Gateaux and Fréchet Derivative in Intuitionistic Fuzzy Normed Linear spaces

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Abstract

Intuitionistic Fuzzy derivative, Intuitionistic Fuzzy Gateaux derivative, Intuitionistic Fuzzy Fréchet derivative are defined and a few of their properties are studied. The relation between Intuitionistic Fuzzy Gateaux derivative and Intuitionistic Fuzzy Fréchet derivative are emphasized.

Keywords : *Intuitionistic fuzzy differentiation, intuitionistic fuzzy continuity, intuitionistic fuzzy Gateaux derivative, intuitionistic fuzzy Fréchet derivative.*
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1 Introduction

Fuzzy set theory is a useful tool to describe the situation in which data are imprecise or vague or uncertain. Intuitionistic fuzzy set theory handle the situation by attributing a degree of membership and a degree of non-membership to which a certain object belongs to a set. It has a wide range of application in the field of population dynamics [6], chaos control [16], computer programming [17], medicine [5] etc.
The concept of intuitionistic fuzzy set, as a generalisation of fuzzy sets [27] was introduced by Atanassov in [1]. The concept of fuzzy norm was introduced by Katsaras [21] in 1984. In 1992, Felbin[13] introduced the idea of fuzzy norm on a linear space. Cheng-Moderson [7] introduced another idea of fuzzy norm on a linear space whose associated metric is same as the associated metric of Kramosil-Michalek [22]. Latter on Bag and Samanta [3] modified the definition of fuzzy norm of Cheng-Moderson [7] and established the concept of continuity

and boundednes of a linear operator with respect to their fuzzy norm in [4]. Many authors in [12, 15, 20, 23] discuss fuzzy derivatives in many approach. After studying continuities and boundedness of linear operator in fuzzy environment in [8, 18, 9, 10, 11], we introduce intuitionistic fuzzy Gateaux derivative and intuitionistic fuzzy Fréchet derivative of linear operator.
In this paper we define intuitionistic fuzzy derivative in **R**, intuitionistic fuzzy Gateaux derivative and intuitionistic fuzzy Fréchet derivative of linear operator and we study some of their properties. Thereafter we show that in **R** intuitionistic fuzzy derivative and intuitionistic fuzzy Fréchet derivative are equivalent. We also show that intuitionistic fuzzy Fréchet derivative implies intuitionistic fuzzy Gateaux derivative.

2 Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 *[25] A binary operation* $\ast : [0\,,\,1] \times [0\,,\,1] \longrightarrow [0\,,\,1]$ *is continuous t - norm if* $\ast$ *satisfies the following conditions :*
$(\,i\,)$ $\ast$ *is commutative and associative ,*
$(\,ii\,)$ $\ast$ *is continuous ,*
$(\,iii\,)$ $a \ast 1 = a \quad \forall\; a\; \varepsilon\; [\,0\,,\,1\,]$,
$(\,iv\,)$ $a \ast b \leq c \ast d$ *whenever* $a \leq c\,,\, b \leq d$ *and* $a\,,\, b\,,\, c\,,\, d\; \varepsilon\; [\,0\,,\,1\,]$.

A few examples of continuous t-norm are $a \ast b = ab$, $a \ast b = min\{a, b\}$, $a \ast b = max\{a + b - 1, 0\}$.

Definition 2.2 *[25]. A binary operation* $\diamond : [0\,,\,1] \times [0\,,\,1] \longrightarrow [0\,,\,1]$ *is continuous t-conorm if* $\diamond$ *satisfies the following conditions :*
$(\,i\,)$ $\diamond$ *is commutative and associative ,*
$(\,ii\,)$ $\diamond$ *is continuous ,*
$(\,iii\,)$ $a \diamond 0 = a \quad \forall\; a \in [\,0\,,\,1\,]$,
$(\,iv\,)$ $a \diamond b \leq c \diamond d$ *whenever* $a \leq c\,,\, b \leq d$ *and* $a\,,\, b\,,\, c\,,\, d \in [\,0\,,\,1\,]$.

A few examples of continuous t-conorm are $a \ast b = a + b - ab$, $a \ast b = max\{a, b\}$, $a \ast b = min\{a + b, 1\}$.

Definition 2.3 *[24] Let* $\ast$ *be a continuous t-norm ,* $\diamond$ *be a continuous t-conorm and* V *be a linear space over the field* $F\,(= \mathbf{R}$ *or* $\mathbf{C}\,)$. *An* ***intuitionistic fuzzy norm*** *on* V *is an object of the form*
$A = \{\, (\,(\,x\,,\,t\,)\,,\, \mu(\,x\,,\,t\,)\,,\, \nu(\,x\,,\,t\,)\,) : (\,x\,,\,t\,) \in V \times \mathbf{R}^+ \,\}$ *, where* $\mu\,,\,\nu$ *are fuzzy sets on* $V \times \mathbf{R}^+$ *,* μ *denotes the degree of membership and* ν

denotes the degree of non - membership $(x, t) \in V \times \mathbf{R}^+$ *satisfying the following conditions* :

(i) $\mu(x, t) + \nu(x, t) \leq 1 \quad \forall (x, t) \in V \times \mathbf{R}^+$;
(ii) $\mu(x, t) > 0$;
(iii) $\mu(x, t) = 1$ *if and only if* $x = \theta$, θ *is null vector* ;
(iv) $\mu(cx, t) = \mu(x, \frac{t}{|c|}) \quad \forall c \in F$ *and* $c \neq 0$;
(v) $\mu(x, s) * \mu(y, t) \leq \mu(x + y, s + t)$;
(vi) $\mu(x, \cdot)$ *is non-decreasing function of* $\mathbf{R}^+$ *and* $\lim_{t \to \infty} \mu(x, t) = 1$;
(vii) $\nu(x, t) < 1$;
$(viii)$ $\nu(x, t) = 0$ *if and only if* $x = \theta$;
(ix) $\nu(cx, t) = \nu(x, \frac{t}{|c|}) \quad \forall c \in F$ *and* $c \neq 0$;
(x) $\nu(x, s) \diamond \nu(y, t) \geq \nu(x + y, s + t)$;
(xi) $\nu(x, \cdot)$ *is non-increasing function of* $\mathbf{R}^+$ *and* $\lim_{t \to \infty} \nu(x, t) = 0$.

Definition 2.4 *[24] If* A *is an intuitionistic fuzzy norm on a linear space* V *then* (V, A) *is called an intuitionistic fuzzy normed linear space.*

For the intuitionistic fuzzy normed linear space (V, A), we further assume that $\mu, \nu, *, \diamond$ satisfy the following axioms :
(xii) $\left.\begin{matrix} a \diamond a = a \\ a * a = a \end{matrix}\right\}$, for all $a \in [0, 1]$.
$(xiii)$ $\mu(x, t) > 0$, for all $t > 0 \Rightarrow x = \theta$.
(xiv) $\nu(x, t) < 1$, for all $t > 0 \Rightarrow x = \theta$.
(xv) For $x \neq \theta$, $\mu(x, .)$ is a continuous function of $\mathbf{R}$ and strictly increasing on the subset $\{t : 0 < \mu(x, t) < 1\}$ of $\mathbf{R}$.
(xvi) For $x \neq \theta$, $\nu(x, .)$ is a continuous function of $\mathbf{R}$ and strictly decreasing on the subset $\{t : 0 < \nu(x, t) < 1\}$ of $\mathbf{R}$.

Definition 2.5 *[24] A sequence* $\{x_n\}_n$ *in an intuitionistic fuzzy normed linear space* (V, A) *is said to* ***converge*** *to* $x \in V$ *if for given* $r > 0,\ t > 0,\ 0 < r < 1$, *there exist an integer* $n_0 \in \mathbf{N}$ *such that*
$\mu(x_n - x, t) > 1 - r$ *and* $\nu(x_n - x, t) < r$ *for all* $n \geq n_0$.

Definition 2.6 *[24] Let,* (U, A) *and* (V, B) *be two intuitionistic fuzzy normed linear space over the same field* F. *A mapping* f *from* (U, A) *to* (V, B) *is said to be* ***intuitionistic fuzzy continuous*** *at* $x_0 \in U$, *if for any given* $\epsilon > 0, \alpha \in (0,1), \exists\, \delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0,1)$ *such that for all* $x \in U$,

$$\mu_U(x - x_0, \delta) > 1 - \beta \Rightarrow \mu_V(f(x) - f(x_0), \epsilon) > 1 - \alpha$$

$$\nu_U(x - x_0, \delta) < \beta \Rightarrow \nu_V(f(x) - f(x_0), \epsilon) < \alpha .$$

3 Intuitionistic fuzzy Gateaux derivative

In this section, we shall consider $(\mathbf{R}, \mu_R, \nu_R, *, \diamond)$ as an intuitionistic fuzzy normed linear space over the field $\mathbf{R}$ (the set of all real numbers).

Definition 3.1 *Let $(\mathbf{R}, \mu_1, \nu_1, *, \diamond)$ and $(\mathbf{R}, \mu_2, \nu_2, *, \diamond)$ be two intuitionistic fuzzy normed linear space over the same field $\mathbf{R}$. A mapping f from $(\mathbf{R}, \mu_1, \nu_1, *, \diamond)$ to $(\mathbf{R}, \mu_2, \nu_2, *, \diamond)$ is said to be* **intuitionistic fuzzy differentiable** *at $x_0 \in \mathbf{R}$, if for any given $\epsilon > 0$, $\alpha \in (0,1)$, $\exists\ \delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for all $x (\neq x_0) \in \mathbf{R}$,*

$$\mu_1(x - x_0\ ,\ \delta)\ >\ 1 - \beta\ \Rightarrow\ \mu_2\left(\frac{f(x) - f(x_0)}{x - x_0}\ -\ f'(x_0)\,,\ \epsilon\right)\ >\ 1 - \alpha$$

$$\nu_1(x - x_0\ ,\ \delta)\ <\ \beta\ \Rightarrow\ \nu_2\left(\frac{f(x) - f(x_0)}{x - x_0}\ -\ f'(x_0)\,,\ \epsilon\right)\ <\ \alpha\ .$$

We denote intuitionistic fuzzy derivative of f at x_0 by $f'(x_0)$.

Alternative definition: Let $(\mathbf{R}, \mu_1, \nu_1, *, \diamond)$ and $(\mathbf{R}, \mu_2, \nu_2, *, \diamond)$ be two intuitionistic fuzzy normed linear space over the same field $\mathbf{R}$. A mapping f from $(\mathbf{R}, \mu_1, \nu_1, *, \diamond)$ to $(\mathbf{R}, \mu_2, \nu_2, *, \diamond)$ is said to be **intuitionistic fuzzy differentiable** at $x_0 \in \mathbf{R}$, if for every $t > 0$

$$\lim_{\mu_1(x - x_0, t)\ \to\ 1}\ \mu_2\left(\frac{f(x) - f(x_0)}{x - x_0}\ -\ f'(x_0)\,,\ t\right)\ =\ 1$$

$$\lim_{\nu_1(x - x_0, t)\ \to\ 0}\ \nu_2\left(\frac{f(x) - f(x_0)}{x - x_0}\ -\ f'(x_0)\,,\ t\right)\ =\ 0$$

$f'(x_0)$ is called intuitionistic fuzzy derivative of f at x_0 .

Note 3.2 *It is easy to see that these two definitions are equivalent.*

Note 3.3 *If the intuitionistic fuzzy derivative of f, be $f'(x_0)$, the intuitionistic fuzzy derivative of $f'(x_0)$ at x_0 is called second order intuitionistic fuzzy derivative of f at x_0 and is denoted by $f''(x_0)$. Similarly, the n-th order intuitionistic fuzzy derivative of f at x_0 exists if $f^{n-1}(x_0)$ is intuitionistic fuzzy differentiable at x_0 and this derivative is denoted by $f^n(x_0)$.*

Theorem 3.4 *Let $f : (\mathbf{R}, \mu_1, \nu_1, *, \diamond) \rightarrow (\mathbf{R}, \mu_2, \nu_2, *, \diamond)$ and $g : (\mathbf{R}, \mu_1, \nu_1, *, \diamond) \rightarrow (\mathbf{R}, \mu_2, \nu_2, *, \diamond)$ be intuitionistic fuzzy differentiable at x_0, $(\mathbf{R}, \mu_1, \nu_1, *, \diamond)$ and $(\mathbf{R}, \mu_2, \nu_2, *, \diamond)$ satisfies the condition (xii). Then for $K \in \mathbf{R}$, $Kf + g$ is intuitionistic fuzzy differentiable at x_0 and $(Kf + g)'(x_0) = Kf'(x_0) + g'(x_0)$.*

Proof. Since f and g are intuitionistic fuzzy differentiable at x_0, therefore we have for any given $\epsilon > 0$, $\alpha \in (0,1)$, $\exists\, \delta = \delta(\alpha,\epsilon) > 0$, $\beta = \beta(\alpha,\epsilon) \in (0,1)$ such that for all $x(\neq x_0) \in \mathbf{R}$,

$$\mu_1(x-x_0\,,\,\delta) > 1-\beta \Rightarrow \mu_2\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\,,\,\epsilon\right) > 1-\alpha$$

$$\nu_1(x-x_0\,,\,\delta) < \beta \Rightarrow \nu_2\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\,,\,\epsilon\right) < \alpha\,.$$

and

$$\mu_1(x-x_0\,,\,\delta) > 1-\beta \Rightarrow \mu_2\left(\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0)\,,\,\epsilon\right) > 1-\alpha$$

$$\nu_1(x-x_0\,,\,\delta) < \beta \Rightarrow \nu_2\left(\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0)\,,\,\epsilon\right) < \alpha\,.$$

Now,

$$\mu_2\left(\frac{(Kf+g)(x)-(Kf+g)(x_0)}{x-x_0} - (Kf'(x_0)+g'(x_0)),\epsilon\right)$$
$$= \mu_2\left(\frac{Kf(x)+g(x)-Kf(x_0)-g(x_0)}{x-x_0} - Kf'(x_0)-g'(x_0),\epsilon\right)$$
$$\geq \mu_2\left(\frac{Kf(x)-Kf(x_0)}{x-x_0} - Kf'(x_0),\frac{\epsilon}{2}\right) * \mu_2\left(\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0),\frac{\epsilon}{2}\right)$$
$$= \mu_2\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0),\frac{\epsilon}{2|K|}\right) * \mu_2\left(\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0),\frac{\epsilon}{2}\right)$$
$> (1-\alpha)*(1-\alpha) = (1-\alpha)$, whenever $\mu_1(x-x_0\,,\,\delta) > 1-\beta$.
and

$$\nu_2\left(\frac{(Kf+g)(x)-(Kf+g)(x_0)}{x-x_0} - (Kf'(x_0)+g'(x_0)),\epsilon\right)$$
$$= \nu_2\left(\frac{Kf(x)+g(x)-Kf(x_0)-g(x_0)}{x-x_0} - Kf'(x_0)-g'(x_0),\epsilon\right)$$
$$\geq \nu_2\left(\frac{Kf(x)-Kf(x_0)}{x-x_0} - Kf'(x_0),\frac{\epsilon}{2}\right) \diamond \nu_2\left(\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0),\frac{\epsilon}{2}\right)$$
$$= \nu_2\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0),\frac{\epsilon}{2|K|}\right) \diamond \nu_2\left(\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0),\frac{\epsilon}{2}\right)$$
$< \alpha * \alpha = \alpha$, whenever $\nu_1(x-x_0\,,\,\delta) < \beta$.
So, $Kf+g$ is intuitionistic fuzzy differentiable at $x_0 \in \mathbf{R}$ and $(Kf+g)'(x_0) = Kf'(x_0) + g'(x_0)$.

Definition 3.5 *Let* (U, A) *and* (V, B) *be two intuitionistic fuzzy normed linear space over the same field* $k\,(=\mathbf{R}\,or\,\mathbf{C})$. *An operator* T *from* (U, A) *to* (V, B) *is said to be* **intuitionistic fuzzy Gateaux differentiable** *at* $x_0 \in U$, *if there exists an intuitionistic fuzzy continuous linear operator* $G:\ (U, A) \longrightarrow (V, B)$ *(generally depends upon* x_0*) and for any given* $\epsilon > 0, \alpha \in (0,1), \exists\, \delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0,1)$ *such that for every* $x \in U$ *and* $s(\neq 0) \in \mathbf{R}$,

$$\mu_R(s, \delta) > 1 - \beta \Rightarrow \mu_V\left(\frac{T(x_0 + sx) - T(x_0)}{s} - G(x), \epsilon\right) > 1 - \alpha$$

$$\nu_R(s, \delta) < \beta \Rightarrow \nu_V\left(\frac{T(x_0 + sx) - T(x_0)}{s} - G(x), \epsilon\right) < \alpha .$$

In this case, the operator G *is called intuitionistic fuzzy Gateaux derivative of* T *at* x_0 *and it is denoted by* $D_{f(x_0)}$.

Alternative definition: Let (U, A) and (V, B) be two intuitionistic fuzzy normed linear space over the same field $k\,(=\mathbf{R}\,or\,\mathbf{C})$. An operator T from (U, A) to (V, B) is said to be **intuitionistic fuzzy Gateaux differentiable** at $x_0 \in U$, if there exists an intuitionistic fuzzy continuous linear operator $G:\ (U, A) \longrightarrow (V, B)$ (generally depends upon x_0) such that for every $x \in U$, $t > 0$ and $s(\neq 0) \in \mathbf{R}$

$$\lim_{\mu_R(s,\delta) \to 1} \mu_V\left(\frac{T(x_0 + sx) - T(x_0)}{s} - G(x), t\right) = 1$$

$$\lim_{\nu_R(s,\delta) \to 0} \nu_V\left(\frac{T(x_0 + sx) - T(x_0)}{s} - G(x), t\right) = 0$$

In this case, the operator G is called intuitionistic fuzzy Gateaux derivative of T at x_0 and it is denoted by $D_{f(x_0)}$.

Note 3.6 *It is easy to see that these two definitions are equivalent.*

Theorem 3.7 *Let* $T : (U, A) \longrightarrow (V, B)$ *be a linear operator, where* (U, A) *and* (V, B) *are two intuitionistic fuzzy normed linear space satisfying* $(xiii)$ *and* (xiv). *If* T *is intuitionistic fuzzy Gateaux differentiable at* x_0 *then it is unique at* x_0 .

Proof. Let G_1, G_2 be two intuitionistic fuzzy Gateaux derivative of T at x_0. Then for for any given $\epsilon > 0, \alpha \in (0,1), \exists\, \delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for every $x \in U$ and $s(\neq 0) \in \mathbf{R}$,

$$\mu_R(s, \delta) > 1 - \beta \Rightarrow \mu_V\left(\frac{T(x_0 + sx) - T(x_0)}{s} - G_1(x), \epsilon\right) > 1 - \alpha$$

$$\nu_R(s,\delta) \,<\, \beta \,\Rightarrow\, \nu_V\left(\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_1(x)\,,\,\epsilon\right) \,<\, \alpha\,.$$

and

$$\mu_R(s,\delta) \,>\, 1-\beta \,\Rightarrow\, \mu_V\left(\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_2(x)\,,\,\epsilon\right) \,>\, 1-\alpha$$

$$\nu_R(s,\delta) \,<\, \beta \,\Rightarrow\, \nu_V\left(\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_2(x)\,,\,\epsilon\right) \,<\, \alpha\,.$$

$\mu_V(\,G_1(x)-G_2(x)\,,\,t\,)$

$$= \mu_V\left(\{\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_1(x)\} \,-\, \{\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_2(x)\}\,,\,t\right)$$

$$\geq\, \mu_V\left(\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_1(x)\,,\,\frac{t}{2}\right) * \mu_V\left(\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_2(x)\,,\,\frac{t}{2}\right)$$

$> \,(1-\alpha)*(1-\alpha) \,=\, (1-\alpha) \quad \forall\,\alpha\in(0,1).$

Therefore, $\mu_V(\,G_1(x)-G_2(x)\,,\,t\,) \,>\, 0 \quad \forall\,t>0$ (1)
and

$\nu_V(\,G_1(x)-G_2(x)\,,\,t\,)$

$$\leq\, \nu_V\left(\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_1(x)\,,\,\frac{t}{2}\right) \diamond \mu_V\left(\frac{T(x_0+sx)-T(x_0)}{s} \,-\, G_2(x)\,,\,\frac{t}{2}\right)$$

$< \,\alpha\diamond\alpha \,=\, \alpha \quad \forall\,\alpha\in(0,1).$

Therefore, $\nu_V(\,G_1(x)-G_2(x)\,,\,t\,) \,<\, 1 \quad \forall\,t>0$ (2)
From (1) and (2) we have $G_1(x)-G_2(x) \,=\theta$. Thus $G_1(x) \,=\, G_2(x)$.

Theorem 3.8 *If* T_1 *and* T_2 *have intuitionistic fuzzy Gateaux derivative at* x_0 *then* $T \,=\, cT_1 + T_2$ *has intuitionistic fuzzy Gateaux derivative at* x_0*, where* c *is a scaler.*

Proof. Straight forward.

4 Intuitionistic fuzzy Fréchet derivative

Definition 4.1 *Let (U, A) and (V, B) be two intuitionistic fuzzy normed linear space over the same field $k\,(=\mathbf{R}\,or\,\mathbf{C})$. An operator T from (U, A) to (V, B) is said to be* **intuitionistic fuzzy Fréchet differentiable** *at an interior $x_0 \in U$, if there exists a continuous linear operator $F : (U, A) \longrightarrow (V, B)$ (in general depends on x_0) and if for any given $\epsilon > 0, \alpha \in (0,1), \exists\, \delta = \delta(\alpha, \epsilon) > 0, \beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for all $x \in U$,*

$$\mu_U(x - x_0\,,\,\delta) \;>\; 1-\beta \;\Rightarrow\; \mu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{1 - \mu_U(x - x_0\,,\,t)}\,,\,\epsilon\right) > 1-\alpha$$

$$\nu_U(x - x_0\,,\,\delta) \;<\; \beta \;\Rightarrow\; \nu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{\nu_U(x - x_0\,,\,t)}\,,\,\epsilon\right) < \alpha\,.$$

In this case, F is called intuitionistic fuzzy Fréchet derivative of T at x_0 and is denoted by $DT(x_0)$.

Alternative definition: Let (U, A) and (V, B) be two intuitionistic fuzzy normed linear space over the same field $k\,(=\mathbf{R}\,or\,\mathbf{C})$. An operator T from (U, A) to (V, B) is said to be **intuitionistic fuzzy Fréchet differentiable** at an interior $x_0 \in U$, if there exists a continuous linear operator $F : (U, A) \longrightarrow (V, B)$ (in general depends on x_0) such that for every $t > 0$

$$\lim_{\mu_U(x - x_0, t) \,\to\, 1} \mu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{1 - \mu_U(x - x_0\,,\,t)}\,,\,t\right) \;=\; 1$$

$$\lim_{\nu_U(x - x_0, t) \,\to\, 0} \mu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{\nu_U(x - x_0\,,\,t)}\,,\,t\right) \;=\; 0$$

In this case, F is called intuitionistic fuzzy Fréchet derivative of T at x_0 and is denoted by $DT(x_0)$.

Note 4.2 *It is easy to see that these two definitions are equivalent.*

Theorem 4.3 *Let $T : (U, A) \longrightarrow (V, B)$ be a linear operator, where (U, A) and (V, B) are two intuitionistic fuzzy normed linear space satisfying $(xiii)$ and (xiv). If T is intuitionistic fuzzy Fréchet differentiable at x_0 then it is unique at x_0.*

Proof. Straight forward.

Example 4.4 *Let* $U = V = \mathbf{R}$ *and* $[a, b]$ *be an interval of* $\mathbf{R}$ *and* $T : [a, b] \longrightarrow \mathbf{R}$. *For all* $t > 0$ *define* $\mu(x,t) = \frac{t}{t + |x|}$, $\nu(x,t) = \frac{|x|}{t + |x|}$ *then the intuitionistic fuzzy Fréchet derivative of* T *at* x_0 *is intuitionistic fuzzy derivative.*

Proof. If T is intuitionistic fuzzy Fréchet differentiable at x_0 then for any given $\epsilon > 0$, $\alpha \in (0,1)$, $\exists\, \delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for all $x \in U$,

$$\mu_U(x - x_0\,,\ \delta) \ > \ 1 - \beta \ \Rightarrow \ \mu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{1 - \mu_U(x - x_0\,,\ t)}\,,\,\epsilon\right) > 1 - \alpha$$

$$\Rightarrow \ \mu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{|\,x - x_0\,|}\,,\,\frac{\epsilon}{t + |\,x - x_0\,|}\right) > 1 - \alpha$$

$$\Rightarrow \ \mu_V\left(\frac{T(x) - T(x_0)}{x - x_0} \ - \ F\,,\,\frac{\epsilon}{t + |\,x - x_0\,|}\right) > 1 - \alpha$$

and

$$\nu_U(x - x_0\,,\ \delta) \ < \ \beta \ \Rightarrow \ \nu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{\nu_U(x - x_0\,,\ t)}\,,\,\epsilon\right) < \alpha\,.$$

$$\Rightarrow \ \nu_V\left(\frac{T(x) - T(x_0) - (x - x_0)\,F}{|\,x - x_0\,|}\,,\,\frac{\epsilon}{t + |\,x - x_0\,|}\right) < \alpha$$

$$\Rightarrow \ \nu_V\left(\frac{T(x) - T(x_0)}{x - x_0} \ - \ F\,,\,\frac{\epsilon}{t + |\,x - x_0\,|}\right) < \alpha$$

Hence, intuitionistic fuzzy Fréchet derivative of T at x_0 implies intuitionistic fuzzy derivative T at x_0 and $T'(x_0) = DT(x_0)$.

Theorem 4.5 *An operator* T *from* $(\,U\,,\,A\,)$ *to* $(\,V\,,\,B\,)$ *is intuitionistic fuzzy Fréchet differentiable at* $x_0 \in U$ *then* T *is intuitionistic fuzzy Gateaux differentiable at* x_0 .

Proof. Since T is intuitionistic fuzzy Fréchet differentiable at x_0 , therefore we have for $t > 0$
$\mu_V\left(\frac{T(x_0+h) - T(x_0) - DT(x_0)h}{1 - \mu_U(h,t)}\,,\,t\right) > 1 - \alpha$, $\quad \nu_V\left(\frac{T(x_0+h) - T(x_0) - DT(x_0)h}{\nu_U(h,t)}\,,\,t\right) < \alpha$
Now,

$$\mu_V\left(\frac{T(x_0 + h) - T(x_0) - DT(x_0)h}{1 - \mu_U(h,t)}\,,\,t\right) > 1 - \alpha$$

$$\Rightarrow \ \mu_V\left(\frac{T(x_0 + sh) - T(x_0) - sDT(x_0)h}{1 - \mu_U(sh,t)}\,,\,t\right) > 1 - \alpha, \quad \textit{Putting } h = sh,\ s \neq 0$$

$$\Rightarrow \mu_V\left(\frac{\frac{T(x_0+sh)-T(x_0)}{s}-DT(x_0)h}{\frac{1}{s}(1-\mu_U(h,\frac{t}{|s|}))}\,,\,t\right) > 1-\alpha$$

$$\Rightarrow \mu_V\left(\frac{T(x_0+sh)-T(x_0)}{s}-DT(x_0)h\,,\,\frac{t}{|s|}(1-\mu_U(h,\frac{t}{|s|})\right) > 1-\alpha$$

$$\Rightarrow \mu_V\left(\frac{T(x_0+sh)-T(x_0)}{s}-DT(x_0)h\,,\,t_1\right) > 1-\alpha, \quad where\ t_1 = \frac{t}{|s|}(1-\mu_U(h,\frac{t}{|s|}))$$

and

$$\nu_V\left(\frac{T(x_0+h)-T(x_0)-DT(x_0)h}{\nu_U(h,t)}\,,\,t\right) > 1-\alpha$$

$$\Rightarrow \nu_V\left(\frac{T(x_0+sh)-T(x_0)-sDT(x_0)h}{\nu_U(sh,t)}\,,\,t\right) > 1-\alpha, \quad Putting\ h = sh,\ s \neq 0$$

$$\Rightarrow \nu_V\left(\frac{\frac{T(x_0+sh)-T(x_0)}{s}-DT(x_0)h}{\frac{1}{s}\nu_U(h,\frac{t}{|s|}\ )}\,,\,t\right) > 1-\alpha$$

$$\Rightarrow \nu_V\left(\frac{T(x_0+sh)-T(x_0)}{s}-DT(x_0)h\,,\,\frac{t}{|s|}\nu_U(h,\frac{t}{|s|}\right) > 1-\alpha$$

$$\Rightarrow \nu_V\left(\frac{T(x_0+sh)-T(x_0)}{s}-DT(x_0)h\,,\,t_2\right) > 1-\alpha, \quad where\ t_2 = \frac{t}{|s|}\nu_U(h,\frac{t}{|s|})$$

Hence, T is intuitionistic fuzzy Gateaux differentiable at x_0 and $D_{T(x_0)h} = DT(x_0)h$.

Theorem 4.6 *Let* $P : U \subset X \longrightarrow V \subset Y$ *and* $Q : V \longrightarrow Z$ *be two linear operator. Suppose* P *is intuitionistic fuzzy continuous and has intuitionistic fuzzy Gateaux derivative at* $x_o \in U$ *and* Q *has intuitionistic fuzzy Fréchet derivative at* $y_0 = P(x_0)$*. Then* $R = QP$ *has intuitionistic fuzzy Gateaux derivative at* x_0 *and* $D_{R(x_0)} = DQ(y_0)\,D_{P(x_0)}$.

Proof. We write $G = D_{P(x_0)}$ and $F = DQ(y_0)$ for shortness. Let $x \in X$ and we further write $\triangle y = P(x_0+sx) - P(x_0)$. Then

$$\mu\left(\frac{R(x_0+sx)-R(x_0)}{s}-FG\,,\,t\right) = \mu\left(\frac{QP(x_0+sx)-QP(x_0)}{s}-FG\,,\,t\right)$$

$$= \mu\left(\frac{F(\triangle y)+A(\triangle y)}{s}-FG\,,\,t\right) \quad ,where\ A(\triangle y) = Q(y_0+\triangle y)-Q(y_0)-F(\triangle y)$$

$$= \mu\left(F\frac{P(x_0+sx)-P(x_0)}{s}+\frac{A(\triangle y)}{s}-FG\,,\,t\right)$$

$$\geq \ \mu\left(F\frac{P(x_0+sx)-P(x_0)}{s} - FG\,,\ \frac{t}{2}\right) * \mu\left(\frac{A(\triangle y)}{\mu(\triangle y, t_1)}\,\frac{\mu(P(x_0+sx)-P(x_0), t_1)}{s}\,,\ \frac{t}{2}\right)$$

$$= \mu\left(\frac{P(x_0+sx)-P(x_0)}{s} - G\,,\ \frac{t}{2\mu(F,t_2)}\right) * \mu\left(\frac{A(\triangle y)}{\mu(\triangle y, t_1)}\,,\ \frac{ts}{2\mu(P(x_0+sx)-P(x_0), t_1)}\right)$$

$$= \mu\left(\frac{P(x_0+sx)-P(x_0)}{s} - G\,,\ \frac{t}{2\mu(F,t_2)}\right)$$

$$* \, \mu\left(\frac{Q(y_0+\triangle y)-Q(y_0)-F(\triangle y)}{\mu(\triangle y, t_1)}\,,\ \frac{ts}{2\mu(P(x_0+sx)-P(x_0), t_1)}\right)$$

$$> \ (1-\alpha) * (1-\alpha) \ = \ (1-\alpha)$$

since P has intuitionistic fuzzy Gateaux derivative and Q has intuitionistic fuzzy Fréchet derivative.
and

$$\nu\left(\frac{R(x_0+sx)-R(x_0)}{s} - FG\,,\ t\right) \ = \ \nu\left(\frac{QP(x_0+sx)-QP(x_0)}{s} - FG\,,\ t\right)$$

$$= \nu\left(\frac{F(\triangle y)+A(\triangle y)}{s} - FG\,,\ t\right) \quad , where\ A(\triangle y) \ = \ Q(y_0{+}\triangle y){-}Q(y_0){-}F(\triangle y)$$

$$= \nu\left(F\frac{P(x_0+sx)-P(x_0)}{s} + \frac{A(\triangle y)}{s} - FG\,,\ t\right)$$

$$\leq \ \nu\left(F\frac{P(x_0+sx)-P(x_0)}{s} - FG\,,\ \frac{t}{2}\right) \diamond \nu\left(\frac{A(\triangle y)}{1-\nu(\triangle y, t_1)}\,\frac{1-\nu(P(x_0+sx)-P(x_0), t_1)}{s}\,,\ \frac{t}{2}\right)$$

$$= \nu\left(\frac{P(x_0+sx)-P(x_0)}{s} - G\,,\ \frac{t}{2\nu(F,t_2)}\right)$$

$$\diamond \, \nu\left(\frac{A(\triangle y)}{1-\nu(\triangle y, t_1)}\,,\ \frac{ts}{2(1-\nu(P(x_0+sx)-P(x_0), t_1))}\right)$$

$$= \nu\left(\frac{P(x_0+sx)-P(x_0)}{s} - G\,,\ \frac{t}{2\nu(F,t_2)}\right)$$

$$\diamond \, \nu\left(\frac{Q(y_0+\triangle y)-Q(y_0)-F(\triangle y)}{1-\nu(\triangle y, t_1)}\,,\ \frac{ts}{2(1-\nu(P(x_0+sx)-P(x_0), t_1))}\right)$$

$$< \ \alpha \diamond \alpha \ = \ \alpha\,.$$

Since P has intuitionistic fuzzy Gateaux derivative and Q has intuitionistic fuzzy Fréchet derivative. Hence $R \ = \ QP$ has intuitionistic fuzzy Gateaux derivative at x_0 and $D_{R(x_0)} \ = \ DQ(y_0)\,D_{P(x_0)}$.

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