The Bergman property for endomorphism monoids of some Fraïssé limits

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Abstract

Based on an idea of Y. Péresse and some results of Maltcev, Mitchell and Ruškuc, we present sufficient conditions under which the endomorphism monoid of an ultrahomogeneous first-order structure has the Bergman property. This property has played a prominent role both in the theory of infinite permutation groups and, more recently, in semigroup theory. As byproducts of our considerations, we establish exactly which ultrahomogeneous structures are homomorphism-homogeneous, and apply our conclusions to questions concerning relative ranks of endomorphism semigroups within the corresponding transformation semigroups.

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1 Introduction

1.1 The Bergman property and Fraïssé limits

Let S be a semigroup. For $A \subseteq S$ and $n \geqslant 1$ we denote $A^n = \{a_1 \cdots a_n : a_1, \dots, a_n \in A\}$. If Γ is a generating set for $S = \langle \Gamma \rangle$, then, by definition, $S = \bigcup_{n=1}^{\infty} \Gamma^n$. However, it might turn out that only a finite portion of the latter infinitary union suffices to obtain the whole S, that is, $S = \bigcup_{n=1}^{m} \Gamma^n$ holds for some $m \geqslant 1$. In such a case we say that S is *semigroup Cayley bounded* with respect to Γ. For a group G and its (group) generator Δ we have that G is generated as a semigroup by $\Delta \cup \Delta^{-1}$; thus we say that G is *group Cayley bounded* with respect to Δ if it is semigroup Cayley bounded with respect to $\Gamma = \Delta \cup \Delta^{-1}$ (that is, the Cayley graph of G with respect to Γ is of finite diameter). A well-known result of George Bergman [2] asserts that for any (infinite) set X, the symmetric group $\mathrm{Sym}(X)$ is group Cayley bounded with respect to *every* its generating set. Hence, the term 'the Bergman property' quickly established itself [23] to describe the property of groups of being group Cayley bounded with respect to every generator. To distinguish between groups and semigroups, we refer to this remarkable property as the *group* Bergman property; the analogous property for semigroups—the main subject of investigation in a recent contribution by Maltcev, Mitchell and Ruškuc [26]—is called the *semigroup* Bergman property. For a group G, the semigroup Bergman property obviously implies the group Bergman property. It is still unknown, however, whether the converse is true.

The other principal theme of this paper are the fascinating objects from model theory called *Fraïssé limits*. Namely, if $\mathscr{C}$ is a class of finitely generated first-order structures (of

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a fixed signature) which is closed for taking (isomorphic copies of) substructures, has the *joint embedding property* (JEP) and the *amalgamation property* (AP), then a celebrated result of Roland Fraïssé [13, 21] guarantees the existence and uniqueness of a countable structure F such that:

(i) F is *$\mathscr{C}$-universal*, that is, any $A \in \mathscr{C}$ embeds into F, and

(ii) F is *ultrahomogeneous*, which means that for any isomorphism $\alpha : A \to A'$ between finitely generated substructures A, A' of F there is an $\widehat{\alpha} \in \mathrm{Aut}(F)$ which extends α, i.e., $\widehat{\alpha}|_A = \alpha$.

Following [21], such structure F is called the *Fraïssé limit of* $\mathscr{C}$ and denoted by $\mathrm{Flim}(\mathscr{C})$. Moreover, any countably infinite ultrahomogeneous structure arises in this way: it is the Fraïssé limit of the class of all of its finitely generated substructures. A class of finitely generated structures that satisfies the premises of the Fraïssé theorem is called a *Fraïssé class*. It is not difficult to show that if $\mathscr{C}$ is a Fraïssé class, and if $\overline{\mathscr{C}}$ denotes the class of all countable structures all of whose finitely generated substructures are contained in $\mathscr{C}$, then in fact any member of $\overline{\mathscr{C}}$ embeds into $\mathrm{Flim}(\mathscr{C})$. Historically, the first Fraïssé limits discovered were the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$ [38] as the limit of all finite metric spaces with rational distances, and $\mathbb{Q}$, the limit of all finite chains [13]. Other classical examples of Fraïssé classes and their limits include:

- finite simple graphs and the *random graph* R [5, 6],
- finite posets and the *generic poset* $\mathbb{P}$ [36],
- finite semilattices and the countable universal ultrahomogeneous semilattice Ω [11],
- finite distributive lattices and the countable universal ultrahomogeneous distributive lattice $\mathbb{D}$ [12],
- finite Boolean algebras and the countable atomless Boolean algebra $\mathbb{A}$.

In the abundance of examples provided in [26] of well-known semigroups both with and without the Bergman property—a significant part of them being semigroups of various mappings, or even morphisms of some structure—one particular result contained in Theorem 4.2 of that paper caught author's special attention and interest. Namely, this is the assertion that $\mathrm{End}(R)$ has the Bergman property. This claim remained unproved in [26]: the theorem itself was formulated as a consequence of Lemma 2.4 of that paper (see Lemma 1.1 below) and two earlier publications [1, 32], which indeed account for all the assertions contained in the theorem except for the one about $\mathrm{End}(R)$. However, later I learned hat the Bergman property for $\mathrm{End}(R)$ is a consequence of a result in the recent doctoral thesis of Y. Péresse [34] (a student of Mitchell's) and the already mentioned Lemma 2.4 of [26]. The present note is centered around a series of remarks leading to the conclusion that the convenient and clever trick presented in [34] can be in fact generalized to a whole class of ultrahomogeneous structures (that is, Fraïssé limits), thus yielding the Bergman property for their endomorphism monoids. This conclusion is reached in Corollary 4.5, with the purpose of complementing the results of [26]. In the course of proving our main results, we will record an exact description of Fraïssé classes whose limits are homomorphism-homogeneous [8], accompanied with a number of examples. As a application of this approach, in the final section we obtain some information on the relative ranks of $\mathrm{End}(F)$ within $\mathcal{T}_F$, where F is a Fraïssé limit.

Now, let us briefly review the aforementioned trick from [34] and the other ingredients of the argument proving that $\mathrm{End}(R)$ has the Bergman property.

1.2 Strong distortion, coproducts, homomorphism extensions

The results we are about to revisit are based on another motif connected to the Bergman property, and it traces back to an old, classical result of W. Sierpiński, who proved in [37] that if X is an infinite set, then any countable set $\{f_i : i \geqslant 0\}$ of self-maps (transformations) $X \to X$ is contained in a 2-generated subsemigroup of $\mathcal{T}_X$, the semigroup of all self-maps of X. Following this landmark example, we say that a semigroup S has *Sierpiński index* $n < \omega$ if n is the least positive integer with the property that for any countable $A \subseteq S$ there exists $s_1, \dots, s_n \in S$ such that $A \subseteq \langle s_1, \dots, s_n \rangle$. If no such n exists, the Sierpiński index of S is said to be *infinite*. Of course, the Sierpiński index of a countable semigroup is simply its *rank*, the minimum size of its generating set, so that the notion is particularly interesting for uncountable semigroups. So, the result of Sierpiński asserts that $\mathcal{T}_X$ has Sierpiński index 2. Some recent results concerning the Sierpiński index of certain classical transformation semigroups—along with a thorough historical introduction to the topic—can be found in [31].

A slight modification of this notion produces a convenient method for proving the Bergman property for semigroups. Namely, in several proofs of the finiteness of the Sierpiński index for various semigroups it turns out—after selecting a countable set $A = \{a_i : i < \omega\}$ and $s_1, \dots, s_n$ such that $A \subseteq \langle s_1, \dots, s_n \rangle$—that in the representation

$$a_i = \mathbf{w}_i(s_1, \dots, s_n)$$

the length of the word $\mathbf{w}_i$ does not depend on the particular choice of a_i, but that it is determined only by the index i. In other words, the Sierpiński property occurs in some sense in a "uniform" way. More formally, call a semigroup S *strongly distorted* if there exists a sequence of natural numbers $(\ell_n)_{n<\omega}$ and $M < \omega$ such that for any sequence $(a_n)_{n<\omega}$ of elements of S there exist $s_1, \dots, s_M \in S$ and a sequence of words $(\mathbf{w}_n)_{n<\omega}$ (over an M-letter alphabet) such that $|\mathbf{w}_n| \leqslant \ell_n$ and $a_n = \mathbf{w}_n(s_1, \dots, s_M)$ for all $n < \omega$. Here is the result that puts strongly distorted semigroups into the context of the initial motivation of this paper.

Lemma 1.1 ([26, Lemma 2.4]) *If S is a non-finitely generated and strongly distorted semigroup, then S has the Bergman property.*

Therefore, any strongly distorted uncountable semigroup has the Bergman property. This observation is the link showing that Lemma 3.10.3 and the proof of Theorem 3.10.4 in [34] in fact establish the Bergman property for $\mathrm{End}(R)$. However, the good thing about the latter theorem is that it is not really about the random graph, as it very easily admits a generalization that we present here. But first recall the classical category-theoretical notion of a *coproduct*. If $\{A_i : i \in I\}$ is a family of first-order structures belonging to a concrete category $\mathbf{C}$ (where objects are structures and morphisms are their homomorphisms), then their coproduct (or *free sum*), denoted by $\coprod^*_{i\in I} A_i$, is a structure $S \in \mathbf{C}$ with the following properties:

(a) there are embeddings $\iota_i : A_i \to S$ for any $i \in I$;

(b) for any $B \in \mathbf{C}$ and any homomorphisms $\varphi_i : A_i \to B$, $i \in I$, there exists a unique homomorphism $\varphi : S \to B$ such that $\varphi\iota_i = \varphi_i$ holds for all $i \in I$.

(In this paper, mappings are composed right-to left, so that fg is a function for which $fg(x)$ means $f(g(x)$. For a set X, $\mathbf{1}_X$ will always denote the identity mapping on X.)

Whenever it exists, the coproduct is unique up to an isomorphism, and it is generated by $\bigcup_{i\in I} \iota_i(A_i)$.

So, here is the "abstract" version of Theorem 3.10.4 from [34].

Theorem 1.2 (Mitchell [28], Péresse [34]) *Let A be an infinite structure with a substructure B satisfying the following conditions:*

(i) $B \cong \coprod_{n<\omega} A_n$*, where* $A_n \cong A$ *for each* $n < \omega$*;*

(ii) any homomorphism $\varphi : B \to A$ *can be extended to* $\widehat{\varphi} \in \mathrm{End}(A)$*.*

Then $\mathrm{End}(A)$ *is strongly distorted. In addition, the Sierpiński index of* $\mathrm{End}(A)$ *is* $\leqslant 3$*.*

Proof. Let $f_0, f_1, \dots$ be any countable sequence of endomorphisms of A. We construct $g_1, g_2, g_3 \in \mathrm{End}(A)$ such that $f_k \in \langle g_1, g_2, g_3\rangle$ for any $k \geqslant 0$. Also, we will freely assume that each A_n is actually contained in B.

First of all, let $g_1 : A \to A_0$ be any isomorphism. Furthermore, let $h_n : A_n \to A_{n+1}$, $n < \omega$, be isomorphisms. By (i) and the definition of the coproduct, since each h_n maps into B, there is a homomorphism $h : B \to B$ (that is, $h \in \mathrm{End}(B)$) such that $h|_{A_n} = h_n$ for any $n < \omega$. By condition (ii), h can be extended to an endomorphism of A, which we denote by g_2. Then for any $n < \omega$ we have $g_2|_{A_n} = h_n$ and $t_n = g_2^n g_1 : A \to A_n$ is an isomorphism.

Now we define the key endomorphism g_3, which can be informally thought of as a "compressed form" of the sequence $\{f_k\}_{k\geqslant 0}$, where each f_k is "packed up" into the copy A_k of A. Since $t_k^{-1} : A_k \to A$ is an isomorphism, it follows that $\psi_k = f_k t_k^{-1}$ is a homomorphism of A_k into A. Similarly as above, by (i) there exists a homomorphism $\psi : B \to A$ such that $\psi|_{A_k} = \psi_k$ for each $k \geqslant 0$. However, by (ii) there is an extension of ψ to $g_3 \in \mathrm{End}(A)$. Note that $g_3|_{A_k} = \psi_k$ holds as well.

It remains to recover f_k from g_3. Indeed, since t_k maps into A_k,

$$g_3 g_2^k g_1 = g_3 t_k = \psi_k t_k = f_k t_k^{-1} t_k = f_k,$$

as wanted. So, not only that $f_k \in \langle g_1, g_2, g_3\rangle$, but we uniformly have that the length of the product representing f_k is $\ell_k = k + 2$. □

Of course, the coproduct in the category of (simple) graphs is just the disjoint union of the given family of graphs. Hence, $\coprod_{n<\omega} R$ exists, and, since it is a countably infinite graph, it embeds into R. In addition, as established in Lemma 3.10.3. of [34], R has the remarkable property that for any countable graph G there is an induced subgraph G' of R, isomorphic to G, such that any homomorphism $\varphi : G' \to R$ extends to an endomorphism $\widehat{\varphi}$ of R. As $\mathrm{End}(R)$ is known to be uncountable, it follows that it has the Bergman property.

Our main goal here is to see to which extent we can utilize the above theorem in order to cover the cases of some of the most important infinite structures arising as Fraïssé limits. At this stage, it is more or less clear that the condition (i) for $\mathrm{Flim}(\mathscr{C})$ is related to the existence of coproducts in the concrete category to which the members of $\overline{\mathscr{C}}$ belong. However, the really intriguing condition here is (ii), the possibility of extending a homomorphism from a certain substructure of $\mathrm{Flim}(\mathscr{C})$. More precisely, we will be interested in the conditions under which for any $A \in \overline{\mathscr{C}}$ there exists a substructure A' of $F = \mathrm{Flim}(\mathscr{C})$ such that $A' \cong A$ and any homomorphism $\varphi : A' \to F$ can be extended to an endomorphism of F. These conditions must be nontrivial, as the next example shows.

Example 1.3 Let H_n denote the *Henson graph* [16, 24], that is, the Fraïssé limit of the class of all finite simple graphs omitting K_n, the complete graph (clique) on n vertices, $n \geqslant 3$. Now, H_n clearly contains copies of K_{n-1} and $\overline{K_{n-1}}$ and any bijection of vertices $f : \overline{K_{n-1}} \to K_{n-1}$ is a graph homomorphism. However, f cannot be extended to an endomorphism $\widehat{f}$ of H_n because there is a vertex u adjacent to each vertex of the anti-clique $\overline{K_{n-1}}$, so that $\widehat{f}(u)$ would be adjacent to each vertex of the clique K_{n-1}, which is impossible by the definition of H_n. Therefore, H_n is not homomorphism-homogeneous. In addition, as shown in [33], every endomorphism of H_n is injective, so not every endomorphism of a (finite) subgraph of H_n can be extended to a member of $\mathrm{End}(H_n)$ (just take two non-adjacent vertices u, v and map them both into u).

2 Preliminaries

In order to handle Fraïssé limits efficiently, we require some auxiliary tools connected to them. However, instead of just referring to some of the standard textbooks on model theory (such as [21]) for the basics of amalgamation theory and the general construction of the limit of a Fraïssé class, for our purpose we will need to be slightly more precise and explicit in certain details of that construction. Therefore, to keep the note reasonably self-contained and complete, we will recall in this section several facts that are otherwise regarded as folklore in model theory.

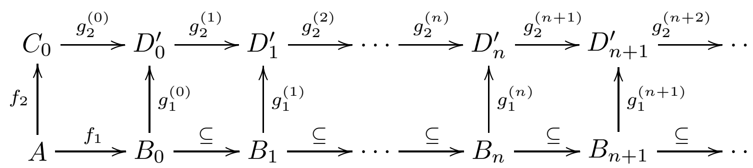
2.1 The amalgamation property and amalgamated free sums

Recall that an *amalgam* is a quintuple (A, B, C, f_1, f_2) consisting of structures A, B, C together with two embeddings $f_1 : A \to B$ and $f_2 : A \to C$. If $A, B, C \in \mathscr{C}$ for some class $\mathscr{C}$, then we have an amalgam *in* $\mathscr{C}$. The *amalgamation property* for $\mathscr{C}$, mentioned earlier, asserts that any amalgam in $\mathscr{C}$ can be embedded into a structure $D \in \mathscr{C}$, i.e. that there are embeddings $g_1 : B \to D$ and $g_2 : C \to D$ such that $g_1 f_1 = g_2 f_2$. If $\mathscr{C}$ is a class of finitely generated structures with the AP (for example, a Fraïssé class), then it is known that the statement of the AP extends to non-finitely generated members of $\overline{\mathscr{C}}$ in the following sense.

Lemma 2.1 *Let (A, B, C, f_1, f_2) be an amalgam such that $A \in \mathscr{C}$ and $B, C \in \overline{\mathscr{C}}$. Then it can be embedded into some structure $D \in \overline{\mathscr{C}}$.*

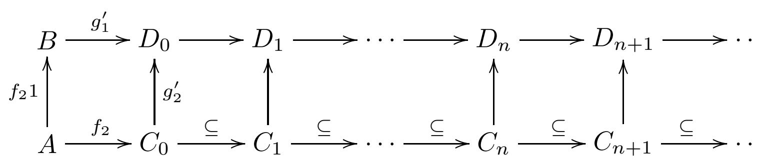
Proof. Both B and C can be written as the union of a chain of their finitely generated substructures: $B = \bigcup_{n<\omega} B_n$ and $C = \bigcup_{n<\omega} C_n$; to see this, just enumerate the elements of these structures as $B = \{b_0, b_1, \dots\}$ and $C = \{c_0, c_1, \dots\}$ and define B_n to be the substructure of B generated by $\{b_0, \dots, b_n\}$ (and similarly for C). In addition, since A is finitely generated, there is no loss of generality in assuming that each B_n contains $f_1(A)$ and that each C_n contains $f_2(A)$.

Thus (A, B_0, C_0, f_1, f_2) is an amalgam in $\mathscr{C}$, so it can be embedded into some $D'_0 \in \mathscr{C}$ by means of $g_1^{(0)} : B_0 \to D'_0$ and $g_2^{(0)} : C_0 \to D'_0$. Now $(B_0, B_1, D'_0, \mathbf{1}_{B_0}, g_1^{(0)})$ is an amalgam in $\mathscr{C}$, so there are a structure $D'_1 \in \mathscr{C}$ and embeddings $g_1^{(1)} : B_1 \to D'_1$ and $g_2^{(1)} : D'_0 \to D'_1$ such that $g_1^{(1)}|_{B_0} = g_2^{(1)} g_1^{(0)}$. We proceed by induction; what we obtain is the following infinite commutative diagram:



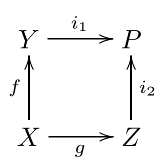
By taking $D' = \lim_{n\to\infty} D'_n$ (a structure containing isomorphic copies D''_n of each D'_n such that D''_n, $n < \omega$, form a chain and $D' = \bigcup_{n<\omega} D''_n$) we obtain embeddings $g'_1 : B \to D'$ and $g'_2 : C_0 \to D'$ such that $g'_1 f_1 = g'_2 f_2$. Moreover, $D' \in \overline{\mathscr{C}}$ because every finitely generated substructure of D' embeds into D'_n for some n, and $\mathscr{C}$ is closed for taking substructures.

The argument presented so far shows that any amalgam (A, B, C, f_1, f_2) with $A, C \in \mathscr{C}$ and $B \in \overline{\mathscr{C}}$ can be embedded into some $D' \in \overline{\mathscr{C}}$. Now starting with (A, B, C_0, f_1, f_2) and $D_0 = D'$, from this assertion we inductively construct the following diagram:

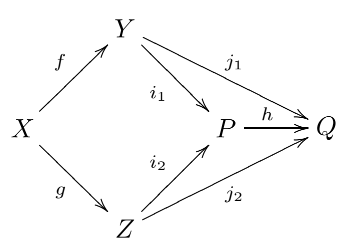


where all unlabelled arrows are embeddings. Again, by taking D to be the limit of $\{D_n : n \geqslant 0\}$ we obtain a structure which embeds the amalgam (A, B, C, f_1, f_2). $\square$

It would be very useful in what follows to fix a *canonical* way for embedding an amalgam (A, B, C, f_1, f_2) such that $A \in \mathscr{C}$ and $B, C \in \overline{\mathscr{C}}$ into a structure from $\overline{\mathscr{C}}$. Such possibility is provided by the standard categorical notion of the *pushout* (see [25] for a background in basic category theory). Recall that if $f : X \to Y$ and $g : X \to Z$ are two morphisms, then their pushout is an object P together with two morphisms $i_1 : Y \to P$ and $i_2 : Z \to P$ such that the following diagram commutes:



while for any object Q and morphisms $j_1 : Y \to Q$ and $j_2 : Z \to Q$ there exists a unique morphism $h : P \to Q$ such that $j_1 = hi_1$, $j_2 = hi_2$:



In concrete categories of structures we often consider the case when f, g are embeddings, whence in the presence of the AP the homomorphisms i_1, i_2 must be injective as well. Hence, P can be thought of as the "smallest" structure embedding the amalgam (X, Y, Z, f, g).

Accordingly, a structure P will be called the *amalgamated free sum* of Y and Z with respect to X if there exist embeddings $i_1 : Y \to P$ and $i_2 : Z \to P$ such that P (with i_1 and i_2) is the pushout of the amalgam (X, Y, Z, f, g). If so, we write $P = Y *_X Z$. It is easy to check that the amalgamated free sum, if it exists, is unique up to an isomorphism, and that it is generated by $i_1(Y) \cup i_2(Z)$. We will consider Fraïssé classes satisfying the following condition, which is a rather strong form of the AP:

(†) For any amalgam (A, B, C, f_1, f_2) such that $A \in \mathscr{C}$, $B, C \in \overline{\mathscr{C}}$, the amalgamated free sum $B *_A C$ exists and belongs to $\overline{\mathscr{C}}$.

For example, if $\overline{\mathscr{C}}$ belongs to a concrete category in which coproducts and coequilizers always exist, then (†) is automatically satisfied.

2.2 From amalgamated sums to the construction of Fraïssé limits

Let $\mathscr{C}$ be a Fraïssé class satisfying (†). Equipped with the construction of the amalgamated free sum, we first describe a particular extension $A^\star$ for an arbitrary structure $A \in \overline{\mathscr{C}}$. This is in fact a generalization of one of the standard constructions of the random graph described in [5] and an adaptation of the general approach from [21].

First of all, recall that a structure C is a *one-point extension* of its substructure B of there is an element $x \in C \setminus B$ such that C is generated by $B \cup \{x\}$. Trivially, if B is finitely generated, so is C.

Now let $\{(B_i, C_i) : \ i < \omega\}$ be the enumeration of all pairs of structures such that $B_i \in \mathscr{C}$ is a finitely generated substructure of A, while C_i is a one-point extension of B_i belonging to $\mathscr{C}$; for each isomorphism type we take one such extension. We construct a chain of structures A_i, $i \geqslant 0$, by successive amalgamations of these extensions. More precisely, let $A_0 = A$ and assume that A_n has already been constructed for some $n \geqslant 0$ such that $A \subseteq A_n \in \overline{\mathscr{C}}$. Then B_n is a substructure of A and so of A_n, whence $(B_n, A_n, C_n, \mathbf{1}_{B_n}, \mathbf{1}_{B_n})$ is an amalgam such that $B_n, C_n \in \mathscr{C}$ and $A_n \in \overline{\mathscr{C}}$. By Lemma 2.1, there exists a structure $A_{n+1} \in \overline{\mathscr{C}}$ which embeds this amalgam. Due to (†), we may eliminate unnecessary degrees of freedom and let $A_{n+1} = A_n *_{B_n} C_n$. Clearly, there is no loss of generality in assuming that $A_n \subseteq A_{n+1}$, so that A is a substructure of A_{n+1}. Finally, we let

$$A^\star = \bigcup_{n<\omega} A_n.$$

Of course, this construction can be iterated, so that we apply $\aleph_0$ successive rounds of amalgamation. Namely, let $A^{(0)} = A$ and define $A^{(n+1)} = (A^{(n)})^\star$ for all $n \geqslant 0$. We set

$$\mathfrak{F}(A) = \bigcup_{n<\omega} A^{(n)},$$

which is an extension of A. Clearly, any finitely generated substructure of $\mathfrak{F}(A)$ must belong to some $A^{(m)}$, and since $A^{(m)} \in \overline{\mathscr{C}}$ the finitely generated structure in question belongs to $\mathscr{C}$; hence, $\mathfrak{F}(A) \in \overline{\mathscr{C}}$.

Proposition 2.2 *For any $A \in \overline{\mathscr{C}}$, $\mathfrak{F}(A)$ is isomorphic to the Fraïssé limit of $\mathscr{C}$.*

Proof. This is a consequence of another well-known result of Fraïssé [14]: namely, it suffices to prove that $\mathfrak{F}(A)$ *realizes all one-point extensions in* $\mathscr{C}$ (effectively, this says that $\mathrm{Flim}(\mathscr{C})$ is the unique existentially closed structure in $\overline{\mathscr{C}}$). This means that for each finitely generated substructure B of $\mathfrak{F}(A)$ and its one-point extension $C \in \mathscr{C}$ there should be an embedding $f : C \to \mathfrak{F}(A)$ such that $f|_B = \mathbf{1}_B$.

However, this is very easy to check. Namely, as already remarked, B must be a substructure of $A^{(m)}$ for some $m \geqslant 0$. Hence, if $\{(B_i^{(m)}, C_i^{(m)}) : i < \omega\}$ is the enumeration of pairs of structures required for the construction of $A^{(m+1)} = (A^{(m)})^\star$, then $(B, C) = (B_j^{(m)}, C_j^{(m)})$ for some j. So, in the course of producing $A_{j+1}^{(m)}$ from $A_j^{(m)}$ we embed the amalgam $(B, A_j^{(m)}, C, \mathbf{1}_B, \mathbf{1}_B)$ into $A_{j+1}^{(m)}$, and so into $A^{(m+1)}$. The restriction of the latter embedding to C is precisely the required one. □

For example, it is straightforward to see that the amalgamated free sum of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ sharing a common induced subgraph on $V = V_1 \cap V_2$ is simply the graph on $V_1 \cup V_2$ whose edges are $E_1 \cup E_2$ (in a somewhat simplified form, one may say that the free sum of the amalgam is the amalgam itself). Therefore, for an arbitrary countable graph $G = (V, E)$, the graph $G^\star$ is obtained by adjoining a vertex u_A for each finite subset $A \subseteq V$ such that u_A is joined by an edge to $v \in V$ if and only if $v \in A$. By iterating this construction, we build a countable graph $R(G)$ "around" the initial graph G. As remarked in [4, 5], regardless of the choice of G, we always end up with $R(G) \cong R$, the countably infinite random graph.

3 Homomorphism-homogeneous Fraïssé limits

In an attempt to put the notion of ultrahomogeneity into a more general setting of arbitrary homomorphisms of first-order structures, Cameron and Nešetřil introduced in [8] the property of *homomorphism-homogeneity*. Namely, a structure A is homomorphism-homogeneous if any homomorphism $B \to C$ between its finitely generated substructures can be extended to an endomorphism of A. Recent results concerning characterizations of this property in various classes of structures include [7, 9, 22, 27, 35]. In this section we make a brief pause towards our aim to record a condition equivalent to homomorphism-homogeneity of a Fraïssé limit.

To this end, we introduce yet another property that a Fraïssé class $\mathscr{C}$ may or may not satisfy. We say that $\mathscr{C}$ satisfies the *one-point homomorphism extension property (1PHEP)* if for any $B, B', C \in \mathscr{C}$ such that C is a one-point extension of B, $C = \langle B \cup \{x\} \rangle$, any surjective homomorphism $\varphi : B \to B'$ can be extended to a homomorphism $\varphi' : C \to C'$ for some $C' \in \mathscr{C}$ containing B'. If we require that φ' is surjective as well, then it is clear that either $C' = B'$, or C' is a one-point extension of B', namely $C' = \langle B' \cup \{\varphi'(x)\} \rangle$.

Remark 3.1 It is quite easy to show that for any class of finitely generated structures of a fixed signature, the 1PHEP is equivalent to the seemingly more general *homo-amalgamation property (HAP)*, which, even though it is not explicitly formulated, transpires from the treatment in Section 4 of [8]. Namely, the HAP is the assertion that for any $A, B_1, B_2 \in \mathscr{C}$, any homomorphism $\varphi : A \to B_1$ and any embedding $f : A \to B_2$ there is a structure $D \in \mathscr{C}$, an embedding $f' : B_1 \to D$ and a homomorphism $\varphi' : B_2 \to D$ such that $f'\varphi = \varphi' f$. However, the more specific form of the 1PHEP might be slightly easier to check, as the following examples show.

Example 3.2 The class of all finite simple graphs has the 1PHEP. Indeed, let $\varphi : G \to H$ be a surjective graph homomorphism, and let G' be a graph obtained from G by adjoining a new vertex x (and some new edges involving x). Construct a new graph H' obtained by adjoining a new vertex x' to H, while for $v \in V(H)$ we set that $(x', v) \in E(H')$ if and only if $(x, u) \in E(G')$ for some $u \in V(G)$ such that $\varphi(u) = v$. Then it is easily verified that $\varphi' : G' \to H'$ obtained by extending φ by $\varphi'(x) = x'$ is a (surjective) graph homomorphism.

Example 3.3 We have already seen that the Fraïssé class of all K_n-free finite simple graphs fails to satisfy the 1PHEP: a bijection from the vertices of an anti-clique of size $n-1$ to a clique of the same size cannot be extended within the considered class to a vertex adjacent to all vertices of the anti-clique.

On the other hand, the "complementary" Fraïssé class to the above one, that of all $\overline{K_n}$-free finite simple graphs has the 1PHEP: it is quite straightforward to check that the construction from the previous example will work for this class as well.

Example 3.4 The class of all finite posets has the 1PHEP. To see this, let B be a finite poset, $C = B \cup \{x\}$ its one-point extension, and $\varphi : B \to B'$ an order-preserving map (a poset homomorphism). Let

$$L = \{b \in B : \ b < x\} \quad \text{and} \quad U = \{b \in B : \ x < b\}.$$

Since $\ell < u$ holds for any $\ell \in L$ and $u \in U$ we have $\varphi(\ell) \leqslant \varphi(u)$. So, $\varphi(L) \cap \varphi(U)$ is either empty, or a singleton. In the former case, define an extension C' of B' by "inserting" a new element y between $L' = \varphi(L)$ and $U' = \varphi(U)$; this is possible as $\ell' < u'$ holds for any $\ell' \in L'$, $u' \in U'$. It is now a routine to check that the mapping φ' such that $\varphi'|_B = \varphi$ and $\varphi'(x) = y$ is a poset homomorphism $C \to C'$. If, however, $\varphi(L) \cap \varphi(U) = \{x'\}$ then extend φ to $\varphi' : C \to B'$ by defining $\varphi'(x) = x'$; once again, φ' turns out to be a homomorphism.

Recall that *metric spaces* can be viewed as first-order structures over an uncountable language consisting of binary relational symbols indexed by the positive reals such that $(x, y) \in R_\alpha$ ($\alpha \in \mathbb{R}^+$) if and only if $d(x, y) < \alpha$. (Of course, we may as well restrict ourselves to metric spaces with rational distances, thus obtaining a countable signature for such structures.) From such a point of view, homomorphisms of metric spaces are just *non-expanding functions* φ so that we have

$$d(\varphi(x), \varphi(y)) \leqslant d(x, y)$$

for any x, y. Then, naturally, the notion of an automorphism coincides with that of an *isometry*, a distance-preserving permutation.

Lemma 3.5 *The class of finite metric spaces has the 1PHEP. The same applies to (the Fraïssé class of) finite metric spaces with rational distances.*

Proof. Let $\varphi : M \to M'$ be a surjective homomorphism of finite metric spaces, and let M_1 be a one-point extension of M, with y being the new point. Our aim is to prove that there exists a metric space $M_1' = M' \cup \{y'\}$, a one-point extension of M', such that for each $x \in M$ we have $d(y', \varphi(x)) \leqslant d(y, x)$. Then we can extend φ to a homomorphism $\widehat{\varphi} : M_1 \to M_1'$ by defining $\widehat{\varphi}(y) = y'$.

Let $M = \{x_i : i < n\}$, and for each $i, j < n$, $i \neq j$, denote $d_{ij} = d(x_i, x_j)$ and $a_i = d(y, x_i)$. First of all, we are going to consider the special case when φ, the initial homomorphism, is a bijection. Then denote $x'_i = \varphi(x_i) \in M'$ and $d'_{ij} = d(x'_i, x'_j) \leqslant d_{ij}$ for $i, j < n$, $i \neq j$. We are looking for a sequence of positive real numbers b_i, $i < n$, such that a (hypothetical) point y' with $d(y', x'_i) = b_i$ for all $i < n$ satisfies all the triangle inequalities with the already existing points of M'. In other words, the required conditions are:

(1) $b_i + b_j \geqslant d'_{ij}$,

(2) $b_i + d'_{ij} \geqslant b_j$

with $i, j < n$, $i \neq j$, in both cases. In addition, we need the third condition

(3) $b_i \leqslant a_i$ for all $i < n$.

There is no loss of generality in assuming that $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-1}$. Now consider the sequence defined by $b_0 = a_0$ and

$$b_i = \min_{0 \leqslant k < i} \{a_i, a_k + d'_{ki}\}$$

for $0 < i < n$. We claim that these numbers constitute a solution of the system of inequalities (1)–(3) above. Indeed, the condition (3) is immediately satisfied. For (1), we distinguish three subcases. If $b_i = a_k + d'_{ki}$ and $b_j = a_m + d'_{mj}$ for some $k < i$ and $m < j$, then

$$b_i + b_j = d'_{ik} + a_k + a_m + d'_{mj} \geqslant d'_{ik} + d_{km} + d'_{mj} \geqslant d'_{ik} + d'_{km} + d'_{mj} \geqslant d'_{ij},$$

since M, M' are metric spaces and φ is non-expanding. On the other hand, if $b_i = a_i$ and $b_j = a_j$, then $b_i + b_j = a_i + a_j \geqslant d_{ij} \geqslant d'_{ij}$. Finally, if $b_i = a_k + d'_{ki}$ for some $k < i$ and $b_j = a_j$ (the symmetric case is analogous), then

$$b_i + b_j = a_j + a_k + d'_{ki} \geqslant d_{jk} + d'_{ki} \geqslant d'_{jk} + d'_{ki} \geqslant d'_{ji} = d'_{ij}.$$

Concerning (2), assume first that $i < j$. Then if $b_i = a_i$ we have $b_i + d'_{ij} = a_i + d'_{ij} \leqslant b_j$ by the definition of b_j; if, however, $b_i = a_k + d'_{ki}$ for some $k < i$ then

$$b_i + d'_{ij} = a_k + d'_{ki} + d'_{ij} \geqslant a_k + d'_{kj} \geqslant b_j,$$

as $k < i < j$. So, it remains to discuss the possibility $i > j$. If $b_i = a_k + d'_{ki}$ for some $k < i$, and, in addition, we have $k < j$ as well, then again $b_i + d'_{ij} = a_k + d'_{ki} + d'_{ij} \geqslant a_k + d'_{kj} \geqslant b_j$. Otherwise, either $k \geqslant j$, or $b_i = a_i$, both cases implying $b_i \geqslant a_k \geqslant a_j \geqslant b_j$, so (2) holds. This completes the case when φ is injective, since M'_1 is obtained by adjoining a point y' to M' such that $d(y', x'_i) = b_i$ for all $i < n$.

Turning to the general case, when φ is not necessarily a bijection, for any $z \in M'$ choose a point $\mathbf{x}_z \in \varphi^{-1}(z) \subseteq M$ whose distance to y is minimal among all elements of $\varphi^{-1}(z)$ (that is, we have $d(y, \mathbf{x}_z) \leqslant d(y, x)$ for all $x \in M$ such that $\varphi(x) = z$). Let $M_0 = \{\mathbf{x}_z : z \in M'\}$. Now $\varphi|_{M_0} : M_0 \to M'$ is a bijective homomorphism of finite metric spaces, so by the previous considerations it follows that there is a one-point extension M'_1 of M' and a homomorphism $\psi : M_0 \cup \{y\} \to M'_1$ extending $\varphi|_{M_0}$. But then $\widehat{\varphi} = \psi \cup \varphi$ is the required extension of φ, since for any $x \in M$ we have

$$d(\widehat{\varphi}(y), \widehat{\varphi}(x)) = d(y', \varphi(x)) = d(\psi(y), \psi(\mathbf{x}_{\varphi(x)})) \leqslant d(y, \mathbf{x}_{\varphi(x)}) \leqslant d(y, x),$$

as wanted. It remains to note that if all a_i, d_{ij}, d'_{ij} are rational numbers, so are all b_i, thus the second part of the assertion follows, too. $\square$

The 1PHEP occurs in algebraic structures as well, where it is intimately related to the *congruence extension property (CEP)*, see [15]. Recall that an algebra A has the CEP if for any subalgebra B of A and any congruence ρ of B there exists a $\theta \in \mathrm{Con}\, A$ whose restriction to B is precisely ρ, that is, $\theta \cap (B \times B) = \rho$.

Lemma 3.6 *Let $\mathscr{C}$ be a class of algebras closed for taking homomorphic images. If all members of $\mathscr{C}$ have the CEP, then $\mathscr{C}$ has the 1PHEP.*

Proof. Let $C = \langle B \cup \{x\} \rangle$ and let $\varphi : B \to B'$ be a surjective homomorphism, where $B, B', C \in \mathscr{C}$. Then $\rho = \ker \varphi$ is a congruence of B (such that $B/\ker \varphi \cong B'$), so by the CEP there exists a congruence θ of C such that $\theta \cap (B \times B) = \ker \varphi$. Now consider the natural homomorphism $\nu_\theta : C \to C/\theta$. The image of B, $\nu_\theta(B)$, is isomorphic to $B/(\theta \cap (B \times B))$, which is by the given conditions $\cong B'$. Therefore, $C/\theta \in \mathscr{C}$ can be considered as an extension of B', whence ν_θ is an extension of φ. $\square$

By invoking the fact that the varieties of semilattices, distributive lattices, Boolean algebras and vector spaces over a given field $\mathbb{F}$ all possess the CEP, we obtain the following conclusion.

Corollary 3.7 *Each of the Fraïssé classes of all finite semilattices, all finite distributive lattices, all finite Boolean algebras and all finite-dimensional vector spaces over a field $\mathbb{F}$ have the 1PHEP.*

Now we provide a characterization of homomorphism-homogeneous Fraïssé limits. It reduces a property of the intricate structure of such a limit to a "local" property of finitely generated structures from $\mathscr{C}$ which usually have much more transparent features. A related result is contained in [8, Proposition 4.1].

Proposition 3.8 *Let $\mathscr{C}$ be a Fraïssé class. Then the Fraïssé limit of $\mathscr{C}$ is homomorphism-homogeneous if and only if $\mathscr{C}$ has the 1PHEP.*

Proof. Throughout the proof, let $F = \mathrm{Flim}(\mathscr{C})$.

($\Rightarrow$) Let $B, B', C \in \mathscr{C}$ be such that C is a one-point extension (or any extension of finite relative rank, for that matter) of B, and let $\xi : B \to B'$ be a surjective homomorphism. By the properties of the Fraïssé limit, there is no loss of generality if we assume that B, B', C are in fact substructures of F. However, by the given conditions then there is a $\widehat{\xi} \in \mathrm{End}(F)$ extending ξ, so that $\xi' = \widehat{\xi}|_C : C \to \widehat{\xi}(C)$ is the homomorphism required by the 1PHEP.

($\Leftarrow$) Let A be a finitely generated substructure of F, while $\varphi : A \to F$ is a homomorphism. Then, since F is countable, there exists a chain $\{F_i : \ i < \omega\}$ of finitely generated substructures of F such that $F_0 = A$, F_{i+1} is a one-point extension of F_i for each $i \geqslant 0$, and $F = \bigcup_{i<\omega} F_i$. We construct by induction a chain of homomorphisms $\varphi_i : F_i \to F$ starting with $\varphi_0 = \varphi$. By the 1PHEP, given φ_j for some $j \geqslant 0$, there exists a finitely generated structure $B'_{j+1} \in \mathscr{C}$, which is an extension of $B_j = \varphi_j(F_j)$, and a homomorphism $\psi_{j+1} : F_{j+1} \to B'_{j+1}$ that extends φ_j (i.e. $\psi_{j+1}|_{B_j} = \varphi_j$). Now since F is the Fraïssé limit of $\mathscr{C}$, there exists an embedding $f_{j+1} : B'_{j+1} \to F$ which is the identity mapping on B_j; define $B_{j+1} = f_{j+1}(B'_{j+1}) \subseteq F$. Whence, $\varphi_{j+1} = f_{j+1}\psi_{j+1}$ is a homomorphism $F_{j+1} \to F$ that extends φ_j. It remains to define

$$\widehat{\varphi} = \bigcup_{i<\omega} \varphi_i$$

to obtain an endomorphism of F that extends φ. □

By combining the previous proposition, Corollary 3.7, Lemma 3.5 and the examples that precede it, we arrive at the following result.

Corollary 3.9 *Each of the following Fraïssé limits is homomorphism-homogeneous: R, $\overline{H_n}$ for all $n \geqslant 3$, $\mathbb{U}_{\mathbb{Q}}$, $\mathbb{P}$, Ω, $\mathbb{D}$, $\mathbb{A}$, and $V_\infty^{\mathbb{F}}$, the $\aleph_0$-dimensional vector space over a fixed field $\mathbb{F}$.*

Remark 3.10 Based on Lemma 3.5 and an analogous approach as in Proposition 3.8, it is now quite easy to prove that the *Urysohn space* $\mathbb{U}$ [38], the completion of $\mathbb{U}_{\mathbb{Q}}$, is homomorphism-homogeneous as well. Namely, if $X \subseteq \mathbb{U}$ is a finite metric space and $\varphi : X \to \mathbb{U}$ is a homomorphism, then one can select a countable dense subspace $Y \subseteq \mathbb{U}$ isometric to $\mathbb{U}_{\mathbb{Q}}$ and use Lemma 3.5 to obtain a homomorphism $\varphi' : X \cup Y \to \mathbb{U}$ extending φ. Now it remains to remark that: (a) $\mathbb{U}$ is the completion of $X \cup Y$, and (b) every homomorphism of metric spaces is a uniformly continuous mapping (since it is in fact a Lipschitz function with constant 1), whence the properties of the completion of a metric space yield an endomorphism $\widehat{\varphi}$ of $\mathbb{U}$ extending φ.

4 Homomorphism extensions and the Bergman property

We start immediately with a condition ensuring that an instance of a Fraïssé limit—as constructed in Subsection 2.2—satisfies the condition (ii) from Theorem 1.2, related to the possibility of extending arbitrary homomorphisms (i.e. partial endomorphisms) into such a limit.

Theorem 4.1 *Let $\mathscr{C}$ be a Fraïssé class satisfying (†) and the 1PHEP. Then every homomorphism $\varphi : A \to \mathfrak{F}(A)$ can be extended to a $\widehat{\varphi} \in \mathrm{End}(\mathfrak{F}(A))$.*

Proof. Let $\varphi : A \to \mathfrak{F}(A)$ be any homomorphism. Our aim is to obtain a sequence of homomorphisms $\varphi^{(0)} = \varphi \subseteq \varphi^{(1)} \subseteq \varphi^{(2)} \subseteq \dots$, where $\varphi^{(n)} : A^{(n)} \to \mathfrak{F}(A)$, whence

$$\widehat{\varphi} = \bigcup_{n<\omega} \varphi^{(n)}$$

will be the desired endomorphism of $\mathfrak{F}(A)$. Therefore, we start with the assumption that the required sequence has already been constructed up to $\varphi^{(n)}$ for some $n \geqslant 0$.

In addition, recall that $A^{(n+1)}$ has been obtained from $A^{(n)}$ by successive amalgamations of all possible (up to isomorphism) one-point $\mathscr{C}$-extensions

$$\{(B_i^{(n)}, C_i^{(n)}) : \ i < \omega\}$$

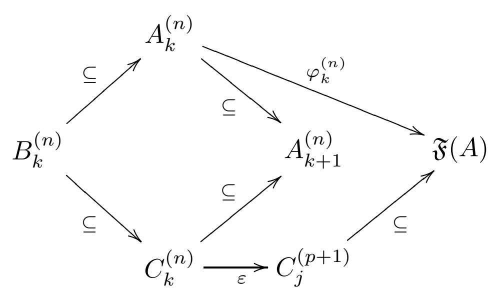
of finitely generated substructures of $A^{(n)}$. This results in a sequence of structures $A_0^{(n)} = A^{(n)} \subseteq A_1^{(n)} \subseteq \dots$ whose limit is $(A^{(n)})^\star = A^{(n+1)}$. Accordingly, we construct a tower of homomorphisms $\varphi_i^{(n)} : A_i^{(n)} \to \mathfrak{F}(A)$, $i \geqslant 0$, as follows, starting with $\varphi_0^{(n)} = \varphi^{(n)}$ and assuming that $\varphi_k^{(n)}$ has already been constructed.

Now, since $\mathscr{C}$ satisfies (†), we know that $A_{k+1}^{(n)}$ is obtained as the amalgamated free sum of

$$(B_k^{(n)}, A_k^{(n)}, C_k^{(n)}, \mathbf{1}_{B_k^{(n)}}, \mathbf{1}_{B_k^{(n)}}).$$

For brevity, denote $B' = \varphi^{(n)}(B_k^{(n)})$ and consider the homomorphism between finitely generated $\mathscr{C}$-structures $\phi = \varphi^{(n)}|_{B_k^{(n)}} : B_k^{(n)} \to B'$. Since B' is finitely generated, there exists an index $p < \omega$ such that $B' \subseteq A^{(p)}$. By the 1PHEP, there exist a structure $C' \in \mathscr{C}$—that is either B', or its one-point extension—and a surjective homomorphism $\varepsilon : C_k^{(n)} \to C'$ agreeing with $\varphi^{(n)}$ (that is, with ϕ) on $B_k^{(n)}$. Moreover, if $C' \neq B'$, then the extension (B', C') can be identified (up to isomorphism) with $(B_j^{(p+1)}, C_j^{(p+1)})$ for some j. In any case, we may assume that $\varepsilon(C_k^{(n)})$ is a (finitely generated) substructure of $A^{(p+1)}$.

What we have right now is depicted in the following diagram:



By (†) and the choice of $A_{k+1}^{(n)}$, there exist a homomorphism $\varphi_{k+1}^{(n)} : A_{k+1}^{(n)} \to \mathfrak{F}(A)$ completing the above diagram to a commutative one. In particular, $\varphi_{k+1}^{(n)}$ is an extension of $\varphi_k^{(n)}$. Finally,

$$\varphi^{(n+1)} = \bigcup_{i<\omega} \varphi_i^{(n)}$$

is a homomorphism $A^{(n+1)} \to \mathfrak{F}(A)$, and so we are done. □

Corollary 4.2 *If $\mathscr{C}$ is a Fraïssé class satisfying (†) and the 1PHEP, then for any $A \in \overline{\mathscr{C}}$ there exists a substructure A' of $F = \mathrm{Flim}(\mathscr{C})$ such that $A' \cong A$ and any homomorphism $\varphi : A' \to F$ can be extended to an endomorphism of F.*

We now get back to applying Lemma 1.1 and Theorem 1.2 to the Bergman property for endomorphism monoids of Fraïssé limits. Consider the following property related to Fraïssé classes, which is a close relative of (†):

(‡) For any countable family $\{A_i : i \in I\}$ of structures from $\overline{\mathscr{C}}$, their free sum $\coprod_{i\in I}^* A_i$ exists and belongs to $\overline{\mathscr{C}}$.

Corollary 4.3 *Let $\mathscr{C}$ be a Fraïssé class satisfying (†), (‡) and the 1PHEP. Then* $\mathrm{End}(F)$ *is strongly distorted and has Sierpiński index $\leqslant 3$, where $F = \mathrm{Flim}(\mathscr{C})$. If, in addition,* $\mathrm{End}(F)$ *is not finitely generated, then it has the Bergman property.*

Note that one may equivalently replace 'not finitely generated' in the above corollary by 'uncountable': indeed, if $\operatorname{End}(F)$ would be countable, then if would be necessarily finitely generated, because of the finite Sierpiński index. However, it is again Theorem 4.1 that admits to easily establish $|\operatorname{End}(F)| > \aleph_0$, $F = \operatorname{Flim}(\mathscr{C})$, for certain classes $\mathscr{C}$, as the latter inequality will follow from the existence of a structure $A \in \overline{\mathscr{C}}$ such that $|\operatorname{End}(A)| > \aleph_0$. We record the following remark, which is of independent interest as well. We say that a semigroup T *divides* a semigroup S if T is a homomorphic image of a subsemigroup of S.

Lemma 4.4 *Let $\mathscr{C}$ be a Fraïssé class with (†) and the 1PHEP, and let $F = \operatorname{Flim}(\mathscr{C})$. Then for any structure $A \in \overline{\mathscr{C}}$ we have that $\operatorname{End}(A)$ divides $\operatorname{End}(F)$.*

Proof. By Corollary 4.2 we have that F contains an isomorphic copy A' of A such that any endomorphism of A' can be extended to en endomorphism of F. Now consider only those endomorphisms f of F that induce (by restriction) an endomorphism of A', that is, $f(A') \subseteq A'$. Such endomorphisms form a subsemigroup S of $\operatorname{End}(F)$. Now for $f, g \in S$ let $(f, g) \in \rho$ if and only if $f|_{A'} = g|_{A'}$. It is easily seen that ρ is a congruence on S; by the given conditions, $S/\rho \cong \operatorname{End}(A') \cong \operatorname{End}(A)$. □

Corollary 4.5 *Let $\mathscr{C}$ be a Fraïssé class satisfying (†), (‡) and the 1PHEP. If there exists a structure $A \in \overline{\mathscr{C}}$ such that $\operatorname{End}(A)$ is uncountable, then $\operatorname{End}(F)$ has the Bergman property, where $F = \operatorname{Flim}(\mathscr{C})$.*

In particular, the endomorphism monoid of any of R, $\mathbb{P}$, Ω, $\mathbb{D}$, $\mathbb{A}$ and $V_\infty^{\mathbb{F}}$ has the Bergman property, and is divided by $\mathcal{T}_{\aleph_0}$.

Proof. For R, the countably infinite anti-clique works as A, since now any self-map is an endomorphism of A. Similarly, the countably infinite anti-chain A shows the required assertions about $\operatorname{End}(\mathbb{P})$. Finally, for limits of Fraïssé classes of algebras it suffices to note that any self-map on X, the set of free generators of the corresponding free algebra $F(X)$, induces an endomorphism of $F(X)$, whence we let X to be countably infinite. □

For various reasons, a number of Fraïssé classes and their corresponding limits remain outside the scope of this approach. As we have seen, some of them, such as the finite K_n-free simple graphs, fail to have the 1PHEP. Other classes, such as the finite $\overline{K_n}$-free simple graphs and the finite linear orders have the 1PHEP, and they even have (both amalgamated and free) sums in certain broader concrete categories (of simple graphs and posets, respectively), but these sums fail to be $\overline{K_n}$-free in the former case, or linearly ordered in the latter. Finally, some structures simply do not have coproducts and/or amalgamated free sums. For example, there seems to be no meaningful notion of a coproduct for (rational) metric spaces. This stems from the fact that when we are given a finite metric space M and we wish to add an a new point x, then the set of possible vectors of distances of x to the existing elements of M is in general an unbounded subset of $\mathbb{R}^m$, where $m = |M|$ (or $\mathbb{Q}^m$, if we go for rational distances), thus rendering impossible the choice of the "farthest point" from M—something which would be required should the coproduct of M and a singleton space exist. So, since the linear order $\mathbb{Q}$ and the universal rational metric space $\mathbb{U}_\mathbb{Q}$ are historically the 'oldest' examples of Fraïssé limits, it is natural to ask the following questions.

Problem 4.6 *Does the monoid of all order-preserving self-maps of $\mathbb{Q}$ have the Bergman property? More generally, what is the case with doubly homogeneous linear orders [10]?*

Problem 4.7 *Does the endomorphism monoid of* $\mathbb{U}_\mathbb{Q}$ *have the Bergman property? What about the monoid of all Lipschitz functions of* $\mathbb{U}_\mathbb{Q}$*?*

Problem 4.8 *Do the endomorphism monoids of ultrahomogeneous graphs* H_n *and* $\overline{H_n}$*,* $n \geqslant 3$*, have the Bergman property?*

Also, in connection with Corollary 4.3 above, the following tantalizing problem arises.

Problem 4.9 *Determine the Sierpiński index of* $\mathrm{End}(R)$ *exactly: is it 2 or 3? The same question applies to any Fraïssé limit under the scope of Corollary 4.3.*

5 Relative ranks

In this section we present an application of results from the previous section to *relative ranks* of endomorphism monoids of certain Fraïssé limits F within the full transformation monoid $\mathcal{T}_F$. Namely, for a semigroup S, the *rank* of S is just the minimum cardinality of a generating set of S. While this is an important invariant for countable semigroups, for any uncountable semigroup S its rank is $|S|$, so that the notion becomes vacuous. Instead, it is useful to consider the rank of S with respect to a distinguished subset $A \subseteq S$: the *relative rank of S modulo A*, denoted $\mathrm{rank}(S : A)$, is the minimum cardinality of $B \subseteq S$ such that $\langle A \cup B \rangle = S$. Alternatively, we say that $\mathrm{rank}(S : A)$ is the *relative rank of A in S*. Recently, this notion gained a considerable interest: while [20] can be viewed as a seminal paper in this vein, some newer important results are contained in [17, 18, 19, 29, 30].

Relative ranks are closely related to the notion of the Sierpiński index; namely, from the definition of the latter it is easy to deduce the following statement.

Proposition 5.1 *Let S be a semigroup with a finite Sierpiński index n, and let $A \subseteq S$. Then either* $\mathrm{rank}(S : A) \leqslant n$*, or* $\mathrm{rank}(S : A)$ *is uncountable.*

In particular, since the Sierpiński index of $\mathcal{T}_X$ is 2 for any infinite X, the following holds.

Corollary 5.2 *Let X be any infinite set. For any subset A of $\mathcal{T}_X$ either* $\mathrm{rank}(\mathcal{T}_X : A) \leqslant 2$*, or* $\mathrm{rank}(\mathcal{T}_X : A) > \aleph_0$*.*

For example, in [18] a full characterization was given of all infinite posets P such that $\mathrm{rank}(\mathcal{T}_P : \mathrm{End}(P)) \leqslant 2$. In addition, if P is a chain that is either countable, or well-ordered, then $\mathrm{rank}(\mathcal{T}_P : \mathrm{End}(P)) = 1$, see [19].

The previous corollary can now be combined with homomorphism extension results just obtained to show that for a number of Fraïssé limits F the first of the two possibilities takes place with $\mathrm{End}(F)$ in the role of A, that is, $\mathrm{rank}(\mathcal{T}_F : \mathrm{End}(F)) \leqslant 2$. The key link is the observation that Lemma 4.1 of [18] (and Lemma 4.10 of [19]) is not really a result about posets: it will work fine for other structures as well.

Lemma 5.3 *Let D be an infinite structure.*

(i) If A is a substructure of D such that $|A| = |D|$, $\mathrm{rank}(\mathcal{T}_A : \mathrm{End}(A)) \leqslant 2$*, and any endomorphism of A extends to an endomorphism of D, then* $\mathrm{rank}(\mathcal{T}_D : \mathrm{End}(D)) \leqslant 2$*.*

(ii) If there is a subset $X \subseteq D$, $|X| = |D|$, such that any mapping $X \to D$ can be extended to an endomorphism of D, then $\operatorname{rank}(\mathcal{T}_D : \operatorname{End}(D)) \leqslant 1$.

Proof. (i) The proof of Lemma 4.1 from [18] applies almost verbatim, but we reproduce it here for the sake of completeness and comparison to part (ii). So, assume that $\langle \operatorname{End}(A) \cup \{f, g\} \rangle = \mathcal{T}_A$ for some $f, g \in \mathcal{T}_A$. Let $\beta : D \to A$ be a bijection, and let $\alpha \in \mathcal{T}_D$ be arbitrary. Then $\beta\alpha\beta^{-1} \in \mathcal{T}_A$, thus we have

$$\beta\alpha\beta^{-1} = \gamma_1 \cdots \gamma_m,$$

where $m \geqslant 1$ and $\gamma_1, \ldots, \gamma_m \in \operatorname{End}(A) \cup \{f, g\}$. By the given conditions, for any $\gamma_i \in \operatorname{End}(A)$ there is a $\widehat{\gamma}_i \in \operatorname{End}(D)$ such that $\widehat{\gamma}_i|_A = \gamma_i$, while for $\gamma_i \in \{f, g\}$ we define $\widehat{\gamma}_i \in \{\widehat{f}, \widehat{g}\}$ to be any function $D \to D$ extending γ_i. By remarking that we have $(\widehat{\gamma}_1 \cdots \widehat{\gamma}_m)|_A = \gamma_1 \cdots \gamma_m$ it follows that

$$\alpha = \beta^{-1}\widehat{\gamma}_1 \cdots \widehat{\gamma}_m\beta.$$

It remains to define δ to be an arbitrary element of $\mathcal{T}_D$ such that $\delta|_A = \beta^{-1}$ to obtain $\alpha = \delta\widehat{\gamma}_1 \cdots \widehat{\gamma}_m\beta$ and so $\langle \operatorname{End}(D) \cup \{\widehat{f}, \widehat{g}, \beta, \delta\} \rangle = \mathcal{T}_D$. Since now $\operatorname{rank}(\mathcal{T}_D : \operatorname{End}(D)) \leqslant 4$, Corollary 5.2 implies $\operatorname{rank}(\mathcal{T}_D : \operatorname{End}(D)) \leqslant 2$.

(ii) Let $\beta : D \to X$ be a bijection and let $\alpha \in \mathcal{T}_D$ be arbitrary. As in (i), we have $\beta\alpha\beta^{-1} \in \mathcal{T}_X$. The given conditions imply that for any $f \in \mathcal{T}_X$ there is an endomorphism of D whose restriction to X is f; therefore, there is a $\varphi \in \operatorname{End}(D)$ such that $\varphi|_X = \beta\alpha\beta^{-1}$. This readily implies $\alpha = \beta^{-1}\varphi\beta$. Now $\beta^{-1} : X \to D$ extends to $\psi \in \operatorname{End}(D)$, so that $\alpha = \psi\varphi\beta$. Hence, $\langle \operatorname{End}(D) \cup \{\beta\} \rangle = \mathcal{T}_D$. □

As a direct consequence of the previous lemma and Corollary 4.2 we obtain the following result.

Corollary 5.4 *Let $\mathscr{C}$ be a Fraïssé class satisfying* (†) *and the 1PHEP, while* $F = \operatorname{Flim}(\mathscr{C})$.

(i) If there exists an infinite structure $A \in \overline{\mathscr{C}}$ such that $\operatorname{rank}(\mathcal{T}_A : \operatorname{End}(A)) \leqslant 2$*, then*

$$\operatorname{rank}(\mathcal{T}_F : \operatorname{End}(F)) \leqslant 2.$$

(ii) If there exists a structure $A \in \overline{\mathscr{C}}$ generated by a countably infinite set X such that any element of $\mathcal{T}_X$ extends to an endomorphism of A, then

$$\operatorname{rank}(\mathcal{T}_F : \operatorname{End}(F)) \leqslant 1.$$

In particular, if F is any of R, $\mathbb{P}$, Ω, $\mathbb{D}$, $\mathbb{A}$ and $V_\infty^{\mathbb{F}}$, then $\operatorname{rank}(\mathcal{T}_F : \operatorname{End}(F)) = 1$.

Remark 5.5 The conclusion of item (ii) holds also for some Fraïssé limits that fail to satisfy the assumptions of the previous corollary: for example, Theorem 2.1 of [19] shows that $\operatorname{rank}(\mathcal{T}_\mathbb{Q} : \operatorname{End}(\mathbb{Q})) = 1$.

Remark 5.6 There is alternative way to establish that $\operatorname{rank}(\mathcal{T}_F : \operatorname{End}(F)) = 1$ holds for a Fraïssé limit F, following from the combination of [3, Theorem 11] and [20, Theorem 4.1]. This combination works if the following conditions are met: (1) the pointwise stabilizer in $\operatorname{Aut}(F)$ of every finite subset of F has an infinite orbit; (2) there is an injective endomorphism of F with a coinfinite image; (3) there is a surjective endomorphism φ of F such that $\varphi^{-1}(x)$ is infinite for infinitely many $x \in F$. For example, it is known that these conditions hold if F is either the random graph R, or the linear order of $\mathbb{Q}$.

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