# AFFINE STRATIFICATIONS FROM FINITE MISÈRE QUOTIENTS

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Abstract. Given a morphism from an affine semigroup $Q$ to an arbitrary commutative monoid, it is shown that every fiber possesses an affine stratification: a partition into a finite disjoint union of translates of normal affine semigroups. The proof rests on mesoprimary decomposition of monoid congruences [KM10] and a novel list of equivalent conditions characterizing the existence of an affine stratification. The motivating consequence of the main result is a special case of a conjecture due to Guo and the author [GM10], which in general states that (the set of winning positions of) any lattice game possesses an affine stratification. The special case proved here assumes that the lattice game has finite misère quotient, in the sense of Plambeck and Siegel [Pla05, PS07].

## 1. Introduction

Lattice games encode finite impartial combinatorial games—and winning strategies for them—in terms of lattice points in rational convex polyhedra [GM10]; see Section 4. The concept grew out of Plambeck's theory of misère quotients [Pla05], as developed by Plambeck and Siegel [PS07] (see also Siegel's lecture notes [Sie06], particularly Figure 7 in Lecture 5 there). Their purpose was to provide data structures for recording and computing winning strategies of combinatorial games, such as octal games, under the misère play condition, where the last player to move loses. In that spirit, Guo and the author conjectured that the lattice points encoding the strategy of any lattice game have a particularly well-behaved presentation, called an affine stratification: a partition into a finite disjoint union of translates of affine semigroups [GM10, Conjecture 8.9].

The most successful applications of misère quotients thus far have occurred when the quotient is finite, because of amenability to algorithmic computation. Bridging misère quotient theory and lattice games in the case of finite quotients is therefore one of the primary intents of this note, whose motivating result is the existence of affine stratifications for lattice games with finite misère quotients (Theorem 4.6).

The proof comes via a more general main result concerning morphisms from affine semigroups to arbitrary commutative monoids: the fibers of such morphisms possess affine stratifications (Theorem 3.1). That result, in turn, follows from two additional

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results of independent interest. The first is a host of equivalent conditions characterizing the existence of an affine stratification (Theorem 2.6). The second is the existence of combinatorial mesoprimary decompositions for congruences on finitely generated commutative monoids, a theory whose development occupies a full-length paper of its own [KM10]. One of our motivations for defining mesoprimary decompositions in the first place was their anticipated relevance to combinatorial game theory, via the results presented here and via potential further applications toward the existence of affine stratifications for general lattice games.

Beyond mesoprimary decomposition and other more elementary theory of commutative monoids, the reasoning in this note involves elementary polyhedral geometry, including subdivisions and Minkowski sums, as well as combinatorial commutative algebra of finely graded modules over affine semigroup rings, particularly filtrations thereof.

## 2. Affine stratifications

The goal is to decompose certain sets of lattice points in polyhedra in particularly nice ways, following [GM10, §8], where this definition originates.

**Definition 2.1.** An *affine stratification* of a subset $\mathcal{W} \subseteq \mathbb{Z}^d$ is a finite partition

$$\mathcal{W} = \biguplus_{i=1}^{r} W_i$$

of $\mathcal{W}$ into a finite disjoint union of sets $W_i$, each of which is a *finitely generated module* for an affine semigroup $A_i \subseteq \mathbb{Z}^d$; that is, $W_i = F_i + A_i$, where $F_i \subseteq \mathbb{Z}^d$ is a finite set.

For the coming sections, it will be helpful to specify, in Theorem 2.6, some alternative decompositions equivalent to affine stratifications. For that, we need four lemmas as stepping stones. In the first, a *normal* affine semigroup is the intersection of a rational polyhedral cone in $\mathbb{R}^d$ with a sublattice of $\mathbb{Z}^d$.

**Lemma 2.2.** *Every affine semigroup $A \subseteq \mathbb{Z}^d$ possesses an affine stratification in which each stratum is a translate $f_i + A_i$ of a normal affine semigroup $A_i \subseteq \mathbb{Z}^d$.*

*Proof.* Let $\overline{A} = \mathbb{R}_+ A \cap \mathbb{Z}A$ denote the *saturation* of $A$: the set of lattice points lying in the intersection of the real cone generated by $A$ with the group generated by $A$. Then $A$ contains a translate $\mathbf{a} + \overline{A}$ of its saturation by [MS05, Exercise 7.15]. Transferring this statement to the language of monoid algebras, the affine semigroup ring $\mathbb{C}[A]$ has a $\mathbb{C}[A]$-submodule $\mathbf{x}^{\mathbf{a}}\mathbb{C}[\overline{A}] \subseteq \mathbb{C}[A]$. The quotient $M = \mathbb{C}[A]/\mathbf{x}^{\mathbf{a}}\mathbb{C}[\overline{A}]$ is a finitely generated $\mathbb{Z}^d$-graded $\mathbb{C}[A]$-module. The module $M$ is therefore *toric*, in the sense of [MMW05, Definition 4.5], by [MMW05, Example 4.7]. This means that $M$ has a *toric filtration* $0 = M_0 \subset M_1 \subset \cdots \subset M_{\ell-1} \subset M_\ell = M$, in which $M_j/M_{j-1}$ is, for each $j$, a $\mathbb{Z}^d$-graded translate $\mathbf{x}^{\mathbf{a}_j}\mathbb{C}[F_j]$ of the affine semigroup ring $\mathbb{C}[F_j]$ for some face $F_j$ of $A$

and some $\mathbf{a}_j \in \mathbb{Z}^d$. Transferring this statement back into the language of lattice points, we find that

$$A = (\mathbf{a} + \overline{A}) \uplus \biguplus_j (\mathbf{a}_j + F_j)$$

is a disjoint union of a translated normal affine semigroup $\mathbf{a} + \overline{A}$ and a disjoint union of translates $\mathbf{a}_j + F_j$ of faces of $A$. But $\dim M < \dim \mathbb{C}[A]$, since $\dim \mathbf{x}^{\mathbf{a}}\mathbb{C}[\overline{A}] = \dim \mathbb{C}[A]$, so each face $F_j$ that appears is a proper face of $A$. Therefore the proof is done by induction on $\dim \mathbb{C}[A]$, the case of dimension 0 being trivial, since then $A = \{0\}$. □

In the next lemma, keep in mind that the polyhedra need not be bounded.

**Lemma 2.3.** *Any finite union of (rational) convex polyhedra in $\mathbb{R}^d$ can be expressed as a disjoint union of finitely many sets, each of which is the relative interior of a (rational) convex polyhedron.*

*Proof.* The polyhedra in the given union $U$ are defined as intersecions of finitely many halfspaces. The totality of all hyperplanes involved subdivide the ambient space into finitely many closed—but perhaps unbounded—polyhedral regions. This union of regions is a *polyhedral complex* (see [Zie95, Section 5.1] for the definition) with finitely many faces, some of which may be unbounded. By construction, the relative interior of each face is either contained in $U$ or disjoint from $U$, proving the lemma. □

The following will be used in the proofs of both Lemma 2.5 and Theorem 2.6.

**Lemma 2.4.** *If $\Pi = P + C$ is a rational convex polyhedron in $\mathbb{R}^d$, expressed as the Minkowski sum of a polytope $P$ and a cone $C$* [Zie95, Theorem 1.2]*, and $\Pi^\circ$ is its relative interior, then $\Pi \cap \mathbb{Z}^d$ and $\Pi^\circ \cap \mathbb{Z}^d$ are finitely generated modules for $A = C \cap \mathbb{Z}^d$.*

*Proof.* Suppose that $\Pi = \bigcap_j \{\mathbf{x} \in \mathbb{R}^d \mid \phi_j(\mathbf{x}) \geq c_j\}$ for rational linear functions $\phi_j$ and rational constants $c_j$. The case of $\Pi^\circ$ follows from that of $\Pi$ itself: the lattice points in $\Pi^\circ$ are the same as those in the closed polyhedron $\Pi_\varepsilon = \bigcap_j \{\mathbf{x} \in \mathbb{R}^d \mid \phi_j(\mathbf{x}) \geq c_j + \varepsilon\}$ obtained from $\Pi$ by moving each of its bounding hyperplanes inward by a small rational distance, where $\varepsilon$ is less than any nonzero positive value of $\phi_j$ on $\mathbb{Z}^d$ for all $j$. (The rationality of $\phi_j$ guarantees that $\phi_j(\mathbb{Z}^d)$ is a discrete subset of the rational numbers $\mathbb{Q}$.)

Given the closed polyhedron $\Pi$, consider its *homogenization* [Zie95, Section 1.5]: the closure $\overline{\Pi}$ of the cone over a copy of $\Pi = \Pi \times \{1\}$ placed at height 1 in $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$. The intersection of $\overline{\Pi}$ with the first factor $\mathbb{R}^d = \mathbb{R}^d \times \{0\}$ is the cone $C$, and $\overline{\Pi} \cap \mathbb{Z}^d = A$ is a face of the affine semigroup $\overline{A} = \overline{\Pi} \cap \mathbb{Z}^{d+1}$. The intersection $\Pi \cap \mathbb{Z}^d$ is isomorphic, as a module over $A$, to the intersection $M = \overline{\Pi} \cap (\mathbb{Z}^d \times \{1\}) = \overline{A} \cap (\mathbb{Z}^d \times \{1\})$ with the copy of $\mathbb{Z}^d$ at height 1. The result now follows from [Mil02, Eq. (1) and Lemma 2.2] or [MS05, Theorem 11.13], where $M$ is identified as the set of (exponent vectors of) monomials annihilated by the prime ideal $\mathfrak{p}_{\overline{A}} \subseteq \mathbb{C}[\overline{\Pi}]$ modulo an irreducible monomial ideal of $\Bbbk[\overline{\Pi}]$. (The prime ideal $\mathfrak{p}_{\overline{A}}$ is the kernel of the surjection $\mathbb{C}[\overline{\Pi}] \twoheadrightarrow \mathbb{C}[\overline{A}]$; this argument is taken from the proof of [DMM09, Proposition 2.13].) □

**Lemma 2.5.** *If $\mathcal{W} \subseteq \mathbb{Z}^d$ is a finite union of sets $W_i$, each a translate of a normal affine semigroup $A_i = \mathbb{R}_+ A_i \cap L_i$ for some sublattice $L_i \subseteq \mathbb{Z}^d$, then $\mathcal{W}$ can be expressed as such a union in which $L_i = L$ for all $i$ is a fixed sublattice of finite index in $\mathbb{Z}^d$.*

*Proof.* Suppose that $A_i = \mathbb{R}_+ A_i \cap L_i$ is given for all $i$. Taking a direct sum with a complementary sublattice, we may assume that $L_i$ has finite index in $\mathbb{Z}^d$ for all $i$. Now set $L = \bigcap_i L_i$. Then $A_i = \bigcup_{\lambda \in L_i/L} A_i \cap (\lambda + L)$ is a finite union of sets obtained by intersecting a coset of $L$ with $A_i$. Each such set $A_i \cap (\lambda + L)$ is a finitely generated module over $A_i \cap L$ by Lemma 2.4. The desired union is therefore achievable using translates of the normal affine semigroups $A_i \cap L$. □

**Theorem 2.6.** *The following are equivalent for a set $\mathcal{W} \subseteq \mathbb{Z}^d$ of lattice points.*

1. *$\mathcal{W}$ possesses an affine stratification.*
2. *$\mathcal{W}$ is a finite (not necessarily disjoint) union of sets $W_i$, each of which is a finitely generated module for an affine semigroup $A_i \subseteq \mathbb{Z}^d$.*
3. *$\mathcal{W}$ is a finite (not necessarily disjoint) union of sets $W_i$, each of which is a translate $f_i + A_i$ of an affine semigroup $A_i \subseteq \mathbb{Z}^d$.*
4. *$\mathcal{W}$ is a finite (not necessarily disjoint) union of sets $W_i$, each of which is a translate $f_i + A_i$ of a normal affine semigroup $A_i \subseteq \mathbb{Z}^d$.*
5. *$\mathcal{W}$ is a finite disjoint union of sets $W_i$, each of which is a translate $f_i + A_i$ of a normal affine semigroup $A_i \subset \mathbb{Z}^d$.*
6. *$\mathcal{W}$ is a finite disjoint union of sets $W_i$, each of which is a translate $f_i + A_i$ of a (not necessarily normal) affine semigroup $A_i \subset \mathbb{Z}^d$.*

*Proof.* By definition it follows that $1 \Rightarrow 2 \Rightarrow 3$ and that $5 \Rightarrow 6 \Rightarrow 1$. It therefore remains only to show that $3 \Rightarrow 4 \Rightarrow 5$. The first of these implications is Lemma 2.2.

For the second, begin by choosing the union to satisfy the conclusion of Lemma 2.5. For each $\lambda \in \mathbb{Z}^d/L$, let $\mathcal{W}_\lambda = \mathcal{W} \cap (\lambda + L)$ be the intersection of $\mathcal{W}$ with the corresponding coset of $L$ in $\mathbb{Z}^d$. Then $\mathcal{W}_\lambda$ is the intersection of $\lambda + L$ with the union $U_\lambda$ of those polyhedra $f_i + \mathbb{R}_+ A_i$ for which $f_i \in \lambda + L$. By Lemma 2.3, it suffices to show that if $W$ is the intersection of $L$ with the relative interior of a polyhedron, then $W$ possesses an affine stratification in which every stratum is a translate of a normal affine semigroup. Replacing $L$ with $\mathbb{Z}^d$, we may as well assume that $L = \mathbb{Z}^d$. Lemma 2.4 implies that $W$ is a finitely generated module over a normal affine semigroup $A$. Thus the vector space over $\mathbb{C}$ with basis $W$ constitutes a finitely generated $\mathbb{Z}^d$-graded submodule $M \subseteq \mathbb{C}[\mathbb{Z}^d]$ over the affine semigroup ring $\mathbb{C}[A]$. The result now follows by a simpler version of the argument in the proof of Lemma 2.2, using a toric filtration: in the present case, dimension is not an issue, and $M_j/M_{j-1}$ is already a $\mathbb{Z}^d$-graded translate of a normal affine semigroup ring, because every face of $A$ is normal. □

**Corollary 2.7.** *Fix a linear map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^d$. If $\mathcal{W} \subseteq \mathbb{Z}^n$ possesses an affine stratification then so does $\varphi(\mathcal{W}) \subseteq \mathbb{Z}^d$.*

*Proof.* The image of any translate of an affine semigroup in $\mathbb{Z}^n$ is a translate of an affine semigroup in $\mathbb{Z}^d$, so use a stratification of $\mathcal{W}$ as in Theorem 2.6.6: nothing guarantees that the images of the strata are disjoint, but that is irrelevant by Theorem 2.6.3. □

**Corollary 2.8.** *If each of finitely many given subsets of $\mathbb{Z}^d$ possesses an affine stratification, then so does their union.*

*Proof.* Use the equivalence of (for example) Theorem 2.6.1 and Theorem 2.6.2. □

**Remark 2.9.** The reason for choosing Definition 2.1 as the fundamental concept instead of the other conditions in Theorem 2.6 is that Definition 2.1 likely results in the most efficient data structure for algorithmic purposes; see [GM10′].

## 3. Fibers of affine presentations of commutative monoids

This section serves as an elementary example of the theory of mesoprimary decomposition and as a bridge to combinatorial game theory in the presence of finite misère quotients considered in Section 4. The main observation in this section is that affine stratifications exist for fibers of affine presentations of arbitrary commutative monoids. The final claim of the theorem invokes the notion of *associated lattice* for congruences on $\mathbb{N}^n$, which form part of the theory of mesoprimary congruences and combinatorial mesoprimary decomposition [KM10]; for the definition, see [KM10, §2-§4]

**Theorem 3.1.** *If $\varphi : A \twoheadrightarrow Q$ is a surjection of a commutative monoids with $A \subseteq \mathbb{Z}^d$ an affine semigroup, then every fiber of $\varphi$ possesses an affine stratification. If $d = n$ and $A = \mathbb{N}^n$, then the stratification can be chosen so that each of its affine semigroups is $L \cap \mathbb{N}^n$ for some intersection $L$ of associated lattices of the congruence defining $\varphi$.*

The proof, included after Proposition 3.5, requires some preliminary results on affine stratifications in simpler situations than Theorem 3.1.

**Lemma 3.2.** *If $M$ is an ideal in an affine semigroup $A$, then $M$ possesses an affine stratification in which each stratum is a translate $f_i + A_i$ of a face $A_i$ of $A$.*

*Proof.* Use a toric filtration as in the proof of Lemma 2.2, where $M$ there is replaced by the $\mathbb{C}[A]$-module $\mathbb{C}\{M\}$ that has the ideal $M \subseteq A$ as a vector space basis over $\mathbb{C}$. □

**Example 3.3.** Every ideal in $\mathbb{N}^n$ has a *Stanley decomposition* [Sta82]: an expression as a finite disjoint union of translates of faces $\mathbb{N}^J$ of $\mathbb{N}^n$ for $J \subseteq \{1, \ldots, n\}$; see [Mil09, §2].

**Lemma 3.4.** *Fix a normal affine semigroup $A$. The intersection $(\mathbf{u} + L) \cap M$ of any coset of a lattice $L \subseteq \mathbb{Z}A$ with an ideal $M \subseteq A$ is a finitely generated module for $L \cap A$.*

*Proof.* The ideal $M$ is finitely generated, and the intersection $(\mathbf{u} + L) \cap (\mathbf{a} + A)$ is finitely generated as a module over $L \cap A$ for any $\mathbf{a} \in A$ by Lemma 2.4. □

In the brief remainder of this section, concepts and notation from [KM10, §2–§4], where all of the terms are defined, are used freely without further comment.

**Proposition 3.5.** *If $L \subseteq \mathbb{Z}^J$ is the associated lattice of a mesoprimary congruence $\sim$ on $\mathbb{N}^n$, then every congruence class under $\sim$ is a finite union of sets $(\mathbf{u} + L) \cap \mathbb{N}^n$.*

*Proof.* Let $\overline{Q} = \mathbb{N}^n/{\sim}$. If $J$ is the associated index set, then it suffices to show that every element of $\overline{Q}[\mathbb{Z}^J]$, each viewed as a subset of $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$, is a finite union of cosets of $L$. This is the content of the semifreeness of the action of $\mathbb{Z}^J/L$ on $\overline{Q}[\mathbb{Z}^J]$: under the action of $\mathbb{Z}^J$ on $\overline{Q}[\mathbb{Z}^J]$, the stabilizer of every non-nil element $\overline{q} \in \overline{Q}[\mathbb{Z}^J]$ is $L$. Viewing $\overline{q}$ as a subset of $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$, this means that the intersection of $\overline{q}$ with any single coset of $\mathbb{Z}^J$ is a single coset of $L$. Hence the result follows from finiteness of the number of $\mathbb{Z}^J/L$-orbits. □

*Proof of Theorem 3.1.* By choosing a presentation $\mathbb{N}^n \twoheadrightarrow A$, we reduce to the case where $A = \mathbb{N}^n$ by Corollary 2.7. Let $\sim$ be the congruence on $\mathbb{N}^n$ induced by the surjection $\mathbb{N}^n \twoheadrightarrow Q$, so that $\mathbb{N}^n/{\sim} = Q$. The fiber over the nil of $Q$, if there is one, is an ideal of $\mathbb{N}^n$, which has an affine stratification because it has a Stanley decomposition (Lemma 3.2 and Example 3.3). To treat the the non-nil fibers, fix a combinatorial mesoprimary decomposition of $\sim$, the existence of which is one of the main results of [KM10]. Let $L_i \subseteq \mathbb{Z}^{J_i}$ for $i = 1, \ldots, r$ be the lattices associated to $\sim$ with corresponding index sets $J_i \subseteq [n]$, and write $\overline{Q}_i$ for the quotient of $\mathbb{N}^n$ modulo the $i^{\text{th}}$ mesoprimary congruence in the decomposition.

Every class $\overline{q} \in \overline{Q}$ is the intersection of the $r$ mesoprimary classes $\overline{q}_i \in \overline{Q}_i$ containing $\overline{q}$ because $\sim$ is the common refinement of its mesoprimary components. Furthermore, as long as $\overline{q}$ is not nil, at least one of the mesoprimary classes $\overline{q}_i$ is not nil. For such a class $\overline{q}_i$, Proposition 3.5 guarantees a finite set $U_i$ such that $\overline{q}_i = \bigcup_{\mathbf{u} \in U_i} (\mathbf{u} + L_i) \cap \mathbb{N}^n$. Renumbering for convenience, assume that $\overline{q}_i$ is non-nil for $i \leq k$ and nil for $i > k$. Then $\overline{q}' := \overline{q}_1 \cap \cdots \cap \overline{q}_k$ is the intersection with $\mathbb{N}^n$ of a finite union of cosets of the lattice $L_1 \cap \cdots \cap L_k$, and $\overline{q}'' := \overline{q}_{k+1} \cap \cdots \cap \overline{q}_r$ is an ideal of $\mathbb{N}^n$. Since $\overline{q} = \overline{q}' \cap \overline{q}''$, the result follows from Lemma 3.4. □

## 4. Lattice games and misère quotients

Fix a pointed rational convex polyhedron $\Pi \subseteq \mathbb{R}^d$ with recession cone $C$ of dimension $d$. The pointed hypothesis means that $\Pi = P + C$ for some polytope (i.e., bounded convex polyhedron) $P$. Write $\Lambda = \Pi \cap \mathbb{Z}^d$ for the set of integer points in $\Pi$. The following definitions summarize [GM10, Definition 2.3, Definition 2.9, and Lemma 3.5].

**Definition 4.1.** A finite subset $\Gamma \subset \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$ is a *rule set* if

1. there exists a linear function on $\mathbb{R}^d$ that is positive on $\Gamma \cup C \smallsetminus \{\mathbf{0}\}$; and
2. there is finite set $F \subset \Lambda$ such that every position $p \in \Lambda$ has a $\Gamma$-*path in* $\Lambda$ *to* $F$: a sequence $p = p_r, \ldots, p_0 \in \Lambda$ with $p_0 \in F$ and $p_k - p_{k-1} \in \Gamma$ for $k = \{1, \ldots, r\}$.

For the next definition, it is important to observe that the rule set $\Gamma$ induces a partial order $\preceq$ on $\Lambda$ in which $p \preceq q$ if $q - p$ lies in the monoid $\mathbb{N}\Gamma$ generated by $\Gamma$ [GM10, Lemma 2.8]. An *order ideal* under this (or any) partial order $\preceq$ is a subset $S$ closed under going down: $p \preceq q$ and $q \in S \Rightarrow p \in S$.

**Definition 4.2.** Fix a rule set $\Gamma$.

- A *game board* $B$ is the complement in $\Lambda$ of a finite $\Gamma$-order ideal in $\Lambda$ called the set of *defeated positions*.
- A *lattice game* $G = (\Gamma, B)$ is defined by a rule set $\Gamma$ and a game board $B$.
- The *P-positions* of $G$ form a subset $\mathcal{P} \subseteq B$ such that $(\mathcal{P} + \Gamma) \cap B = B \smallsetminus \mathcal{P}$.
- An *affine stratification* of $G$ is an affine stratification of its set of $P$-positions.

The $P$-positions of $G$ are uniquely determined by the rule set and game board [GM10, Theorem 4.6].

Following Plambeck and Siegel [Pla05, PS07], every lattice game possesses a unique quotient that optimally collapses $\Lambda$ while faithfully recording the interaction of the $P$-positions with its additive structure.

**Definition 4.3.** Two positions $p, q \in B$ are *indistinguishable*, written $p \sim q$, if

$$(p + C) \cap \mathcal{P} = p - q + (q + C) \cap \mathcal{P}.$$

In other words, $p + r \in \mathcal{P} \Leftrightarrow q + r \in \mathcal{P}$ for all $r$ in the recession cone $C$ of $B$. The *misère quotient* of the lattice game with winning positions $\mathcal{P}$ is the quotient $\Lambda/\!\sim$ of the polyhedral set $\Lambda$ modulo indistinguishability.

Geometrically, indistinguishability means that the P-positions in the cone above $p$ are the same as those above $q$, up to translation by $p - q$. It is elementary to verify that indistinguishability is an equivalence relation, and that it is additive, in the sense that $p \cong q \Rightarrow p+r \cong q+r$ for all $r \in C \cap \mathbb{Z}^d$. Thus, when $B = \Lambda = C \cap \mathbb{Z}^d$ is a monoid, indistinguishability is a congruence, so the quotient of $B$ modulo indistinguishability is again a monoid.

**Lemma 4.4.** *Every fiber of the projection $\Lambda \to \Lambda/\!\sim$ either consists of $P$-positions or has empty intersection with $\mathcal{P}$.*

*Proof.* If $p \sim q$, then by definition either $p$ and $q$ are both $P$-positions or neither is. $\square$

**Corollary 4.5.** *Fix a lattice game $G = (\Gamma, B)$ played on a cone, meaning that the game board is the complement of the defeated positions in a normal affine semigroup $\Lambda$. If the misère quotient $\Lambda/\!\sim$ is finite, then $G$ admits an affine stratification.*

*Proof.* The set $\mathcal{P}$ of $P$-positions is a union of fibers of the projection. If the quotient is finite, then the union is finite. Now apply Theorem 3.1. $\square$

Although it is useful to record Corollary 4.5, which treats the case of finite misère quotient *monoids*, where the game is played on a cone, the extension to arbitrary finite misère quotients requires little additional work.

**Theorem 4.6.** *Fix a lattice game $G = (\Gamma, B)$ played on a polyhedral set $\Lambda = \Pi \cap \mathbb{Z}^d$. If the misère quotient $\Lambda/\!\sim$ is finite, then $G$ admits an affine stratification.*

*Proof.* The set $\mathcal{P}$ of $P$-positions is a union of fibers of the projection $\Lambda \to \Lambda/\!\sim$. Since the quotient is finite, the union is finite. Therefore it suffices to show that every fiber $\Phi \subseteq \Lambda$ of the projection possesses an affine stratification.

The pointed hypothesis on $\Pi$ implies that $\Lambda = F + A$, where $F \subseteq \mathbb{Z}^d$ is finite and $A = C \cap \mathbb{Z}^d$ is a normal affine semigroup. The fiber $\Phi$ is a finite union $\Phi = \bigcup_{f \in F} \Phi \cap (f + A)$. By Corollary 2.8, it therefore suffices to show that for each lattice point $f \in F$, every fiber of the map $f + A \to \Lambda/\!\sim$ possesses an affine stratification. But the composite map $A \to f + A \to \Lambda/\!\sim$ induces a congruence on $A$ whose classes are the fibers, to which Theorem 3.1 applies. □

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