Labeled Ballot Paths and the Springer Numbers

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Abstract. The Springer numbers are defined in connection with the irreducible root systems of type B_n, which also arise as the generalized Euler and class numbers introduced by Shanks. Combinatorial interpretations of the Springer numbers have been found by Purtill in terms of André signed permutations, and by Arnol'd in terms of snakes of type B_n. We introduce the inversion code of a snake of type B_n and establish a bijection between labeled ballot paths of length n and snakes of type B_n. Moreover, we obtain the bivariate generating function for the number $B(n,k)$ of labeled ballot paths starting at $(0,0)$ and ending at (n,k). Using our bijection, we find a statistic α such that the number of snakes π of type B_n with $\alpha(\pi) = k$ equals $B(n,k)$. We also show that our bijection specializes to a bijection between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$.

Keywords: Springer number, snake of type B_n, labeled ballot path, labeled Dyck path, bijection

AMS Subject Classifications: 05A05, 05A19

1   Introduction

The Springer numbers are introduced by Springer [10] in the study of irreducible root system of type B_n. Let S_n denote the n-th Springer number. The sequence $\{S_n\}_{n\geq0}$ is listed as entry $A001586$ in OEIS [9]. The first few values of S_n are

$$1, 1, 3, 11, 57, 361, 2763, 24611, \ldots.$$

To be more specific, S_n can be defined as follows. Let V be a real vector space, R be a root system of type B_n in V, and W be the Weyl group of R. It is known that for a fixed simple root set S of R, any $\alpha \in R$ is either a positive or a negative linear combination of elements of S, denoted by $\alpha > 0$ or $\alpha < 0$. For a subset $I \subset S$, let $\sigma(I, S)$ denote the number of elements $w \in W$ such that $w\alpha > 0$ for any $\alpha \in I$ and $w\alpha < 0$ for any $\alpha \in S - I$. Then the Springer number S_n can be defined as the maximum value of $\sigma(I, S)$ among $I \subset S$. Springer derived the following generating function,

$$\sum_{n\geq0} S_n \frac{x^n}{n!} = \frac{1}{\cos x - \sin x}. \tag{1.1}$$

On the other hand, Hoffman [5] pointed out that the Springer numbers also arise as the generalized Euler and class numbers $s_{m,n}$ $(n \geq 0)$ for $m = 2$, where the numbers $s_{m,n}$ are introduced by Shanks [8] based on the Dirichlet series

$$L_m(s) = \sum_{k=0}^{\infty} \left(\frac{-m}{2k+1} \right) \frac{1}{(2k+1)^s}.$$

Note that the above notation $(-m/(2k+1))$ is the Jacobi symbol. To be precise, the generalized Euler and class numbers $s_{2,n}$ are defined by

$$s_{2,n} = \begin{cases} c_{2,\frac{n}{2}}, & \text{if } n \text{ is even;} \\ d_{2,\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

where the numbers $c_{2,n}$ and $d_{2,n}$ are given by

$$c_{2,n} = \frac{(2n)!}{\sqrt{2}} \left(\frac{\pi}{4} \right)^{-2n-1} L_2(2n+1),$$

$$d_{2,n} = \frac{(2n-1)!}{\sqrt{2}} \left(\frac{\pi}{4} \right)^{-2n} L_{-2}(2n).$$

According to the following recurrence relations for $c_{2,n}$ and $d_{2,n}$ derived by Shanks [8],

$$\sum_{i=0}^{n} (-4)^i \binom{2n}{2i} c_{2,n-i} = (-1)^n,$$

$$\sum_{i=0}^{n-1} (-4)^i \binom{2n-1}{2i} d_{2,n-i} = (-1)^{n-1},$$

one sees that the numbers $s_{2,n}$ are integers. In fact, the above recurrence relations lead to the following formulas

$$\sum_{n \geq 0} c_{2,n} \frac{x^{2n}}{(2n)!} = \sec 2x \cos x,$$

$$\sum_{n \geq 1} d_{2,n} \frac{x^{2n-1}}{(2n-1)!} = \sec 2x \sin x.$$

Shanks raised the question of finding combinatorial interpretations for the Euler and class numbers $s_{m,n}$. For $m = 2$, $s_{2,n}$ is the n-th Springer number. Purtill [6] gave an interpretation of the Springer numbers in terms of the André signed permutations on $[n] = \{1, 2, \ldots, n\}$. Arnol'd [1] found another interpretation of the Springer numbers in terms of snakes of type B_n. Recall that a snake of type B_n is an alternating signed permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ on $[n]$ such that

$$0 < \pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots \pi_n. \tag{1.2}$$

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For example, $1\bar{3}2$ is a snake of type B_3. Intuitively, a signed permutation on $[n]$ can be viewed as an ordinary permutation $\pi_1 \pi_2 \cdots \pi_n$ with some elements associated with minus signs. An element i with a minus sign is often written as $\bar{i}$. The above alternating or up-down condition (1.2) is based on the following natural order:

$$\bar{n} < \ldots < \bar{1} < 1 < \ldots < n.$$

Arnol'd [1] proved that the Springer number S_n equals the number of snakes of type B_n. Hoffman [5] showed that the exponential generating function for the number of snakes of type B_n also equals the right hand side of (1.1), that is, the generating function of the Springer numbers. Recently, Chen, Fan and Jia [3] obtained a formula for the generating function of $s_{m,n}$ for arbitrary m, which in principle leads to a combinatorial interpretation of the numbers $s_{m,n}$ in terms of alternating augmented m-signed permutations. Note that for $m = 2$, alternating augmented 2-signed permutations are exactly snakes of type B_n.

The objective of this paper is to give a combinatorial interpretation for the Springer numbers in terms of labeled ballot paths. In fact, we shall introduce the inversion code of a snake of type B_n. By using the inversion code, we construct a bijection between the set of snakes of type B_n and the set of labeled ballot paths of length n. Let $B(n,k)$ denote the number of labeled ballot paths starting at $(0,0)$ and ending at (n,k). Then the numbers $B(n,k)$ can be viewed as a refinement of the Springer numbers. Using the recurrence relation of $B(n,k)$, we obtain the generating function for $B(n,k)$ for any given n.

Using our bijection, we find a statistic α on snakes of type B_n such that the number of snakes π of type B_n with $\alpha(\pi) = k$ equals $B(n,k)$. A labeled ballot path that eventually returns to the x-axis is called a labeled Dyck path. When $k = 0$, $B(2n,0)$ is the number of labeled Dyck paths of length $2n$. We find that $B(2n,0)$ and the number E_{2n} of alternating permutations on $[2n]$ have the same generating function, and we show that our bijection for labeled ballot paths and snakes of type B_n reduces to a bijection between labeled Dyck paths and alternating permutations.

The paper is organized as follows. In Section 2, we give descriptions of the map Φ from snakes of type B_n to labeled ballot paths of length n, and the map Ψ from labeled ballot paths of length n to snakes of type B_n. In Section 3, we prove that the maps Φ and Ψ are well defined, and they are inverses of each other. The last section is devoted to the bivariate generating function for the numbers $B(n,k)$ and the classification of snakes of type B_n in accordance with the numbers $B(n,k)$. We also show that the map Ψ restricted to labeled Dyck paths serves as a combinatorial interpretation of the fact that $B(2n,0)$ equals E_{2n}.

2 The bijection

In this section, we define a class of labeled ballot paths and establish a bijection between such labeled ballot paths of length n and snakes of type B_n. Recall that a ballot path of

length n is a lattice path with n steps from the origin consisting of up steps $u = (1, 1)$ and down steps $d = (1, -1)$ that do not go below the x-axis. As a special case, a Dyck path is a ballot path of length $2n$ that ends at the x-axis. A ballot path is also called a partial Dyck path [2]. The height of a step of a ballot path is defined to be the smaller y-coordinate of its endpoints. By a labeled ballot path we mean a ballot path for which each step is endowed with a nonnegative integer that is less than or equal to its height. A labeled ballot path $P = p_1 p_2 \cdots p_n$ for which step p_i is labeled by w_i is denoted by $(P; W)$, where $W = w_1 w_2 \cdots w_n$.

For example, for a ballot path $P = uuudduu$, there are 216 labelings. Figure 1 gives a labeling of the ballot path P.
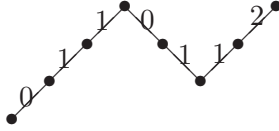

Figure 1: A labeled ballot path $(uuudduu; 0110112)$ of length 7.

For $n = 3$, there are 3 ballot paths $P_1 = uuu, P_2 = uud$ and $P_3 = udu$. There are 6 labelings for P_1, 4 labelings for P_2 and 1 labeling for P_3. On the other hand, there are 11 snakes of type B_3 as listed below:

$$1\bar{2}3, 1\bar{3}2, 1\bar{3}\bar{2}, 213, 2\bar{1}3, 2\bar{3}1, 2\bar{3}\bar{1}, 312, 3\bar{1}2, 3\bar{2}1, 3\bar{2}\bar{1}.$$

In order to establish a bijection between ballot paths of length n and snakes of type B_n, we introduce the inversion code of a snake π of type B_n. Write $\pi = \pi_1 \cdots \pi_n$. We define $c_i(\pi)$ as follows

$$c_i(\pi) = \begin{cases} \#\{(\pi_{2k}, \pi_{2k+1}) | 1 \le k \le (n-1)/2, i < 2k, \pi_{2k} < \pi_i < \pi_{2k+1}\}, & \text{if } n \text{ is odd;} \\ \#\{(\pi_{2k-1}, \pi_{2k}) | 1 \le k \le n/2, i < 2k-1, \pi_{2k} < \pi_i < \pi_{2k-1}\}, & \text{if } n \text{ is even.} \end{cases}$$

The sequence $(c_1(\pi), c_2(\pi), \ldots, c_n(\pi))$, denoted $c(\pi)$, is called the inversion code of π. For example, let $n = 7$ and $\pi = 3\bar{5}21\bar{4}76$. Then the inversion code of π is $(2, 1, 2, 1, 1, 0, 0)$. For $n = 8$ and $\pi = 538\bar{2}\bar{1}476$, the inversion code of π is $(1, 1, 0, 1, 0, 0, 0, 0)$.

We are now ready to describe the map Φ from a snake $\pi = \pi_1 \pi_2 \cdots \pi_n$ of type B_n to a labeled ballot path $(P; W) = (p_1 p_2 \cdots p_n; w_1 w_2 \cdots w_n)$. Suppose that $p_1, p_2, \ldots, p_{k-1}$ and their labels $w_1, w_2, \ldots, w_{k-1}$ have been determined, we proceed to demonstrate how to determine p_k and its label w_k. If we were in Step 1, namely, for $k = 1$, we would locate the element n or $\bar{n}$ in π, and would assume that $\pi_i = n$ or $\bar{n}$. Suppose that we are in Step k. Now we look for the element $n - k + 1$ or $\overline{n - k + 1}$ in π. Here are two cases.

Case 1. Assume that $\pi_i = n - k + 1$. If i is odd, then set $p_k = u$; if i is even, then set $p_k = d$. Set $w_k = c_i(\pi)$.

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Case 2. Assume that $\pi_i = \overline{n-k+1}$. If i is odd, then set $p_k = d$; if i is even, then set $p_k = u$. Set $w_k = h_k - c_i(\pi)$, where h_k denotes the height of the k-th step p_k in the ballot path $p_1 p_2 \cdots p_k$.

For example, let $n = 7$ and $\pi = 2\bar{1}547\bar{6}\bar{3}$. The construction of $\Phi(\pi)$ is illustrated in Figure 2.
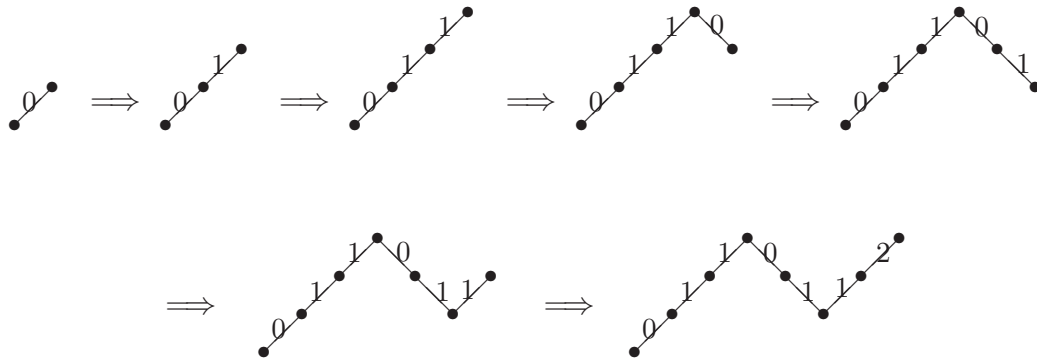

Figure 2: The construction of $\Phi(\pi)$ for $\pi = 2\bar{1}547\bar{6}\bar{3}$.

We now turn to the inverse map Ψ from a labeled ballot path $(P; W) = (p_1 p_2 \cdots p_n; w_1 w_2 \cdots w_n)$ to a snake $\pi = \pi_1 \pi_2 \cdots \pi_n$ of type B_n.

We shall construct a sequence of permutations $\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_n$, such that $\Gamma_0 = \emptyset$ and $\Gamma_n = \pi$ is the desired snake of type B_n. To reach this goal, we generate a sequence of labeled ballot paths $(P_1; W_1), (P_2; W_2), \cdots, (P_{n-1}; W_{n-1})$, where $(P_1; W_1) = (P; W)$, and P_{i+1} is obtained from P_i by contracting a certain step p_{r_i} of P_i into a single point, and W_{i+1} is obtained from W_i by deleting the label of the step p_{r_i} and updating the labels of the other steps. Notice that P_i has $n - i + 1$ steps and W_i has $n - i + 1$ elements for $1 \le i \le n$. Below is a procedure to determine $(P_{i+1}; W_{i+1})$ and Γ_i from $(P_i; W_i)$ and Γ_{i-1}. Let us consider two cases.

Case 1: P_i has an odd number of steps.

If there exists a down step in P_i whose label equals its height, then we assume that p_{r_i} is the leftmost among such down steps. Contract p_{r_i} into a single point to form a ballot path P_{i+1} and add 1 to the labels of all down steps of P_{i+1}. Let $(P_{i+1}; W_{i+1})$ denote the resulting labeled ballot path and set $\Gamma_i = \overline{n - r_i + 1}\Gamma_{i-1}$.

For the case that the label of any down step P_i is less than its height, as will be shown, there must exist at least one up step labeled by 0. We assume that p_{r_i} is the rightmost among such up steps. Contract p_{r_i} into a single point to form a ballot path P_{i+1}. Then subtract 1 from the labels of up steps of P_{i+1} that are originally to the right of p_{r_i} and add 1 to the labels of down steps of P_{i+1} that are originally to the left of p_{r_i}. Denote the resulting labeled ballot path by $(P_{i+1}; W_{i+1})$ and set $\Gamma_i = (n - r_i + 1)\Gamma_{i-1}$.

Case 2: P_i has an even number of steps.

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If there exists a down step of P_i whose label equals 0, we assume that p_{r_i} is the leftmost among such down steps. Contract p_{r_i} into a single point to form a ballot path P_{i+1}. Then add 1 to the labels of up steps of P_{i+1} which are originally to the right of p_{r_i} and subtract 1 from the labels of down steps of P_{i+1} which are originally to the left of p_{r_i}. Denote the resulting labeled ballot path by $(P_{i+1}; W_{i+1})$ and set $\Gamma_i = (n - r_i + 1)\Gamma_{i-1}$.

For the case that there are no down steps in P_i labeled by 0, as can be seen, there must exist at least one up step whose label equals its height. We assume that p_{r_i} is the rightmost among such up steps. Contract p_{r_i} into a single point to form a ballot path P_{i+1}. Then subtract 1 from the labels of all down steps of P_{i+1}. Denote the resulting path by $(P_{i+1}; W_{i+1})$ and set $\Gamma_i = \overline{n - r_i + 1}\Gamma_{i-1}$.

For the labeled ballot path $(P; W) = (uuudduu; 0110112)$ in Figure 1, the construction of $\Psi(P; W)$ is shown in Figure 3. The indices of the steps p_{r_i} that are contracted are listed below: $r_1 = 5, r_2 = 2, r_3 = 1, r_4 = 4, r_5 = 3, r_6 = 7, r_7 = 6$. The labeled ballot paths $(P_i; W_i)$ are given in Figure 3, and the permutations Γ_i are given as follows:

$$\Gamma_0 = \emptyset, \Gamma_1 = \bar{3}, \Gamma_2 = \bar{6}\bar{3}, \Gamma_3 = 7\bar{6}\bar{3}, \Gamma_4 = 47\bar{6}\bar{3}, \Gamma_5 = 547\bar{6}\bar{3}, \Gamma_6 = \bar{1}547\bar{6}\bar{3}, \Gamma_7 = 2\bar{1}547\bar{6}\bar{3}.$$

$$P_1 = p_1p_2p_3p_4p_5p_6p_7 \qquad P_2 = p_1p_2p_3p_4p_6p_7$$

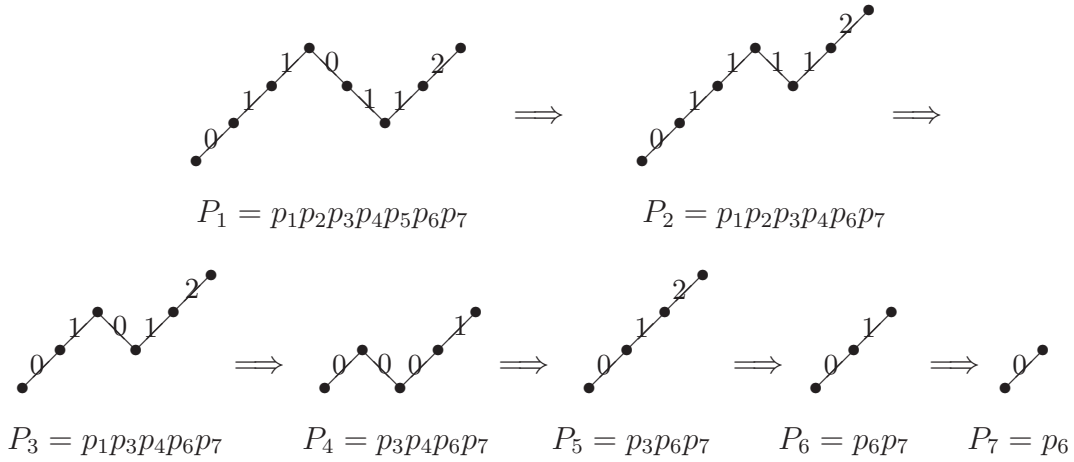$$P_3 = p_1p_3p_4p_6p_7 \qquad P_4 = p_3p_4p_6p_7 \qquad P_5 = p_3p_6p_7 \qquad P_6 = p_6p_7 \qquad P_7 = p_6$$

Figure 3: The construction of $\Psi(P; W)$ for the labeled ballot path in Figure 1.

3   The proof

In this section, we shall show that the map Φ described in the previous section is indeed a bijection.

Theorem 3.1 *The map Φ is a bijection between labeled ballot paths of length n and snakes of type B_n.*

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Proof. As the first step, we verify that Φ is well-defined, that is, for any snake $\pi = \pi_1 \pi_2 \cdots \pi_n$ of type B_n, $\Phi(\pi)$ is a labeled ballot paths.

Before we show that $\Phi(\pi) = (P; W)$ is a labeled ballot path, it is necessary to prove that $P = p_1 p_2 \cdots p_n$ is a ballot path, that is, for any $1 \leq k \leq n$, the number of up steps is not less than the number of down steps among the first k steps of P. By the definition of Φ, we have $p_1 = u$. Assume that in Step k in the implementation of Φ, we have already constructed $p_1, p_2, \ldots, p_{k-1}$ which form a ballot path. The task of this step is to locate $n - k + 1$ or $\overline{n - k + 1}$ in π in order to get p_k. We consider two cases.

If $\pi_i = n - k + 1$ and i is odd or $\pi_i = \overline{n - k + 1}$ and i is even, then we set $p_k = u$. Clearly, $p_1 p_2 \cdots p_k$ is a ballot path. Otherwise, we set $p_k = d$ and we wish to show that the height h_k of p_k is nonnegative. Consider the case $\pi_i = n - k + 1$ and i is odd. Observe that the height of p_k is the number of up steps among $p_1, p_2, \cdots, p_{k-1}$ subtracts the number of down steps among $p_1, p_2, \cdots, p_k$. By the definition of Φ, we have

$$
\begin{aligned}
h_k = {} & \#\{1 \leq j \leq n \mid \pi_j > 0, n - k + 1 < \pi_j \text{ and } j \text{ is odd}\} \\
& + \#\{1 \leq j \leq n \mid \pi_j < 0, n - k + 1 < |\pi_j| \text{ and } j \text{ is even}\} \\
& - \#\{1 \leq j \leq n \mid \pi_j < 0, n - k + 1 \leq |\pi_j| \text{ and } j \text{ is odd}\} \\
& - \#\{1 \leq j \leq n \mid \pi_j > 0, n - k + 1 < \pi_j \text{ and } j \text{ is even}\}. \tag{3.3}
\end{aligned}
$$

In view of the alternating property of π, if there is a negative element π_{2i+1} at an odd position of π, then π_{2i} must be negative as well and $\pi_{2i+1} > \pi_{2i}$. Consequently,

$$
\begin{aligned}
& \#\{1 \leq j \leq n \mid \pi_j < 0, n - k + 1 \leq |\pi_j| \text{ and } j \text{ is odd}\} \\
& \quad \leq \#\{1 \leq j \leq n \mid \pi_j < 0, n - k + 1 < |\pi_j| \text{ and } j \text{ is even}\}.
\end{aligned}
$$

On the other hand, if there is a positive element π_{2i} at an even position of π, then π_{2i-1} must be positive as well and $\pi_{2i} < \pi_{2i-1}$. This yields that

$$
\begin{aligned}
& \#\{1 \leq j \leq n \mid \pi_j > 0, n - k + 1 < \pi_j \text{ and } j \text{ is even}\} \\
& \quad \leq \#\{1 \leq j \leq n \mid \pi_j > 0, n - k + 1 < \pi_j \text{ and } j \text{ is odd}\}.
\end{aligned}
$$

Thus we deduce that whenever there is a negative term contributing to h_k, there is at least one positive term. So we conclude that $h_k \geq 0$. A similar argument applies to the case that $\pi_i = \overline{n - k + 1}$ and i is even. Hence we have shown that P is a ballot path.

We next prove that the label of any step in $\Phi(\pi)$ is nonnegative and it does not exceed its height. Let $\pi = \pi_1 \cdots \pi_i \cdots \pi_n$. Assume we are in the Step k and we have determined $(p_1 \ldots p_{k-1}; w_1, \ldots, w_{k-1})$, which is a labeled ballot path of length $k - 1$. We proceed to locate $n - k + 1$ or $\overline{n - k + 1}$ in π in order to determine p_k and its label w_k. Suppose that $\pi_i = \overline{n - k + 1}$ and i is odd. In this case, by the definition of Φ, we have $p_k = d$ and $w_k = h_k - c_i(\pi)$. We claim that $c_i(\pi) \leq h_k$. In computing h_k by using formula (3.3), we shall split the range of j into two cases: one case is $1 \leq j \leq i$ and the other case is $i + 1 \leq j \leq n$. In other words, we shall consider the contributions of $\pi_1 \pi_2 \ldots \pi_i$ and $\pi_{i+1} \ldots \pi_n$ to the value of h_k.

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We claim that $c_i(\pi)$ is less than or equal to the contribution of $\pi_{i+1} \ldots \pi_n$ to h_k. Suppose that n is odd. By the definition of $c_i(\pi)$, a pair (π_{2j}, π_{2j+1}) of consecutive elements of π with $i < 2j \leq n-1$ contributes 1 to the value of $c_i(\pi)$ if $\pi_{2j} < \pi_i < \pi_{2j+1} < 0$ or $\pi_{2j} < \pi_i < 0$ and $\pi_{2j+1} > 0$. If there is a pair (π_{2j}, π_{2j+1}) with $\pi_{2j} < \pi_i < \pi_{2j+1} < 0$, then this pair contributes 1 to both h_k and $c_i(\pi)$. If there is a pair (π_{2j}, π_{2j+1}) with $\pi_{2j} < \pi_i < 0$ and $\pi_{2j+1} > 0$, then this pair contributes 1 or 2 to h_k (depends on whether $|\pi_{2j+1}|$ is greater than $n-k+1$), while contributes exactly 1 to $c_i(\pi)$. It is straightforward to check that if a pair (π_{2j}, π_{2j+1}) does not contribute to $c_i(\pi)$, then it contributes 0 or 1 to h_k. On the hand hand, because $\pi_1 \ldots \pi_i$ contributes 0 to $c_i(\pi)$, it remains to show that the contribution of $\pi_1 \ldots \pi_i$ to h_k is nonnegative. Let

$$
\begin{aligned}
g_i(\pi) = {}& \#\{1 \leq j \leq i \,|\, \pi_j > 0, n - k + 1 < \pi_j \text{ and } j \text{ is odd}\} \\
& + \#\{1 \leq j \leq i \,|\, \pi_j < 0, n - k + 1 < |\pi_j| \text{ and } j \text{ is even}\} \\
& - \#\{1 \leq j \leq i \,|\, \pi_j < 0, n - k + 1 \leq |\pi_j| \text{ and } j \text{ is odd}\} \\
& - \#\{1 \leq j \leq i \,|\, \pi_j > 0, n - k + 1 < \pi_j \text{ and } j \text{ is even}\}.
\end{aligned}
$$

By the same reasoning as in the proof of $h_k \geq 0$, we can verify that $g_i(\pi) \geq 0$. Thus we have completed the proof for the case that $\pi_i = n - k + 1$ and both n, i are odd. All the other cases depending on the sign of π_i and the parities of n and i can be treated in the same manner. Hence the details are omitted.

Our next task is to show that the map Ψ is well-defined, namely, for any labeled ballot path $(P; W)$ of length n, the signed permutation $\pi = \Gamma_n = \pi_1 \pi_2 \cdots \pi_n$ is a snake of type B_n, i.e., $0 < \pi_1 > \pi_2 < \pi_3 > \cdots \pi_n$.

Suppose that at the i-th step we have already constructed a labeled ballot path $(P_i; W_i)$. We first consider the case that P_i has an odd number of steps. In this case we aim to show that after contracting a certain step of P_i, we can get a ballot path P_{i+1}. By our construction of Ψ, if there is a down step in P_i whose label equals its height, then we contract the leftmost such down step in P_i. In this case, we automatically get a ballot path P_{i+1}. Otherwise, we consider the case that there exist no such down steps. In particular, this implies that there are no down steps with height 0. By our construction Ψ, we shall contract the rightmost up step labeled by 0. After we contract this up step, it is easily seen that every step in P_{i+1} has nonnegative height since we know that there are no down steps that touch the x-axis. So we also get a ballot path P_{i+1} in this case. One can check that after we update the labels of the steps in P_{i+1}, each step will have a nonnegative label that is less than or equal to its height.

The case that P_i has an even number of steps can be dealt with by the same argument as for the case that P_i has an odd number of steps. Thus we conclude that once we have accomplished the mission in step i, we are led to a labeled ballot path (P_{i+1}, W_{i+1}) and a signed permutation Γ_i.

We now turn to the proof of the alternating property of π. It is apparent from the construction of Ψ that $\pi_1 > 0$. Now we prove that $\pi_1 > \pi_2 < \pi_3 > \ldots \pi_n$. Suppose that in step $i - 1$ we have already constructed a signed permutation Γ_{i-1} and a labeled

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ballot path $(P_i; W_i)$. To determine Γ_i, by our construction, we are supposed to contract a certain step p_{r_i} in P_i to form a ballot path P_{i+1} and to set $\Gamma_i = (n - r_i + 1)\Gamma_{i-1}$ or $\Gamma_i = \overline{n - r_i + 1}\Gamma_{i-1}$ depending on whether p_{r_i} is an up step or a down step.

To determine Γ_{i+1}, by our construction, we are supposed to contract a certain step $p_{r_{i+1}}$ of P_{i+1} to form a ballot path P_{i+2} and to set $\Gamma_{i+1} = (n - r_{i+1} + 1)\Gamma_i$ or $\Gamma_{i+1} = \overline{n - r_{i+1} + 1}\Gamma_i$ depending on whether $p_{r_{i+1}}$ is an up step or a down step. For notational convenience, set $t_i = n - r_i + 1$ and $t_{i+1} = n - r_{i+1} + 1$. There are four possibilities for the construction of Γ_{i+1}, namely, $t_{i+1}t_i\Gamma_{i-1}, \bar{t}_{i+1}\bar{t}_i\Gamma_{i-1}, \bar{t}_{i+1}t_i\Gamma_{i-1}$ and $t_{i+1}\bar{t}_i\Gamma_{i-1}$.

We only consider the case that P_i has an odd number of steps and so t_i is at an odd position of π. To prove the alternating property of Γ_n, it is necessary to verify that $t_{i+1} < t_i$, $\bar{t}_{i+1} < \bar{t}_i$ and $\bar{t}_{i+1} < t_i$. And the situation that $\Gamma_{i+1} = t_{i+1}\bar{t}_i\Gamma_{i-1}$ can never happen.

In this case, in the i-th step, suppose that we contract a down step p_{r_i} of P_i, and in the $(i+1)$-st step, suppose that we contract an up step $p_{r_{i+1}}$ of P_{i+1}. By the construction of Ψ, we have $\Gamma_{i+1} = \bar{t}_{i+1}\bar{t}_i\Gamma_{i-1}$. We claim that $t_i < t_{i+1}$, i.e., $r_i > r_{i+1}$. Otherwise, we may assume that $r_i < r_{i+1}$. Once the down step p_{r_i} is contracted, the height of all steps to the right of the step p_{r_i} will increase by 1, but by the construction of Ψ, the labels of up steps remain unchanged. This implies that the labels of up steps to the right of p_{r_i} cannot be equal to their heights. Therefore, the up step $p_{r_{i+1}}$ cannot be chosen in the $(i+1)$-st step, which is a contradiction. So we deduce that $\bar{t}_{i+1} < \bar{t}_i$. The discussions for the cases that $t_{i+1} < t_i$, $\bar{t}_{i+1} < t_i$ and the situation that $\Gamma_{i+1} = t_{i+1}\bar{t}_i\Gamma_{i-1}$ can never happen are similar.

We are now left with the case that the number of steps of P_i is even to complete the proof of the alternating property. But the argument in this case is analogous to that for the case that $n - i$ is even. Hence we have reached the conclusion that $\Gamma_n = \pi_1\pi_2\cdots\pi_n$ is a snake of type B_n.

Finally, we wish to confirm that the maps Φ and Ψ are inverses of each other. Because both Φ and Ψ are carried out in n steps, it suffices to verify that the i-th step of Φ and the i-th step of Ψ are inverses of each other. Let $\pi = \pi_1\pi_2\cdots\pi_n$. For $1 \leq i \leq n$, let $\Pi_i = \pi_1\pi_2\cdots\pi_i$. Define $\Phi(\Pi_i)$ to be the labeled ballot path by applying Φ to the standardization of Π_i. The standardization of a snake Π_i is a snake obtained by keeping the sign of each element unchanged and replacing the smallest element by 1, the second smallest element by 2, and so forth. Note that one can apply Φ to $\Pi_1, \cdots, \Pi_n$ step by step and finally obtain $\Phi(\pi) = \Phi(\Pi_n)$. By the construction of Ψ, one can check that $\Phi(\Pi_i) = (P_{n-i+1}; W_{n-i+1})$ for $1 \leq i \leq n$. That is, the inverse procedure to derive $\Phi(\Pi_{i+1})$ from $\Phi(\Pi_i)$ coincides with the procedure to construct $\Psi(P_{n-i+2}; W_{n-i+2})$ from $\Psi(P_{n-i+1}; W_{n-i+1})$. Therefore Φ and Ψ are inverses of each other.

In summary, we have shown that that the map Φ is a bijection between labeled ballot paths of length n and snakes of type B_n. ∎

4 A refinement

In this section, we obtain the bivariante generating function for the number $B(n, k)$ of labeled ballot paths of length n that end at a given point (n, k), where $0 \leq k \leq n$. The numbers $B(n, k)$ can be considered as a refinement of the Springer numbers. By restriction of the bijection Ψ, we also obtain a bijection between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$. By considering the last step of a labeled ballot path, it is easy to derive the following recurrence relation.

Theorem 4.1 *For $1 \leq k \leq n$, we have*
$$B(n, k) = (k+1)B(n-1, k+1) + kB(n-1, k-1). \tag{4.4}$$

Note that in the above recurrence relation we need the convention that $B(n, k) = 0$ for $n < k$. Moreover, since a ballot path can never ends at a point (m, n) where $m + n$ is odd, so $B(n, k) = 0$ if $n + k$ is odd.
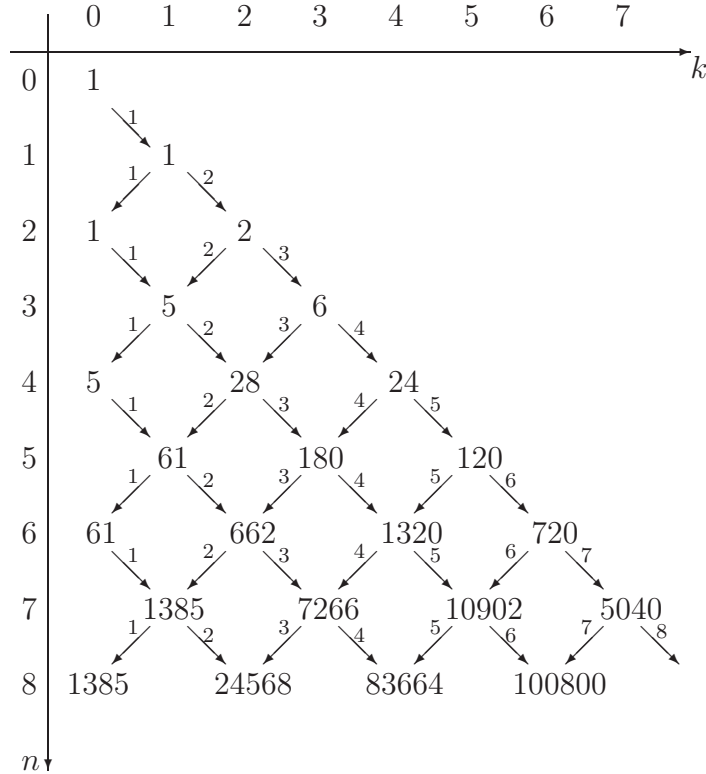

Figure 4: The recurrence relation for $B(n, k)$

Note that when $k = 0$, $B(2n, 0)$ is the number of labeled Dyck paths of length $2n$, where a labeled Dyck path of length $2n$ is a labeled ballot path of length $2n$ that ends

with a point on the x-axis. It is worth mentioning that the numbers $B(2n, 0)$ are in fact the secant numbers and they are closely related to alternate level codes of ballots, see Strehl [12]. Recall that an alternate level code of ballots of length n is an integer sequence $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$ such that $\lambda_1 = 1$, and for $2 \leq j \leq n$,

$$\lambda_{j-1} + 1 \geq \lambda_j \geq 1.$$

Denote by Λ_n the set of alternate level codes of ballots of length n. For example,

$$\Lambda_3 = \{111, 112, 121, 122, 123\}.$$

Rosen [7] derived the following formula

$$\sum_{n \geq 0} \left(\sum_{\lambda \in \Lambda_n} \prod_{i=1}^{n} \lambda_i(\lambda_i + 1) \right) \frac{x^n}{n!} = \tan x. \tag{4.5}$$

Strehl [12] deduced the secant companion equation of (4.5):

$$\sum_{n \geq 0} \left(\sum_{\lambda \in \Lambda_n} \prod_{i=1}^{n} \lambda_i^2 \right) \frac{x^n}{n!} = \sec x. \tag{4.6}$$

To make a connection between labeled Dyck paths and alternate level codes of ballots, we need the following bijection, see Stanley [11, Ex. 6.19].

Theorem 4.2 *There is a bijection between the set of Dyck paths of length $2n$ and the set of alternate level codes of ballots of length n.*

Proof. Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n \in \Lambda_n$ be an alternate level code of ballots of length n. For convenience, we set $\lambda_{n+1} = 1$. We shall construct a Dyck path P of length $2n$ from λ. Let $P = P_1 P_2 \cdots P_n$, where $P_i = ud^k$ for some $k \geq 0$, that is, P_i consists of an up step followed by k down steps. For $1 \leq i \leq n$, if $\lambda_i = \lambda_{i+1} - 1$ then $k = 0$, $P_i = u$. If $\lambda_i \geq \lambda_{i+1}$ then $P_i = ud^{\lambda_i - \lambda_{i+1} + 1}$. It is necessary to show that the above construction generates a Dyck path of length $2n$. That is, after the i-th step, the number of down steps is less than or equal to the number of up steps. That is, we wish to show that

$$\sum_{j=1}^{i} (\lambda_j - \lambda_{j+1} + 1) \leq i.$$

Since $\lambda_1 = 1$ and $\lambda_{i+1} \geq 1$, the above inequality is apparently true. Moreover, it is easy to check that there are n down steps, namely,

$$\sum_{j=1}^{n} (\lambda_j - \lambda_{j+1} + 1) = n.$$

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Conversely, given a Dyck path of length $2n$, let λ_i be the height of the i-th up step plus one. It is readily verified that $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$ is an alternate level code of ballots of length n. This completes the proof. ∎

For instance, let $\lambda = 122 \in \Lambda_3$. Then the Dyck path corresponding to λ is $uududd$. Using the above bijection we are led to a connection between the number $B(2n, 0)$ and alternate level codes of ballots.

Corollary 4.3 *We have*

$$B(2n, 0) = \sum_{\lambda \in \Lambda_n} \prod_{i=1}^{n} \lambda_i^2. \tag{4.7}$$

Proof. Relation (4.7) follows from the observation that for a given Dyck path, the number of labelings equals the product of squares of the elements of the corresponding alternate level code of ballots. ∎

In passing, we mention that Getu, Shapiro and Woen [4] have considered a generalization of the formula of Rosen [7] on tangent numbers, namely, equation (4.5). More precisely, for a given ballot path, they defined the weight of the path to be the product of the y-coordinate of all the endpoints, except for the last point. Let $T(n, k)$ denote the sum of weights of ballot paths from $(1, 1)$ to (n, k). It is easily checked that

$$T(n, k) = (k - 1)T(n - 1, k - 1) + (k + 1)T(n - 1, k + 1).$$

When $k = 1$, $T(n, 1)$ is the tangent number, that is,

$$\sum_{n \geq 1} T(n, 1) \frac{x^n}{n!} = \tan x.$$

They gave a table for $T(n, k)$ similar to the table in Figure 4, where the first column consists of the tangent numbers. For $k \geq 1$, they obtained the generating function

$$\sum_{n \geq 1} T(n, k) \frac{x^n}{n!} = \frac{\tan^k x}{k}.$$

By replacing the first column of their table by the secant numbers they introduced another number $E(n, k)$, and they considered the following recurrence relation

$$E(n, k) = (k - 1)E(n - 1, k - 1) + kE(n - 1, k + 1),$$

where $E(0, 1) = 1$, $E(1, 2) = E(2, 1) = 1$ and $E(n, k) = 0$ for $n < k - 1$ or $k < 1$. When $k = 1$, $E(n, 1)$ is the secant number. However, no combinatorial interpretation was given

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for the numbers $E(n, k)$. Using the recurrence relation of $E(n, k)$, Getu, Shapiro and Woen [4] derived the exponential generating function

$$\sum_{n \geq k} E(n, k)\frac{x^n}{n!} = \tan^{k-1} x \sec x. \tag{4.8}$$

Comparing the recurrence relations and initial values of $B(n, k)$ and $E(n, k)$, it became apparent that

$$B(n, k) = E(n, k + 1).$$

Therefore $B(n, k)$ can be viewed as a combinatorial explanation for $E(n, k)$. Moreover we obtain the generating functions $G_n(y)$ for the rows of the table for $B(n, k)$. Let

$$G_n(y) = \sum_{0 \leq k \leq n} B(n, k)y^k. \tag{4.9}$$

Note that

$$G_n(1) = \sum_{0 \leq k \leq n} B(n, k)$$

equals the n-th Springer number. Let $B(x, y)$ be the generating function for $G_n(y)$, that is,

$$B(x, y) = \sum_{n \geq 0} G_n(y)\frac{x^n}{n!}.$$

Then we have the following formula.

Theorem 4.4

$$B(x, y) = \frac{1}{\cos x - y \sin x}. \tag{4.10}$$

Proof. Let

$$F_k(x) = \sum_{n \geq k} B(n, k)\frac{x^n}{n!} = \sum_{n \geq k} E(n, k + 1)\frac{x^n}{n!} = \tan^k x \sec x.$$

Therefore,

$$B(x, y) = \sum_{n \geq 0}\sum_{0 \leq k \leq n} B(n, k)y^k\frac{x^n}{n!} = \sum_{k \geq 0} F_k(x)y^k = \frac{1}{\cos x - y \sin x},$$

as required. ∎

To conclude this paper, we give two applications of the bijection Φ. More precisely, we obtain a classification of snakes of type B_n, and we establish a connection between labeled Dyck paths and alternating permutations.

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Define the following statistic

$$\begin{aligned}
\alpha(\pi) = & \#\{1 \leq j \leq n \mid \pi_j > 0 \text{ and } j \text{ is odd}\} \\
& + \#\{1 \leq j \leq n \mid \pi_j < 0 \text{ and } j \text{ is even}\} \\
& - \#\{1 \leq j \leq n \mid \pi_j < 0 \text{ and } j \text{ is odd}\} \\
& - \#\{1 \leq j \leq n \mid \pi_j > 0 \text{ and } j \text{ is even}\}.
\end{aligned}$$

Then we have the following classification of snakes of type B_n.

Theorem 4.5 *For $0 \leq k \leq n$, $B(n, k)$ equals the number of snakes $\pi = \pi_1 \pi_2 \cdots \pi_n$ with $\alpha(\pi) = k$.*

In particular, let us consider the implication of the above theorem for $k = 0$. Recall that $B(2n, 0)$ is the number of labeled Dyck paths of length $2n$. By (4.6) and (4.7), we have

$$\sum_{n \geq 0} B(2n, 0) \frac{x^n}{n!} = \sec x.$$

It now comes to our mind that $\sec x$ is the generating function for the number E_{2n} of alternating permutations on $[2n]$. This indicates that $B(2n, 0)$ equals E_{2n}. The following theorem asserts that the restriction of the bijection Φ to labeled Dyck paths serves as a combinatorial interpretation of the fact that $B(2n, 0) = E_{2n}$. Roughly speaking, when restricted to labeled Dyck paths the map Ψ does not involve any negative elements and when restricted to alternating permutations the map Φ generates labeled Dyck paths.

The following theorem is concerned with the restriction of the map Ψ. It is not difficult to see that the restriction of Ψ to labeled Dyck path is the inverse of the restriction of Φ to alternating permutations.

Theorem 4.6 *The map Ψ induces a bijection between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$.*

Proof. Let $(P; W) = (p_1 \ldots p_{2n}; w_1 \ldots w_{2n})$ be a labeled Dyck path of length $2n$. We wish to show that $\pi = \Psi(P; W) = \pi_1 \ldots \pi_{2n}$ contains no negative elements. In the first step of Ψ, since $(P; W)$ is a labeled Dyck path, there must exist down steps labeled by 0. Assume that p_{r_1} is the leftmost among such down steps. Applying the map Ψ, we are supposed to contract p_{r_1} into a single point to form a ballot path P_2. Then we are supposed to add 1 to the labels of up steps of P_2 which are originally to the right of p_{r_1} and subtract 1 from the labels of down steps of P_2 which are originally to the left of p_{r_1}. Hence we get a labeled ballot path $(P_2; W_2)$ and a permutation $\Gamma_1 = (n - r_1 + 1)\Gamma_0 = (n - r_1 + 1)$, which contains no negative elements.

Similarly, in Step 2, in the labeled ballot path $(P_2; W_2)$, there does not exist any down step of P_2 whose label equals its height. So we can find an up step of P_2 labeled

by 0. Suppose that p_{r_2} is the leftmost up step of P_2 with label 0. Contracting p_{r_2} gives a ballot path P_3. Then subtract 1 from the labels of up steps of P_3 that are originally to the right of p_{r_2} and add 1 to the labels of down steps of P_3 that are originally to the left of p_{r_2}. Thus we obtain a labeled ballot path $(P_3; W_3)$ and a permutation $\Gamma_2 = (n - r_2 + 1)\Gamma_1 = (n - r_2 + 1)(n - r_1 + 1)$ without negative elements.

Since P_1 is a Dyck path of length $2n$, and an up step in P_1 and a down step in P_2 are contracted, there are $n - 1$ up steps and $n - 1$ down steps in P_3. It follows that $(P_3; W_3)$ is a labeled Dyck path. Continuing the above process, we eventually obtain an alternating permutation.

Conversely, given an alternating permutation $\pi = \pi_1 \pi_2 \ldots \pi_{2n}$ of length $2n$, we wish to show that $\Phi(\pi) = p_1 \ldots p_{2n}$ is a labeled Dyck path of length $2n$. Since $\Phi(\pi)$ is a labeled ballot path already, it is enough to show that it has the same number of up steps as down steps. In Step k of the map Φ, we are supposed to find the location of the element $n - k + 1$ in π. Assume that $\pi_i = n - k + 1$. Carrying out the construction of $\Phi(\pi)$, this is what happens: if i is odd, then we have $p_k = u$, and if i is even, then we have $p_k = d$. Since π has $2n$ elements, so we conclude with n up steps as well as n down steps. Therefore $\Phi(\pi)$ is a labeled Dyck path. This completes the proof. ∎

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