FINITENESS THEOREMS FOR DEFORMATIONS OF COMPLEXES

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Abstract. We consider deformations of bounded complexes of modules for a profinite group G over a field of positive characteristic. We prove a finiteness theorem which provides some sufficient conditions for the versal deformation of such a complex to be represented by a complex of G-modules that is strictly perfect over the associated versal deformation ring.

Contents

1. Introduction

The object of this paper is to determine the versal deformation rings and versal deformations of bounded complexes of modules for a profinite group. Our main result shows that under certain hypotheses, the versal deformation may be represented by a bounded complex of modules which are finitely generated over the versal deformation ring. This is evidence for the idea that complexes of modules which arise from arithmetic should have versal deformations with this property.

Suppose that k is a field of characteristic $p > 0$ and that G is a profinite group. In [12], Mazur developed a deformation theory of finite dimensional representations of G over k using work of Schlessinger in [14]. In [1, 2], we generalized Mazur's deformation theory by considering, instead of k-representations of G, objects $V^\bullet$ in the derived category $D^-(k[[G]])$ of bounded above complexes of pseudocompact modules over the completed group algebra $k[[G]]$ of G over k. The case of k-representations amounts to studying complexes that have exactly one non-zero cohomology group.

As in [2], we will assume $V^\bullet$ is bounded and has finite dimensional cohomology groups, and that G has a certain finiteness property so as to be able to apply Schlessinger's work. The calculation of the versal deformation ring $R(G, V^\bullet)$ would in principle require an infinite number of first order obstruction calculations, as discussed in [3]. For this reason we will study a different approach, which can be seen as a counterpart for complexes of the method of de Smit and Lenstra in [5]. They first considered lifts of matrix representations of groups; these are called framed deformations by Kisin in [10, §2.3.4]. One then considers the natural morphism of functors from framed deformations to deformations. When this idea is applied to complexes $V^\bullet$, a new issue arises:

Question 1.1. *Is $U(G, V^\bullet)$ represented by a complex of modules for the completed group ring $R(G, V^\bullet)[[G]]$ that is strictly perfect as a complex of $R(G, V^\bullet)$-modules?*

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Here a strictly perfect complex of $R(G,V^\bullet)$-modules is a bounded complex of finitely generated projective $R(G,V^\bullet)$-modules. The answer to Question 1.1 is yes (and obvious) when $V^\bullet$ has only one non-zero cohomology group, corresponding to the classical case. But we do not know the answer in general, even when $V^\bullet$ has only two non-zero cohomology groups.

We view Question 1.1 as a finiteness problem because when G is topologically finitely generated, a complex of $R(G,V^\bullet)[[G]]$-modules representing $U(G,V^\bullet)$ that is strictly perfect as a complex of $R(G,V^\bullet)$-modules can be described by a finite number of finite matrices with coefficients in $R(G,V^\bullet)$. An a priori result showing the existence of such a description, especially with explicit bounds on the sizes of the matrices, can be very useful in determining the ring $R(G,V^\bullet)$ via matrices with indeterminate entries. The proof of Theorem 4.2 gives an example of this method.

It is not very difficult to show that under our hypotheses on G, there is a theory of framed deformations for $V^\bullet$, in the following sense. One can represent $V^\bullet$ by a fixed choice of a bounded complex of pseudocompact $k[[G]]$-modules each of which is finite dimensional over k. Fix a choice of ordered k-basis for each term of $V^\bullet$. By a framed deformation over A one means a complex of pseudocompact $A[[G]]$-modules $M^\bullet$ along with ordered bases for the terms of $M^\bullet$ as free finitely generated A-modules such that there is an isomorphism of complexes $k\hat{\otimes}_A M^\bullet \to V^\bullet$ which carries the chosen ordered bases for the terms of $M^\bullet$ to the chosen ordered bases for the terms of $V^\bullet$. Isomorphisms of framed deformations must be isomorphisms of complexes which respect ordered bases. One can show, using Schlessinger's criteria, that under the hypotheses on G we make in §2, there is a versal deformation ring for the resulting functor. There is a natural transformation from this framed deformation functor to the functor $\hat{F}_{V^\bullet}$. The issue in Question 1.1 is whether this natural transformation will be surjective if we choose the ranks of the terms of $V^\bullet$ to be sufficiently large. This amounts to asking whether a single framed deformation functor has the derived category deformation functor $\hat{F}_{V^\bullet}$ as a quotient.

It is not hard to show that $U(G,V^\bullet)$ is represented by a bounded above complex of projective modules for $R(G,V^\bullet)[[G]]$. The difficulty is that the standard results concerning truncations of such complexes do not readily produce quasi-isomorphic complexes of $R(G,V^\bullet)[[G]]$-modules that are strictly perfect as complexes of modules for $R(G,V^\bullet)$, which is a much smaller ring than $R(G,V^\bullet)[[G]]$.

A fundamental problem in the subject appears to us to be whether Question 1.1 always has an affirmative answer if $V^\bullet$ arises from arithmetic, in a suitable sense. We will prove the following result concerning this question:

Theorem 1.2. *Suppose G is either*

(i) *topologically finitely generated and abelian, or*
(ii) *the tame fundamental group of the spectrum of a regular local ring S whose residue field is finite of characteristic different from p with respect to a divisor with strict normal crossings.*

Then $U(G,V^\bullet)$ is represented by a complex of $R(G,V^\bullet)[[G]]$-modules that is strictly perfect as a complex of $R(G,V^\bullet)$-modules.

In §4 we will apply this Theorem to compute $U(G,V^\bullet)$ and $R(G,V^\bullet)$ for some natural examples in which S in Theorem 1.2 is the ℓ-adic integers $\mathbb{Z}_\ell$ for some prime $\ell \neq p$. These examples pertain to the deformation of elements of order 2 in the Brauer group of $\mathbb{Q}_\ell$. Examples of this kind were first considered in [2], where we determined the associated universal flat deformation rings. We will produce some examples in which the versal deformation ring is strictly larger than the versal proflat deformation ring. Finding explicit arithmetic constructions of the associated versal deformations leads to interesting number theoretical questions, and is a good test of any general theory for determining deformations of complexes of modules for a profinite group.

We now give an outline of this paper.

In §2 we recall the definitions needed to state the main result of [2] concerning the existence of versal and universal deformations of objects $V^\bullet$ in $D^-(k[[G]])$. In §3 we give a proof of Theorem 1.2. The argument outlined in §3.1 proceeds by improving the representative for the versal deformation in question by three steps. In the first step one works from right to left to produce a complex whose individual terms have large annihilators. In the second step, one works from left to right and uses an Artin-Rees argument to produce a complex whose terms are finitely generated over the versal deformation ring. Finally in the last step one works from right to left to refine these terms so they become finitely generated and projective over the versal deformation ring. In §4 we conclude with some examples pertaining to the element of order 2 in the Brauer group of $\mathbb{Q}_\ell$.

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2. Quasi-lifts and deformation functors

Let G be a profinite group, let k be a field of characteristic $p > 0$, and let W be a complete local commutative Noetherian ring with residue field k. Define $\hat{\mathcal{C}}$ to be the category of complete local commutative Noetherian W-algebras with residue field k. The morphisms in $\hat{\mathcal{C}}$ are continuous W-algebra homomorphisms that induce the identity on k. Let $\mathcal{C}$ be the subcategory of Artinian objects in $\hat{\mathcal{C}}$.

Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$. Then $R[[G]]$ denotes the completed group algebra of the usual abstract group algebra $R[G]$ of G over R, i.e. $R[[G]]$ is the projective limit of the ordinary group algebras $R[G/U]$ as U ranges over the open normal subgroups of G. We have that R is a pseudocompact ring and $R[[G]]$ is a pseudocompact R-algebra.

Pseudocompact rings, algebras and modules have been studied, for example, in [6, 7, 4]. Recall that a pseudocompact ring is a topological ring Λ that is complete and Hausdorff and admits a basis of open neighborhoods of 0 consisting of two-sided ideals J for which Λ/J is an Artinian ring. Let Λ be a pseudocompact ring. Then Λ is the projective limit of Artinian quotient rings having the discrete topology. A pseudocompact Λ-module is a complete Hausdorff topological Λ-module M which has a basis of open neighborhoods of 0 consisting of submodules N for which M/N has finite length as Λ-module. Put differently, a Λ-module is pseudocompact if and only if it is the projective limit of Λ-modules of finite length having the discrete topology. If R is a commutative pseudocompact ring and Λ is a complete Hausdorff topological ring, then Λ is called a pseudocompact R-algebra provided Λ is an R-algebra in the usual sense and Λ admits a basis of open neighborhoods of 0 consisting of two-sided ideals J for which Λ/J has finite length as R-module. Note that every pseudocompact R-algebra is a pseudocompact ring, and a module over a pseudocompact R-algebra has finite length if and only if it has finite length as R-module.

Remark 2.1. Let Λ be a pseudocompact ring and let R be a commutative pseudocompact ring. Denote the category of pseudocompact left Λ-modules by $\mathrm{PCMod}(\Lambda)$.

Recall that a pseudocompact Λ-module M is said to be topologically free on a set $X = \{x_i\}_{i\in I}$ if M is isomorphic to the product of a family $(\Lambda_i)_{i\in I}$ where $\Lambda_i = \Lambda$ for all i.

(i) The category $\mathrm{PCMod}(\Lambda)$ is an abelian category with exact projective limits. Since every topologically free pseudocompact Λ-module is a projective object in $\mathrm{PCMod}(\Lambda)$ and since every pseudocompact Λ-module is the quotient of a topologically free Λ-module, $\mathrm{PCMod}(\Lambda)$ has enough projective objects.

(iii) If M and N are pseudocompact Λ-modules, then we define the right derived functors $\mathrm{Ext}^n_\Lambda(M, N)$ by using a projective resolution of M.

(iv) Suppose Λ is a pseudocompact R-algebra, and let $\hat{\otimes}_\Lambda$ denote the completed tensor product in the category $\mathrm{PCMod}(\Lambda)$ (see [4, §2]). If M is a right (resp. left) pseudocompact Λ-module, then $M\hat{\otimes}_\Lambda-$ (resp. $-\hat{\otimes}_\Lambda M$) is a right exact functor.

If M is finitely generated as a pseudocompact Λ-module, it follows from [4, Lemma 2.1(ii)] that the functors $M \otimes_\Lambda -$ and $M\hat{\otimes}_\Lambda-$ (resp. $- \otimes_\Lambda M$ and $-\hat{\otimes}_\Lambda M$) are naturally isomorphic.

(v) Suppose Λ is a pseudocompact R-algebra and M is a right (resp. left) pseudocompact Λ-module. Recall that M is said to be topologically flat, if the functor $M\hat{\otimes}_\Lambda-$ (resp. $-\hat{\otimes}_\Lambda M$)

is exact. By [4, Lemma 2.1(iii)] and [4, Prop. 3.1], M is topologically flat if and only if M is projective.

If $\Lambda = R$ and M is a pseudocompact R-module, it follows from [7, Proof of Prop. 0.3.7] and [7, Cor. 0.3.8] that M is topologically flat if and only if M is topologically free if and only if M is abstractly flat. In particular, if R is Artinian, a pseudocompact R-module is topologically flat if and only if it is abstractly free.

Remark 2.2. Let Λ be a pseudocompact ring.

(i) Suppose $f : M \to N$ is a homomorphism of pseudocompact Λ-modules. Since $\mathrm{PCMod}(\Lambda)$ has exact projective limits, it follows that the image of f is closed in N and is therefore a pseudocompact Λ-submodule of N.

In particular, if I is a two-sided ideal of Λ and N is a pseudocompact left Λ-module such that both I and N are finitely generated as abstract left Λ-modules, then $I\,N$ is a closed pseudocompact Λ-submodule of N, since it is the image of a homomorphism $f : M \to N$ of pseudocompact Λ-modules in which M is a topologically free pseudocompact Λ-module on a finite set of cardinality equal to the product of the cardinalities of generating sets for I and N.

(ii) By [4, Lemma 1.1], if $f : M \to N$ is an epimorphism in $\mathrm{PCMod}(\Lambda)$, i.e. a surjective homomorphism of pseudocompact Λ-modules, then there is a continuous section $s : N \to M$ such that $f \circ s$ is the identity morphism on N. In particular, a homomorphism $f : M \to N$ of pseudocompact Λ-modules is an isomorphism in $\mathrm{PCMod}(\Lambda)$ if and only if it is bijective.

(iii) Suppose M is a pseudocompact Λ-module that is free and finitely generated as an abstract Λ-module. Since a topologically free pseudocompact Λ-module on a finite set X is isomorphic to an abstractly free Λ-module on X, one sees that M is a topologically free pseudocompact Λ-module on a finite set.

If Λ is a pseudocompact ring, let $C^-(\Lambda)$ be the abelian category of complexes of pseudocompact Λ-modules that are bounded above, let $K^-(\Lambda)$ be the homotopy category of $C^-(\Lambda)$, and let $D^-(\Lambda)$ be the derived category of $K^-(\Lambda)$. Let $[1]$ denote the translation functor on $C^-(\Lambda)$ (resp. $K^-(\Lambda)$, resp. $D^-(\Lambda)$), i.e. $[1]$ shifts complexes one place to the left and changes the sign of the differential. Note that by Remark 2.2(ii), a homomorphism in $C^-(\Lambda)$ is a quasi-isomorphism if and only if the induced homomorphisms on all the cohomology groups are bijective.

Hypothesis 1. Throughout this paper, we assume that $V^\bullet$ is a complex in $D^-(k[[G]])$ that has only finitely many non-zero cohomology groups, all of which have finite k-dimension.

Remark 2.3. Let $X^\bullet, Y^\bullet \in \mathrm{Ob}(K^-(R[[G]]))$ and consider the double complex $K^{\bullet,\bullet}$ of pseudocompact $R[[G]]$-modules with $K^{p,q} = (X^p \hat{\otimes}_R Y^q)$ and diagonal G-action. We define the total tensor product $X^\bullet \hat{\otimes}_R Y^\bullet$ to be the simple complex associated to $K^{\bullet,\bullet}$, i.e.

$$(X^\bullet \hat{\otimes}_R Y^\bullet)^n = \bigoplus_{p+q=n} X^p \hat{\otimes}_R Y^q$$

whose differential is $d(x \,\hat{\otimes}\, y) = d_X(x) \,\hat{\otimes}\, y + (-1)^x \, x \,\hat{\otimes}\, d_Y(y)$ for $x \,\hat{\otimes}\, y \in K^{p,q}$. Since homotopies carry over the completed tensor product, we have a functor

$$\hat{\otimes}_R : K^-(R[[G]]) \times K^-(R[[G]]) \to K^-(R[[G]]).$$

Using [16, Thm. 2.2 of Chap. 2 §2], we see that there is a well-defined left derived completed tensor product $\hat{\otimes}^{\mathbf{L}}_R$. Moreover, if $X^\bullet$ and $Y^\bullet$ are as above, then $X^\bullet \hat{\otimes}^{\mathbf{L}}_R Y^\bullet$ may be computed in $D^-(R[[G]])$ in the following way. Take a bounded above complex $Y'^\bullet$ of topologically flat pseudocompact $R[[G]]$-modules with a quasi-isomorphism $Y'^\bullet \to Y^\bullet$ in $K^-(R[[G]])$. Then this quasi-isomorphism induces an isomorphism between $X^\bullet \hat{\otimes}_R Y'^\bullet$ and $X^\bullet \hat{\otimes}^{\mathbf{L}}_R Y^\bullet$ in $D^-(R[[G]])$.

Definition 2.4. (a) We will say that a complex $M^\bullet$ in $K^-(R[[G]])$ has *finite pseudocompact R-tor dimension*, if there exists an integer N such that for all pseudocompact R-modules

S, and for all integers $i < N$, $\mathrm{H}^i(S\hat{\otimes}^{\mathbf{L}}_R M^\bullet) = 0$. If we want to emphasize the integer N in this definition, we say $M^\bullet$ has *finite pseudocompact R-tor dimension at N*.

(b) A *quasi-lift* of $V^\bullet$ over an object R of $\hat{\mathcal{C}}$ is a pair $(M^\bullet, \phi)$ consisting of a complex $M^\bullet$ in $D^-(R[[G]])$ that has finite pseudocompact R-tor dimension together with an isomorphism $\phi : k\hat{\otimes}^{\mathbf{L}}_R M^\bullet \to V^\bullet$ in $D^-(k[[G]])$. Two quasi-lifts $(M^\bullet, \phi)$ and $(M'^\bullet, \phi')$ are *isomorphic* if there is an isomorphism $f : M^\bullet \to M'^\bullet$ in $D^-(R[[G]])$ with $\phi' \circ (k\hat{\otimes}^{\mathbf{L}} f) = \phi$.

(c) Let $\hat{F} = \hat{F}_{V^\bullet} : \hat{\mathcal{C}} \to \mathrm{Sets}$ be the functor which sends an object R of $\hat{\mathcal{C}}$ to the set $\hat{F}(R)$ of all isomorphism classes of quasi-lifts of $V^\bullet$ over R, and which sends a morphism $\alpha : R \to R'$ in $\hat{\mathcal{C}}$ to the set map $\hat{F}(R) \to \hat{F}(R')$ induced by $M^\bullet \mapsto R'\hat{\otimes}^{\mathbf{L}}_{R,\alpha} M^\bullet$. Let $F = F_{V^\bullet}$ be the restriction of $\hat{F}$ to the subcategory $\mathcal{C}$ of Artinian objects in $\hat{\mathcal{C}}$.

Let $k[\varepsilon]$, where $\varepsilon^2 = 0$, denote the ring of dual numbers over k. The set $F(k[\varepsilon])$ is called the *tangent space* to F, denoted by t_F.

Remark 2.5. Suppose $M^\bullet$ is a complex in $K^-([[RG]])$ of topologically flat, hence topologically free, pseudocompact R-modules that has finite pseudocompact R-tor dimension at N. Then the bounded complex $M'^\bullet$, which is obtained from $M^\bullet$ by replacing M^N by $M'^N = M^N/\delta^{N-1}(M^{N-1})$ and by setting $M'^i = 0$ if $i < N$, is quasi-isomorphic to $M^\bullet$ and has topologically free pseudocompact terms over R.

Theorem 2.6. *Suppose that* $\mathrm{H}^i(V^\bullet) = 0$ *unless* $n_1 \le i \le n_2$. *Every quasi-lift of* $V^\bullet$ *over an object* R *of* $\hat{\mathcal{C}}$ *is isomorphic to a quasi-lift* $(P^\bullet, \phi)$ *for a complex* $P^\bullet$ *with the following properties:*

(i) *The terms of* $P^\bullet$ *are topologically free* $R[[G]]$*-modules.*

(ii) *The cohomology group* $\mathrm{H}^i(P^\bullet)$ *is finitely generated as an abstract* R*-module for all* i*, and* $\mathrm{H}^i(P^\bullet) = 0$ *unless* $n_1 \le i \le n_2$.

(iii) *One has* $\mathrm{H}^i(S\hat{\otimes}^{\mathbf{L}}_R P^\bullet) = 0$ *for all pseudocompact* R*-modules* S *unless* $n_1 \le i \le n_2$.

Proof. Part (i) follows from [2, Lemma 2.9]. Assume now that the terms of $P^\bullet$ are topologically free $R[[G]]$-modules, which means in particular that the functors $-\hat{\otimes}^{\mathbf{L}}_R P^\bullet$ and $-\hat{\otimes}_R P^\bullet$ are naturally isomorphic. Let m_R denote the maximal ideal of R, and let n be an arbitrary positive integer. By [2, Lemmas 3.1 and 3.8], $\mathrm{H}^i((R/m_R^n)\hat{\otimes}_R P^\bullet) = 0$ for $i > n_2$ and $i < n_1$. Moreover, for $n_1 \le i \le n_2$, $\mathrm{H}^i((R/m_R^n)\hat{\otimes}_R P^\bullet)$ is a subquotient of an abstractly free (R/m_R^n)-module of rank $d_i = \dim_k \mathrm{H}^i(V^\bullet)$, and $(R/m_R^n)\hat{\otimes}_R P^\bullet$ has finite pseudocompact (R/m_R^n)-tor dimension at $N = n_1$. Since $P^\bullet \cong \varprojlim_n (R/m_R^n)\hat{\otimes}_R P^\bullet$ and since by Remark 2.1(i), the category $\mathrm{PCMod}(R)$ has exact projective limits, it follows that for all pseudocompact R-modules S

$$\mathrm{H}^i(S\hat{\otimes}_R P^\bullet) = \varprojlim_n \mathrm{H}^i\left((S/m_R^n S)\hat{\otimes}_{R/m_R^n}\left((R/m_R^n)\hat{\otimes}_R P^\bullet\right)\right)$$

for all i. Hence Theorem 2.6 follows. □

Definition 2.7. A profinite group G has *finite pseudocompact cohomology*, if for each discrete $k[[G]]$-module M of finite k-dimension, and all integers j, the cohomology group $\mathrm{H}^j(G, M) = \mathrm{Ext}^j_{k[[G]]}(k, M)$ has finite k-dimension.

Theorem 2.8. ([2, Thm. 2.14]) *Suppose that* G *has finite pseudocompact cohomology.*

(i) *The functor* F *has a pro-representable hull* $R(G, V^\bullet) \in \mathrm{Ob}(\hat{\mathcal{C}})$ *(c.f.* [14, Def. 2.7] *and* [13, §1.2]*), and the functor* $\hat{F}$ *is continuous (c.f.* [13]*).*

(ii) *There is a* k*-vector space isomorphism* $h : t_F \to \mathrm{Ext}^1_{D^-(k[[G]])}(V^\bullet, V^\bullet)$.

(iii) *If* $\mathrm{Hom}_{D^-(k[[G]])}(V^\bullet, V^\bullet) = k$*, then* $\hat{F}$ *is represented by* $R(G, V^\bullet)$.

Remark 2.9. By Theorem 2.8(i), there exists a quasi-lift $(U(G, V^\bullet), \phi_U)$ of $V^\bullet$ over $R(G, V^\bullet)$ with the following property. For each $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, the map $\mathrm{Hom}_{\hat{\mathcal{C}}}(R(G, V^\bullet), R) \to \hat{F}(R)$ induced by

$\alpha \mapsto R\hat{\otimes}^{\mathbf{L}}_{R(G,V^\bullet),\alpha} U(G,V^\bullet)$ is surjective, and this map is bijective if R is the ring of dual numbers $k[\varepsilon]$ over k where $\varepsilon^2 = 0$.

In general, the isomorphism type of the pro-representable hull $R(G,V^\bullet)$ is unique up to non-canonical isomorphism. If $R(G,V^\bullet)$ represents $\hat{F}$, then $R(G,V^\bullet)$ is uniquely determined up to canonical isomorphism.

Definition 2.10. Using the notation of Theorem 2.8 and Remark 2.9, we call $R(G,V^\bullet)$ the *versal deformation ring* of $V^\bullet$ and $(U(G,V^\bullet),\phi_U)$ a *versal deformation* of $V^\bullet$.

If $R(G,V^\bullet)$ represents $\hat{F}$, then $R(G,V^\bullet)$ will be called the *universal deformation ring* of $V^\bullet$ and $(U(G,V^\bullet),\phi_U)$ will be called a *universal deformation* of $V^\bullet$.

Remark 2.11. If $V^\bullet$ consists of a single module V_0 in dimension 0, the versal deformation ring $R(G,V^\bullet)$ coincides with the versal deformation ring studied by Mazur in [12, 13]. In this case, Mazur assumed only that G satisfies a certain finiteness condition (Φ_p), which is equivalent to the requirement that $\mathrm{H}^1(G,M)$ have finite k-dimension for all discrete $k[[G]]$-modules M of finite k-dimension. Since the higher G-cohomology enters into determining lifts of complexes $V^\bullet$ having more than one non-zero cohomology group, the condition that G have finite pseudocompact cohomology is the natural generalization of Mazur's finiteness condition in this context.

We conclude this section by recalling a result from [3].

Proposition 2.12. ([3, Prop. 4.2]) *Suppose G has finite pseudocompact cohomology and K is a closed normal subgroup of G which is a pro-p' group, i.e. the projective limit of finite groups that have order prime to p. Let $\Delta = G/K$, and suppose $V^\bullet$ is isomorphic to the inflation $\mathrm{Inf}_\Delta^G V_\Delta^\bullet$ of a bounded above complex $V_\Delta^\bullet$ of pseudocompact $k[[\Delta]]$-modules. Then the two deformation functors $\hat{F}^G = \hat{F}^G_{V^\bullet}$ and $\hat{F}^\Delta = \hat{F}^\Delta_{V_\Delta^\bullet}$ which are defined according to Definition 2.4(c) are naturally isomorphic. In consequence, $R(G,V^\bullet) \cong R(\Delta, V_\Delta^\bullet)$ and $(U(G,V^\bullet),\phi_U) \cong (\mathrm{Inf}_\Delta^G U(\Delta,V_\Delta^\bullet), \mathrm{Inf}_\Delta^G \phi_U)$.*

3. Finiteness questions

In this section, we consider the question of when every quasi-lift of $V^\bullet$ over a ring A in $\hat{\mathcal{C}}$ can be represented by a bounded complex of abstractly finitely generated free A-modules with continuous actions by G. Recall from Remark 2.2(iii) that if a pseudocompact module is abstractly finitely generated free, then it is topologically free on a finite set. As before, k has positive characteristic p. We distinguish two cases:

Case A: G is topologically finitely generated and abelian; and

Case B: G is the tame fundamental group of the spectrum of a regular local ring S whose residue field $k(S)$ is finite of characteristic $\ell \neq p$ with respect to a divisor D with strict normal crossings.

We recall the structure of G as in case B (see [8, 15]). Let $Y = \mathrm{Spec}(S)$, and let $D_Y = D = \sum_{i=1}^r \mathrm{div}_Y(f_i)$ for a subset $\{f_i\}_{i=1}^r$ of a system of local parameters for the maximal ideal m_S of S. Let $X = \mathrm{Spec}(S^h)$ be the strict henselization of Y, so that S^h is local, its residue field is equal to the separable closure $k(S^h) = k(S)^s$ of $k(S)$, and m_{S^h} is generated by m_S. The divisor $D_X = \sum_{i=1}^r \mathrm{div}_X(f_i)$ has normal crossings on X. We have an exact sequence

$$1 \to \pi_1^t(X,D_X) \to \pi_1^t(Y,D_Y) \to \mathrm{Gal}(k(S)^s/k(S)) \to 1 \tag{3.1}$$

in which $G = \pi_1^t(Y,D_Y)$ and $\pi_1^t(X,D_X)$ are tame fundamental groups. There is a Kummer isomorphism

$$\pi_1^t(X,D_X) \cong \prod_{i=1}^r \hat{\mathbb{Z}}^{(\ell')}(1) \tag{3.2}$$

in which $\hat{\mathbb{Z}}^{(\ell')}(1) = \varprojlim_{\ell \nmid m} \mu_m$. The group $\mathrm{Gal}(k(S)^s/k(S))$ is procyclic and is topologically generated by the Frobenius automorphism $\Phi_{k(S)}$ relative to the finite field $k(S)$. Explicitly, if we define

$f \geq 1$ so $k(S) = \mathbb{F}_{\ell^f}$ has order ℓ^f, then a lift $\Phi \in \pi_1^t(Y, D_Y)$ of $\Phi_{k(S)}$ acts on each factor of $\pi_1^t(X, D_X)$ which is isomorphic to $\hat{\mathbb{Z}}^{(\ell')}(1)$ via the map $\zeta \to \zeta^{\ell^f}$. Since the procyclic group $\langle \Phi \rangle$, which is topologically generated by the lift Φ, is isomorphic to the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$, and this maps isomorphically to $\mathrm{Gal}(k(S)^s/k(S))$, we see that (3.1) is a split exact sequence and $G = \pi_1^t(Y, D_Y)$ is the semidirect product of $\langle \Phi \rangle$ with $\pi_1^t(X, D_X)$.

Suppose $V^\bullet \in D^-(k[[G]])$ is as in Hypothesis 1, i.e. $V^\bullet$ has only finitely many non-zero cohomology groups, all of which have finite k-dimension. Assume that $\mathrm{H}^i(V^\bullet) = 0$ unless $n_1 \leq i \leq n_2$. Theorem 1.2 states that for G as in case A or case B, the versal deformation $(U(G, V^\bullet), \phi_U)$ is represented in $D^-(R(G, V^\bullet)[[G]])$ by a complex that is strictly perfect as a complex of $R(G, V^\bullet)$-modules. This is a consequence of the following result.

Theorem 3.1. *Let A be an object of $\hat{\mathcal{C}}$. Suppose $(P^\bullet, \phi)$ is a quasi-lift of $V^\bullet$ over A such that $P^\bullet$ has properties (i), (ii) and (iii) of Theorem 2.6. There is a bounded complex $Q^\bullet$ of pseudocompact $A[[G]]$-modules which is isomorphic to $P^\bullet$ in $D^-(A[[G]])$ for which each term Q^i is an abstractly finitely generated free A-module and $Q^i = 0$ unless $n_1 \leq i \leq n_2$.*

The proof of Theorem 3.1 is outlined in the next section and carried out in subsequent sections.

3.1. **Outline of the proof of Theorem 3.1.** We begin with a reduction.

Lemma 3.2. *There is a pro-p' closed normal subgroup K of G with the following properties:*

(i) *The complex $V^\bullet$ is inflated from a complex for $\Delta = G/K$.*

(ii) *In case A, $\Delta = \mathbb{Z}_p^s \times Q \times Q'$ where Q (resp. Q') is a finite abelian p-group (resp. p'-group). Let $w_{1,j}$ for $1 \leq j \leq s$ be topological generators for the $\mathbb{Z}_p$-factors in this description.*

(iii) *In case B, let $\hat{\mathbb{Z}}^{(\ell',p')}(1)$ be the unique maximal pro-p' subgroup of $\hat{\mathbb{Z}}^{(\ell')}(1)$. Let K_1 be the maximal subgroup of*

$$N_1 = \prod_{i=1}^{r} \hat{\mathbb{Z}}^{(\ell',p')}(1) \subset \prod_{i=1}^{r} \hat{\mathbb{Z}}^{(\ell')}(1) = \pi_1^t(X, D_X)$$

that acts trivially on all of the terms of $V^\bullet$. Then K_1 is closed and normal in G and

$$\Delta_1 = \pi_1^t(X, D_X)/K_1 = \left(\prod_{i=1}^{r} \mathbb{Z}_p(1) \right) \times \tilde{\Delta}_1$$

for a finite abelian group $\tilde{\Delta}_1$ which is of order prime to p and ℓ. Let $N_0 \subset \langle \Phi \rangle$ be the kernel of the action of $\langle \Phi \rangle$ on Δ_1, and view $\langle \Phi \rangle$ as a subgroup of G via a choice of Frobenius Φ in G. Define K_0 to be the maximal subgroup of N_0 that acts trivially on all of the terms of $V^\bullet$. The group K generated by K_0 and K_1 is the semidirect product $K_1.K_0$ and is normal in G. The group $\Delta = G/K$ is the semidirect product of Δ_1 with the quotient $\Delta_0 = \langle \Phi \rangle / K_0$. Let $\overline{\Phi}$ be the image of Φ in Δ_0. The group Δ_0 is isomorphic to the product $\langle \overline{\Phi}^d \rangle \times \tilde{\Delta}_0$, where $\tilde{\Delta}_0$ is cyclic of order d prime to p and $\langle \overline{\Phi}^d \rangle$ is isomorphic to $\mathbb{Z}_p$. Define $w_1 = \overline{\Phi}^d$, and let $\{w_{2,j}\}_{j=1}^r$ be topological generators for the $\mathbb{Z}_p(1)$-factors in Δ_1.

Proof. Case A is clear, since G is abelian in this case. To prove this for $G = \pi_1^t(Y, D_Y)$ as in case B, one lets d' be the smallest integer such that $\ell^{fd'} \equiv 1 \bmod p$. In particular, d' is relatively prime to p. Writing $\langle \Phi^{d'} \rangle$ as a product $d' \prod_q \mathbb{Z}_q$ as q ranges over all primes, one shows that the kernel of the action of $\langle \Phi \rangle$ on $\mathbb{Z}_p(1)$ is equal to $d' \prod_{q \neq p} \mathbb{Z}_q$. It follows that N_0 is the subgroup of $d' \prod_{q \neq p} \mathbb{Z}_q$ that acts trivially on the characteristic subgroup $\tilde{\Delta}_1$ of Δ_1. Since $\tilde{\Delta}_1$ is finite and K_0 is the maximal subgroup of N_0 that acts trivially on all of the terms of $V^\bullet$, this implies that K_0 has finite index in $d' \prod_{q \neq p} \mathbb{Z}_q$. Thus K_0 has finite index d which is prime to p in $\prod_{q \neq p} \mathbb{Z}_q$. One obtains that

$$\langle \Phi \rangle = \mathbb{Z}_p \times \left(\prod_{q \neq p} \mathbb{Z}_q \right) \supset \{0\} \times d \left(\prod_{q \neq p} \mathbb{Z}_q \right) = K_0,$$

which proves that $\Delta_0 = \langle\Phi\rangle/K_0 = \mathbb{Z}_p \times \tilde{\Delta}_0$, where $\tilde{\Delta}_0$ is finite and cyclic of order d prime to p. The remaining statements in the lemma now follow. □

The following result is a consequence of Lemma 3.2 and Proposition 2.12.

Corollary 3.3. *It suffices to prove Theorem 3.1 when G is replaced by the group Δ described in Lemma 3.2.*

Let A be an object of $\hat{\mathcal{C}}$. Since $A[[\mathbb{Z}_p^s]]$ is isomorphic to a power series algebra over A in s commuting variables, it follows that $A[[\Delta]]$ is left and right Noetherian for Δ as in Lemma 3.2(ii). For Δ as in Lemma 3.2(iii), one considers the subgroup $\tilde{\Delta}$ of finite index in Δ that is topologically generated by $w_1 = \overline{\Phi}^d$ and by $w_{2,j}$, $1 \le j \le r$. By embedding $\tilde{\Delta}$ as a closed subgroup of block diagonal matrices with blocks of size 2 inside $\mathrm{GL}_{2r}(\mathbb{Z}_p)$, one sees that $\tilde{\Delta}$ is a compact p-adic analytic group. Hence it follows from Lazard's result [11, Prop. 2.2.4 of Chap. V] that $\mathbb{Z}_p[[\tilde{\Delta}]]$ is left and right Noetherian. Since Lazard's arguments also work if $\mathbb{Z}_p$ is replaced by A, we obtain the following result.

Lemma 3.4. *If A is an arbitrary object of $\hat{\mathcal{C}}$ and Δ is as in Lemma 3.2, then the ring $B = A[[\Delta]]$ is both left Noetherian and right Noetherian.*

For the remainder of this section, let A be an object of $\hat{\mathcal{C}}$ and let $B = A[[\Delta]]$. To better explain the main ideas of the proof without having to use multiple subscripts, we will at first assume that if Δ is as in Lemma 3.2(iii) then $r = 1$. In this case we will write w_2 instead of $w_{2,1}$. We will show in §3.7 how to generalize the proofs to work for $r > 1$.

The proof of Theorem 3.1 depends on the following results.

Proposition 3.5. *Suppose Δ is as in Lemma 3.2(iii) and $r = 1$. Define $w_2 = w_{2,1}$. For positive integers N, N', let $J = B \cdot (w_2^N - 1)^{N'}$. Then J is a closed two-sided ideal of B and the quotient ring $\overline{B} = B/J$ is a pseudocompact A-algebra. Moreover, J is a topologically free rank one left B-module and a topologically free rank one right B-module.*

Proposition 3.6. *Suppose Δ is as in Lemma 3.2(iii) and $r = 1$. Define $w_2 = w_{2,1}$. Let M be a pseudocompact B-module that is finitely generated as an abstract A-module. Then there exist positive integers N, N' such that $(w_2^N - 1)^{N'} \cdot M = \{0\}$.*

Proposition 3.7. *Let J be a two-sided ideal in B of the following form:*

(i) *If Δ is as in Lemma 3.2(ii), let $J = \{0\}$.*
(ii) *If Δ is as in Lemma 3.2(iii) and $r = 1$, let $J = B \cdot (w_2^N - 1)^{N'}$, where $w_2 = w_{2,1}$ and N, N' are positive integers.*

If $\Lambda = B/J$, then Λ is a pseudocompact A-algebra. Suppose M is a pseudocompact Λ-module that is finitely generated as an abstract Λ-module. Let T be a pseudocompact Λ-submodule of M that is finitely generated as an abstract A-module. Then there is a pseudocompact Λ-submodule M' of M such that $M' \cap T = \{0\}$ and M/M' is finitely generated as an abstract A-module.

Proposition 3.8. *Let Ω be a pseudocompact ring that is left Noetherian. Let $P^\bullet$ be a complex in $D^-(\Omega)$ whose terms P^i are free and finitely generated as abstract Ω-modules such that $P^i = 0$ if $i > 0$. Suppose that for $i \le 0$, I_i is a closed two-sided ideal in Ω with the following properties.*

(a) *The cohomology group $\mathrm{H}^i(P^\bullet)$ is annihilated by I_i for $i \le 0$.*
(b) *For $i \le 0$, the two-sided ideal $J_i = I_i \cdot I_{i+1} \cdots I_1 \cdot I_0$ is free and finitely generated as an abstract left Ω-module.*

Then $P^\bullet$ is isomorphic in $D^-(\Omega)$ to a complex $Q^\bullet$ such that $Q^i = 0$ for $i > 0$ and Q^i is annihilated by J_i for $i \le 0$.

Proposition 3.9. *Suppose Δ is one of the groups in Lemma 3.2, where we assume $r = 1$ when Δ is as in Lemma 3.2(iii). Let M be a pseudocompact B-module that is finitely generated as an abstract A-module. Then there exists a pseudocompact B-module F that is free and finitely generated as an abstract A-module and a surjective homomorphism $\varphi : F \to M$ of pseudocompact B-modules.*

Remark 3.10. Let Ω be a pseudocompact ring that is left Noetherian, and let $M^\bullet$ be a bounded above complex of pseudocompact Ω-modules such that $M^i = 0$ for $i > n$ and the cohomology groups $\mathrm{H}^i(M^\bullet)$ are finitely generated as abstract Ω-modules. The construction given by Hartshorne in [9, III Lemma 12.3] shows that there is a quasi-isomorphism $\rho : L^\bullet \to M^\bullet$ in $C^-(\Omega)$, where $L^\bullet$ is a bounded above complex of pseudocompact Ω-modules that are free and finitely generated as abstract Ω-modules and $L^i = 0$ for $i > n$. Moreover, we can require $\rho^{n-1} : L^{n-1} \to M^{n-1}$ to be surjective.

We first show how Theorem 3.1 follows from these results when G is replaced by Δ and, if Δ is as in Lemma 3.2(iii), we assume $r = 1$. As before, we write w_2 instead of $w_{2,1}$.

Suppose $P^\bullet$ has properties (i), (ii) and (iii) of Theorem 2.6. Without loss of generality we will suppose that $n_2 = 0$, so that $P^i = 0$ if $i > 0$.

Step 1: The complex $P^\bullet$ is isomorphic in $D^-(B)$ to a complex $Q^\bullet$ such that $Q^i = 0$ if $i > 0$ or $i < n_1$ and such that if $n_1 \le i \le 0$ then Q^i is annihilated by a closed two-sided ideal J in B of the form described in Proposition 3.7.

Proof of Step 1. If Δ is as in Lemma 3.2(ii), we can define $Q^\bullet$ to be the complex obtained from $P^\bullet$ by replacing P^{n_1} by $P^{n_1}/\mathrm{B}^{n_1}(P^\bullet)$ and P^i by 0 for $i < n_1$.

Suppose now that Δ is as in Lemma 3.2(iii) and $r = 1$. Using Remark 3.10, we can assume that the terms of $P^\bullet$ are free and finitely generated as abstract B-modules and that $P^i = 0$ for $i > 0$. For $i \le 0$, we apply Proposition 3.6 to $M = \mathrm{H}^i(P^\bullet)$ to see that there are integers $N(i), N'(i) \ge 1$ such that the left ideal $I_i = B \cdot (w_2^{N(i)} - 1)^{N'(i)}$ annihilates $\mathrm{H}^i(P^\bullet)$. Proposition 3.5 shows that I_i is a closed two-sided ideal of B that is a topologically free rank one right B-module and a topologically free rank one left B-module. Therefore for $i \le 0$, the ideal $J_i = I_i \cdot I_{i+1} \cdots I_1 \cdot I_0$ is a topologically and abstractly free rank one left B-module. The hypotheses of Proposition 3.8 are now satisfied when we let $\Omega = B$. Therefore $P^\bullet$ is isomorphic in $D^-(B)$ to a complex $Q^\bullet$ such that $Q^i = 0$ for $i > 0$ and Q^i is annihilated by J_i for $i \le 0$. Since $\mathrm{H}^i(Q^\bullet) = \mathrm{H}^i(P^\bullet) = 0$ if $i < n_1$, we may replace Q^{n_1} by $Q^{n_1}/\mathrm{B}^{n_1}(Q^\bullet)$ and Q^i by 0 for $i < n_1$. Let $N = \prod_{i=n_1}^0 N(i)$ and let $N' = \sum_{i=n_1}^0 N'(i)$ and define $J = B \cdot (w_2^N - 1)^{N'}$. Then J is a closed two-sided ideal which lies inside J_{n_1}. Since J_{n_1} annihilates Q^i for all i, step 1 follows.

Step 2: We can assume that the complex $Q^\bullet$ from step 1 has the property that all of the Q^i are finitely generated as abstract A-modules.

Proof of Step 2. Let J be the ideal from step 1. By Remark 3.10, $Q^\bullet$ is isomorphic in $D^-(B/J)$ to a complex $Q'^\bullet$ whose terms are zero in positive degrees and free and finitely generated as abstract B/J-modules in non-positive degrees. Let $Q''^\bullet$ be the complex obtained from $Q'^\bullet$ by replacing Q'^{n_1} by $Q'^{n_1}/\mathrm{B}^{n_1}(Q'^\bullet)$ and Q'^i by 0 for $i < n_1$. By replacing $Q^\bullet$ by $Q''^\bullet$, we can assume that all of the terms Q^i are finitely generated as abstract B/J-modules.

Suppose by induction that n_0 is an integer such that Q^i is finitely generated as an abstract A-module for all integers $i < n_0$. This hypothesis certainly holds when $n_0 = n_1$, since $Q^i = 0$ for $i < n_1$. Since $\mathrm{B}^{n_0}(Q^\bullet) = \mathrm{Image}(Q^{n_0-1} \to Q^{n_0})$ and $\mathrm{H}^{n_0}(Q^\bullet)$ are finitely generated as abstract A-modules, also $\mathrm{Z}^{n_0}(Q^\bullet) = \mathrm{Ker}(Q^{n_0} \to Q^{n_0+1})$ is finitely generated as an abstract A-module. We apply Proposition 3.7 to the modules $M = Q^{n_0}$ and $T = \mathrm{Z}^{n_0}(Q^\bullet)$, where, as arranged above, Q^{n_0} is finitely generated as an abstract B/J-module. This shows that there is a pseudocompact B/J-submodule M' of M such that $M' \cap \mathrm{Z}^{n_0}(Q^\bullet) = \{0\}$ and Q^{n_0}/M' is finitely generated as an abstract A-module. The restriction of the differential $\delta^{n_0} : Q^{n_0} \to Q^{n_0+1}$ to M' is therefore injective. This implies that we have an exact sequence in $C^-(B/J)$

$$0 \to Q_2^\bullet \to Q^\bullet \to Q_1^\bullet \to 0$$

in which $Q_2^\bullet$ consists of the two-term complex $M' \to \delta^{n_0}(M')$ in degrees n_0 and $n_0 + 1$, and the morphism $Q_2^\bullet \to Q^\bullet$ results from the natural inclusions of these terms into Q^{n_0} and Q^{n_0+1}, respectively. Since $Q_2^\bullet$ is acyclic, $Q^\bullet \to Q_1^\bullet$ is a quasi-isomorphism. The term Q_1^i is Q^i if $i < n_0$, and if $i = n_0$ then $Q_1^{n_0} = Q^{n_0}/M'$ which is finitely generated as an abstract A-module. One now replaces $Q^\bullet$ by $Q_1^\bullet$ and continues by ascending induction on n_0. Hence step 2 follows.

Step 3: The complex $Q^\bullet$ from step 2 is isomorphic in $D^-(B)$ to a complex $L^\bullet$ such that $L^i = 0$ for $i > 0$ and L^i is free and finitely generated as an abstract A-module for $i \le 0$.

Proof of Step 3. We construct $L^\bullet$ using Proposition 3.9 together with a modification of the procedure described in [9, III Lemma 12.3].

If $n \le 0$ is an integer, let $Q^{>n}$ be the truncation of $Q^\bullet$ which results by setting to 0 all terms in degrees $\le n$. Suppose by induction that $L^{>n}$ is a complex in $D^-(B)$ with the following properties. The terms of $L^{>n}$ are free and finitely generated as abstract A-modules and these terms are 0 in dimensions $\le n$ and in dimensions > 0. Moreover, there is a morphism $\pi^{>n} : L^{>n} \to Q^{>n}$ in $C^-(B)$ which induces isomorphisms $\mathrm{H}^i(L^{>n}) \to \mathrm{H}^i(Q^\bullet)$ for $i > n+1$ and for which the induced map $\mathrm{Z}^{n+1}(L^{>n}) \to \mathrm{H}^{n+1}(Q^\bullet)$ is surjective. We can certainly construct such an $L^{>n}$ for $n = 0$ since $Q^i = 0$ for $i > 0$.

The pseudocompact B-module $\mathrm{Z}^n(Q^\bullet)$ is finitely generated as an abstract A-module since it is a submodule of Q^n and A is Noetherian. Therefore, by Proposition 3.9, there exists a pseudocompact B-module L_1^n that is free and finitely generated as an abstract A-module together with a surjection $\tau_1 : L_1^n \to \mathrm{Z}^n(Q^\bullet)$. Let $\pi^{n+1} : L^{n+1} \to Q^{n+1}$ be the morphism defined by $\pi^{>n}$. Define M to be the pullback:

$$\begin{array}{ccc} M & \longrightarrow & (\pi^{n+1})^{-1}(\mathrm{B}^{n+1}(Q^\bullet)) \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi^{n+1}} \\ Q^n & \xrightarrow{\ \delta^n\ } & \mathrm{B}^{n+1}(Q^\bullet). \end{array} \tag{3.3}$$

Because $(\pi^{n+1})^{-1}(\mathrm{B}^{n+1}(Q^\bullet))$ is contained in L^{n+1}, it is finitely generated as an abstract A-module. Since Q^n is also finitely generated as an abstract A-module, it follows that the pseudocompact B-module M is finitely generated as an abstract A-module. Note that the top horizontal morphism in (3.3) is surjective because the lower horizontal morphism is surjective.

By Proposition 3.9, there exists a pseudocompact B-module L_2^n that is free and finitely generated as an abstract A-module together with a surjection $\tau_2 : L_2^n \to M$ of pseudocompact B-modules. This and (3.3) lead to a diagram of the following kind:

$$\begin{array}{ccc} L^n = L_1^n \oplus L_2^n & \xrightarrow{\ d^n\ } & L^{n+1} \\ {\scriptstyle \pi^n}\big\downarrow & & \big\downarrow{\scriptstyle \pi^{n+1}} \\ Q^n & \xrightarrow{\ \delta^n\ } & Q^{n+1}. \end{array} \tag{3.4}$$

Here the restriction of $d^n : L_1^n \oplus L_2^n \to L^{n+1}$ to L_1^n is trivial, and the restriction of d^n to L_2^n is the composition of the surjection $\tau_2 : L_2^n \to M$ with the morphism $M \to \pi_{n+1}^{-1}(\mathrm{B}^{n+1}(Q^\bullet))$ in the top row of (3.3) followed by the inclusion of $\pi_{n+1}^{-1}(\mathrm{B}^{n+1}(Q^\bullet))$ into L^{n+1}. The restriction of the left downward morphism $\pi^n : L^n = L_1^n \oplus L_2^n \to Q^n$ to L_1^n is the composition of $\tau_1 : L_1^n \to \mathrm{Z}^n(Q^\bullet)$ with the inclusion of $\mathrm{Z}^n(Q^\bullet)$ into Q^n, and the restriction of this morphism to L_2^n results from the surjection $\tau_2 : L_2^n \to M$ followed by the left downward morphism in (3.3).

By construction, the diagram (3.4) is commutative, and gives a morphism $\pi^{>(n-1)} : L^{>(n-1)} \to Q^{>(n-1)}$ in $C^-(B)$. We assumed that the morphism $\mathrm{Z}^{n+1}(L^\bullet) \to \mathrm{H}^{n+1}(Q^\bullet)$, which is induced by $\pi^{>n}$, is surjective. Since the top horizontal morphism in (3.3) is surjective, the image of $d^n : L^n \to L^{n+1}$ is $(\pi^{n+1})^{-1}(B^{n+1}(Q^\bullet)) \subset L^{n+1}$. It follows that $\pi^{>(n-1)} : L^{>(n-1)} \to Q^{>(n-1)}$ induces an isomorphism

$$\mathrm{H}^{n+1}(L^{>(n-1)}) \to \mathrm{H}^{n+1}(Q^\bullet).$$

Because $L_1^n \subset \mathrm{Z}^n(L^{>(n-1)})$, we also have that $\pi^n : \mathrm{Z}^n(L^{>(n-1)}) \to \mathrm{Z}^n(Q^\bullet)$ is surjective. So since L^n is free and finitely generated as an abstract A-module, we conclude by induction that we can construct a bounded above complex $L^\bullet$ in $D^-(B)$ whose terms are free and finitely generated as abstract A-modules together with a quasi-isomorphism $L^\bullet \to Q^\bullet$ in $C^-(B)$. This completes the proof of step 3.

Since $L^\bullet$ from step 3 is isomorphic to $P^\bullet$ in $D^-(B)$, $L^\bullet$ satisfies hypotheses (ii) and (iii) of Theorem 2.6. By Definition 2.4(a), this implies that $L^\bullet$ has finite pseudocompact A-tor dimension at n_1. Since all the terms of $L^\bullet$ are topologically free by Remark 2.1(v), it follows by Remark 2.5 that the bounded complex $C^\bullet$ that is obtained from $L^\bullet$ by replacing L^{n_1} by $L^{n_1}/\mathrm{B}^{n_1}(L^\bullet)$ and L^i by 0 for $i < n_1$, is quasi-isomorphic to $L^\bullet$ and has topologically free pseudocompact terms over A. By Remark 2.1(v) and step 3, this implies that all terms of $C^\bullet$ are free and finitely generated as abstract A-modules.

Because of Corollary 3.3, this completes the proof of Theorem 3.1, assuming Propositions 3.5 - 3.9 and assuming $r = 1$ if G is as in case B. We will prove these propositions in §3.2 - §3.6 and discuss the case $r > 1$ for G as in case B in §3.7.

3.2. **Proof of Proposition 3.5.** Suppose Δ is as in Lemma 3.2(iii) and $r = 1$. Write w_2 instead of $w_{2,1}$, and let $J = B \cdot (w_2^N - 1)^{N'}$ be as in the statement of Proposition 3.5. The key to proving this proposition is to uniquely express each element in $B = A[[\Delta]]$ by a unique convergent power series as in Lemma 3.11 below.

We first note that the left ideal $J = B \cdot (w_2^N - 1)^{N'}$ is a two-sided ideal in B, since

$$(w_2^N - 1)^{N'}\, \overline{\Phi}^{-1} = \overline{\Phi}\, (w_2^{\ell^f N} - 1)^{N'} = \overline{\Phi} \left(\sum_{i=0}^{\ell^f - 1} w_2^{iN} \right)^{N'} (w_2^N - 1)^{N'}. \tag{3.5}$$

Suppose that in the description of Δ_0 in Lemma 3.2(iii), the finite cyclic p'-group $\tilde{\Delta}_0$ of order d is generated by $\sigma \in \Delta$.

Lemma 3.11. *Write $N = p^s t$ where $s \geq 0$ and t is prime to p. Then $w_2^N - 1 = (w_2^{p^s} - 1) \cdot v$ where v is a unit of B commuting with w_2, so $J = B \cdot (w_2^{p^s} - 1)^{N'}$. Every element f of B can be written in a unique way as a convergent power series*

$$f = \sum z_{u,a,\xi,b,c}\, \sigma^u\, (w_1 - 1)^a\, \xi\, w_2^b\, (w_2^{p^s} - 1)^c, \tag{3.6}$$

in which the sum ranges over all tuples (u, a, ξ, b, c) with $0 \leq u \leq d-1$, $a \geq 0$, $\xi \in \tilde{\Delta}_1$, $0 \leq b \leq p^s - 1$ and $c \geq 0$, and each $z_{u,a,\xi,b,c}$ lies in A. Moreover, any choice of $z_{u,a,\xi,b,c} \in A$ defines an element $f \in B$.

Proof. A cofinal system of closed normal finite index subgroups of Δ is given by the groups $H(m, m')$ that are topologically generated by $w_2^{p^{s+m}}$ and $w_1^{p^{m'}}$, where $m \geq 0$ is arbitrary and m' is chosen so that $\ell^{f d p^{m'}} \equiv 1 \bmod p^{s+m}$ and the order of the automorphism of the finite group $\tilde{\Delta}_1$ induced by the pro-p element w_1 divides $p^{m'}$. Note that these requirements on m' ensure that each $H(m, m')$ is normal in Δ. Define $\Gamma(m, m') = \Delta/H(m, m')$.

In B, we have

$$w_2^N - 1 = w_2^{p^s t} - 1 = (w_2^{p^s} - 1) \cdot v,$$

where $v = 1 + w_2^{p^s} + \cdots + w_2^{p^s \cdot (t-1)}$ is congruent to t mod the two-sided ideal $B \cdot (w_2 - 1)$. Since $t \not\equiv 0$ mod p, v has invertible image in $B/(B \cdot (w_2 - 1) \cap pB)$. Since the two-sided ideal $B \cdot (w_2 - 1) \cap pB$ has nilpotent image in $A'[\Gamma(m, m')]$ for all discrete Artinian quotients A' of A, this implies that v is a unit in B.

Since $B = A[[\Delta]]$ is the projective limit of the quotient rings $A'[\Gamma(m, m')]$, as A' ranges over all discrete Artinian quotients of A and (m, m') ranges over all pairs of integers satisfying the above conditions, it follows that every $f \in B$ can be written in a unique way as a power series as in (3.6) and every such power series converges to an element in B. □

Remark 3.12. By a similar argument, every element f of B can be written in a unique way as a convergent power series

$$f = \sum \omega_{u,a,\xi,b,c}\ (w_2^{p^s} - 1)^c\ w_2^b\ \xi\ (w_1 - 1)^a\ \sigma^u, \tag{3.7}$$

in which the sum ranges over all tuples (u, a, ξ, b, c) with $0 \le u \le d-1$, $a \ge 0$, $\xi \in \tilde{\Delta}_1$, $0 \le b \le p^s - 1$ and $c \ge 0$, and each $\omega_{a,b,\xi,c,u}$ lies in A. Moreover, any choice of $\omega_{a,b,\xi,c,u} \in A$ defines and element in B.

To prove Proposition 3.5, let $J = B \cdot (w_2^N - 1)^{N'}$. By (3.5), J is a two-sided ideal in B. By Remark 2.2(i), J is closed in B, which implies that $\overline{B} = B/J$ is a pseudocompact A-algebra. By using Lemma 3.11 (resp. Remark 3.12), we see that right (resp. left) multiplication with $(w_2^{p^s} - 1)^{N'}$ is an injective homomorphism $B \to B$. This shows that J is an abstractly free rank one left (resp. right) B-module. By Remark 2.2(iii), this proves Proposition 3.5.

3.3. **Proof of Proposition 3.6.** Suppose Δ is as in Lemma 3.2(iii) and $r = 1$. Write w_2 instead of $w_{2,1}$. We will prove Proposition 3.6 by proving Lemmas 3.13 and 3.15 below, which enable us to essentially reduce to the case when A is a field.

Lemma 3.13. *Let L be a field and let M be a pseudocompact $L[[\Delta]]$-module that is finite dimensional as L-vector space. There exist positive integers N, N' which are bounded functions of $\dim_L M$ such that $(w_2^N - 1)^{N'} \cdot M = \{0\}$.*

Proof. The action of w_2 on M defines an automorphism in $\mathrm{Aut}_L(M) \subset \mathrm{End}_L(M)$. This implies that there is a monic polynomial $h(x) \in L[x]$ of degree less than or equal to $(\dim_L M)^2$ such that $h(w_2) \cdot M = 0$. Since w_2 is a unit, we can assume that $h(x)$ is not divisible by x. Since $w_1 h(w_2) w_1^{-1} = h(w_2^{\ell^{fd}})$, it follows that $h(w_2^{\ell^{fd}})$ also annihilates M.

Let I be the ideal of $L[x]$ that is (abstractly) generated by $\{h(x^{\ell^{fdn}}) \mid n \ge 0\}$. Then $f(w_2) \cdot M = 0$ for all $f(x) \in I$. Since $L[x]$ is a principal ideal domain, I is generated by a single polynomial $d(x) \in L[x]$. Moreover, since x does not divide $h(x)$, x also does not divide $d(x)$. Because $d(x^{\ell^{fd}}) \in I$, $d(x)$ divides $d(x^{\ell^{fd}})$. This means that if $\{\rho_1, \ldots, \rho_m\}$ are the roots of $d(x)$, then for each $1 \le i \le m$, $\{\rho_i^{\ell^{fdn}} \mid n \ge 0\}$ is contained in $\{\rho_1, \ldots, \rho_m\}$. Note that $m \le \deg d(x) \le \deg h(x) \le (\dim_L M)^2$. This implies that there exists a positive integer s, which is a bounded function of $\dim_L M$, such that each ρ_i is a root of unity of finite order bounded by ℓ^{fds}. Thus $d(x)$ divides a polynomial of the form $(x^N - 1)^{N'}$ where N, N' are bounded functions of $\dim_L M$. □

Corollary 3.14. *Suppose M is a pseudocompact $A[[\Delta]]$-module that is finitely generated as an abstract A-module. There exist positive integers N, N'' that are bounded functions of the number of abstract generators of M over A such that $(w_2^N - 1)^{N''}$ annihilates $k(\mathfrak{p}) \hat{\otimes}_A M$ for all prime ideals $\mathfrak{p}$ of A, where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$.*

Proof. Note that $\dim_{k(\mathfrak{p})}(k(\mathfrak{p}) \otimes_A M)$ is less than or equal to the number of generators of M as an abstract A-module. Hence we can use Lemma 3.13 with $k(\mathfrak{p})$ for L and $k(\mathfrak{p}) \hat{\otimes}_A M$ for M. □

Lemma 3.15. *Let M be as in Corollary 3.14. Suppose $f \in \mathrm{End}_A(M)$ and that for all prime ideals $\mathfrak{p}$ of A we have*

$$f(M)_{\mathfrak{p}} \subseteq \mathfrak{p} \cdot M_{\mathfrak{p}} \tag{3.8}$$

where the subscript $\mathfrak{p}$ means localization at the prime ideal $\mathfrak{p}$. Then f is nilpotent.

Proof. Let first $\mathfrak{p}$ be a prime ideal of A of codimension 0. Then $\dim A_{\mathfrak{p}} = 0$ so that $A_{\mathfrak{p}}$ is Artinian. Because $f(M)$ is finitely generated as an abstract A-module, this implies that $f(M)_{\mathfrak{p}}$ is an Artinian $A_{\mathfrak{p}}$-module. Since by assumption, $f(M)_{\mathfrak{p}} \subseteq \mathfrak{p} \cdot M_{\mathfrak{p}}$, we obtain for all positive integers n that $f^n(M)_{\mathfrak{p}} \subseteq \mathfrak{p}^n \cdot M_{\mathfrak{p}}$. Thus there is a positive integer $n(\mathfrak{p})$ with $f^{n(\mathfrak{p})}(M)_{\mathfrak{p}} = 0$. Since A is Noetherian, there are only finitely many prime ideals of A of codimension 0. Hence there is a positive integer n_0 such that $f^{n_0}(M)_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of A of codimension 0.

Now let $t \geq 1$, and suppose by induction that there is an integer n_{t-1} such that $f^{n_{t-1}}(M)_{\mathfrak{q}} = 0$ for all prime ideals $\mathfrak{q}$ of A of codimension at most $t-1$. In particular, for each prime ideal $\mathfrak{q}$ of A of codimension at most $t-1$ there exists an element $b(\mathfrak{q}) \in A$ such that $b(\mathfrak{q}) \notin \mathfrak{q}$ and $b(\mathfrak{q}) \cdot f^{n_{t-1}}(M) = 0$. Let I_{t-1} be the ideal of A that is abstractly generated by all elements $b(\mathfrak{q})$ as $\mathfrak{q}$ ranges over all prime ideals of A of codimension at most $t-1$.

Let $\mathfrak{p}$ be a prime ideal of A of codimension t. Using that the non-zero prime ideals of $(A/I_{t-1})_{\mathfrak{p}} \cong A_{\mathfrak{p}}/I_{t-1}A_{\mathfrak{p}}$ correspond to the prime ideals of $A_{\mathfrak{p}}$ containing $I_{t-1}A_{\mathfrak{p}}$ and that the prime ideals of $A_{\mathfrak{p}}$ correspond to the prime ideals of A contained in $\mathfrak{p}$, one shows that $(A/I_{t-1})_{\mathfrak{p}}$ has dimension 0. Since $f^{n_{t-1}}(M)$ is finitely generated as an abstract (A/I_{t-1})-module, one shows, similarly to the first paragraph of this proof, that there is a positive integer $n(\mathfrak{p})$ with $f^{n(\mathfrak{p})}(M)_{\mathfrak{p}} = 0$. Since $f^{n_{t-1}}(M)$ is supported in codimension t, it follows that there is a positive integer n_t such that $f^{n_t}(M)_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of A of codimension at most t.

Since A is Noetherian, the codimensions of all prime ideals of A are bounded above by a fixed non-negative integer. Hence we obtain that there is a positive integer n such that $f^n(M)_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of A, which implies $f^n(M) = 0$. $\square$

To prove Proposition 3.6, let M be a pseudocompact B-module that is finitely generated as an abstract A-module. By Corollary 3.14, there exist positive integers N, N'' that are bounded functions of the number of abstract generators of M such that $(w_2^N - 1)^{N''}$ annihilates $k(\mathfrak{p})\hat{\otimes}_A M$ for all prime ideals $\mathfrak{p}$ of A. Letting f be the endomorphism of M defined by the action of $(w_2^N - 1)^{N''}$ on M, this is equivalent to condition (3.8) for all prime ideals $\mathfrak{p}$ of M. Hence Lemma 3.15 implies that f is nilpotent, i.e. there exists an integer N' such that $(w_2^N - 1)^{N'}$ annihilates M. This proves Proposition 3.6.

3.4. **Proof of Proposition 3.7.** Let J, Λ, M and T be as in the statement of Proposition 3.7. The main idea for proving this proposition is to use the Artin-Rees Lemma to construct the almost complement M' for T. This works directly if Δ is abelian. If Δ is as in Lemma 3.2(iii) and $r = 1$, we first use the Artin-Rees Lemma for the case when the exponent N' in the definition of the ideal J is equal to 1 and then use an inductive argument in the general case. Note that J is a closed two-sided ideal of B by Proposition 3.5, so $\Lambda = B/J$ is a pseudocompact A-algebra.

Suppose first that Δ is as in Lemma 3.2(ii), i.e. $J = 0$ and $\Lambda = B = A[[\Delta]]$ is commutative. For $1 \leq j \leq s$, the action of $w_{1,j}$ on T defines an automorphism in $\mathrm{Aut}_A(T) \subset \mathrm{End}_A(T)$. Since T is finitely generated as an abstract A-module, it follows that the same is true for $\mathrm{End}_A(T)$. Hence there exists a monic polynomial $F_j(x) \in A[x]$ such that $F_j(w_{1,j})$ annihilates T. Let I be the ideal in the commutative Noetherian ring Λ that is abstractly generated by $F_j(w_{1,j})$ for $1 \leq j \leq s$. By the Artin-Rees Lemma, there is an integer $q >> 0$ such that $T \cap (I^{q+1} \cdot M) = I \cdot (T \cap (I^q \cdot M))$. However, I annihilates T by construction, so we conclude that $T \cap (I^{q+1} \cdot M) = \{0\}$. Since Λ is commutative and I^{q+1} is abstractly finitely generated, it follows that $I^{q+1} \cdot M$ is a pseudocompact Λ-submodule of M by Remark 2.2(i). The quotient $M/(I^{q+1} \cdot M)$ is finitely generated as an abstract module for the ring Λ/I^{q+1}, and this ring is finitely generated as an abstract A-module, since I contains a monic polynomial in $w_{1,j}$ for each $1 \leq j \leq s$ and $\Delta/\langle w_{1,1}, \ldots, w_{1,s}\rangle$ is finite. Hence $M/(I^{q+1} \cdot M)$ is finitely generated as an abstract A-module and Proposition 3.7 is proved if Δ is as in Lemma 3.2(ii).

Suppose now that Δ is as in Lemma 3.2(iii) and $r = 1$. Write w_2 instead of $w_{2,1}$. Then $J = B \cdot (w_2^N - 1)^{N'}$ for positive integers N, N' and $\Lambda = B/J$. Suppose first that $N' = 1$. Then $J = B \cdot (w_2^{p^s} - 1)$ by Lemma 3.11, and hence $\Lambda = B/J = A[[\overline{\Delta}]]$, where $\overline{\Delta}$ is the quotient of Δ by the closed normal subgroup that is topologically generated by $w_2^{p^s}$. The conjugation action of w_1 on the finite normal abelian subgroup $\left(\langle w_2\rangle \times \tilde{\Delta}_1\right)/\langle w_2^{p^s}\rangle$ of $\overline{\Delta}$ gives an automorphism of finite order. Thus w_1^z is in the center of $\overline{\Delta}$, and of Λ, if $z \geq 1$ is sufficiently divisible. Similarly to the case when Δ is as in Lemma 3.2(ii), one finds an ideal I in $A[[\langle w_1^z\rangle]]$ that is abstractly generated by a monic polynomial in w_1^z such that $T \cap (I^{q+1} \cdot M) = \{0\}$ for an integer $q >> 0$. Since M is a pseudocompact Λ-module and I^{q+1} is generated by a single element that lies in the center of Λ, it follows that $I^{q+1} \cdot M$ is a pseudocompact Λ-submodule of M by Remark 2.2(i). The quotient $M/(I^{q+1} \cdot M)$ is finitely generated as an abstract module for the ring Λ/I^{q+1}. This ring is finitely generated as an abstract A-module, since I contains a monic polynomial in w_1, since $w_2^{p^s} = 1$ in Λ, and since $\tilde{\Delta}_0$ and $\tilde{\Delta}_1$ are finite. So $M/(I^{q+1} \cdot M)$ is finitely generated as an abstract A-module and Proposition 3.7 is proved if $N' = 1$.

We now suppose that $N' \geq 1$ are arbitrary. In this case, we break the proof into several steps given by Lemma 3.16, Corollary 3.17 and Lemma 3.18 below. For simplicity, let $\epsilon = (w_2^{p^s} - 1)$ so that $J = B \cdot \epsilon^{N'}$. For $m \geq 1$, define

$$M(\epsilon^m) = \{\alpha \in M \mid \epsilon^m \cdot \alpha = 0\}. \tag{3.9}$$

Since by Proposition 3.5, $\Lambda \cdot \epsilon^m$ is a two-sided ideal of Λ, it follows that $M(\epsilon^m)$ is a pseudocompact Λ-submodule of M.

Lemma 3.16. *The module M is a left Noetherian Λ-module. If $M(\epsilon)$ is not finitely generated as an abstract A-module, then there exists a non-zero pseudocompact Λ-submodule Y of $M(\epsilon)$ such that $T \cap Y = 0$.*

Proof. Since M is finitely generated as an abstract Λ-module, where $\Lambda = B/J$, and B is left Noetherian, M must be a left Noetherian Λ-module. By (3.9), $M(\epsilon)$ is annihilated by ϵ. Thus $M(\epsilon)$ is a pseudocompact Λ_1-module, where

$$\Lambda_1 = \Lambda/\Lambda\epsilon = B/B \cdot (w_2^{p^s} - 1),$$

and $T_1 = T \cap M(\epsilon)$ is a pseudocompact Λ_1-submodule of $M(\epsilon)$. Because T is finitely generated as an abstract A-module, T_1 is also finitely generated as an abstract A-module. By what we proved in the case when $N' = 1$, we can therefore conclude that there is a pseudocompact Λ-submodule Y of $M(\epsilon)$ such that $T \cap Y = T_1 \cap Y = \{0\}$ and $M(\epsilon)/Y$ is finitely generated as an abstract A-module. If $M(\epsilon)$ is not finitely generated as an abstract A-module, this forces Y to be non-zero. □

Corollary 3.17. *There is a pseudocompact Λ-submodule M' of M such that $T \cap M' = 0$ and $(M/M')(\epsilon)$ is finitely generated as an abstract A-module.*

Proof. Suppose we have constructed for some integer $n \geq 0$ a strictly increasing sequence of pseudocompact Λ-submodules $M_0 \subset M_1 \subset \cdots \subset M_n$ of M such that $M_0 = \{0\}$ and $T \cap M_n = \{0\}$. If $(M/M_n)(\epsilon)$ is finitely generated as an abstract A-module, then we let $M' = M_n$ and we are done. Otherwise, observe that T injects into M/M_n. We can apply Lemma 3.16 to this inclusion and to the module M/M_n to conclude that there a non-zero pseudocompact Λ-submodule Y of $(M/M_n)(\epsilon)$ such that $T \cap Y = 0$. The inverse image of Y in M is a pseudocompact Λ-submodule M_{n+1} which properly contains M_n and for which $T \cap M_{n+1} = \{0\}$. Since M is left Noetherian by Lemma 3.16, the process stops at some n, meaning that $(M/M_n)(\epsilon)$ is finitely generated as an abstract A-module, and we can let $M' = M_n$. □

Lemma 3.18. *If $M(\epsilon)$ is finitely generated as an abstract A-module, then M is finitely generated as an abstract A-module.*

Proof. We show this by proving by increasing induction on m that $M(\epsilon^m)$ is finitely generated as an abstract A-module for all $m \geq 1$. When $m = 1$, this statement holds by assumption. Suppose now that it is true for some $m \geq 1$. We have an exact sequence of A-modules

$$0 \to M(\epsilon) \to M(\epsilon^{m+1}) \to M(\epsilon^m)$$

in which the A-linear map $M(\epsilon^{m+1}) \to M(\epsilon^m)$ is multiplication by ϵ. Since $M(\epsilon)$ and $M(\epsilon^m)$ are finitely generated as abstract A-modules by induction, this proves that $M(\epsilon^{m+1})$ is finitely generated as an abstract A-module. Since $\epsilon^{N'} = 0$ in Λ, we conclude that $M(\epsilon^{N'}) = M$ is finitely generated as an abstract A-module. □

3.5. **Proof of Proposition 3.8.** As in the statement of Proposition 3.8, let Ω be a pseudocompact left Noetherian ring and let $P^\bullet$ be a complex in $D^-(\Omega)$ whose terms P^i are free and finitely generated as abstract Ω-modules such that $P^i = 0$ for $i > 0$. For $i \le 0$, assume that I_i is a closed two-sided ideal in Ω that annihilates $\mathrm{H}^i(P^\bullet)$ such that $J_i = I_i \cdot I_{i+1} \cdots I_1 \cdot I_0$ is free and finitely generated as an abstract left Ω-module. We need to prove that $P^\bullet$ is isomorphic in $D^-(\Omega)$ to a complex $Q^\bullet$ such that $Q^i = 0$ for $i > 0$ and Q^i is annihilated by J_i for $i \le 0$. We will prove this by constructing $Q^\bullet$ inductively from right to left.

Let $j \le 0$ be an integer. Suppose by induction that $Q^{>j}$ is a complex which is isomorphic to $P^\bullet$ in $D^-(\Omega)$ with the following properties. The terms Q^i are zero for $i > 0$ and free and finitely generated as abstract Ω-modules for $i \le j$. Also, for $j+1 \le i \le 0$, Q^i is annihilated by J_i and is finitely generated as an abstract Ω-module. We can certainly construct such a complex $Q^{>j}$ when $j = 0$ since then we can simply let $Q^{>0} = P^\bullet$.

Claim 1: The complex $Q^{>j}$ is isomorphic in $D^-(\Omega)$ to a complex $Q_1^\bullet$ such that $Q_1^i = Q^i$ for $i > j$, Q_1^j is annihilated by J_j and Q_1^i is finitely generated as an abstract Ω-module for $i \le j$.

Proof of Claim 1. The differential $\delta^j : Q^j \to Q^{j+1}$ of $Q^{>j}$ induces an exact sequence of pseudocompact Ω-modules

$$\begin{array}{ccccccc} 0 \longrightarrow & \dfrac{\mathrm{Z}^j(Q^{>j})}{\mathrm{B}^j(Q^{>j})} & \longrightarrow & \dfrac{Q^j}{\mathrm{B}^j(Q^{>j})} & \xrightarrow{\overline{\delta^j}} & Q^{j+1}. \\ & \| & & & & \\ & \mathrm{H}^j(Q^{>j}) & & & & \end{array} \tag{3.10}$$

Since $\mathrm{H}^j(Q^{>j}) = \mathrm{H}^j(P^\bullet)$ has been assumed to be annihilated by I_j and Q^{j+1} is annihilated by $J_{j+1} = I_{j+1} I_{j+2} \cdots I_0$ by induction, (3.10) shows that $Q^j/\mathrm{B}^j(Q^{>j})$ is annihilated by $J_j = I_j(I_{j+1} \cdots I_0)$. Hence $J_j\, Q^j$ lies in $\mathrm{B}^j(Q^{>j})$ and we obtain a short exact sequence of pseudocompact Ω-modules

$$0 \to \mathrm{Z}^{j-1}(Q^{>j}) \to (\delta^{j-1})^{-1}(J_j\, Q^j) \xrightarrow{\delta^{j-1}} J_j\, Q^j \to 0. \tag{3.11}$$

Since, by assumption, J_j is a two-sided ideal which is free and finitely generated as an abstract left Ω-module and since, by induction, Q^j is free and finitely generated as an abstract Ω-module, $J_j\, Q^j$ is also free and finitely generated as an abstract Ω-module. By Remark 2.2(i), $J_j\, Q^j$ is a pseudocompact Ω-submodule of Q^j. By Remark 2.2(iii), $J_j\, Q^j$ is a topologically free pseudocompact Ω-module. Thus there is a homomorphism $s : J_j\, Q^j \to (\delta^{j-1})^{-1}(J_j\, Q^j)$ of pseudocompact Ω-modules such that $\delta^{j-1} \circ s$ is the identity on $J_j\, Q^j$. In particular, $s(J_j\, Q^j)$ is a pseudocompact Ω-submodule of $(\delta^{j-1})^{-1}(J_j\, Q^j)$, and hence of Q^{j-1}, such that

$$s(J_j\, Q^j) \cap \mathrm{Z}^{j-1}(Q^{>j}) = \{0\}. \tag{3.12}$$

The restriction of the differential $\delta^{j-1} : Q^{j-1} \to Q^j$ to $s(J_j\, Q^j)$ is therefore injective. This implies that we have an exact sequence in $C^-(\Omega)$

$$0 \to Q_2^\bullet \to Q^{>j} \to Q_1^\bullet \to 0$$

in which $Q_2^\bullet$ is the two-term complex $s(J_j\, Q^j) \xrightarrow{\delta^{j-1}} J_j\, Q^j$ concentrated in degrees $j-1$ and j, and the morphism $Q^{>j} \to Q_1^\bullet$ results from the natural inclusions of these terms into Q^{j-1} and Q^j, respectively. Since $Q_2^\bullet$ is acyclic, $Q^{>j} \to Q_1^\bullet$ is a quasi-isomorphism. The terms Q_1^i are equal to Q^i for $i > j$, and if $i = j$ then $Q_1^j = Q^j/J_j\, Q^j$. Moreover, since all terms of $Q^{>j}$ are finitely generated as abstract Ω-modules, the same is true for $Q_1^\bullet$. This proves claim 1.

Claim 2: Let $Q_1^{\le(j-1)}$ be the truncation of $Q_1^\bullet$ which results by setting to 0 all terms in degrees $> j-1$. There is a quasi-isomorphism $\rho : L^{\le(j-1)} \to Q_1^{\le(j-1)}$ in $C^-(\Omega)$, where $L^{\le(j-1)}$ is a bounded above complex of pseudocompact Ω-modules that are free and finitely generated as abstract Ω-modules and $L^i = 0$ for $i > j$. Moreover, $\rho^{j-1} : L^{j-1} \to Q_1^{j-1}$ is surjective.

Proof of Claim 2. This immediately follows from Remark 3.10.

Claim 3: The complex $Q_1^\bullet$ from claim 1 is isomorphic in $D^-(\Omega)$ to a complex $T^\bullet$ such that the terms T^i are zero for $i > 0$ and free and finitely generated as abstract Ω-modules for $i \le j-1$. Also, for $j \le i \le 0$, T^i is annihilated by J_i and is finitely generated as an abstract Ω-module.

Proof of Claim 3. We use the complex $L^{\le(j-1)}$ and the quasi-isomorphism $\rho : L^{\le(j-1)} \to Q_1^{\le(j-1)}$ from claim 2 to prove this. Define $T^\bullet$ to be the complex with terms

$$T^i = Q_1^i \quad \text{for } i \ge j \quad \text{and} \quad T^i = L^i \quad \text{for } i \le j-1. \tag{3.13}$$

Let the differentials d_T^i be given by

$$d_T^i = d_{Q_1}^i \quad \text{for } i \ge j, \quad d_T^{j-1} = d_{Q_1}^{j-1} \circ \rho^{j-1} \quad \text{and} \quad d_T^i = d_L^i \quad \text{for } i \le j-2. \tag{3.14}$$

Define $\tau : T^\bullet \to Q_1^\bullet$ to be the map such that

$$\tau^i = \text{ identity on } Q_1^i \quad \text{for } i \ge j \quad \text{and} \quad \tau^i = \rho^i \quad \text{for } i \le j-1. \tag{3.15}$$

We claim that τ is a quasi-isomorphism in $C^-(\Omega)$.

It follows from the definition of $T^\bullet$ and τ in (3.13), (3.14) and (3.15) that τ is a homomorphism in $C^-(\Omega)$. Since $\tau^{j-1} = \rho^{j-1}$ is surjective by claim 2, it follows from (3.14) that

$$\mathrm{B}^j(T^\bullet) = d_T^{j-1}(T^{j-1}) = d_{Q_1}^{j-1}(Q_1^{j-1}) = \mathrm{B}^j(Q_1^\bullet). \tag{3.16}$$

Thus the definition of $\tau : T^\bullet \to Q_1^\bullet$ in (3.15), together with claim 2, show that τ induces an isomorphism $\mathrm{H}^i(T^\bullet) \to \mathrm{H}^i(Q_1^\bullet)$ for $i \le j-2$ and for $i \ge j$. So the only issue is the case $i = j-1$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathrm{H}^{j-1}(T^\bullet) & & \mathrm{H}^{j-1}(L^{\le(j-1)}) & & & & \\
 & & \| & & \| & & & & \\
0 & \longrightarrow & \dfrac{\mathrm{Z}^{j-1}(T^\bullet)}{\mathrm{B}^{j-1}(T^\bullet)} & \longrightarrow & \dfrac{T^{j-1}}{\mathrm{B}^{j-1}(T^\bullet)} & \xrightarrow{\overline{d_T^{j-1}}} & B^j(T^\bullet) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{H}^{j-1}(\tau)} & & \Big\downarrow{\scriptstyle \overline{\tau^{j-1}}} & & \Big\downarrow{\scriptstyle \tau^j} & & \\
0 & \longrightarrow & \dfrac{\mathrm{Z}^{j-1}(Q_1^\bullet)}{\mathrm{B}^{j-1}(Q_1^\bullet)} & \longrightarrow & \dfrac{Q_1^{j-1}}{\mathrm{B}^{j-1}(Q_1^\bullet)} & \xrightarrow{\overline{d_{Q_1}^{j-1}}} & B^j(Q_1^\bullet) & \longrightarrow & 0 \\
 & & \| & & \| & & & & \\
 & & \mathrm{H}^{j-1}(Q_1^\bullet) & & \mathrm{H}^{j-1}(Q_1^{\le(j-1)}) & & & &
\end{array} \tag{3.17}$$

The rightmost vertical homomorphism in (3.17), which is induced by τ^j, is an isomorphism by (3.16). The middle vertical homomorphism in (3.17), which is induced by $\tau^{j-1} = \rho^{j-1}$, is equal to the isomorphism $\mathrm{H}^{j-1}(\rho) : \mathrm{H}^{j-1}(L^{\le(j-1)}) \to \mathrm{H}^{j-1}(Q_1^{\le(j-1)})$. So the left vertical homomorphism $\mathrm{H}^{j-1}(\tau)$ in (3.17) must be an isomorphism by the five lemma. This proves claim 3.

It follows from claims 1 and 3 that we can let $Q^{>(j-1)} = T^\bullet$. Thus we proceed by descending induction to construct a bounded above complex $Q^\bullet$ which is isomorphic to $P^\bullet$ in $D^-(\Omega)$ such that $Q^i = 0$ for $i > 0$ and Q^i is annihilated by J_i for $i \le 0$. This proves Proposition 3.8.

3.6. **Proof of Proposition 3.9.** Let M be a pseudocompact B-module that is finitely generated as an abstract A-module, as in the statement of Proposition 3.9. The key to proving this proposition is to use the Weierstrass preparation theorem in a suitable power series algebra over A to construct a pseudocompact B-module F that is free and finitely generated as an abstract A-module together with a surjective homomorphism $F \to M$ of pseudocompact B-modules.

Suppose first that Δ is as in Lemma 3.2(ii), i.e. $\Delta = \mathbb{Z}_p^s \times Q \times Q'$. For $1 \le j \le s$, the action of $w_{1,j}$ on M defines an automorphism in $\mathrm{Aut}_A(M) \subset \mathrm{End}_A(M)$. Since M is finitely generated as an abstract A-module, the same is true for $\mathrm{End}_A(M)$. Hence there exists a monic polynomial $g_j(x) \in A[x]$ such that $g_j(w_{1,j})$ annihilates M for all j. Let I be the ideal in B that is abstractly generated by $g_j(w_{1,j})$ for $1 \le j \le s$. Then I is a closed ideal of B by Remark 2.2(i). For $1 \le j \le s$, let $x_{1,j} = w_{1,j} - 1$, so that $A[[\langle w_{1,1}, \ldots, w_{1,s}\rangle]] \cong A[[x_{1,1}, \ldots, x_{1,s}]]$. We can rewrite the polynomials

$g_j(w_{1,j})$ as monic polynomials $f_j(x_{1,j})$ in $x_{1,j}$ with coefficients in A. By the Weierstrass preparation theorem, one has $f_j(x_{1,j}) = h_j(x_{1,j})\cdot u_j(x_{1,j})$, where $h_j(x)$ is a monic polynomial in $A[x]$ whose non-leading coefficients lie in the maximal ideal of A and $u_j(x)$ is a unit power series in $A[[x]]$. Since $x_{1,j}$ lies in every maximal ideal of B and $u_j(x_{1,j})$ has invertible image in $B/B\cdot x_{1,j}$, it follows that $u_j(x_{1,j})$ is a unit in B. Hence $h_j(x_{1,j})$ annihilates M for all j. Let $B_{f_1,\ldots,f_s} = A[[x_{1,1},\ldots,x_{1,s}]]/I_{h_1,\ldots,h_s}$, where $I_{h_1,\ldots,h_s}$ is the ideal in $A[[x_{1,1},\ldots,x_{1,s}]]$ generated by $h_{1,j}(x_{1,j})$ for $1\le j\le s$. Then $B_{f_1,\ldots,f_s}$ is free and finitely generated as an abstract A-module. The ring $D = B/I$ is isomorphic to the group ring $B_{f_1,\ldots,f_s}[Q\times Q']$, which implies that D is free and finitely generated as an abstract A-module. Since, as noted above, I is a closed ideal in B and $D = B/I$, it follows that D is a pseudocompact A-algebra that is a pseudocompact B-module. Because M is a pseudocompact D-module that is finitely generated as an abstract A-module, there is a surjective homomorphism $\bigoplus_{i=1}^{z} D \to M$ of pseudocompact D-modules for some finite number z. Since this homomorphism is also a homomorphism of pseudocompact B-modules, this proves Proposition 3.9 if Δ is as in Lemma 3.2(ii).

Suppose now that Δ is as in Lemma 3.2(iii) and $r = 1$. Write w_2 instead of $w_{2,1}$. Let $\tilde{\Delta}$ be the subgroup of Δ that is topologically generated by $w_1 = \overline{\Phi}^d$ and by w_2. Then $\tilde{\Delta}$ has finite index $d\,|\tilde{\Delta}_1|$ in Δ. Let $\tilde{B} = A[[\tilde{\Delta}]]$. Suppose we prove that there is a pseudocompact $\tilde{B}$-module $\tilde{F}$ that is free and finitely generated as an abstract A-module and a surjective homomorphism $\tilde{\varphi}:\tilde{F}\to M$ of pseudocompact $\tilde{B}$-modules. Then the induced module $F = \mathrm{Ind}_{\tilde{\Delta}}^{\Delta}(\tilde{F})$ is a pseudocompact B-module that is free and finitely generated as an abstract A-module and $\tilde{\varphi}$ induces a surjective homomorphism $\varphi: F\to M$ of pseudocompact B-modules. Hence we are reduced to proving Proposition 3.9 for $\tilde{\Delta}$.

By Proposition 3.6 and Lemma 3.11, there exist integers $s\ge 0$ and $N'\ge 1$ such that $(w_2^{p^s} - 1)^{N'}\cdot M = \{0\}$. By (3.5), the left ideal $\tilde{J} = \tilde{B}\cdot(w_2^{p^s}-1)^{N'}$ is a two-sided ideal in $\tilde{B}$. Moreover, it is closed in $\tilde{B}$ by Remark 2.2(i). Let $x_1 = w_1 - 1$, so that $A[[x_1]]\cong A[[\langle w_1\rangle]]$, and define

$$A_{\tilde{J}} = A[[\langle w_2\rangle]]/\left((w_2^{p^s}-1)^{N'}\right).$$

Since $(w_2^{p^s}-1)^{N'}$ is a monic polynomial in (w_2-1) whose non-leading coefficients lie in the maximal ideal of A, $A_{\tilde{J}}$ is free and finitely generated as an abstract A-module. Every element in $\tilde{D} = \tilde{B}/\tilde{J}$ can be written in a unique way as a convergent power series

$$\sum_{i=0}^{\infty} a_i\, x_1^i, \quad \text{where each } a_i \text{ lies in } A_{\tilde{J}}. \tag{3.18}$$

Moreover, any choice of $a_i\in A_{\tilde{J}}$, for all $i\ge 0$, defines an element in $\tilde{D}$.

Using the Weierstrass preparation theorem in $A[[x_1]]\cong A[[\langle w_1\rangle]]$ and arguing similarly to the case when Δ is as in Lemma 3.2(ii), it follows that there exists a monic polynomial

$$f_1(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in A[x]$$

whose non-leading coeffcients are in the maximal ideal m_A of A such that $f_1(x_1)$ annihilates M.

Let $\tilde{D}\cdot f_1(x_1)$ be the left ideal in $\tilde{D}$ that is generated by $f_1(x_1)$. Consider the natural surjective A-module homomorphism

$$\beta:\quad \bigoplus_{i=0}^{n-1} A_{\tilde{J}}\, x_1^i \longrightarrow \tilde{D}/(\tilde{D}\cdot f_1(x_1))$$

which sends $\sum_{i=0}^{n-1} a_i\,x_1^i$ to the corresponding residue class of $\sum_{i=0}^{n-1} a_i\,x_1^i$ modulo $\tilde{D}\cdot f_1(x_1)$. We claim that β is injective. Suppose there exists an element $t = \sum_{i=0}^{\infty} a_i\,x_1^i$ in $\tilde{D}$ such that

$$t\cdot f_1(x_1) = (a_0 + a_1x_1 + \cdots + a_ix_1^i + \cdots)\cdot(x_1^n + b_{n-1}x_1^{n-1} + \cdots + b_0) \tag{3.19}$$

lies in $\bigoplus_{i=0}^{n-1} A_{\tilde{J}}\,x_1^i$. Since the a_i lie in $A_{\tilde{J}}$ and the b_j lie in $m_A\subset A$, the b_j commute with the a_i. Since all the b_j lie in m_A, we see that all the a_i lie in $m_A\cdot A_{\tilde{J}}$. Iterating this process, it follows, using induction, that all the a_i lie in $(m_A)^c\cdot A_{\tilde{J}}$ for all $c\ge 1$. This means that all the a_i have to be zero.

Thus β is injective, which implies that $\tilde{D}/(\tilde{D}\cdot f_1(x_1)) \cong \bigoplus_{i=0}^{n-1} A_{\tilde{J}}\, x_1^i$ as abstract A-modules. Since we have already noted that $A_{\tilde{J}}$ is free and finitely generated as an abstract A-module, it follows that $\tilde{D}/(\tilde{D}\cdot f_1(x_1))$ is free and finitely generated as an abstract A-module. By Remark 2.2(i), it follows that $\tilde{D}/(\tilde{D}\cdot f_1(x_1))$ is a pseudocompact $\tilde{D}$-module, and hence, since $\tilde{D}=\tilde{B}/\tilde{J}$, also a pseudocompact $\tilde{B}$-module. Because M is a pseudocompact $\tilde{D}$-module that is finitely generated as an abstract A-module, there is a surjective homomorphism $\bigoplus_{i=1}^{z}\tilde{D}\to M$ of pseudocompact $\tilde{D}$-modules for some finite number z. Since this homomorphism is also a homomorphism of pseudocompact $\tilde{B}$-modules, this proves Proposition 3.9 if Δ is as in Lemma 3.2(iii) and $r=1$.

3.7. **The case $r>1$ for Δ as in Lemma 3.2(iii).** In this section, we complete the proof of Theorem 3.1 by considering the case when Δ is as in Lemma 3.2(iii) and $r>1$. As before, let $B=A[[\Delta]]$. We make the following adjustments to Propositions 3.5 - 3.9.

In Proposition 3.5, we consider ideals of the form $J_j = B\cdot (w_{2,j}^{N_j}-1)^{N_j'}$ for positive integers N_j, N_j' for all $1\le j\le r$ and define $J_{(j)} = J_1+\cdots+J_j$. As in Lemma 3.11, it follows that $J_j = B\cdot(w_{2,j}^{p^{s_j}}-1)^{N_j'}$, where p^{s_j} is the maximal power of p dividing N_j. Using (3.5) and Remark 2.2(i), we see that $J_{(j)}$ is a closed two-sided ideal in B. Hence the quotient ring $\overline{B}_j = B/J_{(j)}$ is a pseudocompact A-algebra. Letting $\overline{J}_j = \overline{B}_{j-1}\cdot(w_{2,j}^{N_j}-1)^{N_j'}$ in $\overline{B}_{j-1}$, where we set $\overline{B}_0=B$, we similarly see that $\overline{J}_j = \overline{B}_{j-1}\cdot(w_{2,j}^{p^{s_j}}-1)^{N_j'}$ is a closed two-sided ideal in $\overline{B}_{j-1}$. The last statement to be shown in generalizing Proposition 3.5 is that $\overline{J}_j$ is a topologically free rank one left $\overline{B}_{j-1}$-module and a topologically free rank one right $\overline{B}_{j-1}$-module.

To show this last statement, we use a suitable cofinal system of closed normal finite index subgroups of Δ to prove the following. Every element of $\overline{B}_{j-1}$ can be written in a unique way as a convergent power series

$$\sum z_{u,a,\xi,b_j,c_j,\ldots,c_r}\,\sigma^u(w_1-1)^a\,\xi\left(\prod_{i=j+1}^{r}(w_{2,i}-1)^{c_i}\right)w_{2,j}^{b_j}(w_{2,j}^{p^{s_j}}-1)^{c_j} \tag{3.20}$$

$$\left(\text{resp.}\quad \sum \omega_{u,a,\xi,b_j,c_j,\ldots,c_r}\,(w_{2,j}^{p^{s_j}}-1)^{c_j}w_{2,j}^{b_j}\left(\prod_{i=j+1}^{r}(w_{2,i}-1)^{c_i}\right)\xi\,(w_1-1)^a\sigma^u\right) \tag{3.21}$$

in which the sum ranges over all tuples $(u,a,\xi,b_j,c_j,\ldots,c_r)$ with $0\le u\le d-1$, $a\ge 0$, $\xi\in\tilde{\Delta}_1$, $0\le b_j\le p^{s_j}-1$ and $c_j,\ldots,c_r\ge 0$, and each $z_{u,a,\xi,b_j,c_j,\ldots,c_r}$ (resp. $\omega_{u,a,\xi,b_j,c_j,\ldots,c_r}$) lies in

$$A_{(j-1)} = A[[\langle w_{2,1},\ldots,w_{2,j-1}\rangle]]/\left((w_{2,1}^{p^{s_1}}-1)^{N_1'},\ldots,(w_{2,j-1}^{p^{s_{j-1}}}-1)^{N_{j-1}'}\right).$$

Moreover, any choice of $z_{u,a,\xi,b_j,c_j,\ldots,c_r}$ (resp. $\omega_{u,a,\xi,b_j,c_j,\ldots,c_r}$) in $A_{(j-1)}$ defines an element in $\overline{B}_{j-1}$.

In Proposition 3.6, let M be a pseudocompact B-module that is finitely generated as an abstract A-module. Using the same arguments as in the case when $r=1$, it follows that for each $1\le j\le r$, there exist positive integers N_j,N_j' such that $(w_{2,j}^{N_j}-1)^{N_j'}\cdot M=\{0\}$.

In Proposition 3.7, we replace in part (ii) the ideal J by an ideal of the form $J=J_1+\cdots+J_r$, where for $1\le j\le r$, $J_j = B\cdot(w_{2,j}^{N_j}-1)^{N_j'}$ for certain integers $N_j,N_j'\ge 1$. Then, as before, $J_j = B\cdot(w_{2,j}^{p^{s_j}}-1)^{N_j'}$, where p^{s_j} is the maximal power of p dividing N_j, and J is a closed two-sided ideal in B. Suppose M is a pseudocompact module for $\Lambda=B/J$ that is finitely generated as an abstract Λ-module and T is a pseudocompact Λ-submodule of M that is finitely generated as an abstract A-module. We need to prove the existence of a pseudocompact Λ-submodule M' of M such that $M'\cap T=\{0\}$ and M/M' is finitely generated as an abstract A-module.

To prove this statement, we proceed as for $r=1$ and first consider the case when $N_j'=1$ for all $1\le j\le r$. In this case, $\Lambda=B/J=A[[\overline{\Delta}]]$, where $\overline{\Delta}$ is the quotient of Δ by the closed normal subgroup that is topologically generated by $w_{2,j}^{p^{s_j}}$ for $1\le j\le r$. Using similar arguments as in the case when $r = 1$, we find a pseudocompact Λ-submodule M' of M having the desired properties if $N'_j = 1$ for all $1 \le j \le r$. For arbitrary N'_j, we replace $M(\epsilon^m)$ in (3.9) by

$$M(\epsilon_1^{m_1}, \ldots, \epsilon_r^{m_r}) = \{\alpha \in M \mid \epsilon_j^{m_j} \cdot \alpha = 0 \quad \text{for } 1 \le j \le r\} \tag{3.22}$$

for $m_1, \ldots, m_r \ge 1$, and prove analogous statements to the ones in Lemma 3.16, Corollary 3.17 and Lemma 3.18 to find M'.

Proposition 3.8 stays the same as before. To prove Proposition 3.9 for $r > 1$, we let $\tilde{\Delta}$ be the subgroup of Δ that is topologically generated by w_1 and by $w_{2,j}$ for $1 \le j \le r$. Then $\tilde{\Delta}$ has finite index $d\,|\tilde{\Delta}_1|$ in Δ. One argues as in the case when $r = 1$, that it is enough to prove Proposition 3.9 for $\tilde{B} = A[[\tilde{\Delta}]]$. By Proposition 3.6 and Lemma 3.11, there exist integers $s_j \ge 0$ and $N'_j \ge 1$ such that $(w_{2,j}^{p^{s_j}} - 1)^{N'_j} \cdot M = \{0\}$ for $1 \le j \le r$. Using (3.5) and Remark 2.2(i), it follows that $\tilde{J} = \tilde{J}_1 + \cdots + \tilde{J}_r$ is a closed two-sided ideal in $\tilde{B}$, where $\tilde{J}_j = \tilde{B} \cdot (w_{2,j}^{p^{s_j}} - 1)^{N'_j}$. Define

$$A_{\tilde{J}} = A[[\langle w_{2,1}, \ldots, w_{2,r}\rangle]] / \left((w_{2,1}^{p^{s_1}} - 1)^{N'_1}, \ldots, (w_{2,r}^{p^{s_r}} - 1)^{N'_r} \right).$$

Then $A_{\tilde{J}}$ is free and finitely generated as an abstract A-module, and every element in $\tilde{D} = \tilde{B}/\tilde{J}$ can be written in a unique way as a convergent power series as in (3.18). We can now proceed using the same arguments as in the case when $r = 1$ to complete the proof for the case when $r > 1$.

The proof of Theorem 3.1 in the case when G is replaced by Δ as in Lemma 3.2(iii) and $r > 1$ follows the same three steps as in the case when $r = 1$.

In the proof of step 1, we need to use an inductive argument as follows. As in the case when $r = 1$, suppose $P^\bullet$ has properties (i), (ii) and (iii) of Theorem 2.6 and suppose that $n_2 = 0$, so that $P^i = 0$ if $i > 0$. Since $\mathrm{H}^i(P^\bullet) = 0$ if $i < n_1$, $P^\bullet$ is isomorphic in $D^-(B)$ to the complex $P_0^\bullet$ which is obtained from $P^\bullet$ by replacing P^{n_1} by $P^{n_1}/\mathrm{B}^{n_1}(P^\bullet)$ and P^i by 0 for $i < n_1$. Define $J_{(0)} = \{0\}$ and $\overline{B}_0 = B/J_{(0)}$. Assume by induction that for $1 \le j \le r$, $P^\bullet$ is isomorphic in $D^-(B)$ to a complex $P_{j-1}^\bullet$ such that $P_{j-1}^i = 0$ if $i > 0$ or $i < n_1$ and such that if $n_1 \le i \le 0$ then P_{j-1}^i is annihilated by a closed two-sided ideal $J_{(j-1)} = J_1 + \cdots + J_{j-1}$, where for $1 \le t \le j-1$, $J_t = B \cdot (w_{2,t}^{N_t} - 1)^{N'_t}$ for certain integers $N_t, N'_t \ge 1$. Let $\overline{B}_{j-1} = B/J_{(j-1)}$ and view $P_{j-1}^\bullet$ as a complex in $D^-(\overline{B}_{j-1})$. Using the above adjustments of Propositions 3.5 - 3.8 and Remark 3.10, we find a complex $P_j^\bullet$ which is isomorphic to $P_{j-1}^\bullet$ in $D^-(\overline{B}_{j-1})$ such that $P_j^i = 0$ if $i > 0$ or $i < n_1$ and such that if $0 \le i \le n_1$ then P_j^i is annihilated by a closed two-sided ideal $\overline{J}_j = \overline{B}_{j-1} \cdot (w_{2,j}^{N_j} - 1)^{N'_j}$ for certain integers $N_j, N'_j \ge 1$. Note that if $J_j = B \cdot (w_{2,j}^{N_j} - 1)^{N'_j}$ and $J_{(j)} = J_{(j-1)} + J_j$, then $\overline{B}_j = B/J_{(j)} = \overline{B}_{j-1}/\overline{J}_j$ as pseudocompact rings. Since $P_j^\bullet$ can be viewed as a complex in $D^-(B)$ by inflation, it follows that $P_j^\bullet$ is isomorphic to $P_{j-1}^\bullet$, and thus to $P^\bullet$, in $D^-(B)$. Hence step 1 follows by induction.

Steps 2 and 3 of the proof of Theorem 3.1 are proved in the same way as when $r = 1$, using the above adjustments of Propositions 3.7 and 3.9.

4. An example

In this section, we want to revisit an example that was considered in [2] concerning the deformations of group cohomology elements. Let $\ell > 2$ be a rational prime with $\ell \equiv 3 \mod 4$ and let $G = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Let $k = \mathbb{Z}/2$ and $W = \mathbb{Z}_2$, and let $M = k$ have trivial G-action. Because of the Kummer sequence

$$1 \to \{\pm 1\} \to \overline{\mathbb{Q}}_\ell^* \xrightarrow{\cdot 2} \overline{\mathbb{Q}}_\ell^* \to 1$$

we obtain that $\mathrm{H}^2(G, M) = \mathbb{Z}/2$ has exactly one non-trivial element β. Moreover, it was shown in [2] that the mapping cone $C(\beta)^\bullet$ is isomorphic to $V^\bullet[1]$ for a two-term complex $V^\bullet$ that is concentrated in degrees -1 and 0

$$V^\bullet : \qquad \cdots 0 \to k[G_b] \xrightarrow{d} k[G_a] \to 0 \cdots, \tag{4.1}$$

where $a = \ell$, b is an element of $\mathbb{Z}_\ell^*$ that is not a square mod ℓ, $G_a = \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{a})/\mathbb{Q}_\ell)$, $G_b = \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{b})/\mathbb{Q}_\ell)$ and d is the augmentation map of $k[G_b]$ composed with multiplication by $1+\sigma_a$ when $G_a = \{1, \sigma_a\}$. It was also shown in [2] that the tangent space $\mathrm{Ext}^1_{D^-(k[[G]])}(V^\bullet, V^\bullet)$ is 4-dimensional over k, and that the versal proflat deformation ring of $V^\bullet$ is universal and isomorphic to $R^{\mathrm{fl}}(G, V^\bullet) \cong W[[G^{\mathrm{ab},2}]]\hat{\otimes}_W W[[G^{\mathrm{ab},2}]]$, where $G^{\mathrm{ab},2}$ denotes the abelianized 2-completion of G. Note that the universal proflat deformation ring $R^{\mathrm{fl}}(G, V^\bullet)$ is universal with respect to isomorphism classes of quasi-lifts of $V^\bullet$ over objects R in $\hat{\mathcal{C}}$ whose cohomology groups are topologically flat, and hence topologically free, pseudocompact R-modules.

We now turn to the situation when G is replaced by its maximal abelian quotient G^{ab}. We want to compute the versal deformation rings of several complexes related to the above $V^\bullet$. The complexes we will consider are all inflated from the maximal pro-2 quotient $G^{\mathrm{ab},2}$ of G^{ab}. By Proposition 2.12, it will suffice to determine their versal deformation rings as complexes for $\Gamma = G^{\mathrm{ab},2}$.

Since $\ell \equiv 3 \mod 4$, local class field theory shows that there are topological generators w_1 and w_2 for $\Gamma = \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ab},2}/\mathbb{Q}_\ell)$ with the following properties. The element w_2 has order 2 and $\{\mathrm{id}, w_2\} = \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ab},2}/\mathbb{Q}_\ell^{\mathrm{un},2})$ where $\mathbb{Q}_\ell^{\mathrm{un},2}$ is the maximal unramified pro-2 extension of $\mathbb{Q}_\ell$. The element w_1 is a topological generator of $\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ab},2}/\mathbb{Q}_\ell(\sqrt{\ell})) \cong \mathbb{Z}_2$, and $\Gamma = \langle w_1, w_2\rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}/2$. Note that w_1 (resp. w_2, resp. w_1w_2) acts trivially on the quadratic extension $\mathbb{Q}_\ell(\sqrt{\ell})$ (resp. $\mathbb{Q}_\ell(\sqrt{-1})$, resp. $\mathrm{Q}_\ell(\sqrt{-\ell})$).

As before, let $M = k = \mathbb{Z}/2$ have trivial Γ-action. Since $\langle w_1\rangle = \mathbb{Z}_2$ has cohomological dimension 1, the spectral sequence

$$\mathrm{H}^p(\langle w_1\rangle, \mathrm{H}^q(\langle w_2\rangle, M)) \Longrightarrow \mathrm{H}^{p+q}(\Gamma, M)$$

degenerates and we get a short exact sequence for all $s \geq 1$

$$0 \to \mathrm{H}^1(\langle w_1\rangle, \mathrm{H}^{s-1}(\langle w_2\rangle, M)) \to \mathrm{H}^s(\Gamma, M) \to \mathrm{H}^s(\langle w_2\rangle, M)^{\langle w_1\rangle} \to 0.$$

Since $\mathrm{H}^s(\langle w_2\rangle, M) = k$ for all $s \geq 0$ and $\mathrm{H}^1(\langle w_1\rangle, k) = k$, we obtain that

$$\mathrm{H}^0(\Gamma, M) = k \quad \text{and} \quad \mathrm{H}^s(\Gamma, M) = k \oplus k \text{ for } s \geq 1.$$

This means that there are three non-trivial elements in $\mathrm{H}^2(\Gamma, M)$. Let $x \in \{\ell, -1, -\ell\}$ and consider the element h_x in $\mathrm{H}^1(\Gamma, \{\pm 1\}) = \mathrm{Hom}(\Gamma, \{\pm 1\})$ which corresponds to the augmentation sequence

$$0 \to k \to k[G_x] \to k \to 0 \tag{4.2}$$

where $G_x = \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{x})/\mathbb{Q}_\ell)$. Inflating the cup product $h_a \cup h_b$ for $a, b \in \{\ell, -1, -\ell\}$ to an element in $\mathrm{H}^2(G, \{\pm 1\})$, it follows that $h_a \cup h_b$ corresponds to the Hilbert symbol $(a, b) \in \mathrm{H}^2(G, \{\pm 1\})$. Hence $h_\ell \cup h_\ell$ and $h_\ell \cup h_{-1}$ define non-trivial elements in $\mathrm{H}^2(G, \{\pm 1\})$, whereas $h_\ell \cup h_{-\ell}$ defines a trivial element in $\mathrm{H}^2(G, \{\pm 1\})$. Since the restriction of $h_\ell \cup h_\ell$ to $\langle w_2\rangle$ is non-trivial, whereas the restriction of $h_\ell \cup h_{-1}$ to $\langle w_2\rangle$ is trivial, $h_\ell \cup h_\ell \neq h_\ell \cup h_{-1}$ in $\mathrm{H}^2(\Gamma, k)$. It follows that $h_\ell \cup h_\ell$, $h_\ell \cup h_{-1}$ and $h_\ell \cup h_{-\ell}$ are representatives of the three non-trivial elements in $\mathrm{H}^2(\Gamma, M)$. We obtain three non-split two-term complexes $V_y^\bullet$ in $D^-(k[[\Gamma]])$ that are concentrated in degrees -1 and 0

$$V_y^\bullet : \qquad \cdots 0 \to k[G_y] \xrightarrow{d} k[G_\ell] \to 0 \cdots \tag{4.3}$$

where $y \in \{\ell, -1, -\ell\}$ and d is the augmentation map followed by multiplication with the trace element of G_ℓ. In particular, for $y \in \{\ell, -1\}$, the inflation of $V_y^\bullet$ to G is isomorphic to $V^\bullet$.

Lemma 4.1. *For $y \in \{\ell, -\ell\}$ (resp. $y = -1$), the k-dimension of $\mathrm{Ext}^1_{D^-(k[[\Gamma]])}(V_y^\bullet, V_y^\bullet)$ is at least 3 (resp. at least 4). Moreover, the proflat tangent space $t_{F^{\mathrm{fl}}}$ is isomorphic to the tangent space t_F.*

Proof. Let $y \in \{\ell, -1, -\ell\}$ and consider the triangle in $D^-(k[[\Gamma]])$

$$k^\bullet[1] \xrightarrow{\gamma_y} V_y^\bullet \xrightarrow{\alpha_y} k^\bullet \xrightarrow{\beta_y} k^\bullet[2] \tag{4.4}$$

where $k^\bullet$ stands for the one-term complex with k concentrated in degree 0 and $\beta_y = h_\ell \cup h_y$ is the non-zero element in $\mathrm{H}^2(\Gamma, k)$ associated to $V_y^\bullet$.

The morphism $\mathrm{Ext}^1_{D^-(k[[\Gamma]])}(k^\bullet[1],k^\bullet)\xrightarrow{\circ\beta_y[-1]}\mathrm{Ext}^1_{D^-(k[[\Gamma]])}(k^\bullet[-1],k^\bullet)$ is injective, since it sends the identity in $\mathrm{Ext}^1_{D^-(k[[\Gamma]])}(k^\bullet[1],k^\bullet)=\mathrm{Hom}_{D^-(k[[\Gamma]])}(k^\bullet[1],k^\bullet[1])\cong k$ to $\beta_y[-1]$ where β_y is as in (4.4). Hence it follows from [2, Prop. 9.6] that $t_{F^{\mathrm{fl}}}\cong t_F$.

Using long exact Hom sequences in $D^-(k[[\Gamma]])$ associated to the triangle (4.4), we obtain the following diagram with exact rows and columns, where Hom stands for $\mathrm{Hom}_{D^-(k[[\Gamma]])}$ and Ext stands for $\mathrm{Ext}_{D^-(k[[\Gamma]])}$.

(4.5)

$$\begin{array}{cccccc}
 & \mathrm{Ext}^{-1}(k^\bullet[1],k^\bullet) & \mathrm{Hom}(k^\bullet[1],k^\bullet[1]) & & \mathrm{Hom}(k^\bullet[1],k^\bullet) & \mathrm{Ext}^1(k^\bullet[1],k^\bullet[1]) \\
 & \downarrow & \downarrow & & \downarrow & \downarrow \\
 & \mathrm{Hom}(k^\bullet,k^\bullet) & \mathrm{Ext}^1(k^\bullet,k^\bullet[1]) & & \mathrm{Ext}^1(k^\bullet,k^\bullet) \longrightarrow & \mathrm{Ext}^2(k^\bullet,k^\bullet[1]) \\
 & \downarrow & \downarrow & & \downarrow & \downarrow \\
\mathrm{Hom}(V_y^\bullet,V_y^\bullet) \longrightarrow & \mathrm{Hom}(V_y^\bullet,k^\bullet) \longrightarrow & \mathrm{Ext}^1(V_y^\bullet,k^\bullet[1]) \longrightarrow & \mathrm{Ext}^1(V_y^\bullet,V_y^\bullet) \longrightarrow & \mathrm{Ext}^1(V_y^\bullet,k^\bullet) \longrightarrow & \mathrm{Ext}^2(V_y^\bullet,k^\bullet[1]) \\
 & \downarrow & \downarrow & & \downarrow & \\
 & \mathrm{Hom}(k^\bullet[1],k^\bullet) & \mathrm{Ext}^1(k^\bullet[1],k^\bullet[1]) & & \mathrm{Ext}^1(k^\bullet[1],k^\bullet) & \\
 & & \downarrow & & \downarrow & \\
 & & \mathrm{Ext}^2(k^\bullet,k^\bullet[1]) & & \mathrm{Ext}^2(k^\bullet,k^\bullet) &
\end{array}$$

Because $\mathrm{Ext}^{-1}(k^\bullet[1],k^\bullet)=0=\mathrm{Hom}(k^\bullet[1],k^\bullet)$ in the second column of (4.5), it follows that $\mathrm{Hom}(V_y^\bullet,k^\bullet)\cong k$ in the third row. We conclude that the horizontal morphism in the third row

$$\mathrm{Ext}^1(V_y^\bullet,k^\bullet[1])\to\mathrm{Ext}^1(V_y^\bullet,V_y^\bullet) \tag{4.6}$$

is injective.

Since the vertical morphism $\mathrm{Ext}^1(k^\bullet[1],k^\bullet[1])\to\mathrm{Ext}^2(k^\bullet,k^\bullet[1])$ in the sixth column and the horizontal morphism $\mathrm{Ext}^1(k^\bullet,k^\bullet)\to\mathrm{Ext}^2(k^\bullet,k^\bullet[1])$ in the second row of (4.5) can both be identified with the morphism $\mathrm{H}^1(\Gamma,k)\to\mathrm{H}^3(\Gamma,k)$ that sends h_x to $h_x\cup\beta_y$, it follows that the composition of morphisms $\mathrm{Ext}^1(k^\bullet,k^\bullet)\to\mathrm{Ext}^2(k^\bullet,k^\bullet[1])\to\mathrm{Ext}^2(V_y^\bullet,k^\bullet[1])$ in the second row and sixth column of (4.5) is the zero morphism. We conclude that the horizontal morphism in the third row

$$\mathrm{Ext}^1(V_y^\bullet,V_y^\bullet)\to\mathrm{Ext}^1(V_y^\bullet,k^\bullet) \tag{4.7}$$

is surjective.

Because the vertical morphism in the third column of (4.5)

$$k\cong\mathrm{Hom}(k^\bullet[1],k^\bullet[1])\to\mathrm{Ext}^1(k^\bullet,k^\bullet[1])\cong\mathrm{H}^1(\Gamma,k)\cong k\oplus k$$

has cokernel of k-dimension at least 1, it follows that $\dim_k\mathrm{Ext}^1(V_y^\bullet,k^\bullet[1])$ is at least 1. The vertical morphism $\mathrm{Ext}^1(k^\bullet[1],k^\bullet[1])\to\mathrm{Ext}^2(k^\bullet,k^\bullet[1])$ in the third column of (4.5) can be identified with the morphism $\mathrm{H}^1(\Gamma,k)\to\mathrm{H}^3(\Gamma,k)$ that sends h_x to $h_x\cup\beta_y$. Since $h_{-1}\cup h_{-1}$ is inflated from an element in $\mathrm{H}^2(\langle w_1\rangle,k)$ and $\langle w_1\rangle=\mathbb{Z}_2$ has cohomological dimension 1, it follows that for $y=-1$, the vertical morphism in the third column $\mathrm{Ext}^1(k^\bullet[1],k^\bullet[1])\to\mathrm{Ext}^2(k^\bullet,k^\bullet[1])$ has non-trivial kernel. Hence $\dim_k\mathrm{Ext}^1(V_y^\bullet,k^\bullet[1])$ is at least 2 if $y=-1$.

Since $\mathrm{Hom}(k^\bullet[1],k^\bullet)=0$ and the vertical morphism in the fifth column $\mathrm{Ext}^1(k^\bullet[1],k^\bullet)\to\mathrm{Ext}^2(k^\bullet,k^\bullet)$ is injective, it follows that the vertical morphism $\mathrm{Ext}^1(k^\bullet,k^\bullet)\to\mathrm{Ext}^1(V_y^\bullet,k^\bullet)$ is an isomorphism. Because $\mathrm{Ext}^1(k^\bullet,k^\bullet)\cong\mathrm{H}^1(\Gamma,k)\cong k\oplus k$, this implies that $\mathrm{Ext}^1(V_y^\bullet,k^\bullet)$ has k-dimension 2. Using (4.6) and (4.7), this implies Lemma 4.1. □

Theorem 4.2. *For $y\in\{\ell,-1,-\ell\}$, the versal deformation ring $R(\Gamma,V_y^\bullet)$ and the versal proflat deformation ring $R^{\mathrm{fl}}(\Gamma,V_y^\bullet)$ have the following isomorphism types:*

$$R(\Gamma,V_\ell^\bullet)\cong W[[t_1,t_2,t_3]]/(t_2t_3(2+t_3))\quad\textit{and}\quad R^{\mathrm{fl}}(\Gamma,V_\ell^\bullet)\cong W[[t_1,t_2,t_3]]/(t_3(2+t_3)),$$
$$R(\Gamma,V_{-\ell}^\bullet)\cong R^{\mathrm{fl}}(\Gamma,V_{-\ell}^\bullet)\cong W[[t_1,t_2,t_3]]/(t_3(2+t_3)),$$
$$R(\Gamma,V_{-1}^\bullet)\cong R^{\mathrm{fl}}(\Gamma,V_{-1}^\bullet)\cong W[[t_1,t_2,t_3,t_4]]/(t_2(2+t_2),t_4(2+2t_2-t_3t_4)).$$

Proof. We first give an outline of the proof. We will show that every quasi-lift of $V_y^\bullet$ over a ring A in $\hat{\mathcal{C}}$ can be represented by a two-term complex $P^\bullet : P^{-1} \xrightarrow{d_P} P^0$, concentrated in degrees -1 and 0, in which P^{-1} and P^0 are pseudocompact $A[[\Gamma]]$-modules that are free of rank two over A. We will show further that the action of the topological generators w_1 and w_2 of Γ on P^{-1} and P^0, as well as the differential d_P, are described by 2×2 matrices over A whose entries satisfy certain equations. We construct a candidate for the versal deformation ring $R(\Gamma, V_y^\bullet)$ by taking the completion of the ring obtained by adjoining to W indeterminates corresponding to these matrix entries which are required to satisfy the above equations. We prove that $R(\Gamma, V_y^\bullet)$ is the versal deformation ring by showing that each $P^\bullet$ as above is a specialization of the resulting quasi-lift $(U(\Gamma, V_y^\bullet), \phi_U)$ over $R(\Gamma, V_y^\bullet)$ and by showing that the tangent space of $R(\Gamma, V_y^\bullet)$ has the correct dimension. The last step uses Lemma 4.1.

Let $y \in \{\ell, -1, -\ell\}$. Let A be in $\hat{\mathcal{C}}$ and let $(L^\bullet, \phi_L)$ be a quasi-lift of $V_y^\bullet$ over A. Since $\mathrm{H}^0(V_y^\bullet) = k$, it follows that $\mathrm{H}^0(L^\bullet)$ is a quotient of A. By Theorem 2.6 and Remark 2.5, we can thus assume that $L^\bullet$ is a two-term complex, concentrated in degrees -1 and 0,

$$L^\bullet : \qquad \cdots 0 \to L^{-1} \xrightarrow{d_L} A[[\Gamma]] \to 0 \cdots$$

where L^{-1} is topologically free over A, and such that we have an exact sequence in $C^-(A[[\Gamma]])$

$$0 \to \mathrm{H}^{-1}(L^\bullet) \to L^{-1} \xrightarrow{d_L} A[[\Gamma]] \to \mathrm{H}^0(L^\bullet) \to 0. \tag{4.8}$$

Since $\mathrm{H}^0(L^\bullet)$ is a quotient of A, w_1 (resp. w_2) acts on $\mathrm{H}^0(L^\bullet)$ as a scalar s_1 (resp. s_2) in A^*.

Because w_1 acts on $\mathrm{H}^0(L^\bullet)$ as the scalar s_1, $(w_1 - s_1)$ annihilates $\mathrm{H}^0(L^\bullet)$. Since $A[[\Gamma]](w_1 - s_1)$ is a free rank one pseudocompact $A[[\Gamma]]$-module that is a submodule of $A[[\Gamma]]$, there exists a free rank one pseudocompact $A[[\Gamma]]$-module F that is a submodule of L^{-1} such that d_L maps F isomorphically onto $A[[\Gamma]](w_1 - s_1)$. Hence the exact sequence (4.8) leads to an exact sequence in $C^-(A[[\Gamma]])$

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{H}^{-1}(L^\bullet) \longrightarrow & P^{-1} & \xrightarrow{\quad d_P \quad} & P^0 & \xrightarrow{\quad \mu \quad} & \mathrm{H}^0(L^\bullet) \longrightarrow 0 \\ & & \| & & \| & & \\ & & L^{-1}/F & & A[[\Gamma]]/A[[\Gamma]](w_1 - s_1) & & \end{array} \tag{4.9}$$

where $P^0 \cong A\langle w_2\rangle$ and w_1 acts on P^0 as multiplication by s_1. Since w_2 has order 2, it follows that $\{1, w_2\}$ is an A-basis of P^0. With respect to this A-basis, w_1 (resp. w_2) acts on P^0 as the matrix

$$W_{1,P^0} = \begin{pmatrix} s_1 & 0 \\ 0 & s_1 \end{pmatrix} \quad \left(\text{resp. } W_{2,P^0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right). \tag{4.10}$$

Moreover, $L^\bullet$ is quasi-isomorphic to the two-term complex

$$P^\bullet : \qquad \cdots 0 \to P^{-1} \xrightarrow{d_P} P^0 \to 0 \cdots \tag{4.11}$$

concentrated in degrees -1 and 0, where P^{-1}, P^0 and d_P are as in (4.9). Let $\phi_P : k\hat{\otimes}_A P^\bullet \to V_y^\bullet$ be an isomorphism in $D^-(k[[\Gamma]])$ such that $(L^\bullet, \phi_L)$ and $(P^\bullet, \phi_P)$ are isomorphic quasi-lifts of $V_y^\bullet$ over A. Since $L^\bullet$, and hence $P^\bullet$, has finite pseudocompact A-tor dimension at -1, P^{-1} is topologically flat, and hence topologically free, over A. Because $k\hat{\otimes}_A P^\bullet$ must be isomorphic to $V_y^\bullet$ in $D^-(k[[\Gamma]])$, it follows that $k\hat{\otimes}_A P^{-1}$ has k-dimension 2, and hence P^{-1} is free over A of rank 2.

Let K^0 be the kernel of the morphism $\mu : P^0 \to \mathrm{H}^0(L^\bullet)$ in (4.9). Because w_2 acts on $\mathrm{H}^0(L^\bullet)$ as the scalar $s_2 \in A^*$, K^0 contains the element $-s_2 \cdot 1 + 1 \cdot w_2 = (-s_2, 1)$ which generates a free rank one A-submodule of P^0. Since K^0 is an $A[[\Gamma]]$-submodule of P^0, we also have that

$$w_2 \cdot (-s_2, 1) = (1, -s_2) = -s_2(-s_2, 1) + (1 - s_2^2, 0)$$

is an element of K^0, and thus $(1 - s_2^2, 0) \in K^0$. On the other hand, $k\hat{\otimes}_A K^0$ has k-dimension at most 2, since K^0 is a homomorphic image of P^{-1}. Hence K^0 is generated by one or two elements, depending on whether $\mathrm{H}^0(L^\bullet)$ is flat over A or not. If (c, d) is an arbitrary element in K^0, then

$$(c, d) = d \cdot (-s_2, 1) + (c + ds_2, 0),$$

and hence K^0 is generated by $(-s_2,1)$ and an element of the form $(\lambda,0)$ such that λ divides $(1-s_2^2)$. It follows that $\mathrm{H}^0(L^\bullet)\cong A/(\lambda)$ and $\mathrm{H}^0(L^\bullet)\neq\{0\}$. In particular, $\mathrm{H}^0(L^\bullet)$ is flat over A if and only if $\lambda=0$, in which case we must have $1-s_2^2=0$. Since the image of d_P in (4.9) must be equal to K^0 and since $(-s_2,1)$ generates a free A-module of rank 1, we can lift $(-s_2,1)$ to a basis element z_1 of P^{-1}. If $\lambda=0$, then $\mathrm{H}^{-1}(L^\bullet)=\mathrm{Ker}(d_P)\cong A$ as A-modules, and we choose z_2 to be a basis element of P^{-1} which lies in $\mathrm{Ker}(d_P)$. If $\lambda\neq 0$, then K^0 is generated as A-module by $(-s_2,1)$ and $(\lambda,0)$, and we let z_2 be a preimage under d_P of $(\lambda,0)$. Since $(\lambda,0)$ is not an A-multiple of $(-s_2,1)$ if $\lambda\neq 0$, the homomorphism $k\hat{\otimes}_A P^{-1}\to k\hat{\otimes}_A K^0$ induced by the surjection $P^{-1}\xrightarrow{d_P}K^0$ must be an isomorphism of two-dimensional k-vector spaces. It follows that $\{z_1,z_2\}$ is an A-basis of P^{-1} for all λ.

With respect to the A-basis $\{z_1,z_2\}$ of P^{-1} and the A-basis $\{1,w_2\}$ of P^0, $d_P:P^{-1}\to P^0$ is given by the matrix

$$D_P=\begin{pmatrix}-s_2 & \lambda\\ 1 & 0\end{pmatrix} \tag{4.12}$$

when we write basis vectors as column vectors. In particular, $\mathrm{H}^{-1}(L^\bullet)=\mathrm{Ker}(d_P)\cong\mathrm{Ann}_A(\lambda)$, which implies that $(P^\bullet,\phi_P)$ is a proflat quasi-lift of $V_y^\bullet$ over A if and only if $\lambda=0$. Additionally, if $\lambda=0$ then $1-s_2^2=0$.

Suppose now that $\lambda\neq 0$. In particular, λ is not a unit since $\mathrm{H}^0(L^\bullet)\cong A/(\lambda)$ is not 0. Then $1-s_2^2=\lambda t_2$ for some $t_2\in A$, and K^0 is generated as A-module by $(-s_2,1)$ and $(\lambda,0)$. The action of w_1 on K^0 is given by the scalar matrix s_1. The action of w_2 on P^0 sends $(-s_2,1)$ to

$$(1,-s_2)=-s_2(-s_2,1)+(1-s_2^2,0)=-s_2(-s_2,1)+t_2(\lambda,0)$$

and $(\lambda,0)$ to

$$(0,\lambda)=\lambda(-s_2,1)+s_2(\lambda,0).$$

To obtain the action of w_2 on P^{-1}, we use the matrix representation D_P of d_P from (4.12) with respect to the A-basis $\{z_1,z_2\}$ of P^{-1}. This means that the kernel of d_P is given by $\mathrm{Ann}_A(\lambda)\cdot z_2$. Hence the action of w_1 (resp. w_2) on P^{-1} has the form

$$\begin{pmatrix}s_1 & 0\\ x_1 & s_1+y_1\end{pmatrix}\quad\left(\text{resp. }\begin{pmatrix}-s_2 & \lambda\\ t_2+x_2 & s_2+y_2\end{pmatrix}\right) \tag{4.13}$$

for certain elements $x_1,y_1,x_2,y_2\in\mathrm{Ann}_A(\lambda)$. If t_2 were not a unit, then w_1 and w_2 would both act trivially on $k\hat{\otimes}_A P^{-1}$. Hence $k\hat{\otimes}_A P^\bullet$ would correspond to the cup product of h_ℓ with the trivial character h_1 which defines the trivial element in $\mathrm{H}^2(\Gamma,\{\pm 1\})$. This is a contradiction to $k\hat{\otimes}_A P^\bullet\cong V_y^\bullet$ in $D^-(k[[\Gamma]])$. Thus t_2 must be a unit, which implies $(\lambda)=(1-s_2^2)$. But then the action of w_1 (resp. w_2) on $k\hat{\otimes}_A P^{-1}$ is trivial (resp. non-trivial). Hence $k\hat{\otimes}_A P^\bullet$ corresponds to the cup product $h_\ell\cup h_\ell$. We conclude that if $\lambda\neq 0$ then $y=\ell$.

We now concentrate on the case when $y=\ell$. The cases when $y=-\ell$ or $y=-1$ are treated similarly.

Given an arbitrary quasi-lift of $V_\ell^\bullet$ over a ring A in $\hat{\mathcal{C}}$, we can assume this quasi-lift is given by $(P^\bullet,\phi_P)$ with $P^\bullet$ as in (4.11). The complex $k\hat{\otimes}_A P^\bullet$ defines $h_\ell\cup h_\ell\in\mathrm{H}^2(\Gamma,k)$, and it follows from the construction of $P^\bullet$ in (4.11) that $h_\ell\cup h_\ell=h_\ell\cup h'$, where $h'\in\mathrm{H}^1(\Gamma,k)$ is the class of $0\to k\to k\hat{\otimes}_A P^{-1}\to k\to 0$. Hence $h_\ell\cup h_\ell=h_\ell\cup h'$ implies $h'=h_\ell$. So w_1 (resp. w_2) acts trivially (resp. non-trivially) on $k\hat{\otimes}_A P^{-1}$.

Since w_2 acts non-trivially on $k\hat{\otimes}_A P^{-1}$, it follows that $1\hat{\otimes}z_1$ and $w_2\cdot(1\hat{\otimes}z_1)=1\hat{\otimes}(w_2\cdot z_1)$ form a k-basis of $k\hat{\otimes}_A P^{-1}$. Since A is a commutative local ring, this implies that $\{z_1,w_2\cdot z_1\}$ is an A-basis of P^{-1}. It follows that with respect to the A-basis $\{z_1,w_2\cdot z_1\}$ of P^{-1} and the A-basis $\{1,w_2\}$ of P^0, $d_P:P^{-1}\to P^0$ is given by the matrix

$$\tilde{D}_P=\begin{pmatrix}-s_2 & 1\\ 1 & -s_2\end{pmatrix}. \tag{4.14}$$

Considering the actions of w_1 (resp. w_2) on P^{-1} and P^0, we obtain that with respect to the A-basis $\{z_1, w_2 \cdot z_1\}$ of P^{-1}, w_1 (resp. w_2) acts on P^{-1} as the matrix

$$\tilde{W}_{1,P^{-1}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \left(\text{resp. } \tilde{W}_{2,P^{-1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \tag{4.15}$$

where $a \in A^*$ and $b \in m_A$ satisfy $-s_1 s_2 = -s_2 a + b$ and $s_1 = -s_2 b + a$. These two conditions are equivalent to the two conditions

$$\begin{aligned} b &= s_2(a - s_1), \\ 0 &= (a - s_1)(1 - s_2^2). \end{aligned} \tag{4.16}$$

Since these are the only conditions needed to ensure that $P^\bullet$ defines a quasi-lift of $V_\ell^\bullet$ over A, we obtain the following: For all $s_1, s_2, a \in A^*$ with $(a - s_1)(1 - s_2^2) = 0$, there is a two-term complex

$$Q^\bullet : \qquad Q^{-1} = A \oplus A \xrightarrow{\begin{pmatrix} -s_2 & 1 \\ 1 & -s_2 \end{pmatrix}} A \oplus A = Q^0 \tag{4.17}$$

where w_1 (resp. w_2) acts on Q^0 as the matrix W_{1,P^0} (resp. W_{2,P^0}) from (4.10), and w_1 (resp. w_2) acts on Q^{-1} as the matrix $\tilde{W}_{1,P^{-1}}$ (resp. $\tilde{W}_{2,P^{-1}}$) from (4.15) where $b = s_2(a - s_1)$. Moreover, for all choices of s_1, s_2, a as above, $k\hat{\otimes}_A Q^\bullet$ is equal to the same complex $Z_\ell^\bullet$. By choosing suitable k-bases for the terms of $V_\ell^\bullet$, we see that $V_\ell^\bullet = Z_\ell^\bullet$ in $C^-(k[[\Gamma]])$. Thus for each choice of isomorphism $\phi_Q : V_\ell^\bullet \to V_\ell^\bullet$ in $D^-(k[[\Gamma]])$, we obtain a quasi-lift $(Q^\bullet, \phi_Q)$ of $V_\ell^\bullet$ over A. Analyzing all isomorphisms in $\mathrm{Hom}_{D^-(k[[\Gamma]])}(V_\ell^\bullet, V_\ell^\bullet)$, it follows that if $\phi_Q, \phi_Q' : Z_\ell^\bullet = V_\ell^\bullet \to V_\ell^\bullet$ are isomorphisms in $D^-(k[[\Gamma]])$, then $(Q^\bullet, \phi_Q)$ is isomorphic to $(Q^\bullet, \phi_Q')$ as quasi-lifts of $V_\ell^\bullet$ over A.

Let $S_\ell = W[[t_1, t_2, t_3]]/(t_2 t_3(t_3 + 2))$. We obtain a two-term complex $U^\bullet$ in $C^-(S_\ell[[\Gamma]])$ from $Q^\bullet$ by replacing A by S_ℓ, s_1 by $1 + t_1$, $a - s_1$ by t_2, s_2 by $1 + t_3$ and b by $(1 + t_3)t_2$ in (4.17), in (4.10) and in (4.15). Let $\phi_U : k\hat{\otimes}_A U^\bullet = Z_\ell^\bullet \to V_\ell^\bullet$ be a fixed isomorphism in $D^-(k[[\Gamma]])$. Given a quasi-lift of $V_\ell^\bullet$ over A which is isomorphic to $(Q^\bullet, \phi_Q)$ for $Q^\bullet$ as above, it follows that the morphism $\alpha : S_\ell \to A$ with $\alpha(t_1) = s_1 - 1$, $\alpha(t_2) = a - s_1$ and $\alpha(t_3) = s_2 - 1$ is a morphism in $\hat{\mathcal{C}}$ such that $(Q^\bullet, \phi_Q)$ is isomorphic to $(A\hat{\otimes}_{S_\ell,\alpha} U^\bullet, \phi_U)$ as quasi-lifts of $V_\ell^\bullet$ over A. Because $\max(S_\ell)/(\max^2(S_\ell) + 2S_\ell)$ has k-dimension 3, it follows from Lemma 4.1 that S_ℓ is the versal deformation ring of $V_\ell^\bullet$.

For proflat quasi-lifts of $V_\ell^\bullet$ over a ring A in $\hat{\mathcal{C}}$, the only additional condition is $s_2^2 = 1$. It follows that the versal proflat deformation ring of $V_\ell^\bullet$ is isomorphic to $W[[t_1, t_2, t_3]]/(t_3(t_3 + 2))$. □

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