nu = \nu'$.*

We conjecture a similar claim in the case where $|A|$ is even.

The proof of Theorem 0.1 is based on an explicit computation of $|M|_{\mathcal{C}}$ for the lens spaces $L_k = L_{k,1}$ with $k = 0, 1, 2, \dots$ Recall that L_k is the closed oriented 3-manifold obtained from the 3-sphere S^3 by surgery along a trivial knot in S^3 with framing k. In particular, $L_0 = S^1 \times S^2$, $L_1 = S^3$, and $L_2 = \mathbb{R}P^3$. The manifolds $\{L_k\}_k$ are pairwise non-homeomorphic; they are distinguished by the fundamental group $\pi_1(L_k) = \mathbb{Z}/k\mathbb{Z}$.

To formulate our computation of $|L_k|_{\mathcal{C}}$, recall the notion of a Gauss sum. Let A be a finite abelian group and $\chi : A \times A \longrightarrow S^1$ be a symmetric bilinear form (possibly degenerate). A *quadratic map* associated with χ is a map $\mu : A \to S^1$ such that for all $a, b \in A$,

$$\mu(a+b) = \chi(a,b)\, \mu(a)\, \mu(b).$$

In other words, the coboundary of μ viewed as a 1-cochain on A is equal to χ. Such a μ always exists (see, for example, [5]) and determines the normalized *Gauss sum*

$$\gamma(\mu) = |A|^{-1/2} |A_\chi^\perp|^{-1/2} \sum_{a \in A} \mu(a) \in \mathbb{C},$$

where

$$A_\chi^\perp = \{a \in A \,|\, \chi(a,b) = 1 \text{ for all } b \in A\}$$

is the annihilator of χ. (If χ is a bicharacter, then $A_\chi^\perp = \{0\}$.) The normalization is chosen so that either $\gamma(\mu) = 0$ or $|\gamma(\mu)| = 1$ (see Lemma 2.1 below).

Denote by Q_χ the set of quadratic maps associated with χ. This set has $|A|$ elements; this follows from the fact that any two quadratic maps associated with χ differ by a homomorphism $A \to S^1$. Every integer $k \geq 0$ determines a subgroup $A_k = \{a \in A \,|\, ka = 0\}$ of A and a number

$$\zeta_k(\chi) = |A|^{-1/2} |A_k|^{-1/2} \sum_{\mu \in Q_\chi} \gamma(\mu)^k \in \mathbb{C}.$$

For example, $A_0 = A$ and $\zeta_0(\chi) = 1$.

Theorem 0.3. *Let $\mathcal{C} = \mathcal{TY}(A,\chi,\nu)$ be a Tambara–Yamagami category. For any odd integer $k \geq 1$, we have*

$$|L_k|_{\mathcal{C}} = \frac{|A_k|}{2|A|}. \tag{1}$$

For any even integer $k \geq 0$, we have

$$|L_k|_{\mathcal{C}} = \frac{|A_k| + \nu^{k/2} |A|^{1/2} |A_{k/2}|^{1/2} \zeta_{k/2}(\chi)}{2|A|}. \tag{2}$$

For $k = 0$, Formula (2) gives $|S^1 \times S^2|_{\mathcal{C}} = 1$ which is known to be true for all spherical fusion categories $\mathcal{C}$.

Our proof of Theorem 0.3 is based on two results. The first is the equality $|M|_{\mathcal{C}} = \tau_{\mathcal{Z(C)}}(M)$ recently established in [12]. Here $\mathcal{C}$ is an arbitrary spherical fusion category of non-zero dimension, $\mathcal{Z}(\mathcal{C})$ is the Drinfeld-Joyal-Street center of $\mathcal{C}$, and $\tau_{\mathcal{Z(C)}}(M)$ is the Reshetikhin-Turaev invariant of M. The second result is the computation of the center of $\mathcal{C} = \mathcal{TY}(A,\chi,\nu)$ in [4].

The paper is organized as follows. In Section 1 we recall the Tambara-Yamagami category and its center and prove Theorem 0.3. In Sections 2 and 3 we prove respectively claims (a) and (b) of Theorem 0.1.

This paper was started during the visit of VT to the University of Caen in June 2010. VT would like to thank the University of Caen for hospitality. The work of V. Turaev was partially supported by the NSF grant DMS-0904262.

1. The Tambara-Yamagami categories and their centers

In this section, (A,χ) is a bicharacter pair, $\nu=\pm 1$, and $n=|A|$.

1.1. **The category $\mathcal{TY}(A,\chi,\nu)$.** The simple objects of the Tambara-Yamagami category $\mathcal{C}=\mathcal{TY}(A,\chi,\nu)$ are all elements a of A and an additional object m. All other objects of $\mathcal{C}$ are finite direct sums of the simple objects. The tensor product in $\mathcal{C}$ is determined by the following fusion rules:

$$a\otimes b=a+b \text{ and } a\otimes m=m\otimes a=m \text{ for all } a,b\in A, \text{ and } m\otimes m=\bigoplus_{a\in A} a.$$

This tensor product is associative. For any simple objects U,V,W of $\mathcal{C}$, the associativity isomorphism $\phi_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W)$ is given by the following formulas (where a,b,c run over A):

$$\begin{aligned}
&\phi_{a,b,c}=id_{a+b+c}, && \phi_{a,b,m}=id_m, && \phi_{m,a,b}=id_m,\\
&\phi_{a,m,b}=\chi(a,b)id_m, && \phi_{a,m,m}=\bigoplus_{b\in A} id_b, && \phi_{m,m,a}=\bigoplus_{b\in A} id_b,\\
&\phi_{m,a,m}=\bigoplus_{b\in A}\chi(a,b)id_b, && \phi_{m,m,m}=(\nu n^{-1/2}\chi(a,b)^{-1}id_m)_{a,b}.
\end{aligned}$$

The unit object of $\mathcal{C}$ is the zero element $0\in A$ and the unit isomorphisms are the identity maps. The duality in $\mathcal{C}$ is defined by $a^*=-a$ for all $a\in A$ and $m^*=m$. The left duality morphisms in $\mathcal{C}$ are the identity maps $0\to a\otimes a^*,\ a^*\otimes a\to 0$ for $a\in A$, the inclusion $0\hookrightarrow m\otimes m$ and $\nu n^{1/2}$ times the obvious projection $m\otimes m\to 0$. The right duality morphisms in $\mathcal{C}$ are the identity maps $0\to a^*\otimes a,\ a\otimes a^*\to 0$ for $a\in A$, ν times the inclusion $0\hookrightarrow m\otimes m$ and $n^{1/2}$ times the obvious projection $m\otimes m\to 0$.

We define a fusion category (over $\mathbb{C}$) as a $\mathbb{C}$-linear monoidal category with compatible left and right dualities such that all objects are direct sums of simple objects, the number of isomorphism classes of simple objects is finite, and the unit object is simple. (An object V is simple if $\mathrm{End}(V)=\mathbb{C}\,id_V$.) The condition of sphericity says that the left and right dimensions of all objects are equal. A spherical fusion category has a numerical dimension defined as the sum of the squares of the dimensions of the (isomorphism classes of) simple objects. A basic reference on the theory of fusion categories is [3]. It is easy to see that $\mathcal{C}=\mathcal{TY}(A,\chi,\nu)$ is a spherical fusion category of dimension $2n$.

1.2. **The center.** The center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}=\mathcal{TY}(A,\chi,\nu)$ was computed in [4], Prop. 4.1. The category $\mathcal{Z}(\mathcal{C})$ has three types of simple objects whose description together with the corresponding quantum dimensions and twists is as follows:

(1) $2n$ invertible objects $X(a,\varepsilon)$, where a runs over A and ε runs over complex square roots of $\chi(a,a)^{-1}$. Here $\dim(X(a,\varepsilon))=1$ and $\theta_{X(a,\varepsilon)}=\chi(a,a)^{-1}$;

(2) $\frac{n(n-1)}{2}$ objects $Y(a,b)$ parameterized by unordered pairs (a,b), where a,b are distinct elements of A. Here $\dim(Y(a,b))=2$ and $\theta_{Y(a,b)}=\chi(a,b)^{-1}$;

(3) $2n$ objects $Z(\mu,\Delta)$, where μ runs over Q_χ and Δ runs over the square roots of $\nu\gamma(\mu)$. Here $\dim(Z(\mu,\Delta))=n^{1/2}$ and $\theta_{Z(\mu,\Delta)}=\Delta$.

Denote by I the set of the (isomorphism classes of) simple objects of $\mathcal{Z}(\mathcal{C})$. Then

$$\dim \mathcal{Z}(\mathcal{C}) = \sum_{i\in I} (\dim(i))^2 = 2n \times 1 + \frac{n(n-1)}{2} \times 4 + 2n \times n = 4n^2.$$

We will need the following more general computation.

Lemma 1.1. *For an integer $k \geq 0$, set $\tau_k = \sum_{i\in I} \theta_i^k (\dim(i))^2$, where θ_i and $\dim(i)$ are the twist and the dimension of $i \in I$. If k is odd, then $\tau_k = 2n|A_k|$. If k is even, then $\tau_k = 2n(|A_k| + \nu^{k/2}|A|^{1/2}|A_{k/2}|^{1/2}\zeta_{k/2}(\chi))$.*

Proof. A direct computation shows that $\tau_k = 2u_k + nv_k$, where

$$u_k = \sum_{a\in A} \chi(a,a)^{-k} + \sum_{(a,b)\in A^2, a\neq b} \chi(a,b)^{-k}$$

and $v_k = \sum_{(\mu,\Delta)} \Delta^k$. Since χ is non-degenerate,

$$u_k = \sum_{a,b\in A} \chi(a,b)^{-k} = \sum_{a,b\in A} \chi(a, b^{-k}) = n\,|A_k|.$$

If k is odd, then the contributions of the pairs (μ, Δ) and $(\mu, -\Delta)$ to v_k cancel each other so that $v_k = 0$ and $\tau_k = 2n|A_k|$. For even k,

$$v_k = \sum_{\mu} 2(\nu\gamma(\mu))^{k/2} = 2\nu^{k/2}|A|^{1/2}|A_{k/2}|^{1/2}\zeta_{k/2}(\chi).$$

□

1.3. **Proof of Theorem 0.3.** Since $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ is a spherical fusion category of non-zero dimension, it determines for any closed oriented 3-manifold M a state sum invariant $|M|_{\mathcal{C}} \in \mathbb{C}$, see [11], [1]. By a theorem of Müger [6], the category $\mathcal{Z}(\mathcal{C})$ is modular in the sense of [10]. A modular category endowed with a square root $\mathcal{D}$ of its dimension gives rise to the Reshetikhin-Turaev invariant of any M as above. Let $\tau_{\mathcal{Z}(\mathcal{C})}(M)$ be the RT-invariant of M determined by $\mathcal{Z}(\mathcal{C})$ and the square root $\mathcal{D} = 2n = \tau_1$ of $\dim \mathcal{Z}(\mathcal{C})$. A theorem of Virelizier and Turaev [12] implies that $|M|_{\mathcal{C}} = \tau_{\mathcal{Z}(\mathcal{C})}(M)$ for all M. By [10], Chapter II, 2.2, for all $k \geq 0$,

$$\tau_{\mathcal{Z}(\mathcal{C})}(L_k) = \mathcal{D}^{-2} \sum_{i\in I} \theta_i^k (\dim(i))^2 = 4n^{-2}\tau_k.$$

Substituting the expression for τ_k provided by Lemma 1.1, we obtain the claim of the theorem.

2. Proof of Theorem 0.1(a)

We start with a well known lemma. In this lemma we call a quadratic map $\mu : A \to S^1$ *homogeneous* if $\mu(na) = (\mu(a))^{n^2}$ for all $n \in \mathbb{Z}$ and $a \in A$.

Lemma 2.1. *Let A be a finite abelian group and $\mu : A \to S^1$ be a quadratic map associated with a symmetric bilinear form $\chi : A \times A \to S^1$. Set $A^\perp = A^\perp_\chi \subset A$.*

- *If $\mu(A^\perp) \neq 1$, then $\gamma(\mu) = 0$.*
- *If $\mu(A^\perp) = 1$, then $|\gamma(\mu)| = 1$.*
- *If $\mu(A^\perp) = 1$ and μ is homogeneous, then $\gamma(\mu)$ is an 8-th complex root of unity.*

Proof. We have

$$|A|\,|A^\perp|\,|\gamma(\mu)|^2 = |\sum_{a\in A} \mu(a)|^2 = \sum_{a,b\in A} \mu(a)\overline{\mu(b)} = \sum_{a,b\in A} \mu(a)\mu(b)^{-1}$$

$$= \sum_{a,b\in A} \mu(a+b)\mu(b)^{-1} = \sum_{a,b\in A} \chi(a,b)\mu(a).$$

When b runs over A, the complex number $\chi(a,b)$ runs over a finite subgroup of S^1. We have $\sum_{b\in A}\chi(a,b)=0$ unless this subgroup is trivial. The latter holds if and only if $a\in A^\perp$ and in this case $\sum_{b\in A}\chi(a,b)=|A|$. Therefore,

$$|A|\,|A^\perp|\,|\gamma(\mu)|^2=|A|\sum_{a\in A^\perp}\mu(a).$$

The restriction of μ to $A^\perp$ is a group homomorphism $A^\perp\to S^1$. If $\mu(A^\perp)\neq 1$, then $\sum_{a\in A^\perp}\mu(a)=0$ and therefore $\gamma(\mu)=0$. Suppose now that $\mu(A^\perp)=1$. Then $\sum_{a\in A^\perp}\mu(a)=|A^\perp|$ and therefore $|\gamma(\mu)|=1$. The equality $\mu(A^\perp)=1$ also ensures that μ is the composition of the projection $A\to A'=A/A^\perp$ with a quadratic map $\mu':A'\to S^1$ associated with the non-degenerate symmetric bilinear form $A'\times A'\to S^1$ induced by χ. It follows from the definitions that $\gamma(\mu)=\gamma(\mu')$. If μ is homogeneous, then so is μ'. It is known (see, for instance, [7], Chapter 5, Section 2) that for any homogeneous quadratic map on a finite abelian group associated with a non-degenerate symmetric bilinear form, the corresponding invariant γ is an 8-th root of unity. This implies the last claim of the lemma. □

Lemma 2.2. *For any bicharacter pair (A,χ) and any integer $k\geq 1$, either $\zeta_k(\chi)=0$ or $\zeta_k(\chi)$ is an 8-th root of unity. If $k=1$ or k is divisible by $8|A|$, then $\zeta_k(\chi)=1$.*

Proof. Pick a quadratic map $\mu_0:A\to S^1$ associated with χ. Observe that for every integer k, the function $\mu_0^k:A\to S^1$ carrying any $a\in A$ to $(\mu_0(a))^k$ is a quadratic map associated with the symmetric bilinear form $\chi^k:A\times A\to S^1$ defined by $\chi^k(a,b)=(\chi(a,b))^k$. We claim that for all $k\in\mathbb{Z}$,

$$\zeta_k(\chi)=\gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k. \tag{3}$$

Indeed, since χ is non-degenerate, any quadratic map $\mu:A\to S^1$ associated with χ can be expanded in the form $\mu(a)=\chi(a,c)\,\mu_0(a)$ for a unique $c=c(\mu)\in A$. Since $\chi(a,c)\,\mu_0(a)=\mu_0(a+c)\,\mu_0(c)^{-1}$ for all $a,c\in A$, we have

$$\begin{aligned}\zeta_k(\chi)&=|A|^{-1/2}|A_k|^{-1/2}\sum_{\mu\in Q_\chi}(|A|^{-1/2}\sum_{a\in A}\mu(a))^k\\&=|A|^{-1/2}|A_k|^{-1/2}\sum_{c\in A}(|A|^{-1/2}\sum_{a\in A}\chi(a,c)\,\mu_0(a))^k\\&=\{|A|^{-1/2}|A_k|^{-1/2}\sum_{c\in A}\mu_0(c)^{-k}\}\{|A|^{-1/2}\sum_{b\in A}\mu_0(b)\}^k\\&=\gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k.\end{aligned}$$

In the last equality we use the obvious fact that $A^\perp_{\chi^{-k}}=A_k$.

We can always choose $\mu_0:A\to S^1$ to be homogeneous. Then μ_0^{-k} also is homogeneous. Since χ is non-degenerate, the previous lemma implies that $\gamma(\mu_0)$ is an 8-th root of unity and $\gamma(\mu_0^{-k})$ is either zero or an 8-th root of unity. This implies the first claim of the lemma.

For $k=1$, Formula (3) gives

$$\zeta_1(\chi)=\gamma(\mu_0^{-1})\,\gamma(\mu_0)=\gamma(\overline{\mu_0})\,\gamma(\mu_0)=\overline{\gamma(\mu_0)}\,\gamma(\mu_0)=1,$$

where the overbar is the complex conjugation.

Observe that $\mu_0^{2n}=1$ for $n=|A|$. Indeed, for any $a\in A$,

$$\begin{aligned}1=\mu_0(0)=\mu_0(2na)&=(\mu_0(a))^{2n}\chi(a,a)^{n(n-1)}\\&=(\mu_0(a))^{2n}\chi(na,(n-1)a)=(\mu_0(a))^{2n}.\end{aligned}$$

Therefore for all $k\in 2n\mathbb{Z}$, we have $\gamma(\mu_0^{-k})=1$. If $k\in 8\mathbb{Z}$, then $(\gamma(\mu_0))^k=1$. Hence, if $k\in 8n\mathbb{Z}$, then $\zeta_k(\chi)=\gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k=1$. □

2.1. **Proof of Theorem 0.1(a).** For $k=1$, Formula (1) gives $|L_1|_{\mathcal{C}} = (2|A|)^{-1}$. Thus,

$$|A| = |L_1|_{\mathcal{C}}^{-1}/2 = |L_1|_{\mathcal{C}'}^{-1}/2 = |A'|.$$

This and Formula (1) implies that $|A_k| = |A'_k|$ for all odd $k \geq 1$.

Set $n = |A| = |A'|$. Suppose that $\nu \neq \nu'$. Assume for concreteness that $\nu = -1$ and $\nu' = +1$. Formula (2) with $k=2$ and Lemma 2.2 show that

$$|A_2| - n^{1/2} = 2n|L_2|_{\mathcal{C}} = 2n|L_2|_{\mathcal{C}'} = |A'_2| + n^{1/2}.$$

Thus, $|A_2| - |A'_2| = 2n^{1/2}$. Therefore, $n = m^2$ for an integer $m \geq 1$. Since n is not a positive power of 4, either $m = 1$ or m is not a power of 2. If $m = 1$, then $A = A' = \{0\}$ and so $A_2 = A'_2 = \{0\}$ which contradicts the equality $|A_2| - |A'_2| = 2m$.

Suppose that $m = n^{1/2}$ is not a power of 2. Pick an odd divisor $\ell \geq 3$ of m. Applying Formula (2) to $k = 2\ell$, we obtain that

$$|A_k| - m|A_\ell|^{1/2}\zeta_\ell(\chi) = |A'_k| + m|A'_\ell|^{1/2}\zeta_\ell(\chi').$$

Note that $|A_k| = |A_2|\,|A_\ell|$ and similarly for A'. Since ℓ is odd, we have $|A_\ell| = |A'_\ell|$. Therefore

$$|A_2| - |A'_2| = m|A_\ell|^{-1/2}(\zeta_\ell(\chi') + \zeta_\ell(\chi)).$$

The right-hand side of this equality must be a real number that cannot exceed $2m|A_\ell|^{-1/2}$ by Lemma 2.2. Thus, $|A_2| - |A'_2| \leq 2m|A_\ell|^{-1/2}$. Since ℓ divides n, we have $A_\ell \neq 1$ so that $|A_\ell| \geq 2$. This gives $|A_2| - |A'_2| \leq 2m/\sqrt{2}$ which contradicts the equality $|A_2| - |A'_2| = 2m$. This contradiction shows that $\nu = \nu'$.

2.2. **Remarks.** (i) It is easy to extend the argument above to show that the conclusion of Theorem 0.1(a) holds also for $|A| = 4$.

(ii) Let in the proof above $|A| = |A'| = n$ be a positive power of 2 and $\nu = -1, \nu' = 1$. Formula (2) with $k = 2\ell$, where $\ell \geq 3$ is odd, shows that

$$|A_{2\ell}| - n^{1/2}|A_\ell|^{1/2}\zeta_\ell(\chi) = 2n|L_{2\ell}|_{\mathcal{C}} = 2n|L_{2\ell}|_{\mathcal{C}'} = |A'_{2\ell}| + n^{1/2}|A'_\ell|^{1/2}\zeta_\ell(\chi').$$

But now $A_\ell = \{0\}$, so $|A_\ell| = 1$, $|A_{2\ell}| = |A_2|$ and similarly for A'. This gives $|A_2| - |A'_2| = n^{1/2}(\zeta_\ell(\chi') + \zeta_\ell(\chi))$. Comparing with the equality $|A_2| - |A'_2| = 2n^{1/2}$ obtained above, we conclude that $\zeta_\ell(\chi') + \zeta_\ell(\chi) = 2$. By Lemma 2.2, this is possible if and only if $\zeta_\ell(\chi) = \zeta_\ell(\chi') = 1$ for all odd $l \geq 3$.

(iii) The number $\zeta_k(\chi)$ is closely related to the Frobenius-Schur indicator $\nu_{2k}(m)$ of the object m of the category $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ computed by Shimizu [8]. Indeed, substituting $n = 2k, V = m$ in formula (3) of [8] and taking into account that $\dim(\mathcal{C}) = 2|A|$, $\theta_m = \Delta$, $\dim(m) = |A|^{1/2}$, we obtain

$$\nu_{2k}(m) = \frac{1}{2|A|^{1/2}} \sum_{\mu,\Delta} \Delta^{2k} = |A|^{-1/2} \sum_{\mu \in Q_\chi} [\nu\gamma(\mu)]^k = \nu^k |A_k|^{1/2}\zeta_k(\chi)$$

(our sign ν is equal to Shimizu's $sgn(\tau)$). This and Lemma 2.2 give another proof of the following results of Shimizu (see [8], Theorem 3.5): the number $|A_k|^{-1/2}\nu_{2k}(m)$ is either 0 or an 8-th complex root of unity for all k; this number is 0 if and only if for some (and then for any) $\mu \in Q_\chi$, there is $a = a_\chi \in A_k$ such that $\mu(a)^k = 1$. The latter claim follows from Lemma 2.1, Formula (3), and the equality $A^\perp_{\chi^{-k}} = A_k$.

3. Proof of Theorem 0.1(b)

3.1. **Preliminaries on bicharacters.** Any finite abelian group A splits uniquely as a direct sum $A = \oplus_p A^{(p)}$, where $p \geq 2$ runs over all prime integers and $A^{(p)}$ consists of all elements of A annihilated by a sufficiently big power of p. The group $A^{(p)}$ is a p-group, i.e., an abelian group annihilated by a sufficiently big power of p. Given a bicharacter χ of A, we have $\chi(A^{(p)}, A^{(p')}) = 1$ for any distinct

p, p'. Therefore the restriction, $\chi^{(p)}$, of χ to $A^{(p)}$ is a bicharacter and we have an orthogonal splitting $(A, \chi) = \oplus_p (A^{(p)}, \chi^{(p)})$.

Fix a prime integer $p \geq 2$ and recall the properties of bicharacters on p-groups, see, for example, [2] for a survey. Given a bicharacter χ on a finite abelian p-group A, there is an orthogonal splitting $(A, \chi) = \oplus_{s \geq 1}(A_s, \chi_s)$, where A_s is a direct sum of several copies of $\mathbb{Z}/p^s\mathbb{Z}$ and $\chi_s : A_s \times A_s \to S^1$ is a bicharacter. The rank of A_s as a $\mathbb{Z}/p^s\mathbb{Z}$-module depends only on A and is denoted $r_{p,s}(A)$.

For $p \geq 3$, the splitting $(A, \chi) = \oplus_{s \geq 1}(A_s, \chi_s)$ is unique up to isomorphism and each χ_s is an orthogonal sum of bicharacters on $r_s(A)$ copies of $\mathbb{Z}/p^s\mathbb{Z}$. Using the injection $\mathbb{Z}/p^s\mathbb{Z} \hookrightarrow S^1, z \mapsto e^{2\pi i z/p^s}$, we can view χ_s as a pairing with values in the ring $\mathbb{Z}/p^s\mathbb{Z}$. This allows us to consider the determinant $\det \chi_s \in \mathbb{Z}/p^s\mathbb{Z}$ of χ_s. Since χ_s is non-degenerate, $\det \chi_s$ is coprime with p. Let

$$\sigma_{p,s}(\chi) = (\frac{\det \chi_s}{p}) \in \{\pm 1\}$$

be the corresponding Legendre symbol. Recall that for an integer d coprime with p, the Legendre symbol $(\frac{d}{p})$ is equal to 1 if $d \,(\mathrm{mod}\ p)$ is a quadratic residue and to -1 otherwise. If $r_{p,s}(A) = 0$, then by definition $\sigma_{p,s} = 1$. It is easy to see that the integers $\{r_{p,s}\}_s$ are additive and the signs $\{\sigma_{p,s}\}_s$ are multiplicative with respect to orthogonal summation of bicharacter pairs. A theorem due to H. Minkowski, E. Seifert, and C.T.C. Wall says that these invariants form a complete system: two bicharacters, χ_1 and χ_2, on p-groups A_1 and A_2, respectively, are isomorphic if and only if $r_{p,s}(A_1) = r_{p,s}(A_2)$ and $\sigma_{p,s}(\chi_1) = \sigma_{p,s}(\chi_2)$ for all $s \geq 1$. In the sequel, when p is specified, we denote $r_{p,s}(A)$ and $\sigma_{p,s}(\chi)$ by $r_s(A)$ and $\sigma_s(\chi)$, respectively.

3.2. **Computation of ζ_k.** Consider the $\mathbb{C}$-valued invariants $\{\zeta_k\}_{k \geq 1}$ of bicharacters defined in the introduction. It is easy to deduce from the definitions that $\zeta_k(\chi \oplus \chi') = \zeta_k(\chi)\, \zeta_k(\chi')$ for any bicharacters χ, χ' and any k. Thus, the formula $\chi \mapsto \zeta_k(\chi)$ defines a multiplicative function from the semigroup of bicharacter pairs (with operation being the orthogonal sum $\oplus$) to $\mathbb{C}$.

Fix an odd prime $p \geq 3$. We now compute ζ_k on the bicharacters on p-groups. For any odd integer a, set $\varepsilon_a = i = \sqrt{-1}$ if $a \equiv 3 (\mathrm{mod}\ 4)$ and $\varepsilon_a = 1$ otherwise. For any integers $k, s \geq 1$, we have $gcd(k, p^s) = p^t$ with $0 \leq t \leq s$. Set

$$\alpha_{k,s} = ks + s - t \quad \text{and} \quad \beta_{k,s} = \frac{\varepsilon_{p^s}^k}{\varepsilon_{p^{s-t}}} \, (\frac{h}{p})^{ks+s-t} \, (\frac{k'}{p})^{s-t} \in \{\pm 1, \pm i\},$$

where $h = (p^s + 1)/2 \in \mathbb{Z}$ and $k' = k/p^t \in \mathbb{Z}$. Note that $gcd(h, p) = 1$ so that the Legendre symbol $(\frac{h}{p})$ is defined. If $t < s$, then $gcd(k', p) = 1$ so that the Legendre symbol $(\frac{k'}{p})$ is defined; if $t = s$, then by definition, $(\frac{k'}{p})^{s-t} = 1$.

Lemma 3.1. *For any $k \geq 1$ and any bicharacter χ on a p-group A,*

$$\zeta_k(\chi) = \prod_{s \geq 1} \beta_{k,s}^{r_s(A)} \, [\sigma_s(\chi)]^{\alpha_{k,s}}. \tag{4}$$

Proof. The proof is based on the following classical Gauss formula: for any integer d coprime with p,

$$\sum_{j=0}^{p^s-1} \exp(\frac{2\pi i}{p^s} d j^2) = p^{\frac{s}{2}} \varepsilon_{p^s} (\frac{d}{p})^s. \tag{5}$$

A more general formula holds for any integer d: if $gcd(d,p^s)=p^t$ with $0\le t\le s$ and $d'=d/p^t$, then

$$\sum_{j=0}^{p^s-1}\exp(\frac{2\pi i}{p^s}dj^2)=p^t\sum_{j=0}^{p^{s-t}-1}\exp(\frac{2\pi i}{p^{s-t}}d'j^2)=p^{\frac{s+t}{2}}\varepsilon_{p^{s-t}}(\frac{d'}{p})^{s-t}, \tag{6}$$

where, by definition, for $t=s$, the expression $(\frac{d'}{p})^{s-t}$ is equal to 1.

We now prove (4). It is clear that both sides of (4) are multiplicative with respect to orthogonal summation of bicharacters. The results stated in Section 3.1 allow us to reduce the proof of (4) to the case where $A=\mathbb{Z}/p^s\mathbb{Z}$ for some $s\ge 1$. We must prove that for any bicharacter $\chi: A\times A\to S^1$,

$$\zeta_k(\chi)=\beta_{k,s}\,[\sigma_s(\chi)]^{\alpha_{k,s}}. \tag{7}$$

Set as above $h=(p^s+1)/2$ and $k'=k/p^t$, where $gcd(k,p^s)=p^t$ with $0\le t\le s$. The bicharacter χ is given by $\chi(a,b)=\exp(\frac{2\pi i}{p^s}\Delta ab)$ for all $a,b\in A$, where Δ is an integer coprime with p. Observe that the map $\mu_0: A\to S^1$ carrying any $a\in A$ to $\exp(\frac{2\pi i}{p^s}h\Delta a^2)$ is a quadratic map associated with χ. Formula (5) and the multiplicativity of the Legendre symbol imply that

$$\gamma(\mu_0)=p^{-s/2}\sum_{j=0}^{p^s-1}\exp(\frac{2\pi i}{p^s}h\Delta j^2)=\varepsilon_{p^s}(\frac{h}{p})^s(\frac{\Delta}{p})^s=\varepsilon_{p^s}(\frac{h}{p})^s\,[\sigma_s(\chi)]^s.$$

Similarly, Formula (6) implies that

$$\sum_{a\in A}\mu_0(a)^{-k}=\sum_{j=0}^{p^s-1}\overline{\exp(\frac{2\pi i}{p^s}kh\Delta j^2)}=p^{(s+t)/2}\varepsilon^{-1}_{p^{s-t}}(\frac{h}{p})^{s-t}(\frac{k'}{p})^{s-t}[\sigma_s(\chi)]^{s-t}.$$

Since $|A|=p^s$ and $|A_k|=gcd(k,p^s)=p^t$, we have

$$\gamma(\mu_0^{-k})=|A|^{-1/2}|A_k|^{-1/2}\sum_{a\in A}\mu_0(a)^{-k}=\varepsilon^{-1}_{p^{s-t}}(\frac{h}{p})^{s-t}(\frac{k'}{p})^{s-t}[\sigma_s(\chi)]^{s-t}.$$

These computations and Formula (3) imply that

$$\zeta_k(\chi)=\gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k=\frac{\varepsilon^k_{p^s}}{\varepsilon_{p^{s-t}}}(\frac{h}{p})^{ks+s-t}(\frac{k'}{p})^{s-t}[\sigma_s(\chi)]^{ks+s-t}.$$

This is equivalent to Formula (7). □

Note one special case of Lemma 3.1: if k is divisible by $2|A|$, then $\zeta_k(\chi)=\prod_{s\ge 1}\beta_{k,s}^{r_s(A)}$. Indeed, in this case for all s such that $\mathbb{Z}/p^s\mathbb{Z}$ is a direct summand of A, we have $gcd(k,p^s)=p^s$ and $\alpha_{k,s}=ks\in 2\mathbb{Z}$. For all other s, we have $\sigma_s(\chi)=1$. Therefore $[\sigma_s(\chi)]^{\alpha_{k,s}}=1$ for all s.

3.3. **Proof of Theorem 0.1(b).** We begin with a few remarks concerning the subgroups $(A_k)_k$ of A defined in the introduction. Using the splitting $A=\oplus_p A^{(p)}$, one easily checks that $A_{kl}=A_k\oplus A_l$ for any relatively prime integers k,l. For any prime p, the integers $(|A_{p^m}|)_{m\ge1}$ depend only on the group $A^{(p)}$ and determine the isomorphism class of $A^{(p)}$. Indeed, $A^{(p)}=\oplus_{s\ge1}(\mathbb{Z}/p^s\mathbb{Z})^{r_{p,s}}$ for $r_{p,s}=r_{p,s}(A)\ge 0$. Given $m\ge 1$,

$$A_{p^m}=(A^{(p)})_{p^m}=\oplus_{s=1}^m(\mathbb{Z}/p^s\mathbb{Z})^{r_{p,s}}\ \oplus\ \oplus_{s>m}(\mathbb{Z}/p^m\mathbb{Z})^{r_{p,s}}.$$

Hence,

$$\log_p(|A_{p^{m+1}}|/|A_{p^m}|)=r_{p,m+1}+r_{p,m+2}+\cdots.$$

Therefore, the sequence $(|A_{p^m}|)_{m\ge1}$ determines the sequence $\{r_{p,s}(A)\}_{s\ge1}$ and so determines the isomorphism type of $A^{(p)}$.

Formula (1) and the assumptions of the theorem imply that, for all odd $k \geq 1$,

$$|A_k| = 2n|L_k|_{\mathcal{C}} = 2n|L_k|_{\mathcal{C}'} = |A'_k|,$$

where $n = |A| = |A'|$. By the previous paragraph, $A^{(p)} \cong A'^{(p)}$ for all prime $p \neq 2$, and for all $s \geq 1$,

$$r_{p,s}(A^{(p)}) = r_{p,s}(A) = r_{p,s}(A') = r_{p,s}(A'^{(p)}). \tag{8}$$

Since $n = \prod_{p \geq 2} |A^{(p)}|$, we also have $|A^{(2)}| = |A'^{(2)}|$.

Let $N \geq 2$ be a positive power of 2 annihilating both $A^{(2)}$ and $A'^{(2)}$. Then $A_N = A^{(2)}$ and $A'_N = A'^{(2)}$. For any odd integer $\ell \geq 1$,

$$|A_{N\ell}| = |A_N|\,|A_\ell| = |A^{(2)}|\,|A_\ell| = |A'^{(2)}|\,|A'_\ell| = |A'_N|\,|A'_\ell| = |A'_{N\ell}|.$$

Similarly, $|A_{2N\ell}| = |A'_{2N\ell}|$. Applying (2) to $k = 2N\ell$, we obtain $\zeta_{N\ell}(\chi) = \zeta_{N\ell}(\chi')$.

Fix from now on an odd prime p. The identity (8) shows that to prove that the bicharacter pairs $(A^{(p)}, \chi^{(p)})$ and $(A'^{(p)}, \chi'^{(p)})$ are isomorphic, it is enough to verify that $\sigma_s(\chi^{(p)}) = \sigma_s(\chi'^{(p)})$ for all $s \geq 1$. Set

$$\ell = \frac{|A|}{|A^{(2)}||A^{(p)}|} = \prod_{q \geq 3, q \neq p} |A^{(q)}| = \prod_{q \geq 3, q \neq p} |A'^{(q)}| = \frac{|A'|}{|A'^{(2)}||A'^{(p)}|},$$

where q runs over all odd primes distinct from p. Clearly, ℓ is an odd integer. For any N as above, $\zeta_{N\ell}(\chi) = \zeta_{N\ell}(\chi')$. Observe that

$$\zeta_{N\ell}(\chi) = \zeta_{N\ell}(\chi^{(2)}) \prod_{q \geq 3} \zeta_{N\ell}(\chi^{(q)}),$$

where q runs over all odd primes. Since $N\ell$ is divisible by $2|A^{(q)}|$ for $q \neq p$, the remark at the end of Section 3.2 implies that $\zeta_{N\ell}(\chi^{(q)}) = \zeta_{N\ell}(\chi'^{(q)}) \neq 0$ for all $q \neq p$. Replacing if necessary N by a bigger power of 2, we can assume that N is divisible by $8|A^{(2)}| = 8|A'^{(2)}|$. The last claim of Lemma 2.2 yields $\zeta_{N\ell}(\chi^{(2)}) = \zeta_{N\ell}(\chi'^{(2)}) = 1$. Combining these equalities, we obtain that $\zeta_{N\ell}(\chi^{(p)}) = \zeta_{N\ell}(\chi'^{(p)})$. Expanding both sides as in Formula (4) and using Formula (8) and the inclusions $\sigma_s(\chi^{(p)}), \sigma_s(\chi'^{(p)}) \in \{\pm 1\}$, we obtain that

$$\prod_{s \geq 0} \sigma_{2s+1}(\chi^{(p)}) = \prod_{s \geq 0} \sigma_{2s+1}(\chi'^{(p)}).$$

Replacing in this argument ℓ by ℓp^u, we obtain that more generally

$$\prod_{s \geq 0} \sigma_{2s+1+u}(\chi^{(p)}) = \prod_{s \geq 0} \sigma_{2s+1+u}(\chi'^{(p)})$$

for all $u \geq 0$. These equalities easily imply that $\sigma_s(\chi^{(p)}) = \sigma_s(\chi'^{(p)})$ for all s.

References

[1] J. Barrett, B. Westbury, *Invariants of piecewise-linear* 3*-manifolds*, Trans. Amer. Math. Soc. **348** (1996), 3997–4022.

[2] F. Deloup, *Monoïde des enlacements et facteurs orthogonaux*, Algebraic and Geometric Topology **5** (2005), 419–442.

[3] P. Etingof, D. Nikshych, V. Ostrik, *On fusion categories*, Ann. Math. **162** (2005), 581–642.

[4] S. Gelaki, D. Naidu, D. Nikshych, *Centers of graded fusion categories*, ArXiv:0905.3117v1 [math.QA](2009).

[5] A. Kleppner, *Multipliers on abelian groups*, Math. Ann **158** (1965), 11–34.

[6] M. Müger, *From subfactors to categories and topology, II. The quantum double of tensor categories and subfactors*, J. Pure Appl. Algebra **180** (2003), 159–219.

[7] W. Scharlau, Quadratic and Hermitian forms. Grundlehren der Math. Wissenschaften, **270**. Springer-Verlag, Berlin, 1985.

[8] K. Shimizu, *Frobenius-Schur indicators in Tambara-Yamagami categories*, Preprint arXiv:1005.4500 [math.QA] (2010).

[9] D. Tambara, S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra **209** (2) (1998), 692–707.
[10] V. Turaev, *Quantum invariants of Knots and 3-Manifolds*, de Gruyter Studies in Mathematics, **18** (1994).
[11] V. Turaev, O. Viro, *State sum invariants of* 3*-manifolds and quantum* $6j$*-symbols*, Topology **31** (1992), 865–902.
[12] V. Turaev, A. Virelizier, *On two approaches to 3-dimensional TQFTs*, ArXiv: 1006.3501v1[math.GT](2010).

Vladimir Turaev
Department of Mathematics
Indiana University
Bloomington IN47405
USA
e-mail: vtouraev@indiana.edu

Leonid Vainerman
Department of Mathematics
University of Caen
14000 Caen
France
e-mail: leonid.vainerman@math.unicaen.fr