Exponential Riesz bases, discrepancy of irrational rotations and BMO

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Abstract. Matei and Meyer proved that 'simple quasicrystals' are universal sets of sampling and interpolation for signals with a band-limited spectrum. We ask if the corresponding exponential system is a universal Riesz basis in L^2 on appropriate multiband sets on the circle. We prove that the answer depends on a diophantine property of the quasicrystal. For the proof we extend to BMO a theorem of Kesten on the discrepancy of irrational rotations of the circle.

1. Introduction

1.1. Sampling and interpolation. A band-limited signal is an entire function F of exponential type, square-integrable on the real axis. According to the classical Paley-Wiener theorem, F is the Fourier transform of an L^2-function supported by a bounded (measurable) set $S \subset \mathbb{R}$, which is called the spectrum of F. We shall denote by PW_S the Paley-Wiener space of all functions $F \in L^2(\mathbb{R})$ which are Fourier transforms of functions from $L^2(S)$,

$$F(t) = \int_S f(x)\, e^{-2\pi i t x}\, dx, \quad f \in L^2(S).$$

A discrete set $\Lambda \subset \mathbb{R}$ is called a *set of sampling* for PW_S if every signal with spectrum in S can be reconstructed in a stable way from its 'samples' $\{F(\lambda), \lambda \in \Lambda\}$, that is, there are positive constants A, B such that the inequalities

$$A\|F\|_{L^2(\mathbb{R})} \leqslant \Big(\sum_{\lambda\in\Lambda} |F(\lambda)|^2 \Big)^{1/2} \leqslant B\|F\|_{L^2(\mathbb{R})} \tag{1}$$

hold for every $F \in PW_S$. Also, Λ is called a *set of interpolation* for PW_S if every data $\{c_\lambda\} \in \ell^2(\Lambda)$ can be "transmitted" as samples, which means that there exists at least one function $F \in PW_S$ such that $F(\lambda) = c_\lambda$ ($\lambda \in \Lambda$).

The sampling and interpolation properties can also be formulated in terms of the exponential system

$$\mathcal{E}(\Lambda) = \{\exp 2\pi i \lambda t,\ \lambda \in \Lambda\}.$$

The sampling property of Λ means that $\mathcal{E}(\Lambda)$ is a *frame* in the space $L^2(S)$, while the interpolation holds when $\mathcal{E}(\Lambda)$ is a *Riesz-Fischer* system in this space. It follows that Λ is a set of both sampling and interpolation if and only if $\mathcal{E}(\Lambda)$ forms a *Riesz basis* in $L^2(S)$. See [**29**] for more details.

Sampling and interpolation may also be discussed in the periodic setting. If S is a measurable subset of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, then one can consider the frame or Riesz-Fischer properties in $L^2(S)$ of the exponential system $\mathcal{E}(\Lambda)$, where $\Lambda \subset \mathbb{Z}$.

1.2. Density. The right inequality in (1) follows from the separation condition

$$\inf_{\lambda,\lambda'\in\Lambda,\ \lambda\neq\lambda'} |\lambda-\lambda'|>0,$$

which is also necessary for the interpolation, so this condition will always be assumed below. The lower and upper uniform densities of a separated set Λ are defined respectively by

$$D^-(\Lambda)=\lim_{r\to\infty}\min_{a\in\mathbb{R}}\frac{\#\big(\Lambda\cap(a,a+r)\big)}{r},\quad D^+(\Lambda)=\lim_{r\to\infty}\max_{a\in\mathbb{R}}\frac{\#\big(\Lambda\cap(a,a+r)\big)}{r}.$$

Landau [**15**] proved the following two results:

If Λ is a set of sampling for PW_S, then $D^-(\Lambda)\geqslant \operatorname{mes} S$.

If Λ is a set of interpolation for PW_S, then $D^+(\Lambda)\leqslant \operatorname{mes} S$.

Here S is a bounded measurable set, and $\operatorname{mes} S$ is the Lebesgue measure of S.

For “regularly” distributed sequences the two densities above coincide, and their common value, denoted $D(\Lambda)$, is called the *uniform density* of Λ. It follows that:

If $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(S)$ then Λ has a uniform density $D(\Lambda)=\operatorname{mes} S$.

1.3. Universality problem. Olevskii and Ulanovskii posed in [**21, 22**] the following question: is it possible to find a “universal” set Λ of given density, which provides a stable reconstruction of any signal whose spectrum has a sufficiently small Lebesgue measure? Similarly, does exist Λ of given density which is a set of interpolation in every PW_S with $\operatorname{mes} S$ sufficiently large?

It was proved in [**21, 22**] that no universal set of sampling or interpolation exists if the spectrum S is allowed to be an arbitrary bounded measurable set. On the other hand it was also proved that under some topological restrictions on the spectra, universal sampling and interpolation does exist:

Given any $d>0$ there is a (separated) set $\Lambda\subset\mathbb{R}$ of density $D(\Lambda)=d$, such that:

(i) *Λ is a set of sampling for PW_S for every compact set $S\subset\mathbb{R}$, $\operatorname{mes} S<d$.*

(ii) *Λ is a set of interpolation for PW_S for every open set $S\subset\mathbb{R}$, $\operatorname{mes} S>d$.*

In fact, this is a consequence of the following result:

Theorem A (Olevskii and Ulanovskii [**21, 22**]). *Given any $d>0$ one can find a discrete set $\Lambda\subset\mathbb{R}$ (a perturbation of an arithmetical progression) such that $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(S)$ for every set $S\subset\mathbb{R}$, $\operatorname{mes} S=d$, which is the union of finitely many disjoint intervals such that the lengths of these intervals and the gaps between them are commensurable.*

The latter result shows that one can construct a universal exponential Riesz basis $\mathcal{E}(\Lambda)$, $\Lambda\subset\mathbb{R}$, in the space $L^2(S)$ for a “dense” family of sets $S\subset\mathbb{R}$.

1.4. Simple quasicrystals. A different construction of universal sampling and interpolation sets, termed ‘simple quasicrystals’, was presented by Matei and Meyer in [**17, 18**]. The construction is based on the so-called ‘cut and project’ scheme introduced by Meyer in 1972 (see [**19, 20**]). In what follows we shall mainly focus in the periodic case, in which the simple quasicrystals are constructed as follows.

Let α be an irrational real number, and consider the sequence of points $\{n\alpha\}$, $n\in\mathbb{Z}$, on $\mathbb{T}$. Given an interval $I=[a,b)\subset\mathbb{T}$ define the following subset of $\mathbb{Z}$,

$$\Lambda(\alpha,I):=\{n\in\mathbb{Z}: a\leqslant n\alpha<b\}.$$

It is well-known that the points $\{n\alpha\}$ are equidistributed on the circle $\mathbb{T}$ (and moreover, they are well-distributed, see [**14**]). This implies that the set $\Lambda(\alpha, I)$ has a uniform density $D(\Lambda(\alpha, I)) = |I|$, where $|I|$ is the length of the interval I.

The set $\Lambda(\alpha, I)$ is a set of universal sampling and interpolation:

Theorem B (Matei and Meyer [**17**])**.**

(i) *$\mathcal{E}(\Lambda(\alpha, I))$ is a frame in $L^2(S)$ for every compact set $S \subset \mathbb{T}$, $\operatorname{mes} S < |I|$.*

(ii) *$\mathcal{E}(\Lambda(\alpha, I))$ is Riesz-Fischer in $L^2(S)$ for every open set $S \subset \mathbb{T}$, $\operatorname{mes} S > |I|$.*

This result was further developed in [**18**]. In that paper the authors raised the problem to understand what can be said about the "limiting case" when the measure of S is equal to the density of Λ.

1.5. Riesz bases and quasicrystals. In the present paper we consider the following aspect of the problem. Is it true that the exponential system $\mathcal{E}(\Lambda(\alpha, I))$ is a Riesz basis in L^2 on a certain subset S of the circle $\mathbb{T}$? Moreover, does it provide a universal Riesz basis for an appropriate "dense" family of sets S?

Our first result shows that the question admits a positive answer provided that a particular diophantine condition, relating α and the length of the interval I, holds:

Theorem 1. *Let $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$. Then the exponential system $\mathcal{E}(\Lambda(\alpha, I))$ is a Riesz basis in $L^2(S)$ for every set $S \subset \mathbb{T}$, $\operatorname{mes} S = |I|$, which is the union of finitely many disjoint intervals whose lengths belong to $\mathbb{Z} + \alpha\mathbb{Z}$.*

In Section 3 this theorem will be proved in a more general form. We will also see that the family of sets S in Theorem 1 is "dense" in an appropriate sense. In particular we will prove the following corollary, which shows that properties (i) and (ii) in Theorem B can be strengthened when $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$.

Corollary 1. *Let $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$. Suppose that $U \subset \mathbb{T}$ is an open set, K is compact, $K \subset U$ and $\operatorname{mes} K < |I| < \operatorname{mes} U$. Then one can find a set S, $K \subset S \subset U$, such that $\mathcal{E}(\Lambda(\alpha, I))$ is a Riesz basis in $L^2(S)$.*

Theorem 1 extends results from the papers [**2, 16**] on the existence of exponential Riesz bases in L^2 on 'multiband sets', that is, finite unions of intervals. See Section 6 below for a more detailed discussion.

Our second result complements the picture by clarifying the role of the diophantine assumption $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ in Theorem 1. It turns out that this condition is not only sufficient, but also necessary, for the simultaneous sampling and interpolation property on multiband sets.

Theorem 2. *Suppose that $|I| \notin \mathbb{Z} + \alpha\mathbb{Z}$. Then $\mathcal{E}(\Lambda(\alpha, I))$ is not a Riesz basis in $L^2(S)$, for any set $S \subset \mathbb{T}$ which is the union of finitely many intervals.*

1.6. Discrepancy and Kesten's theorem. The proofs of the above mentioned results are based on the connection of the problem to the theory of equidistribution and discrepancy for the rotation of the circle by an irrational angle α.

Let α be a fixed irrational number. Given an interval $I \subset \mathbb{T}$, we denote by $\nu(n, I)$ the number of integers $0 \leqslant k \leqslant n-1$ such that $k\alpha \in I$. The equidistribution of the points $\{n\alpha\}$ on the circle $\mathbb{T}$ means that

$$\lim_{n\to\infty} \frac{\nu(n, I)}{n} = |I|,$$

for every interval I. A quantitative measurement of this equidistribution is given by the *discrepancy* function, defined by

$$D(n, I) = \nu(n, I) - n|I|.$$

Thus we have $|D(n, I)| = o(n)$ as $n \to \infty$, and, in fact, it is not difficult to show that this estimate holds uniformly with respect to the interval I.

Better estimates for the discrepancy can be obtained based on diophantine properties of the number α. For example, if α is a quadratic irrational number then $|D(n, I)| = O(\log n)$, where the estimate is uniform with respect to I. On the other hand, it is known that for every α the lower bound

$$\sup_{I \subset \mathbb{T}} |D(n, I)| > c \log n$$

holds for infinitely many n, where c is a positive absolute constant (see [**14**]). Compare also to the case where α and I are random [**10, 11**] .

It was discovered, though, that for certain *fixed* intervals I the discrepancy can be bounded. Hecke [**6**] proved that if $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ then $D(n, I) = O(1)$ as $n \to \infty$. Erdös and Szüsz conjectured [**3**] that also the converse to Hecke's result should be true. This conjecture was confirmed by Kesten [**12**] who proved that if $D(n, I) = O(1)$ then $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$. For further developments see [**4, 23, 25**].

We will prove an extension of Kesten's theorem for *bounded mean oscillations* of the discrepancy. We say that a sequence of complex numbers $\{c_n\}$ has bounded mean oscillations, $\{c_n\} \in \mathrm{BMO}$, if for every $n < m$ one has

$$\frac{1}{m-n} \sum_{k=n}^{m-1} \left| c_k - \frac{c_n + \cdots + c_{m-1}}{m-n} \right| \leqslant M, \tag{2}$$

for some constant M independent of n and m. Certainly, every bounded sequence belongs to BMO. On the other hand, there are also unbounded sequences in BMO, for example, the sequence $c_n = \log n$.

We will prove the following generalization of Kesten's theorem.

Theorem 3. *Let α be an irrational number, and $I \subset \mathbb{T}$ be an interval. If the sequence $\{D(n, I)\}$, $n = 1, 2, 3, \dots$, belongs to* BMO*, then $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$.*

This theorem will be proved in a more general form in Section 4.

1.7. Meyer's duality. The link between the sampling and interpolation problems for the sets $\Lambda(\alpha, I)$ and the theory of discrepancy for irrational rotations is an idea due to Meyer, which we refer to as the 'duality principle'. Meyer's duality principle (see Section 2) enables us to reduce the problem about exponential Riesz bases in $L^2(S)$ to a similar problem in $L^2(I)$, where I is a single interval. It is then possible to invoke known results about exponential Riesz bases in $L^2(I)$.

The proof of Theorem 1 (Section 3) consists of three main ingredients: Meyer's duality principle, a theorem of Avdonin [**1**] about exponential Riesz bases in $L^2(I)$ (an extension of Kadec's 1/4 theorem) and Hecke's result about discrepancy. In order to prove Theorem 2 (Section 5) we combine the duality principle with a theorem due to Pavlov [**24**], which describes completely the exponential Riesz bases in $L^2(I)$. Pavlov's theorem allows us to conclude that the discrepancy must be in BMO, and we can then apply an extension of Theorem 3 (Section 4).

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2. Meyer's duality principle

2.1. Let α be a fixed irrational number. For any set $S \subset \mathbb{T}$ one can, in principle, consider the "Meyer set" based on α and S, defined by

$$\Lambda(\alpha, S) = \{n \in \mathbb{Z} : n\alpha \in S\}. \tag{3}$$

Meyer discovered a duality phenomenon connecting the sampling and interpolation properties of the sets $\Lambda(\alpha, S)$. This duality allows to prove results about sampling and interpolation such as Theorem B mentioned in the introduction. Meyer's duality principle lies in the basis of our approach as well.

It will be convenient to introduce the following terminology.

Definition. A set $S \subset \mathbb{T}$ will be called a *multiband set* if it is the union of finitely many disjoint intervals. We will say that a multiband set S is *left semi-closed* if each one of the intervals contains its left endpoint but not its right endpoint. Equivalently, S is left semi-closed if the indicator function $\mathbb{1}_S$ is continuous from the right. We also define a *right semi-closed* multiband set, in a similar way. Finally, S will be called *semi-closed* if it is either left semi-closed or right semi-closed.

We shall need a version of the duality principle which is suitable for our setting. Let $-\Lambda$ denote the set $\{-n : n \in \Lambda\}$. We will prove the following:

Lemma 2.1. *Let U be a semi-closed multiband set, and V be a (not necessarily semi-closed) multiband set. Then:*

(i) *If $\mathcal{E}(\Lambda(\alpha, V))$ is a frame in the space $L^2(U)$, then $\mathcal{E}(-\Lambda(\alpha, U))$ is a Riesz-Fischer system in $L^2(V)$.*

(ii) *If $\mathcal{E}(\Lambda(\alpha, V))$ is a Riesz-Fischer system in $L^2(U)$, then $\mathcal{E}(-\Lambda(\alpha, U))$ is a frame in $L^2(V)$.*

An immediate consequence of Lemma 2.1 is:

Corollary 2.2. *Let U and V be two multiband sets, where U is semi-closed. If the exponential system $\mathcal{E}(\Lambda(\alpha, V))$ is a Riesz basis in $L^2(U)$, then $\mathcal{E}(-\Lambda(\alpha, U))$ is a Riesz basis in $L^2(V)$.*

2.2. We choose and fix a function $\varphi(x)$ on $\mathbb{R}$, infinitely smooth, with compact support contained in the interval $(0, 1)$, and such that $\int_{\mathbb{R}} |\varphi(x)|^2 \, dx = 1$. Let

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x) \, e^{-2\pi i \xi x} \, dx, \quad \xi \in \mathbb{R},$$

be the Fourier transform of φ, which is a smooth and rapidly decreasing function. For each $0 < \varepsilon < 1$ define a function φ_ε on the circle $\mathbb{T}$ by

$$\varphi_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \, \varphi(t/\varepsilon), \quad 0 \leqslant t < 1.$$

It follows that φ_ε is an infinitely smooth function on $\mathbb{T}$, supported by $(0, \varepsilon)$, such that $\int_{\mathbb{T}} |\varphi_\varepsilon(t)|^2 \, dt = 1$, and the Fourier coefficients of φ_ε are given by

$$\widehat{\varphi}_\varepsilon(n) = \sqrt{\varepsilon} \, \widehat{\varphi}(\varepsilon n), \quad n \in \mathbb{Z}.$$

The following two lemmas are essentially due to Matei and Meyer [**17**].

Lemma 2.3. *For every Riemann integrable function f on $\mathbb{T}$,*

$$\lim_{\varepsilon \to 0} \sum_{n \in \mathbb{Z}} \big| f(n\alpha) \, \widehat{\varphi}_\varepsilon(n) \big|^2 = \int_{\mathbb{T}} |f(t)|^2 \, dt.$$

Lemma 2.4. *Let $\{c_n\}$ be a sequence of complex numbers in $\ell^1(\mathbb{Z})$. Then*

$$\lim_{\varepsilon\to 0} \int_S \Big| \sum_{n\in\mathbb{Z}} c_n\, \varphi_\varepsilon(t-n\alpha) \Big|^2 dt = \sum_{n\in\Lambda(\alpha,S)} |c_n|^2,$$

for every left semi-closed multiband set S.

Lemma 2.3 is a consequence of the equidistribution of the points $\{n\alpha\}$ on the circle. Lemma 2.4 is due to the fact that $\varphi_\varepsilon(t-n\alpha)$ is supported by a small right neighborhood of the point $n\alpha$. See [**17**] for a proof sketch. Certainly, one can also get a version of Lemma 2.4 for right semi-closed multiband sets, by choosing the function φ supported by the interval $(-1,0)$ instead of $(0,1)$.

We will also need the following well-known fact:

Lemma 2.5 (See [**29**], p. 155)**.** *The exponential system $\mathcal{E}(\Lambda)$ is Riesz-Fischer in $L^2(S)$ if and only if there is a positive constant C such that the inequality*

$$\sum_{\lambda\in\Lambda} |c_\lambda|^2 \leqslant C \int_S \Big| \sum_{\lambda\in\Lambda} c_\lambda\, e^{2\pi i\lambda t} \Big|^2 dt$$

holds for every finite sequence of scalars $\{c_\lambda\}$.

2.3. Here we give the proof of Lemma 2.1. It is based on the proof from [**17**] but we find it useful to provide the reader with a self contained proof. Below we will assume that U is a left semi-closed multiband set, but as we have remarked one can easily adapt the proof to the case when U is right semi-closed.

Proof of Part (i) of Lemma 2.1. Suppose that $\mathcal{E}(\Lambda(\alpha,V))$ is a frame in the space $L^2(U)$. It is to be proved that $\mathcal{E}(-\Lambda(\alpha,U))$ is a Riesz-Fischer system in $L^2(V)$. By Lemma 2.5 it will be enough to show that, if f is a trigonometric polynomial

$$f(t) = \sum_n c_n\, e^{-2\pi i n t} \tag{4}$$

such that $c_n = 0$ unless $n\alpha \in U$, then

$$\sum_n |c_n|^2 \leqslant C \int_V |f(t)|^2\, dt. \tag{5}$$

Given such a trigonometric polynomial f we define

$$F_\varepsilon(t) = \sum_{n\in\mathbb{Z}} c_n\, \varphi_\varepsilon(t-n\alpha).$$

Since U is left semi-closed, it follows that F_ε is supported by U if ε is sufficiently small. Since $\mathcal{E}(\Lambda(\alpha,V))$ is a frame in $L^2(U)$, there is a constant C such that

$$\int_U |F_\varepsilon(t)|^2\, dt \leqslant C \sum_{n\in\Lambda(\alpha,V)} |\widehat{F}_\varepsilon(n)|^2. \tag{6}$$

Again we take the limit as $\varepsilon\to 0$. Since U is left semi-closed, Lemma 2.4 implies that the left hand side of (6) converges to $\sum |c_n|^2$. On the other hand, it is easy to see that $\widehat{F}_\varepsilon(n) = f(n\alpha)\, \widehat{\varphi}_\varepsilon(n)$. The right hand side of (6) can therefore be written as

$$\sum_{n\in\mathbb{Z}} \big| f(n\alpha)\, \mathbb{1}_V(n\alpha)\, \widehat{\varphi}_\varepsilon(n) \big|^2,$$

and by Lemma 2.3 it converges to $\int_V |f(t)|^2\, dt$ as $\varepsilon\to 0$. This proves the first part of Lemma 2.1. $\square$

Proof of Part (ii) of Lemma 2.1. We suppose that $\mathcal{E}(\Lambda(\alpha, V))$ is a Riesz-Fischer system in $L^2(U)$. It is to be proved that $\mathcal{E}(-\Lambda(\alpha, U))$ is a frame in $L^2(V)$, that is, we must show that the inequality

$$\int_V |f(t)|^2 \, dt \leqslant C \sum_{n \in \Lambda(\alpha, U)} |\widehat{f}(-n)|^2 \tag{7}$$

holds for every $f \in L^2(V)$. Since V is a multiband set, it is actually enough to verify (7) for every infinitely smooth function f supported by V. Given such f, define

$$F_\varepsilon(t) := \sum_{n \in \mathbb{Z}} f(n\alpha) \, \widehat{\varphi}_\varepsilon(n) \exp 2\pi i n t, \quad t \in \mathbb{T}.$$

The fact that f is supported by V implies that only exponentials from $\mathcal{E}(\Lambda(\alpha, V))$ have their coefficient non-zero in the series above. Since $\mathcal{E}(\Lambda(\alpha, V))$ is a Riesz-Fischer system in $L^2(U)$, it follows from Lemma 2.5 that there is a constant C such that

$$\int_{\mathbb{T}} |F_\varepsilon(t)|^2 \, dt = \sum_{n \in \mathbb{Z}} \big| f(n\alpha) \, \widehat{\varphi}_\varepsilon(n) \big|^2 \leqslant C \int_U |F_\varepsilon(t)|^2 \, dt. \tag{8}$$

Now we take the limit of (8) as $\varepsilon \to 0$. Lemma 2.3 implies that the left hand side of (8) tends to $\int |f(t)|^2 \, dt$. On the other hand, substituting f with its Fourier expansion in the definition of F_ε, it is easy to see that

$$F_\varepsilon(t) = \sum_{n \in \mathbb{Z}} \widehat{f}(-n) \, \varphi_\varepsilon(t - n\alpha).$$

The coefficients $\{\widehat{f}(-n)\}$ belong to ℓ^1, since f is smooth. Since U is left semi-closed we may use Lemma 2.4, which implies that the limit as $\varepsilon \to 0$ of the right hand side of (8) is equal to the right hand side of (7). This proved the second part of Lemma 2.1. □

3. Exponential Riesz bases and multiband sets

3.1. In this section we prove Theorem 1. Let I be an interval on $\mathbb{T}$ which is either left or right semi-closed. We will show that if the (necessary) diophantine condition $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ is satisfied, then the exponential system $\mathcal{E}(\Lambda(\alpha, I))$ is a universal Riesz basis in $L^2(S)$ for a family of multiband sets S. In fact we will prove a somewhat more general result than formulated in the introduction.

Theorem 3.1. *Suppose that $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$. Then $\mathcal{E}(\Lambda(\alpha, I))$ is a Riesz basis in $L^2(S)$ for every set $S \subset \mathbb{T}$, $\operatorname{mes} S = |I|$, which satisfies the following condition: the indicator function $\mathbb{1}_S$ can be expressed as a finite linear combination of indicator functions of intervals $I_1, \dots, I_N$ whose lengths belong to $\mathbb{Z} + \alpha\mathbb{Z}$, that is,*

$$\mathbb{1}_S(t) = \sum_{j=1}^{N} c_j \, \mathbb{1}_{I_j}(t), \quad c_j \in \mathbb{Z}, \quad |I_j| \in \mathbb{Z} + \alpha\mathbb{Z} \quad (1 \leqslant j \leqslant N). \tag{9}$$

Condition (9) is certainly satisfied by any set S which is the union of finitely many *disjoint* intervals $I_1, \dots, I_N$ such that $|I_j| \in \mathbb{Z} + \alpha\mathbb{Z}$. However, remark that also other configurations are possible. For example, consider the set S of the form

$$S = I_1 \setminus I_2, \quad I_2 \subset I_1, \quad |I_1|, |I_2| \in \mathbb{Z} + \alpha\mathbb{Z},$$

which certainly satisfies the condition (9), but which is the union of two disjoint intervals whose lengths do not necessarily belong to $\mathbb{Z} + \alpha\mathbb{Z}$.

3.2. We shall suppose, with no loss of generality, that S is a left semi-closed multiband set. The condition (9) plays its key role in the following lemma.

Lemma 3.2. *Let S be a left semi-closed multiband set satisfying* (9)*. Then there is a bounded function $g : \mathbb{T} \to \mathbb{R}$, continuous from the right and with finitely many jump discontinuities, such that*

$$\mathbb{1}_S(t) - \operatorname{mes} S = g(t) - g(t+\alpha), \quad t \in \mathbb{T}. \tag{10}$$

Proof. Due to the linearity of the conditions (9) and (10), it will be enough to prove the lemma in the case when S is a single interval whose length belongs to $\mathbb{Z} + \alpha\mathbb{Z}$. Moreover, by rotation, we may suppose that S is an interval whose left endpoint is zero.

Let therefore $S = [0, n\alpha)$, where $n \in \mathbb{Z}$. For simplicity we shall suppose that n is a positive integer, as the case when n is negative is very similar. Let us denote by $\theta(t)$ the 1-periodic function on $\mathbb{R}$, defined by $\theta(t) = t$ for $0 \leqslant t < 1$. Considered as a function on the circle $\mathbb{T}$, this is a piecewise linear function, with slope $+1$, and with a jump discontinuity of magnitude -1 at $t = 0$. Set

$$g(t) := \sum_{k=1}^{n} \theta(t - k\alpha), \quad t \in \mathbb{T},$$

then g is a bounded function, continuous from the right, and with finitely many jump discontinuities. We have

$$g(t) - g(t+\alpha) = \theta(t - na) - \theta(t).$$

Observe that the function on the right hand side has the following properties: it has a jump of magnitude $+1$ at $t = 0$ and another jump of magnitude -1 at $t = n\alpha$, it has derivative zero at all other points, and has zero integral on $\mathbb{T}$. These properties determine the function uniquely as $\mathbb{1}_S(t) - \operatorname{mes} S$, and so (10) is established. □

Remark. If S is a multiband set, then the condition (9) is not only sufficient, but also necessary, for the existence of a bounded measurable function $g(t)$ satisfying (10). This is a consequence of a result due to Oren [**23**] (see also [**27**]).

3.3. We turn to the proof of Theorem 3.1. The first step in the proof, based on Meyer's duality principle, is to reduce the problem about exponential Riesz bases in $L^2(S)$ to a similar problem in $L^2(I)$, where I is a single interval. This allows us then to use a theorem of Avdonin [**1**] on exponential Riesz bases in $L^2(I)$. Below we formulate a special case of Avdonin's theorem, in a form which will be convenient in our setting.

Theorem 3.3 (Avdonin [**1**])**.** *Let $I \subset \mathbb{R}$ be a bounded interval, and let*

$$\lambda_j = \frac{j + \delta_j + c}{|I|}, \quad j \in \mathbb{Z}, \tag{11}$$

where c is a constant, and $\{\delta_j\}$ is a bounded sequence of real numbers. Suppose that there is a positive integer N such that

$$\sup_{n \in \mathbb{Z}} \Big| \frac{1}{N} \sum_{j=1}^{N} \delta_{n+j} \Big| < \frac{1}{4}. \tag{12}$$

If the sequence $\Lambda = \{\lambda_j,\ j \in \mathbb{Z}\}$ is separated then $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(I)$.

Here the separation condition means that $\inf_{n \neq m} |\lambda_n - \lambda_m| > 0$.

Remark that the famous Kadec 1/4 theorem (see [**29**]) corresponds to the case when $N = 1$ in Avdonin's theorem.

Proof of Theorem 3.1. Let S be a multiband set satisfying (9) and such that $\operatorname{mes} S = |I|$. We must prove that $\mathcal{E}(\Lambda(\alpha, I))$ is a Riesz basis in $L^2(S)$. Recall that I is semi-closed by assumption. Hence by Corollary 2.2, with $U = -I$ and $V = S$, it will be enough to show that $\mathcal{E}(\Lambda(\alpha, S))$ is a Riesz basis for $L^2(-I)$.

There is no loss of generality in assuming that S is left semi-closed. Let us enumerate the set $\Lambda(\alpha, S)$ in an increasing order,

$$\Lambda(\alpha, S) = \{\lambda_j,\ j \in \mathbb{Z}\}, \quad \cdots < \lambda_{-1} < 0 \leqslant \lambda_0 < \lambda_1 < \lambda_2 < \cdots,$$

and take g to be the function from Lemma 3.2. By the definition (3) of $\Lambda(\alpha, S)$ and according to (10), for any two integers $m < n$ we have

$$\#\big(\Lambda(\alpha, S) \cap [m, n)\big) = \sum_{k=m}^{n-1} \mathbb{1}_S(k\alpha) = (n-m) \operatorname{mes} S + g(m\alpha) - g(n\alpha).$$

Using this with $m = 0$, $n = \lambda_j$ $(j \geqslant 0)$ or with $m = \lambda_j$, $n = 0$ $(j < 0)$ we get

$$j = \lambda_j \operatorname{mes} S + g(0) - g(\lambda_j \alpha), \quad j \in \mathbb{Z}.$$

Since $\operatorname{mes} S = |I|$, this implies (11) with $\delta_j = g(\lambda_j \alpha)$ and $c = -g(0)$.

Observe that the perturbations $\{\delta_j\}$ are bounded, since g is bounded. We may assume, by adding a constant to g if necessary, that $\int_S g(t)\, dt = 0$. Hence

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=n+1}^{n+m} g(k\alpha) \mathbb{1}_S(k\alpha) = 0,$$

uniformly with respect to $n \in \mathbb{Z}$ (see [**20**], Chapter V, §6.3). It is then easy to see that (12) holds for a sufficiently large N. Moreover, the sequence $\Lambda(\alpha, S)$ is clearly separated, as the elements λ_j are distinct integers. So the proof is concluded by Theorem 3.3. □

3.4. Let $\mathscr{U}(\alpha)$ denote the collection of sets S which are finite unions of disjoint intervals whose lengths belong to $\mathbb{Z} + \alpha\mathbb{Z}$. In some sense, $\mathscr{U}(\alpha)$ is a 'dense' family of subsets of the circle $\mathbb{T}$. For example, if U is an open set, K is compact, and $K \subset U$, then there is a set $S \in \mathscr{U}(\alpha)$ such that $K \subset S \subset U$. This is due to the fact that the points $\{n\alpha\}$ are dense on $\mathbb{T}$.

In Theorem 1 we have considered the sub-class of sets $S \in \mathscr{U}(\alpha)$ satisfying the additional restriction $\operatorname{mes} S = \beta$, where β is a fixed number belonging to $\mathbb{Z} + \alpha\mathbb{Z}$. It is not immediately obvious whether a similar 'density' property is true for this sub-class. We shall prove, however, that this is true in the following sense:

Proposition 3.4. *Suppose that $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$, $0 < \beta < 1$. Let $U \subset \mathbb{T}$ be an open set, K be compact, such that $K \subset U$ and $\operatorname{mes} K < \beta < \operatorname{mes} U$. Then one can find a set $S \in \mathscr{U}(\alpha)$, $\operatorname{mes} S = \beta$, such that $K \subset S \subset U$.*

Using Theorem 1 this implies Corollary 1 above. For the proof we will need

Lemma 3.5. *Let α be an irrational number, and k be a non-zero integer. Given any $\varepsilon > 0$ one can find real numbers $\gamma_1, \dots, \gamma_s$ in the segment $(0, \varepsilon)$, such that $\gamma_j \in \mathbb{Z} + \alpha\mathbb{Z}$ for each j, and $\sum \gamma_j = \{k\alpha\}$, where $\{k\alpha\}$ is the fractional part of $k\alpha$.*

Proposition 3.4 is an easy consequence of this lemma: given U and K as above, choose numbers $\gamma_1, \dots, \gamma_s \in (0, \varepsilon)$, such that $\gamma_j \in \mathbb{Z} + \alpha\mathbb{Z}$ for each j, and $\sum \gamma_j = \beta$. If ε is sufficiently small then we may cover K by disjoint intervals $I_1, \dots, I_s$ contained in U, such that $|I_j| = \gamma_j$. The proposition follows by taking $S = I_1 \cup \cdots \cup I_s$.

In the proof of Lemma 3.5 we shall use basic facts from the theory of continued fractions, which can be found, for example, in [**13**]. Below $\|x\|$ will denote the distance of a real number x to its nearest integer, $\|x\| = \min |x - n|$, $n \in \mathbb{Z}$.

Proof of Lemma 3.5. It would be enough to prove the claim in the case when $k = 1$, as the general case would follow by considering the irrational number $k\alpha$ in place of α. We may therefore suppose that $k = 1$, and with no loss of generality also that $\alpha \in (0, 1)$. Let $[0; a_1, a_2, \dots]$ be the continued fraction expansion of α, and p_n/q_n be the sequence of convergents of α. Fix a sufficiently large integer n, to be chosen later. We define the numbers $\gamma_1, \dots, \gamma_s$, where $s = p_{n+1} + p_n$, by choosing p_{n+1} of the numbers being equal to $\|q_n\alpha\|$ and the other p_n of the numbers being equal to $\|q_{n+1}\alpha\|$. Then each γ_j belongs to $\mathbb{Z} + \alpha\mathbb{Z}$, since

$$\|q_n\alpha\| = (-1)^n(q_n\alpha - p_n) \in \mathbb{Z} + \alpha\mathbb{Z} \quad (n = 1, 2, \dots).$$

Moreover, we have

$$\begin{aligned}\sum_{j=1}^{s} \gamma_j &= p_{n+1}\|q_n\alpha\| + p_n\|q_{n+1}\alpha\| \\ &= p_{n+1}(-1)^n(q_n\alpha - p_n) + p_n(-1)^{n+1}(q_{n+1}\alpha - p_{n+1}) \\ &= (-1)^n(p_{n+1}q_n - p_nq_{n+1})\alpha = \alpha,\end{aligned}$$

as required. It remains to recall that $\|q_n\alpha\| < 1/q_{n+1}$, so by choosing n sufficiently large we can make sure that all the numbers γ_j are smaller than ε. □

4. Ergodic sums, BMO and Kesten's theorem

In this section we study the bounded mean oscillations of ergodic sums, first in an abstract setting, and then for irrational rotations of the circle. The results obtained will allow us in Section 5 to prove an extension of Theorem 3.

4.1. It will be convenient to start with an abstract setting. Let H be a Hilbert space, and U be a unitary operator on H. Given a vector $f \in H$ we consider its 'ergodic sums' defined by

$$S_n := f + Uf + \cdots + U^{n-1}f.$$

The von Neumann ergodic theorem asserts that the ratios S_n/n converge to the projection of f onto the closed subspace of U-invariant vectors. In particular, f is perpendicular to this subspace if and only if $\|S_n\| = o(n)$ as $n \to \infty$.

A vector f is called a *coboundary* if there exists $g \in H$ such that $f = g - Ug$. In this case the ergodic sums have the form $S_n = g - U^n g$. This, of course, implies the boundedness of the ergodic sums, $\|S_n\| = O(1)$. However, it was proved by Robinson [**26**] that the latter condition is not only necessary but also sufficient for f to be a coboundary. In fact, as the proof in [**26**] shows, the condition $(1/N)\sum_{n=1}^{N} \|S_n\|^2 = O(1)$ implies that f is a coboundary.

We will prove the following

Theorem 4.1. *For a vector f to be a coboundary it is necessary and sufficient that the numbers V_N (so-called 'variances') defined by*

$$V_N := \frac{1}{N}\sum_{n=1}^{N} \left\| S_n - \frac{S_1 + \cdots + S_N}{N} \right\|^2 \tag{13}$$

are bounded for $N = 1, 2, 3, \dots$.

This result strengthens Robinson's theorem, since the assumption $V_N = O(1)$ is weaker than the condition $(1/N)\sum_{n=1}^{N}\|S_n\|^2 = O(1)$. This is true because

$$V_N = \frac{1}{N}\sum_{n=1}^{N}\|S_n\|^2 - \Big\|\frac{1}{N}\sum_{n=1}^{N}S_n\Big\|^2.$$

A key tool in [**26**] as well as in our proof of Theorem 4.1 is the *spectral measure.* Since the sequence $\{\langle U^n f, f\rangle\}$ is positive definite, by Herglotz's theorem (see [**9**]) there exists a positive, finite measure μ_f on the circle $\mathbb{T}$, such that

$$\int_{\mathbb{T}} e^{2\pi int}\, d\mu_f(t) = \langle U^n f, f\rangle, \quad n \in \mathbb{Z}. \tag{14}$$

The measure μ_f is called the spectral measure of f with respect to U. In the paper [**26**] the coboundaries were characterized also in terms of the spectral measure, in the following way: f is a coboundary if and only if the integral

$$\int_{\mathbb{T}} \frac{d\mu_f(t)}{\sin^2 \pi t}$$

(with the integrand taking the value $+\infty$ at $t = 0$) is finite. Theorem 4.1 is obtained by a combination of this result and the following

Lemma 4.2. *For any $f \in H$ we have*

$$\lim_{N\to\infty} V_N = \frac{1}{4}\int_{\mathbb{T}} \frac{d\mu_f(t)}{\sin^2 \pi t}\ , \tag{15}$$

where the integral on the right hand side may be finite or infinite.

Proof. It follows from (14) that

$$\|P(U)f\|^2 = \int_{\mathbb{T}} |P(e^{2\pi it})|^2\, d\mu_f(t)$$

for every polynomial $P(z) = c_0 + c_1 z + \cdots + c_n z^n$. In particular, using (13) we can express the variance V_N in terms of the spectral measure in the form

$$V_N = \int_{\mathbb{T}} Q_N(t)\, d\mu_f(t), \tag{16}$$

where $Q_N(t)$ is the trigonometric polynomial defined by

$$Q_N(t) := \frac{1}{N}\sum_{n=1}^{N}\Big|\sum_{k=0}^{n-1} e^{2\pi ikt} - \frac{1}{N}\sum_{m=1}^{N}\sum_{k=0}^{m-1} e^{2\pi ikt}\Big|^2.$$

By evaluation of the inner sums we get

$$Q_N(t) = \frac{1}{N}\sum_{n=1}^{N}\Big|\frac{1-e^{2\pi int}}{1-e^{2\pi it}} - \frac{1}{N}\sum_{m=1}^{N}\frac{1-e^{2\pi imt}}{1-e^{2\pi it}}\Big|^2,$$

and it follows that

$$\begin{aligned} 4\,Q_N(t)\sin^2\pi t &= \frac{1}{N}\sum_{n=1}^{N}\Big|(1-e^{2\pi int}) - \frac{1}{N}\sum_{m=1}^{N}(1-e^{2\pi imt})\Big|^2 \\ &= \frac{1}{N}\sum_{n=1}^{N}\Big|e^{2\pi int} - \frac{1}{N}\sum_{m=1}^{N}e^{2\pi imt}\Big|^2 = \frac{1}{N}\sum_{n=1}^{N}\Big|e^{2\pi int}\Big|^2 - \Big|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi int}\Big|^2 \\ &= 1 - \frac{\sin^2 \pi Nt}{N^2\sin^2\pi t}\,. \end{aligned}$$

We have therefore obtained the formula

$$Q_N(t) = \frac{1}{4\sin^2 \pi t}\left(1 - \frac{\sin^2 \pi N t}{N^2 \sin^2 \pi t}\right).$$

The conclusion of (15) from this formula is immediate: in the case when the integral on the right hand side of (15) is finite one should merely apply to (16) the dominated convergence theorem, while in the case of divergence of the integral in (15) the result would follow from Fatou's lemma. □

4.2. We shall now consider the special case when $H = L^2(\mathbb{T})$ and the unitary operator U is the irrational rotation,

$$(Uf)(x) = f(x+\alpha), \quad f \in L^2(\mathbb{T}),$$

where α is a fixed irrational number. In this case the ergodic sums have the form

$$S_n(x) = \sum_{k=0}^{n-1} f(x + k\alpha). \tag{17}$$

Much attention has been paid to the case when these ergodic sums are *bounded*, that is, $\sup_n |S_n(x)| < \infty$ for some (or every) $x \in \mathbb{T}$. For more details we refer the reader to the papers [**4, 23, 25, 27**] and to the references therein.

Here we consider the *bounded mean oscillations* of the ergodic sums. In the next theorem we shall see that if f is sufficiently "good", then BMO behavior of the ergodic sums implies that f is a coboundary with respect to the rotation by α.

Theorem 4.3. *Let α be an irrational number, and f be a Riemann integrable function on $\mathbb{T}$. Suppose that for some fixed $x_0 \in \mathbb{T}$, the sequence $\{S_n(x_0)\}$, $n = 1, 2, 3, \dots$, belongs to* BMO*. Then there exists a function $g \in L^2(\mathbb{T})$ such that $f(x) = g(x) - g(x+\alpha)$ almost everywhere.*

Recall the definition (2) of the space BMO given in Section 1. It is well-known that replacing the condition (2) with an appropriate ℓ^p version yields an equivalent definition of BMO. This is a consequence of a classical theorem of John and Nirenberg [**8**] (see also [**5**], Chapter VI §2) in a version adopted for sequences. In the next lemma we formulate this fact for $p = 2$.

Lemma 4.4 ([**8**])**.** $\{c_n\} \in$ BMO *if and only if there is a constant M such that*

$$\left(\frac{1}{m-n}\sum_{k=n}^{m-1}\left|c_k - \frac{c_n + \dots + c_{m-1}}{m-n}\right|^2\right)^{1/2} \leqslant M \tag{18}$$

for every $n < m$.

Proof of Theorem 4.3. Define

$$v_N(x) := \left(\frac{1}{N}\sum_{n=1}^{N}\left|S_n(x) - \frac{S_1(x) + \dots + S_N(x)}{N}\right|^2\right)^{1/2} \qquad (x \in \mathbb{T}).$$

Using the obvious property $S_n(x + j\alpha) = S_{j+n}(x) - S_j(x)$ it follows that

$$v_N(x + j\alpha) = \left(\frac{1}{N}\sum_{n=1}^{N}\left|S_{j+n}(x) - \frac{S_{j+1}(x) + \dots + S_{j+N}(x)}{N}\right|^2\right)^{1/2}.$$

The assumption that $\{S_n(x_0)\} \in$ BMO combined with Lemma 4.4 thus implies the existence of a constant M such that

$$v_N(x_0 + j\alpha) \leqslant M, \quad j = 0, 1, 2, \dots .$$

The function v_N is therefore bounded by M on a dense subset of $\mathbb{T}$. But v_N is a Riemann integrable function, since so is f, hence it follows that

$$\int_{\mathbb{T}} v_N(x)^2 \, dx \leqslant M^2, \quad N = 1, 2, \dots .$$

On the other hand, we have $\int v_N(x)^2 \, dx = V_N$, where V_N is the variance defined by (13). It follows that the variances are bounded for $N = 1, 2, 3, \dots$, so the proof is concluded by Theorem 4.1. □

4.3. We can now prove Theorem 3. In fact, this theorem is a consequence of the following more general result.

Theorem 4.5. *Let α be an irrational number, and $S \subset \mathbb{T}$ be a measurable set whose boundary has Lebesgue measure zero. Let $\nu(n, S)$ denote the number of integers $0 \leqslant k \leqslant n - 1$ such that $k\alpha \in S$. If the sequence $\{\nu(n, S) - n \operatorname{mes} S\}$, $n = 1, 2, 3, \dots$, belongs to* BMO*, then* $\operatorname{mes} S \in \mathbb{Z} + \alpha\mathbb{Z}$.

The proof is a combination of the previous result with an argument due to Furstenberg, Keynes and Shapiro [**4**] and Petersen [**25**].

Proof of Theorem 4.5. The fact that the boundary of S has Lebesgue measure zero implies that the function $f(x) = \mathbb{1}_S(x) - \operatorname{mes} S$ is Riemann integrable. Let $S_n(x)$ be the ergodic sums of f defined by (17). Then we have $\{S_n(0)\} \in \mathrm{BMO}$ by assumption. Theorem 4.3 therefore implies the existence of a function $g \in L^2(\mathbb{T})$ such that $f(x) = g(x) - g(x + \alpha)$ (a.e.). We may suppose that g is real-valued. Consider then the function $\tau(x) = \exp 2\pi i g(x)$. We have

$$\tau(x + \alpha) = e^{2\pi i g(x+\alpha)} = e^{2\pi i (g(x) - f(x))} = \tau(x) \, e^{2\pi i \operatorname{mes} S} \quad \text{(a.e.)},$$

since the function $\mathbb{1}_S$ takes integer values. This means that τ is an eigenfunction of the irrational rotation by α, with eigenvalue $\exp 2\pi i \operatorname{mes} S$. However all the eigenvalues are known to be of the form $\exp 2\pi i j\alpha$, $j \in \mathbb{Z}$, hence $\operatorname{mes} S \in \mathbb{Z} + \alpha\mathbb{Z}$. □

5. Necessary condition for Riesz bases

In this section we prove Theorem 2. We will show that unless the diophantine condition $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ holds, the exponential system $\mathcal{E}(\Lambda(\alpha, I))$ cannot be a Riesz basis in $L^2(S)$ for any multiband set S. The proof consists of three main ingredients: Meyer's duality principle, the results from Section 4 on discrepancy and BMO, and Pavlov's theorem describing the exponential Riesz bases in $L^2(I)$ where I is an interval.

5.1. We start with the formulation of Pavlov's theorem.

Let $f(x)$ be a locally integrable function on $\mathbb{R}$. Given a bounded interval $J \subset \mathbb{R}$ we denote by f_J and $V_f(J)$ respectively the average and the variance of f over J,

$$f_J = \frac{1}{|J|} \int_J f(x) \, dx, \quad V_f(J) = \frac{1}{|J|} \int_J |f(x) - f_J|^2 \, dx.$$

If $\sup V_f(J) < \infty$, where the supremum is taken over all bounded intervals J, then we say that f has bounded mean oscillations, $f \in \mathrm{BMO}(\mathbb{R})$. Remark that the space $\mathrm{BMO}(\mathbb{R})$ is usually defined using L^1 means instead of L^2, but the two definitions are equivalent, similarly to the case of BMO space of sequences which was discussed in Section 4 (see again [**5**], Chapter VI §2).

For a discrete set $\Lambda \subset \mathbb{R}$ we denote by $n_\Lambda(x)$ the 'counting function' satisfying

$$n_\Lambda(b) - n_\Lambda(a) = \#(\Lambda \cap [a, b)), \quad a < b,$$

which is defined uniquely up to an additive constant. We will use the following version of Pavlov's theorem (see [**7**, p. 240]) which is formulated in terms of the counting function n_Λ.

Theorem 5.1 (Hruščev, Nikol'skii, Pavlov [**7**]). *Let $\Lambda = \{\lambda_n, n \in \mathbb{Z}\}$. The exponential system $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(0,a)$, $a > 0$, if and only if*

(i) *Λ is separated, that is, $\inf_{n\neq m} |\lambda_n - \lambda_m| > 0$;*

(ii) *$f(x) = n_\Lambda(x) - ax$ is a function in $\mathrm{BMO}(\mathbb{R})$;*

(iii) *There is $y > 0$ such that the harmonic continuation $U_f(x+iy)$ of f into the upper half plane admits the following representation:*

$$U_f(x+iy) = c + \widetilde{u}(x) + v(x), \quad x \in \mathbb{R},$$

where c is a constant, u, v are bounded measurable functions, $\|v\|_\infty < \frac{1}{4}$, and $\widetilde{u}$ is the Hilbert transform of u.

In the proof below we shall exploit only part (ii) of Theorem 5.1. We therefore do not discuss part (iii) in more detail. See [**7**] for a complete exposition.

5.2. We can finally prove the necessity of the condition $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$.

Proof of Theorem 2. Suppose that $\mathcal{E}(\Lambda(\alpha, I))$ is a Riesz basis in $L^2(S)$, for some multiband set $S \subset \mathbb{T}$. We may suppose that S is semi-closed. Corollary 2.2, with $U = S$ and $V = I$, implies that $\mathcal{E}(\Lambda(\alpha, S))$ is a Riesz basis in $L^2(-I)$. Using part (ii) of Theorem 5.1 it follows that the function

$$f(x) = n_{\Lambda(\alpha,S)}(x) - |I|x$$

belongs to $\mathrm{BMO}(\mathbb{R})$.

Since $\Lambda(\alpha, S)$ is a subset of $\mathbb{Z}$, the function f is linear on each interval $(n-1, n]$ and has slope $-|I|$ there. It is therefore possible to write f as the sum of two functions, $f = g + h$, where $g(x)$ admits the constant value $f(n)$ on each interval $(n-1, n]$, and $h(x)$ is a 1-periodic function which is linear with slope $-|I|$ on each such an interval. It is then easy to see that the functions $g - g_J$ and $h - h_J$ are orthogonal on each interval J of the form $J = (n, m]$, $n < m$, and it follows that $V_f(J) = V_g(J) + V_h(J)$.

However, observe that if $J = (n, m]$ then

$$V_g(J) = \frac{1}{m-n} \sum_{k=n+1}^{m} \Big| f(k) - \frac{f(n+1) + \cdots + f(m)}{m-n} \Big|^2,$$

and, since $f \in \mathrm{BMO}(\mathbb{R})$, we have

$$\sup_{J=(n,m]} V_g(J) \leqslant \sup_{J=(n,m]} V_f(J) < \infty.$$

Hence by Lemma 4.4 the sequence $\{f(n)\}$, $n = 1, 2, 3, \dots$, belongs to BMO.

Recall that Landau's inequalities imply that $\operatorname{mes} S = |I|$. By adding an appropriate constant to the counting function $n_{\Lambda(\alpha,S)}$ we may assume that $n_{\Lambda(\alpha,S)}(0) = 0$. This means that

$$f(n) = \nu(n, S) - n \operatorname{mes} S, \quad n = 1, 2, 3, \dots,$$

where $\nu(n, S)$ denotes the number of integers $0 \leqslant k \leqslant n-1$ such that $k\alpha \in S$. Since $\{f(n)\} \in \mathrm{BMO}$, it follows from Theorem 4.5 that $\operatorname{mes} S \in \mathbb{Z} + \alpha\mathbb{Z}$. We conclude that $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ and so the theorem is proved. $\square$

6. Remarks

6.1. If S is a single interval, then the complete description of the exponential Riesz bases $\mathcal{E}(\Lambda)$, $\Lambda \subset \mathbb{R}$, in $L^2(S)$ is given by Pavlov's theorem. Much less is known, however, in the case when S is the union of more than one interval. In fact, it is unknown in general whether an exponential Riesz basis in $L^2(S)$ exists at all. This existence has been established in the following special cases:

(i) S is a finite union of disjoint intervals with commensurable lengths [**2, 16**].

(ii) S is the union of two general intervals [**28**] (in this paper there is also a partial result for the union of more than two intervals).

Theorem 3.1 exhibits a new family of examples of multiband sets S with a Riesz basis of exponentials, namely, every set S satisfying the condition (9).

It is also interesting to compare our result to the previous ones in the case of intervals with commensurable lengths. In [**2, 16**] the authors establish the existence of a set $\Lambda \subset \mathbb{R}$, a finite union of translates of an arithmetical progression, such that $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(S)$. Theorem 1 allows to construct a different such Λ. By translation and rescaling we may suppose that $S \subset [0, 1]$ is a finite union of disjoint intervals of a common irrational length α. We can then choose Λ to be the set of all integers k such that $\{k\alpha\} < \operatorname{mes} S$ (where $\{k\alpha\}$ is the fractional part of $k\alpha$).

6.2. In connection with 'universal' exponential Riesz bases in the periodic setting, we have proved that if the density is an irrational number then there is an exponential system $\mathcal{E}(\Lambda)$, $\Lambda \subset \mathbb{Z}$, which is a Riesz basis in $L^2(S)$ for a 'dense' family of sets S. Precisely, an immediate consequence of Theorem 1 is:

Given any irrational number $0 < \alpha < 1$ *there is a set* $\Lambda \subset \mathbb{Z}$ *such that* $\mathcal{E}(\Lambda)$ *is a Riesz basis in* $L^2(S)$ *for every set* $S \subset \mathbb{T}$, $\operatorname{mes} S = \alpha$, *which is the union of finitely many disjoint intervals whose lengths belong to* $\mathbb{Z} + \alpha\mathbb{Z}$.

It is interesting to know what can be said if the density is a *rational* number.

6.3. Theorems 1 and 2 show that the condition $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ is necessary and sufficient for the Riesz basis property of $\mathcal{E}(\Lambda(\alpha, I))$ in $L^2(S)$ for some multiband set $S \subset \mathbb{T}$. Remark, however, that we have not classified completely all such multiband sets S. It was proved that the condition (9) is sufficient for S, but the question of whether it is also necessary remains open.

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