Recurrence relation for the $6j$-symbol of $\mathrm{su}_q(2)$ from an eigenvalue problem

Igor Khavkine[1, a)]
Institute for Theoretical Physics, Utrecht, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands

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A new, linear, three-term recurrence relation for the $6j$-symbol of the quantum group $\mathrm{su}_q(2)$ is derived. It is cast in the form of a symmetric eigenvalue problem, generalizing a result of Schulten and Gordon for the classical $6j$-symbol, and is particularly useful as an efficient numerical evaluation algorithm. The derivation is elementary and avoids the use of q-hypergeometric functions, as in the previous work on related recurrence relations by Kachurik and Klimyk. Intermediate calculations are simplified using the diagrammatic spin network formalism of Temperley-Lieb recoupling theory.

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I. INTRODUCTION

Quantum groups first appeared in the study of quantum integrable systems. Since then, they have proven useful in many applications, including among others conformal field theory, statistical mechanics, representation theory and the theory of hypergeometric function, along with exhibiting a rich internal structure. Recently, the quantum group $\mathrm{su}_q(2)$ was used to construct spin foam models of quantum gravity with a positive cosmological constant[1].

The $6j$-symbol, or Racah-Wigner coefficients, play a central role in the representation theory of $\mathrm{su}_q(2)$. An explicit formula for the quantum $6j$-symbol is known[2] and involves a number of operations (additions and multiplications) that is linear in its arguments. In applications where a large number of $6j$-symbols is needed at once, e.g., for all values of one argument with others fixed as is the case in Ref. 1, the total number of operations becomes quadratic in the arguments. If a recurrence relation exists among the set of desired $6j$-symbols, the efficiency of the calculation can be greatly increased by reducing the total operation count to be linear in the arguments.

Recurrence relations have long been known[3] for the classical $6j$-symbol. They have also been obtained in the quantum case[4] using the theory of q-hypergeometric functions. This paper gives an elementary derivation of a particular linear, three-term recurrence relation for the quantum $6j$-symbol in the guise of a symmetric eigenvalue problem, generalizing the classical result of the Appendix of Ref. 3. It is particularly convenient for evaluating the $6j$-symbol for all values of one argument with others held fixed. The notation and calculations are simplified using the diagrammatic spin network formalism[2,5,6], thus obviating the need to appeal to the theory of q-hypergeometric functions. An advantage of the eigenvalue problem formulation is the ability to make use of readily available, robust linear algebra packages, such as LAPACK[7]. The ability to use standard numerical linear algebra software is particularly useful since such software packages automatically take care of many issues of numerical accuracy and stability. That is especially true when $q = 1$ or when q is a primitive root of unity, because the relevant inner product becomes either positive- or negative-definite, allowing the use of even more specialized numerical methods.

Sec. II introduces the basic notions of the spin network formalism. Sec III defines the Kauffman-Lins convention for the $6j$-symbol and summarizes basic identities needed for in Sec. IV, where the recurrence relation is derived. Sec. V connects the Kauffman-Lins convention with the traditional Racah-Wigner convention used in the physics literature and emphasizes the novel aspects of the derivation. Finally, the Appendix conveniently summarizes all formulas needed for a direct computer implementation of the recurrence-based evaluation of the quantum $6j$-symbol.

II. SPIN NETWORKS

In a variety of physical and mathematical applications, one often encounters tensor contraction expressions of the form

$$T^{lm\cdots}_{j\cdots} = A^{kl\cdots}_{ij\cdots} B^{im\cdots}_{k\cdots} \cdots Z^{\cdots}_{\cdots}, \tag{1}$$

where T, A, B, ..., Z are invariant tensors, with each index transforming under a representation of a group or an algebra. The application at hand usually calls for evaluating T, or at least simplifying it. An extensive literature on this subject exists for the classical group $SU(2)$ or its Lie algebra, a subject known as *angular momentum recoupling*[8,9]. It is well known that such tensor contractions can be very efficiently expressed, manipulated, and simplified using diagrams known as *spin networks*[5]. Extensions of these techniques[2,6] are also known for the *quantum* (or *q-deformed*, since they depend on an arbitrary complex number $q \neq 0$) analogs, the quantum

a) Electronic mail: i.khavkine@uu.nl

group $\mathrm{su}_q(2)$ or $U_q(su(2))$. The basics of this diagrammatic formalism, as needed for the derivation of the recurrence relation, are given in this and next sections. All relevant formulas, including explicit spin network evaluations in terms of *quantum integers* are listed in the Appendix.

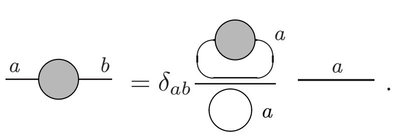
Single spin networks are edge-labeled graphs[10], where each vertex has valence either 1 or 3. General spin networks are formal linear combinations of single spin networks. Edges attached to univalent vertices are called *free*. Spin networks without free edges are called *closed*. Conventionally, the labels are either integers (*spins*) of half-integers (*twice-spins*), which correspond to irreducible representations of $\mathrm{su}_q(2)$. Ref. 2 labels all spin networks with twice-spins. Unless otherwise indicated, all conventions in this paper follow Ref. 2. Two spin networks may be equal even if not represented with identical labeled graphs. A complete description of these identities are given in Refs. 2 and 6; their study constitutes *spin network recoupling* and is what allows us to equate[11] spin networks with $\mathrm{su}_q(2)$-invariant tensors and their contractions.

In this correspondence, each index of a tensor, transforming under an irreducible representation, corresponds to a spin network edge, labeled by the same representation (free indices correspond to free edges). In particular, a closed spin network corresponds to a complex number. Spin networks form a graded algebra over $\mathbb{C}$ (as do tensors). The grading is given by the number of free edges (free indices) and the product is diagrammatic juxtaposition (tensor product).

III. DIAGRAMMATIC IDENTITIES

The spin networks with n free edges with fixed labels (*n-valent* spin networks) form a linear space with a natural bilinear form (or inner product). Suppose that the free edges are ordered in some canonical way, then, given two spin networks, we can reflect one of them in a mirror and connect the free edges in order. The value of the resulting closed spin network defines the bilinear form, which is symmetric and non-degenerate[2]. We use the bra-ket notation for this inner product $\langle s'|s\rangle$, where s and s' are two spin networks. We also let $|s\rangle$ stand for s and $\langle s'|$ for the reflection of s'. The existence of an inner product allows the following identities, whose proofs can be found in Ref. 2. For each identity, the corresponding well known fact of $SU(2)$ representation theory is given.

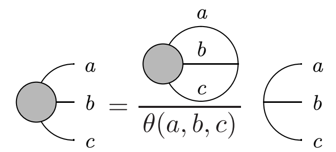
The space of 2-valent spin networks, with ends labeled a and b, is 1-dimensional if $a = b$, and 0-dimensional otherwise. For non-trivial dimension, the single edge gives a complete basis and therefore the *bubble identity*:



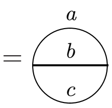
(2)

This identity the diagrammatic analog of Schur's lemma for intertwiners between irreducible representations.

The space of 3-valent spin networks, with ends labeled a, b and c, is also 1-dimensional if the triangle inequalities (A.6) and parity constraints (A.7) are satisfied, and 0-dimensional otherwise, if q is generic. When q is a primitive root of unity, the dimension also vanishes whenever the further r-boundedness constraint (A.8) is violated. In the case of nontrivial dimension, the canonical trivalent vertex gives a complete basis and therefore the *vertex collapse identity*:



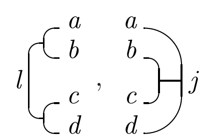
(3)

$\theta(a,b,c) =$ 

(4)

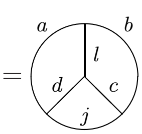
The normalization of the vertex, the value of the θ-network, is evaluated in Eq. (A.5). This identity is the diagrammatic analog of the uniqueness (up to normalization) of the Clebsch-Gordan intertwiner.

Now, consider the space of 4-valent networks with free edges labeled a, b, c and d. There are two natural bases, the vertical $\langle l|$ and the horizontal $|\bar{j}\rangle$:



(5)

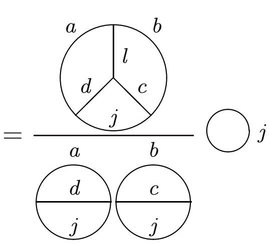
The admissible ranges for j and l, the dimension n of this space, and the conditions on (a, b, c, d) under which $n > 0$ are given by Eqs. (A.16) through (A.27). The transition matrix between the two bases is given by the so-called Tet-network:

$\mathrm{Tet}(a, b, c, d; j, l) =$ 

$= \langle\bar{j}|l\rangle.$ (6)

The coefficients expressing the vertical basis in terms of the horizontal one define the *6j-symbol*, which can be expressed in terms of the Tet-network:

$$|l\rangle = \sum_j \begin{Bmatrix} a & b & j \\ c & d & l \end{Bmatrix}_{KL} |\bar{j}\rangle, \tag{7}$$

$\begin{Bmatrix} a & b & j \\ c & d & l \end{Bmatrix}_{KL} =$ 

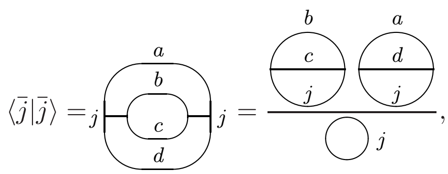
(8)

Note the subscript KL for Kauffman-Lins, since this $6j$-symbol is defined with respect to the conventions of Ref. 2. The relation to the classical Racah-Wigner $6j$-symbol used in the physics literature is given explicitly in Sec. V.

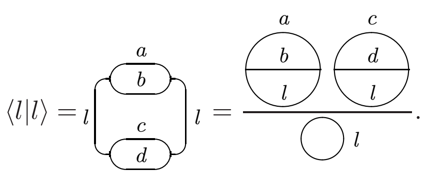
IV. RECURRENCE RELATION FOR THE Tet-NETWORK

The identities given in the previous section allow an elementary derivation of a three-term recurrence relation for the Tet-network.

It is easy to check, using the bubble identity, that both the vertical and horizontal bases are orthogonal and that they are normalized as



(9)

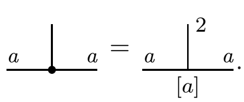


(10)

Curiously, when these normalizations are fully expanded using formulas from the Appendix, they take the form $(-)^{\sigma} P/Q$, where P and Q are products of positive quantum integers. In both cases, $\sigma = (a+b+c+d)/2$, is an integer independent of j or l. When $q = 1$ or when q is a primitive root of unity, positive quantum integers are positive real numbers. Hence the above inner product is real and either positive- or negative-definite. On the other hand, for arbitrary complex q, the normalizations (9) and (10) can be essentially arbitrary complex numbers.

If we can find a linear operator L that is diagonal in one basis, but not in the other, then we can obtain $\langle \bar{j}|l\rangle$ as matrix elements of the diagonalizing transformation. Furthermore, if the non-diagonal form of L is tridiagonal, then the linear equations defining $\langle \bar{j}|l\rangle$ reduce to a three-term recurrence relation.

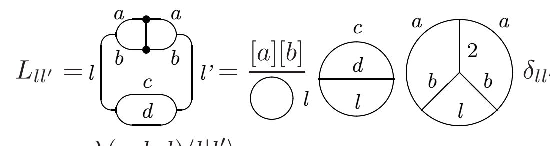
We can construct such an operator by generalizing the argument for the classical case, found in the Appendix of Ref. 3. For brevity of notation, we introduce a special modified version of the trivalent vertex:



(11)

The unlabeled edge implicitly carries twice-spin 2 and the bold dot indicates the multiplicative factor of $[a]$. Using it, we can define a symmetric operator L. Its diagrammatic representation and its matrix elements are given below.

The operator L is diagonal in the $|l\rangle$-basis and its matrix elements $L_{ll'} = \langle l|L|l'\rangle$ are



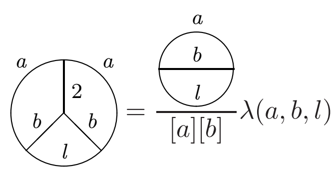
(12)

$$= \lambda(a,b,l)\langle l|l'\rangle, \tag{13}$$

with

$$\lambda(a,b,l) = \frac{\left[\frac{a-b+l}{2}\right]\left[\frac{-a+b+l}{2}\right] - \left[\frac{a+b-l}{2}\right]\left[\frac{a+b+l}{2}+2\right]}{[2]}, \tag{14}$$

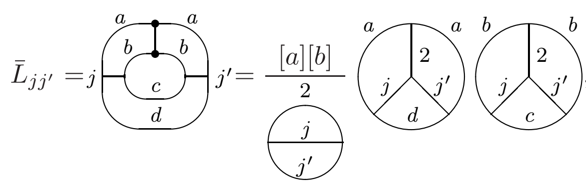
where we have evaluated $\mathrm{Tet}(a,a,b,b;l,2)$ as



(15)

This result may be obtained directly from Eq. (A.9), where the sum reduces to two terms, or from more fundamental considerations[12]. In the limit, $q \to 1$, the eigenvalues simplify to $\lambda(a,b,l) = \frac{1}{4}[l(l+2) - a(a+2) - b(b+2)]$, which shows that the operator L is closely related to the "square of angular momentum" in quantum mechanics, which was used to obtain the classical version of this recurrence relation[3].

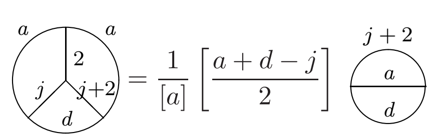
On the other hand, in the $|\bar{j}\rangle$ basis, the operator L is not diagonal and the matrix elements $\bar{L}_{jj'} = \langle \bar{j}|L|\bar{j}'\rangle$, making use of the vertex collapse identity, are



(16)

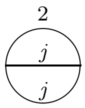
with the special case $\bar{L}_{00} = 0$. Fortunately, though $\bar{L}_{jj'}$ is not diagonal, it is tridiagonal. This property is a consequence of the conditions enforced at the central vertex in both Tet-networks above: the triangle inequality, $|j - j'| \le 2$, and the parity constraint, which forces admissible values of j to change by 2. If these conditions are violated, the matrix element $\bar{L}_{jj'}$ vanishes.

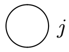
The diagonal elements $\bar{L}_{jj}$ can be evaluated using (15). For the off-diagonal elements $\bar{L}_{j+2,j} = \bar{L}_{j,j+2}$ we also need

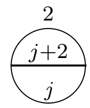


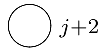
(17)

which can be obtained in the same way as (15). Finally, we need the identities



$= -\frac{[j+2]}{[2][j]}$ 

and 

$=$ 

. (18)

The $|j\rangle$-basis matrix elements can now be expressed as (again, recall the special case $\bar{L}_{00} = 0$)

$$\bar{L}_{jj} = -\langle j|j\rangle \frac{[2]\lambda(a,j,d)\lambda(b,j,c)}{[j][j+2]}, \tag{19}$$

$$\bar{L}_{j,j+2} = \langle j{+}2|j{+}2\rangle \left[\frac{a+d-j}{2}\right]\left[\frac{b+c-j}{2}\right]. \tag{20}$$

The transition matrix elements $\langle\bar{j}|l\rangle$ can now be obtained by solving an eigenvalue problem in the $|\bar{j}\rangle$-basis:

$$\langle\bar{j}|L-\lambda_l|l\rangle = \sum_{j'} \frac{\langle\bar{j}|L-\lambda_l|\bar{j}'\rangle}{\langle\bar{j}'|\bar{j}'\rangle}\langle\bar{j}'|l\rangle, \tag{21}$$

$$0 = \sum_{j'} \left(\frac{\bar{L}_{jj'}}{\langle\bar{j}'|\bar{j}'\rangle} - \lambda_l\delta_{jj'}\right)\langle\bar{j}'|l\rangle, \tag{22}$$

$$0 = \sum_{j'} \left(\bar{L}_{jj'} - \lambda_l\langle\bar{j}'|\bar{j}'\rangle\delta_{jj'}\right)\frac{\langle\bar{j}'|l\rangle}{\langle\bar{j}'|j'\rangle}, \tag{23}$$

where $\lambda_l = \lambda(a,b,l)$. Since $\bar{L}_{jj'}$ is tridiagonal, we obtain a three-term recurrence relation for the $\langle\bar{j}|l\rangle$ transition coefficients. Expanding the expression for $\bar{L}_{jj'}$, we find the following general form of the recurrence relation:

$$\frac{\bar{L}_{j,j-2}}{\langle\overline{j-2}|\overline{j-2}\rangle}\langle\overline{j-2}|l\rangle + \left(\frac{\bar{L}_{jj}}{\langle\bar{j}|\bar{j}\rangle} - \lambda_l\right)\langle\bar{j}|l\rangle + \frac{\bar{L}_{j,j+2}}{\langle\overline{j+2}|\overline{j+2}\rangle}\langle\overline{j+2}|l\rangle = 0, \tag{24}$$

with the provision that $\bar{L}_{jj'}$ vanishes whenever either of the indices fall outside the admissible range or $j = j' = 0$. Finally, the transition coefficients are uniquely determined (up to sign) by requiring the normalization condition

$$\sum_j \frac{\langle l|\bar{j}\rangle\langle\bar{j}|l\rangle}{\langle\bar{j}|\bar{j}\rangle} = \langle l|l\rangle. \tag{25}$$

Practically, it is more convenient to recover the correct normalization of $\langle\bar{j}|l\rangle$ for all j and fixed l, or vice versa, by requiring $\langle\underline{\bar{j}}|l\rangle$, cf. (A.16), to agree with (A.9), where the sum reduces to a single term.

Once the Tet-network has been evaluated recursively, the $6j$-symbol can be obtained from Eq. (7). Alternatively, a linear, three-term recurrence relation directly for the $6j$-symbol follows from (24) and the linear, two-term recurrence relations for the bubble and θ-networks, obvious from (A.4) and (A.5). However, because of the additional normalization factors in Eq. (7), this direct recurrence relation cannot be cast in the form of a symmetric eigenvalue problem like (23).

V. DISCUSSION

In the classical $q = 1$ case, the Kauffman-Lins version of the $6j$-symbol (7) differs from the Racah-Wigner convention used in the physics literature, which preserves the symmetries of the underlying Tet-network. The two $6j$-symbols are related through the formula

$$\begin{Bmatrix} j_1/2 & j_2/2 & j_3/2 \\ J_1/2 & J_2/2 & J_3/2 \end{Bmatrix}_{RW} = \frac{\mathrm{Tet}(J_1,J_2,j_1,j_2;J_3,j_3)}{\sqrt{|\theta(J_1,J_2,j_3)\theta(j_1,j_2,j_3)\theta(J_1,j_2,J_3)\theta(J_2,j_1,J_3)|}}, \tag{26}$$

which can be obtained by comparing the explicit expressions (A.9) and (6.3.7) of Ref. 8. Note that, before the absolute value, the argument of the square root has sign $(-)^{j_3-J_3}$.

Adjusting to the Racah-Wigner convention, the classical version of the recurrence relation (24) reduces to Eqs. (12a-b) of Ref. 3. There, the recurrence was derived in two ways: first by chaining other, more general but less convenient, recurrences and second by obtaining it from a tridiagonal eigenvalue problem. The derivation of Eq. (24) is an elementary generalization of the second method of Ref. 3 to the quantum case.

Other, less convenient, recurrence relations for the quantum $6j$-symbol have previously been obtained by Kachurik and Klimyk[4]. They can be chained together to obtain Eq. (24), generalizing the first method of Ref. 3. However, their derivation is far from elementary, requiring the use of identities for q-hypergeometric functions. Moreover, the interpretation as a symmetric eigenvalue problem, which, as discussed in the Introduction, is advantageous from a numerical point of view, is not obvious even if the recurrence relation (24) is known.

Finally, note that it would have been difficult to obtain (24) directly from the classical version by using the rule given in Ref. 4: replacement of all factorials $m!$ by quantum factorials $[m]!$. This is due to the fact classical integers are not uniquely representable as sums of ratios of factorials. Consider for instance the difference between the "natural" ways of writing $\lambda(a,b,l)$ and its classical limit.

ACKNOWLEDGMENTS

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Appendix: Formulas

For a complex number $q \neq 0$ and an integer n the corresponding *quantum integer* is defined as

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{A.1}$$

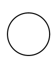
In the limit $q \to 1$, we recover the regular integers, $[n] \to n$. When $q = \exp(i\pi/r)$, for some integer $r > 1$, it is a *primitive root of unity* and the definition reduces to

$$[n] = \frac{\sin(n\pi/r)}{\sin(\pi/r)}, \tag{A.2}$$

This expression is clearly real and positive in the range $0 < n < r$. *Quantum factorials* are direct analogs of classical factorials:

$$[0]! = 1, \quad [n]! = [1][2]\cdots[n]. \tag{A.3}$$

Next, we give the evaluations of some spin networks needed in the paper. They are reproduced from Ch. 9 of Ref. 2. The *bubble diagram* evaluates to



$$j = (-)^j [j+1] \tag{A.4}$$

whenever it is non-vanishing. For generic q, it vanishes if $j < 0$ and if q is a primitive root of unity then it also vanishes when $j > r - 2$. The *θ-network* evaluates to

$$\theta(a,b,c) = \frac{(-)^s [s+1]![s-a]![s-b]![s-c]!}{[a]![b]![c]!}, \tag{A.5}$$

with $s = (a+b+c)/2$, whenever the twice-spins (a,b,c) are admissible and vanishes otherwise. Admissibility consists of the following criteria (besides the obvious $a,b,c \geq 0$):

$$\textit{triangle inequalities} \quad \begin{cases} a \leq b + c \\ b \leq c + a \\ c \leq a + b \end{cases}, \tag{A.6}$$

$$\textit{parity} \quad a + b + c \equiv 0 \pmod 2. \tag{A.7}$$

When q is a primitive root of unity, further constraints needs to be satisfied:

$$r\textit{-boundedness} \quad \begin{cases} a,b,c \leq r-2 \\ a+b+c \leq 2r-4 \end{cases}. \tag{A.8}$$

The *tetrahedral*- or Tet-*network* evaluates to

$$\mathrm{Tet}(a,b,c,d;j,l) = \frac{\mathcal{I}!}{\mathcal{E}!} \sum_S \frac{(-)^S [S+1]!}{\prod_\imath [S-a_\imath]! \prod_\jmath [b_\jmath - S]!}, \tag{A.9}$$

where the summation is over the range $m \leq S \leq M$ and

$$\mathcal{I}! = \prod_{\imath,\jmath} [b_\jmath - a_\imath]!, \qquad \mathcal{E}! = [a]![b]![c]![d]![j]![l]!, \tag{A.10}$$

$$a_1 = (a+d+j)/2, \quad b_1 = (b+d+j+l)/2, \tag{A.11}$$

$$a_2 = (b+c+j)/2, \quad b_2 = (a+c+j+l)/2, \tag{A.12}$$

$$a_3 = (a+b+l)/2, \quad b_3 = (a+b+c+d)/2, \tag{A.13}$$

$$a_4 = (c+d+l)/2, \quad m = \max\{a_\imath\}, \tag{A.14}$$

$$M = \min\{b_\jmath\}. \tag{A.15}$$

The indices $\imath$ and $\jmath$ fully span the defined ranges. Each of the triples of twice-spins (a,b,l), (c,d,l), (a,d,j) and (c,b,j) must be admissible, otherwise the Tet-network vanishes. Then, due to parity constraints, $a_\imath$, $b_\jmath$, m, M, and S are always integers. If the twice-spins (a,b,c,d) are fixed, the admissibility conditions for generic q enforce the ranges of $\underline{j} \leq j \leq \overline{j}$ and $\underline{l} \leq l \leq \overline{l}$ to

$$\underline{j} = \max\{|a-d|, |b-c|\}, \quad \overline{j} = \min\{a+d, b+c\}, \tag{A.16}$$

$$\underline{l} = \max\{|a-b|, |c-d|\}, \quad \overline{l} = \min\{a+b, c+d\}, \tag{A.17}$$

with

$$j \equiv a+b \equiv c+d \pmod 2, \tag{A.18}$$

$$l \equiv a+d \equiv b+c \pmod 2. \tag{A.19}$$

The number of admissible values is the same for j and l and is equal to $n = \max\{0, \bar{n}\}$, where

$$\bar{n} = \min\{m, s-M\} + 1, \quad m = \min\{a,b,c,d\}, \tag{A.20}$$

$$s = (a+b+c+d)/2, \quad M = \max\{a,b,c,d\}. \tag{A.21}$$

This number n is also the dimension of the space of 4-valent spin networks with fixed twice-spins (a,b,c,d) labeling the free edges. This dimension is non-vanishing, $n > 0$, precisely when the twice-spins satisfy the conditions

$$a+b+c+d \leq 2\max\{a,b,c,d\}, \tag{A.22}$$

$$a+b+c+d \equiv 0 \pmod 2. \tag{A.23}$$

When q is a primitive root of unity, the admissible ranges shrink to $\underline{l} \leq j \leq \overline{j}_r$ and $\underline{l} \leq l \leq \overline{l}_r$, where

$$\overline{j}_r = \min\left\{\overline{j}, r-2, 2r-4-\max\{a+d, b+c\}\right\}, \tag{A.24}$$

$$\overline{l}_r = \min\left\{\overline{l}, r-2, 2r-4-\max\{a+b, c+d\}\right\}. \tag{A.25}$$

The number of admissible values in each range is thus restricted to $n = \max\{0, \bar{n}_r\}$, where

$$\bar{n}_r = \min\left\{\bar{n}, r-1-\max\{M, s-m\}\right\}. \tag{A.26}$$

The condition $n > 0$ requires (A.22), (A.23) and

$$a+b+c+d \leq 2\min\{a,b,c,d\} + 2r - 4. \tag{A.27}$$

The above admissibility criteria are well known. However, the consequent explicit expressions for the constraints on (a, b, c, d), the bounds on j and l, and the dimension n are not easily found in the literature.

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[10] Spin networks are actually ribbon graphs, but since all diagrams in this paper are planar, the ribbon structure can be added through blackboard framing.

[11] A down-to-earth guide to this correspondence, for the classical $q = 1$ case, can be found in the Appendix A of Ref. 13. Complete details with proofs can be found in Ref. 6.

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