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ON WEAK MIXING, MINIMALITY AND WEAK DISJOINTNESS OF ALL ITERATES

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Abstract. Given a continuous map f, we study the recurrence properties of $f^{(*m)} = f \times f^2 \times \ldots \times f^m$. In contrast to minimal case, we show that in general there is no relation between topological weak mixing and transitivity of $f^{(*m)}$. In minimal case, we survey some old results related to the topic and show that some of them work also in noninvertible case. That way we answer some open question existing in the literature.

1. Introduction

Let X and Y be compact metric spaces and let $f\colon X \mapsto X$ and $g\colon Y \mapsto Y$ be continuous maps. In the theory of dynamical systems there is well-established and extensively explored notion of *weak disjointness* of dynamical systems given by f and g (see [1, 12, 14, 15]). It was introduced both, in topological dynamics and ergodic theory setting, by Furstenberg in his seminal paper of 1967 [8]. Let us recall, that f and g are *weakly disjoint* if their Cartesian product $f \times g$ is topologically transitive. Weakly disjoint systems are kind of *independent* one from another. It is independence in a rather weak sense as it may happen that f is weakly disjoint from itself (if it is the case, we say that f is *weakly mixing*).

Therefore, it is natural to ask: *Can f be weakly disjoint from some of its iterates, f^m, where $m \geq 2$?* This question is connected with the analysis of the recurrence properties of $f \times f^2 \times \ldots \times f^m$, for $m \geq 2$, and the latter question can be thought of as a topological dynamics counterpart of the problems considered in ergodic theory (see [11]). Here we consider two properties, very similar to weak mixing, namely:

- **($\star$):** for each $m \in \mathbb{N}$ the map $f \times f^2 \times \ldots \times f^m$ is transitive.
- **($\star\star$):** for each $m \in \mathbb{N}$ there is a residual set of $Y \subset X$ such that for every point $x \in Y$ the tuple $(x, \ldots, x) \in X^m$ has a dense orbit in X^m under the map $f \times f^2 \times \ldots \times f^m$.

Following [24], we will say that f is *multi-transitive* if it satisfies ($\star$) and that f is *Δ-transitive* if ($\star\star$) holds.

It is known that both properties presented above are equivalent to weak mixing if f is a minimal homeomorphism. The proof of this equivalence using only elementary notions of topological dynamics is contained in [24]. The implication stating that weak mixing implies Δ-transitivity was earlier proved by Glasner ([11]) with the help of the general structure theorem for minimal homeomorphisms. In [24] the question whether this implication holds for necessarily invertible continuous maps was left open. Here we answer it affirmatively.

Moreover, we solve another open problem stated in [24]. We show that in general there is no connection between weak mixing and multi-transitivity by constructing examples of weakly mixing but not multi-transitive and multi-transitive but not weakly mixing systems. Finally, we offer some remarks regarding the last question of [24]. Moothathu asked if there is a nontrivial minimal system $f\colon X \mapsto X$ such that $f \times f^2 \times \ldots \times f^m\colon X^m \mapsto X^m$ is minimal for some $m \geq 2$.

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2. Preliminaries

Let X be a compact metric space and $f\colon X \mapsto X$ be a continuous map. For every $m \geq 1$ denote the Cartesian product of m copies of X with itself by X^m and define two maps of X^m to itself: $f^{(\times m)} = f \times \ldots \times f$ and $f^{(*m)} = f \times f^2 \times \ldots \times f^m$.

Given any sets $U, V \subset X$ we denote $N(U,V; f) = \{n > 0 \ : \ f^n(U) \cap V \neq \emptyset\}$. If the map f is clear from the context we simply write $N(U,V)$.

A map f is *minimal*, if it has no proper closed invariant set, that is, if $K \subset X$ nonempty, closed and $f(K) \subset K$ then $K = X$. We say that f is *transitive* if $N(U,V) \neq \emptyset$ for any pair of nonempty open sets $U, V \subset X$. A set $S \subset \mathbb{Z}_+$ is *syndetic* if there is a constant $K > 0$ such that for every $n \geq$ we have $[n, n+L] \cap S \neq \emptyset$. Then we say that a map f is *syndetically transitive* if $N(U,V)$ is syndetic for any nonempty open sets $U, V \subset X$. If $f \times f$ is transitive, then we say that f is *weakly mixing*. If for any nonempty open set $U \subset X$ there is $M > 0$ such that $\bigcup_{j=1}^{M} f(U) = X$ then f is said to be *strongly transitive*. It immediately follows from the definition that any strongly transitive map is syndetically transitive.

Let f and g be two continuous surjective maps acting on compact metric spaces X and Y, respectively. We say that a nonempty closed set $J \subset X \times Y$ is *a joining* of f and g if it is invariant for the product map $f \times g$ and its projections on first and second coordinate are X and Y respectively. If $X \times Y$ is the only joining of f and g then we say that f and g are *disjoint*.

The notion of disjointness was first introduced by Furstenberg in [8]. It is well known that if f and g are disjoint then at least one of them is minimal. It is also not so hard to verify that if f, g are both minimal, then they are disjoint if and only if $f \times g$ is minimal.

3. Strong transitivity and Δ-transitivity

The main result of this section (Theorem 5) is obtained as a corollary from Theorem 4 below. The Theorem 4 was proved by [24, Theorem 4] with the additional assumption that f is a homeomorphism. Here we present it with a new proof, which works for any continuous map.

We recall two results from [24], modifying first to a suitable form.

Theorem 1 ([24, Proposition 1])**.** *Let X be a compact metric space. A continuous map $f\colon X \mapsto X$ is Δ-transitive if and only if for each $m \geq 1$ and nonempty open sets $U, V_1, \ldots, V_m \subset X$, there exists $n \geq 1$ such that*

$$U \cap \bigcap_{i=1}^{m} f^{-in}(V_i) \neq \emptyset.$$

Theorem 2 ([24, Corollary 2])**.** *Let X be a compact metric space. If $f\colon X \mapsto X$ is a weakly mixing and syndetically transitive continuous map, then $f^{(*m)}$ is also weakly mixing and syndetically transitive for any $m \geq 1$. In particular, f is multi-transitive.*

The induction step in a proof of Theorem 4 is based on the following:

Lemma 3. *Let X be a compact metric space. If $f\colon X \mapsto X$ is multi-transitive continuous map, then for any $m \geq 1$ and nonempty open sets $V_1, \ldots, V_m \subset X$ there is a sequence of integers $\{k_n\}_{n=0}^{\infty}$ such that for each $n \geq 0$ we have $k_n - n > 0$ and for each $i = 1, \ldots, m$ there is a sequence $\{V_i^{(n)}\}_{n=0}^{\infty}$ of nonempty open subset of V_i such that*

$$f^{ik_j - j}(V_i^{(n)}) \subset V_i$$

for $i = 1, \ldots, m$, and $j = 0, \ldots, n$.

Proof. Let $V_1, \ldots, V_m$ be nonempty open subsets of X. Set $W = V_1 \times \ldots \times V_m$. We proceed by induction on n. From multi-transitivity of f there is $k_0 > 0$ such that $(f^{(*m)})^{k_0}(W) \cap W \neq \emptyset$, or equivalently $f^{-ik_0}(V_i) \cap V_i \neq \emptyset$ for $i = 1, \ldots, m$. Put $V_i^{(0)} = f^{-ik_0}(V_i) \cap V_i \subset V_i$ for $i = 1, \ldots, m$, to complete the base step.

For the induction step, suppose that $n \geq 1$ and we have found a sequence $k_0, \ldots, k_{n-1}$ and for each $i = 1, \ldots, m$ we have nonempty open set $V_i^{(n-1)} \subset V_i$ such that

$$f^{ik_j - j}(V_i^{(n-1)}) \subset V_i \qquad \text{and} \qquad k_j - j > 0, \tag{1}$$

hold for $j = 0, \ldots, n-1$. For $i = 1, \ldots, m$, let $U_i = f^{-n}(V_i^{(n-1)})$. Put $U = U_1 \times \ldots \times U_m$. By multi-transitivity we get an integer k_n such that $k_n - n > 0$ and $(f^{(*m)})^{k_n}(U) \cap W \neq \emptyset$, or equivalently $f^{-ik_n}(V_i) \cap U_i \neq \emptyset$, for $i = 1, \ldots, m$. Fix $1 \leq i \leq m$. We have

$$f^{ik_n}(U_i) \cap V_i = f^{ik_n}(f^{-n}(V_i^{(n-1)})) \cap V_i = f^{ik_n - n}(V_i^{(n-1)}) \cap V_i.$$

By the above, $V_i^{(n)} = V_i^{(n-1)} \cap f^{-ik_n+n}(V_i)$ is nonempty, open, and clearly $f^{ik_n - n}(V_i^{(n)}) \subset V_i$. Moreover, $V_i^{(n)} \subset V_i^{(n-1)}$. Using (1), we conclude that

$$f^{ik_j - j}(V_i^{(n)}) \subset V_i$$

for $j = 0, \ldots, n$. This completes the proof. □

Theorem 4. *Let X be a compact metric space. If $f\colon X \mapsto X$ is a weakly mixing and strongly transitive continuous map, then f is Δ-transitive.*

Proof. First, note that f is multi-transitive by Theorem 2. In particular, it is transitive, and surjective.

To prove that f is Δ-transitive we are going to use the equivalent condition provided by Theorem 1. We will prove by induction on m that for any nonempty open sets $U, V_1, \ldots, V_m \subset X$, there exists $n \geq 1$ such that

$$U \cap \bigcap_{i=1}^{m} f^{-in}(V_i) \neq \emptyset.$$

For $m = 1$ this statement simply follows from transitivity of f. Assume we established the result for some $m \geq 1$. We fix nonempty open sets U and $V_1, \ldots, V_{m+1}$, and we want to show that there are $n > 0$ and $z \in U$ such that $f^{in}(z) \in V_i$ for $i = 1, \ldots, m+1$. By strong transitivity, $\bigcup_{j=1}^{N} f(U) = X$ for some $N > 0$. Lemma 3 gives us nonempty open sets $V_1^{(N)}, \ldots, V_{m+1}^{(N)}$ and integers $k_0, \ldots, k_N$ such that

$$f^{ik_j - j}(V_i^{(N)}) \subset V_i \qquad \text{and} \qquad k_j > j,$$

for $i = 1, \ldots, m+1$ and $j = 0, \ldots, N$. By the induction hypothesis we can find $x \in V_1^{(N)}$ and $n > 0$ such that $f^{in}(x) \in V_{i+1}^{(N)}$ for $i = 1, \ldots, m$. Clearly, there is $y \in X$ such that $f^n(y) = x$. But strong transitivity gives us $f^j(z) = y$ for some $z \in U$ and $0 \leq j \leq N$. ¿From the above we get

$$\begin{aligned} f^{i(n+k_j)}(z) &= f^{i(n+k_j)-j}(y) = f^{ik_j - j}(f^{in}(y)) = \\ &= f^{ik_j - j}(f^{i(n-1)}(x)) \in f^{ik_j - j}(V_i^{(N)}) \subset V_i \end{aligned}$$

for any $i = 1, 2, \ldots, m+1$. We showed that

$$z \in U \cap f^{-n}(V_1) \cap \ldots \cap f^{-n\cdot(m+1)}(V_{m+1}),$$

which completes the proof. □

Theorem 5. *Let X be a compact metric space. If $f\colon X \mapsto X$ is a weakly mixing and minimal continuous map, then f is Δ-transitive.*

Proof. It is well known that any minimal map (invertible or not) on a compact metric space is strongly transitive (see [21, Theorem 2.5(8)] for a proof). We apply Theorem 4 to finish the proof. □

Now we may formulate a general version of [24, Corollary 7], which was stated there for homeomorphisms. Only the implication given by Theorem 5 is new here. The rest of the proof is identical as in [24].

Theorem 6. *Let $f\colon X \mapsto X$ be a minimal continuous map on a compact metric space X. Then the following are equivalent:*

(1) $f \times f^2$ is transitive.
(2) f is multi-transitive.
(3) f is weakly mixing.
(4) f is Δ-mixing.

4. Weak mixing and multi-transitivity

In [24, page 10] T. K. S. Moothathu asked the following question

Question 1. *Are there any implications between weak mixing and multi-transitivity?*

The aim of this section is to show that these notions are not related in a general situation, that is a continuous map can be multi-transitive and not weakly mixing, or weakly mixing and not multi-transitive. As it is often the case, to finish our task we will construct a symbolic systems.

Consider the set $A = \{0, 1\}$ endowed with discrete topology. Let Σ denote the set of all infinite sequences of 0's and 1's regarded as the product of infinitely many copies of A with the product topology. All sequences $x \in \Sigma$ are indexed by nonnegative integers, $x = x_0x_1x_2\ldots$. Then the *shift* transformation is a continuous map $\sigma\colon \Sigma \mapsto \Sigma$ given by $\sigma(x) = y$, where $x = (x_i)$, $y = (y_i)$, and $y_i = x_{i+1}$ for $i = 0, 1, \ldots$. Any closed subset $X \subset \Sigma$ invariant for σ is called a *subshift* of Σ. A *word* is a finite sequence of elements of $\{0, 1\}$. The *length* of a word w is just the number of elements of w, and is denoted $|w|$. We say that a word $w = w_1w_2\ldots w_l$ appears in $x = (x_i) \in \Sigma$ at position t if $x_{t+j-1} = w_j$ for $j = 1, \ldots, l$. If X is a subshift, then the *language* of X is the set $\mathcal{L}(X)$ of all words which appear at some position in some element $x \in X$. For any word w let $[w]_t$ denote the element of the sequence w standing at position t and let $\mathrm{Sp}(w) = \left\{|i - j| \;:\; [w]_i = [w]_j = 1, i \neq j\right\}$. The set $\mathcal{L}_n(X)$ consists of all elements of $\mathcal{L}(X)$ of length n.

Let P be a set of nonnegative integers. We say that a word $w = w_1w_2\ldots w_l$ is *P-admissible* if $w_i = w_j = 1$ for some $1 \le i < j \le l$ implies $|i - j| \in P$, equivalently, if $\mathrm{Sp}(w) \subset P$. Let Σ_P be the subset of Σ consisting of all sequences x such that every word which appears in x is P-admissible. It is easy to see that Σ_P is a subshift, and $\mathcal{L}(\Sigma_P)$ is the set of all P-admissible words. We will write σ_P for σ restricted to Σ_P, and call the dynamical system given by $\sigma_P\colon \Sigma_P \mapsto \Sigma_P$ a *spacing* shift. The class of spacing shifts was introduced by Lau and Zame in [22], and for a detailed exposition of their properties we refer to [4].

Let w be a P-admissible word. By $[w]_P$ we denote the set of all $x \in \Sigma_P$ such that the word w appears at position 0 in x. We call the set $[w]_P$ a *P-admissible cylinder* (a *cylinder* for short). The family of P-admissible cylinders is a base of topology of Σ_P inherited from Σ. It is easy to see that definition of a spacing shift implies that $N([1]_P, [1]_P; \sigma_P) = P$. Moreover, σ_P is weakly mixing if and only if P is a *thick* set (see [22, 4]). A thick set is a subset of integers that contains arbitrarily long intervals (P is thick if and only if for every n, there is some k such that $\{k, k + 1, \ldots, k + n - 1\} \subset P$). If w is a word and $n \ge 1$ then by w^n we denote a word which is a concatenation of n copies of w. If $n = 0$ then w^n is the *empty word*.

4.1. Multi-transitive and not weakly mixing example.

The results of this section generalize construction of totally transitive not weakly mixing spacing shift presented in [4].

We say that a finite set $S \subset \mathbb{N}$ is *q-dispersed*, where $q \ge 2$, if for every $a, b \in S \cup \{0\}$ such that $a \neq b$ we have $|a - b| \ge q$.

Lemma 7. *Let M, N be positive integers such that $M \ge 3$ and let $A \subset \mathbb{N}$ be an M-dispersed finite set. Then there exists an M-dispersed finite set B containing A and such* *that for* $k = \max A + 1$ *and any pair of sequences of words* $u_1, \ldots, u_N$ *and* $v_1, \ldots, v_N$ *from* $\mathcal{L}_k(\Sigma_B)$ *there is* $n \geq 0$ *such that*

$$\sigma^{in}([u_i]_B) \cap [v_i]_B \neq \emptyset \quad \text{for } i = 1, \ldots, N.$$

Proof. Let $k = \max A + 1$. Let $m = |\mathcal{L}_k(\Sigma_A)|^{2N}$ be the cardinality of the set of all N-element sequences of pairs of words from $\mathcal{L}_k(\Sigma_A)$. We enumerate all members of this set as a list $W^{(1)}, \ldots, W^{(m)}$. Hence, each $W^{(j)}$ is an ordered list of N pairs of words from $\mathcal{L}(\Sigma_A)$:

$$W^{(j)} = \left((u_1^{(j)}, v_1^{(j)}), \ldots, (u_N^{(j)}, v_N^{(j)})\right), \qquad \text{for each } j = 1, \ldots, m,$$

where $(u_i^{(j)}, v_i^{(j)}) \in \mathcal{L}_k(\Sigma_A) \times \mathcal{L}_k(\Sigma_A)$ for every $i = 1, \ldots, N$. Choose integers $l_1, \ldots, l_m$ fulfilling the following conditions

$$l_1 \geq 2k + M - 1, \tag{2}$$

$$l_{j+1} \geq (N+1)^j l_j. \tag{3}$$

Given $1 \leq i \leq N$ and $1 \leq j \leq m$ we define

$$w_i^{(j)} = u_i^{(j)} 0^{il_j - k} v_i^{(j)},$$

where $l_1, \ldots, l_m$ are as above. Using (2) and (3), it is easy to see that

$$[il_\alpha - k + 1, il_\alpha + k - 1] \cap [jl_\beta - k + 1, jl_\beta + k - 1] = \emptyset, \tag{4}$$

for $1 \leq \alpha, \beta \leq m$, $\alpha \neq \beta$ and $1 \leq i, j \leq N$. Let

$$B = \bigcup_{j=1}^{m} \bigcup_{i=1}^{N} \mathrm{Sp}(w_i^{(j)}).$$

If $n \in A$ then let $u = 10^{n-1}10^{k-n-1}$. Clearly, $n \in \mathrm{Sp}(u)$ and $u \in \mathcal{L}_k(\Sigma_A)$, since $k = \max A + 1$. This gives $A \subset B$. The construction of $w_i^{(j)}$ implies that for $1 \leq i \leq N$ and $1 \leq j \leq m$ we have

$$\mathrm{Sp}(w_i^{(j)}) \setminus A \subset [il_j - k + 1, il_j + k - 1]. \tag{5}$$

Therefore,

$$\min B \setminus A \geq l_1 - k + 1 \geq M + k. \tag{6}$$

In particular, $\min B = \min A \geq M$. Moreover, we conclude form (4) and (5) that if $r \in B \setminus A$, then there are unique indexes $i(r)$ and $j(r)$ such that $r \in \mathrm{Sp}(w_{i(r)}^{(j(r))})$.

Next, we are going to prove that B is M-dispersed, that is, $|q - p| \geq M$ for each $q, p \in B$, $q \neq p$. We consider three cases:

Case I: Both p and q belong to A.
Case II: Both p and q belong to $B \setminus A$.
Case III: None of the above cases hold.

The first case is clear, since A is M-dispersed. The third case follows from (6). To prove the remaining case, **Case II**, we consider subcases. But first note that in the computations below we use (2 - 5) without further reference. Given $p, q \in B \setminus A$ consider:

Case IIA: $\mathbf{j(p) \neq j(q)}$. Without lost of generality we assume $j(q) > j(p)$. We have

$$\begin{aligned} q &\geq i(q)l_{j(q)} - k + 1 \geq l_{j(q)} - k + 1 \\ &\geq (N+1)l_{j(p)} - k + 1 \geq Nl_{j(p)} - k + 1 + l_1 \\ &\geq i(p)l_{j(p)} + k - 1 + M \geq p + M. \end{aligned}$$

But then

$$q - p \geq M.$$

Case IIB: j(p) = j(q), but i(p) ≠ i(q). Without lost of generality we assume $i(q) > i(p)$. Let $j = j(p) = j(q)$. Then

$$\begin{aligned} q &\geq i(q)l_j - k + 1 \geq (i(p)+1)\cdot l_j - k + 1 \\ &\geq i(p)l_j - k + 1 + l_1 \geq i(p)l_j + k - 1 + M \geq p + M. \end{aligned}$$

Hence,

$$q - p \geq M.$$

Case IIC: j(p) = j(q), and i(p) = i(q). Let $j = j(p) = j(q)$ and $i = i(p) = i(q)$. For $r \in \{p, q\}$ we define

$$s(r) = \min\left\{ s : [w_i^{(j)}]_s = [w_i^{(j)}]_{s+r} = 1 \right\}.$$

Clearly, either $s(p) \neq s(q)$, or $s(p) + p \neq s(q) + q$. We have

$$\begin{aligned} |q - p| &= |(s(q) + q) - s(q) - (s(p) + p - s(p))| \\ &= |(s(q) + q) - (s(p) + p) - (s(q) - s(p))| \\ &\geq ||(s(q) + q) - (s(p) + p)| - |s(q) - s(p)||. \end{aligned}$$

But $|(s(q)+q)-(s(p)+p)|, |s(q)-s(p)| \in A \cup \{0\}$, so either $|(s(q)+q)-(s(p)+p)| \neq |s(q) - s(p)|$ and then

$$|(s(q) + q) - (s(p) + p)| - |s(q) - s(p)| \geq M,$$

or $|(s(q) + q) - (s(p) + p)| = |s(q) - s(p)| \neq 0$, and then

$$|q - p| \geq 2M.$$

It remains to prove that for any pair of sequences of words $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ from $\mathcal{L}_k(\Sigma_B)$ there is $n \geq 0$ such that

$$\sigma^{in}([u_i]_B) \cap [v_i]_B \neq \emptyset \quad \text{for } i = 1, \ldots, k.$$

Observe that $\mathcal{L}_k(\Sigma_B) = \mathcal{L}_k(\Sigma_A)$, since $\min B \setminus A \geq k$, $\max A + 1 = k$, and $A \subset B$. Therefore, according to our notation defined at the beginning of the proof, for any two sequences of words $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ from $L_k(\Sigma_B)$, there is $j = 1, \ldots, m$ such that

$$W^{(j)} = ((u_1, v_1), \ldots, (u_N, v_N)).$$

Let $w_i^{(j)} = u_i 0^{il_j - k} v_i$ as above. Clearly, $w_1^{(j)}, \ldots, w_m^{(j)} \in L(\Sigma_B)$, and from the definition of $w_i^{(j)}$ we conclude that

$$w_i^{(j)} \in \sigma^{in}([u_i]_B) \cap [v_i]_B \qquad \text{for } n = l_j.$$

Hence,

$$\sigma^{in}([u_i]_B) \cap [v_i]_B \neq \emptyset \quad \text{for } i = 1, \ldots, N,$$

where $n = l_j$. □

Theorem 8. *There exists a set $P \subset \mathbb{N}$ such that the spacing shift (Σ_P, σ_P) is multi-transitive but not weakly mixing.*

Proof. Fix any integer $M \geq 3$ and denote $P_0 = \{M\}$. Define a sequence of sets $P_n \subset \mathbb{N}$ ($n \geq 1$) inductively by putting $P_{n+1} = B$, where B is the set obtained for $A = P_n$, $N = n$, and M as above by Lemma 7. Denote

$$P = \bigcup_{n=0}^{\infty} P_n.$$

Easy induction gives $|p - q| \geq M$ for every distinct $p, q \in P$ and $P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots$. In particular P is not thick, so Σ_P is not weakly mixing. We are going to show that $\sigma_P \times \sigma_P^2 \times \ldots \times \sigma_P^m$ is transitive for any $m = 1, 2, \ldots$. Fix any integer $m \geq 1$ and choose any open sets $U_1, \ldots, U_m, V_1, \ldots, V_m \in \Sigma_P$. Without loss of generality, we may assume that fore each $1 \leq i \leq m$ there are words $u_i, v_i \in \mathcal{L}(\Sigma_P)$ such that $[u_i]_P \subset U_i$, and $[v_i]_P \subset V_i$. We may also assume that for each $1 \leq i \leq m$ we have $u_i, v_i \in \mathcal{L}_k(\Sigma_{P_l})$ for some $l \geq m$ and

$k = \max P_l + 1$. The last equality implies that $\mathcal{L}_k(\Sigma_{P_l}) = \mathcal{L}_k(\Sigma_P)$. If $m < l$ then we put $u_j = v_j = u_m$ for $j = m+1, \ldots, l$.

Now, by Lemma 7, there is $j > 0$ such that

$$\begin{aligned} \sigma_P^{ij}(U_i) \cap V_i &\supset \sigma^{ij}([u_i]_P) \cap [v_i]_P \\ &\supset \sigma^{in}([u_i]_{P_l}) \cap [v_i]_{P_l} \neq \emptyset \end{aligned}$$

for $i = 1, \ldots, l$. We have just proved that $\sigma_P \times \sigma_P^2 \times \ldots \times \sigma_P^m$ is transitive for any $m = 1, 2, \ldots$, which in other words means that σ_P is multi-transitive. □

It is clear from the construction of P in Lemma 7, that the spacing shift σ_P from the assertion of Theorem 8 is not syndetically transitive, since the set P, and as a result $N([1]_P, [1]_P)$, have thick complement. Then the following question arises:

Question 2. *Does every multi-transitive and syndetically transitive system have to be weakly mixing?*

4.2. **Weakly mixing and not multi-transitive example.** Fix $m \geq 2$. Let

$$B(m,k) = \{m^{2k-1}, m^{2k-1}+1, \ldots, m^{2k}-1\}, \qquad \text{and} \qquad P(m) = \bigcup_{k=1}^{\infty} B(m,k).$$

Observe that for every $m \geq 2$ the set $P(m)$ has the following property

$$p \in P(m) \implies m \cdot p \notin P(m). \tag{7}$$

Theorem 9. *Let $m \geq 2$ and $P = P(m)$ be as defined above. Then $\tau = \sigma_P \times \ldots \sigma_P^{m-1}$ is transitive, but $\tau \times \sigma_P^m$ is not transitive. In particular, the spacing shift (Σ_P, σ_P) is weakly mixing, but not multi-transitive.*

Proof. It is easy to see that P is thick, hence σ_P is weakly mixing. To prove that $\tau = \sigma_P \times \ldots \sigma_P^{m-1}$ is transitive, we fix open cylinders

$$[u^{(1)}]_P, \ldots, [u^{(m-1)}]_P, [v^{(1)}]_P, \ldots, [v^{(m-1)}]_P \in \mathcal{L}(P).$$

Set $t = m^{2k}$. Without lost of the generality we may assume that $|u^{(i)}| = |v^{(i)}| = t$ for some $k \geq 1$, and any $i = 1, \ldots, m-1$. Set $s = m^{2k+1} + m^{2k}$ and define

$$w^{(i)} = u^{(i)} 0^{is-t} v^{(i)}, \qquad \text{where } i = 1, \ldots, m-1.$$

Clearly,

$$[w^{(i)}]_P \subset \left(\sigma_P^i\right)^{-s} ([v^{(i)}]_P) \cap [u^{(i)}]_P,$$

and therefore

$$\begin{gathered} [w^{(1)}]_P \times \ldots \times [w^{(m-1)}]_P \subset \\ \tau^{-s}\left([v^1]_P \times \ldots \times [v^{(m-1)}]_P\right) \cap \left([u^{(1)}]_P \times \ldots \times [u^{(m-1)}]_P\right), \end{gathered}$$

so it is enough to prove that $[w^{(i)}]_P \neq \emptyset$, that is, $w^{(i)} \in \mathcal{L}(\Sigma_P)$. It follows from definition of $w^{(i)}$ that

$$\mathrm{Sp}(w^{(i)}) = \mathrm{Sp}(u^{(i)}) \cup \mathrm{Sp}(v^{(i)}) \cup \{l - k : (l,k) \in \Delta\},$$

where Δ is some subset of

$$\{0, \ldots, m^{2k}-1\} \times \{i \cdot m^{2k+1} + i \cdot m^{2k}, \ldots, i \cdot m^{2k+1} + (i+1) \cdot m^{2k} - 1\}.$$

Hence, we have

$$l - k \in \{m^{2k+1}, \ldots, m^{2k+2}-1\} \subset B(m, k+1),$$

and $w^{(i)} \in \mathcal{L}(\Sigma_P)$ as desired. We proved that $\tau = \sigma_P \times \ldots \sigma_P^{m-1}$ is transitive. To finish the proof it is enough to show that $\sigma_P \times \sigma_P^m$ is not transitive. Let $U = V = [1]_P \times [1]_P$. It is easy to see from (7) that

$$(\sigma_P \times \sigma_P^m)^n (U) \cap V = \emptyset$$

for every $n \geq 0$, so $\sigma_P \times \sigma_P^m$ cannot be transitive. □

In the literature there are considered other recurrence properties stronger than weak mixing, see e.g. [10]. It is natural to ask if we can replace weak mixing by one of them in Theorem 9. In the view of the above results we would like to pose the following problem.

Question 3. *Is there any nontrivial characterization of multi-transitive weakly mixing systems?*

5. Minimal self-joinings

The last question in [24] asks: *Can* $f \times f^2 \times \ldots \times f^m : X^m \mapsto X^m$ *be minimal if* $m \geq 2$ *and* X *has at least two elements?* Let us call a map $f : X \mapsto X$ providing an affirmative answer to the above question *multi-minimal*. Apparently, Moothathu posing his problem was not aware that the examples of multi-minimal homeomorphisms are known. But since their existence is stated in the language slightly different than terminology used in [24] we find it necessary to add some explanations. In fact the construction of multi-minimal systems is related to the considerations on multiple disjointness.

The first example of a system disjoint from any of its iterates (we are aware of) is the example of a POD (*proximal orbit dense*) minimal homeomorphism given by Furstenberg, Keynes and Shapiro in [9]. By Theorem 2.6 of [23] every POD system has *positive topological minimal self-joinings* (see [23]). It also follows from Proposition 2.1 of [23] that every homeomorphism possessing positive topological minimal self-joinings is multi-minimal, and so is the example from [9]. Furthermore, del Junco's work [16], together with his joint work with Rahe and Swanson [17] shows that Chacon's example [7] is POD, and hence also multi-minimal. In [2] Auslander and Markley introduced the class of *graphic* minimal systems, which generalizes POD homeomorphisms. They also proved that each graphic flow is multi-minimal [2, Corollary 22]. Moreover, as announced in [2, page 490] Markley constructed an example of a graphic homeomorphisms which is not POD, hence it is another kind of multi-minimal homeomorphism.

More information about minimal subsystems of $f \times f^2 \times \ldots \times f^m$ is to be found in [3, 5, 6, 18, 19] to name only a few. There is also in some sense parallel and certainly deep theory of minimal self-joinings (a part of ergodic theory), introduced by Rudolph [25], see Glasner's book [12]. We remark that although every weak mixing minimal map is multi-transitive it is not necessarily multi-minimal. The *discrete horocycle flow* h is an example of a weakly mixing minimal homeomorphism such that h is topologically conjugated to h^2, and hence it is not multi-minimal (see [12, pages 26, and 105-110]). The facts gathered above prompt us to raise following questions:

Question 4. *Is there any nontrivial characterization of multi-minimality in terms of some dynamical properties?*

It is also interesting whether is it possible to characterize multi-minimal systems adding some mild assumptions to Theorem 6. In particular, we don't know the answer for the following question.

Question 5. *Assume that* f *is a weakly mixing map such that* $f \times f^2$ *is minimal. Is* f *necessarily multi-minimal?*

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References

[1] J. Auslander, *Minimal flows and their extensions*, North-Holland Mathematics Studies, 153, Amsterdam, 1988.

[2] J. Auslander, N. Markley, *Graphic flows and multiple disjointness*, Trans. Amer. Math. Soc. **292** (1985), no. 2, 483–499.

[3] J. Auslander, N. Markley, *Isomorphism classes of products of powers for graphic flows* Ergodic Theory Dynam. Systems **17** (1997), no. 2, 297–305.

[4] J. Banks, T.T.D. Nguyen, P. Oprocha and B. Trotta, *Dynamics of Spacing Shifts*, preprint, 2009.

[5] F. Blanchard, E. Glasner, J. Kwiatkowski, *Minimal self-joinings and positive topological entropy*, Monatsh. Math. **120** (1995), no. 3-4, 205–222.

[6] F. Blanchard, J. Kwiatkowski, *Minimal self-joinings and positive topological entropy. II*, Studia Math. **128** (1998), no. 2, 121–133.

[7] R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*, Proc. Amer. Math. Soc. **22** 1969 559–562.

[8] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory, **1** (1967) 1–49.

[9] H. Furstenberg, H. Keynes, L. Shapiro, *Prime flows in topological dynamics*, Israel J. Math. **14** (1973), 26–38.

[10] E. Glasner, *Classifying dynamical systems by their recurrence properties*, Topol. Methods Nonlinear Anal. **24** (2004), no. 1, 21–40.

[11] E. Glasner, *Structure theory as a tool in topological dynamics*, London Math. Soc. Lecture Note Series, 277, (2000), 173-209.

[12] E.Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[13] E. Glasner and D. Maon, *Rigidity in topological dynamics*, Ergodic Theory Dynam. Systems, **9** (1989), 309–320.

[14] W. Huang and X. Ye, *Topological complexity, return times and weak disjointness*, Ergodic Theory Dynam. Systems, **24** (2004), 825–846.

[15] W. Huang and X. Ye, *An explicit scattering, non-weakly mixing example and weak disjointness*, Nonlinearity, **15** (2002), 849–862.

[16] A. del Junco, *A family of counterexamples in ergodic theory*, Israel J. Math. **44** (1983), no. 2, 160–188.

[17] A. del Junco, M. Rahe, L. Swanson, *Chacon's automorphism has minimal self-joinings*, J. Analyse Math. **37** (1980), 276–284.

[18] A. del Junco *On minimal self-joinings in topological dynamics*, Ergodic Theory Dynam. Systems **7** (1987), no. 2, 211–227.

[19] J. King, *A map with topological minimal self-joinings in the sense of del Junco*, Ergodic Theory Dynam. Systems **10** (1990), no. 4, 745–761.

[20] S. Kolyada and L. Snoha, *Some aspects of topological transitivity—a survey*, Grazer Math. Ber., **334** (1997), 3–35.

[21] S. Kolyada, L. Snoha and S. Trofimchuk, *Noninvertible minimal maps*, Fund. Math., **168** (2001), 141–163.

[22] K. Lau and A. Zame, *On weak mixing of cascades*, Math. Systems Theory, **6** (1972/73), 307–311.

[23] N. Markley, *Topological minimal self-joinings*, Ergodic Theory Dynam. Systems **3** (1983), no. 4, 579–599.

[24] T.K.S. Moothathu, *Diagonal points having dense orbit*, Colloq. Math., **120** (2010), 127–138.

[25] D. Rudolph, *An example of a measure preserving map with minimal self-joinings, and applications*, J. Analyse Math. **35** (1979), 97–122.

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