AUTOMORPHISMS OF MODULI SPACES OF SYMPLECTIC BUNDLES

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Abstract. Let X be an irreducible smooth complex projective curve of genus $g \geq 3$. Fix a line bundle L on X. Let $M_{\mathrm{Sp}}(L)$ be the moduli space of symplectic bundles $(E, \varphi : E \otimes E \to L)$ on X, with the symplectic form taking values in L. We show that the automorphism group of $M_{\mathrm{Sp}}(L)$ is generated by automorphisms of the form $E \longmapsto E \otimes M$, where $M^2 \cong \mathcal{O}_X$, and automorphisms induced by automorphisms of X.

1. Introduction

Let X be a smooth complex projective curve of genus g, with $g \geq 3$. A set of generators of the automorphism group of the moduli space of semistable vector bundles over X of rank r with fixed determinant L was obtained by Kouvidakis and Pantev in [KP]. More precisely, they proved that the automorphism group is generated by the automorphisms of X, automorphisms of the form $E \longmapsto E \otimes M$, where M is a line bundle with $M^{\otimes r} \cong \mathcal{O}_X$, and, if r divides $2 \deg L$, automorphisms of the form $E \longmapsto E^{\vee} \otimes N$, where N is a line bundle with $N^{\otimes r} \cong L^{\otimes 2}$. In the same paper they prove a Torelli theorem for these moduli space. The proofs of their results crucially use the Hitchin map defined on the moduli of Higgs bundles.

In [HR], Hwang and Ramanan gave different proof of the above results using Hecke curves, which are minimal rational curves constructed using Hecke transformations.

Fix a holomorphic line bundle L on X, and consider the moduli space $M_{\mathrm{Sp}}(L)$ of stable symplectic bundles $(E, \varphi : E \otimes E \longrightarrow L)$ of rank $2n$ and with values in L. Take a line bundle M on X with $M^{\otimes 2} \cong \mathcal{O}_X$. Fix an isomorphism $\beta : M^{\otimes 2} \longrightarrow \mathcal{O}_X$. Then we have an automorphism of $M_{\mathrm{Sp}}(L)$ defined by $(E, \varphi) \longmapsto (E \otimes M, \varphi \otimes \beta)$.

More generally, let $\sigma : X \longrightarrow X$ be an automorphism and let M be a line bundle on X such that $M^{\otimes 2} \cong L \otimes (\sigma^* L)^{\vee}$. Fix an isomorphism β as above. Then $(E, \varphi) \longmapsto (M \otimes \sigma^* E, \beta \otimes \sigma^* \varphi)$ is an automorphism of $M_{\mathrm{Sp}}(L)$. We remark that, in both cases, the automorphism does not depend on the choice of β.

In Theorem 6.3 we show that these are all the automorphisms of $M_{\mathrm{Sp}}(L)$. More precisely, the automorphism group $\mathrm{Aut}(M_{\mathrm{Sp}}(L))$ fits in a short exact sequence of groups

$$e \longrightarrow J(X)_2 \longrightarrow \mathrm{Aut}(M_{\mathrm{Sp}}(L)) \longrightarrow \mathrm{Aut}(X) \longrightarrow e\,,$$

where $J(X)_2$ is the group of line bundles on X of order two (see Proposition 6.5).

We also prove a Torelli type theorem for this moduli space (Theorem 4.3). This was proved earlier in [BH] by a different method.

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2. Moduli space of symplectic bundles

Let

$$J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$$

be the standard symplectic form on $\mathbb{C}^{2n}$. Define the group

$$\mathrm{Gp}(2n,\mathbb{C}) = \left\{A \in \mathrm{GL}(2n,\mathbb{C}) \,:\, A^t J A = cJ \text{ for some } c \in \mathbb{C}^*\right\}. \tag{2.1}$$

It is an extension of $\mathbb{C}^*$ by the symplectic group $\mathrm{Sp}(2n,\mathbb{C})$

$$e \longrightarrow \mathrm{Sp}(2n,\mathbb{C}) \longrightarrow \mathrm{Gp}(2n,\mathbb{C}) \xrightarrow{q} \mathbb{C}^* \longrightarrow e\,,$$

where $q(A) = c$ for any A and c as in (2.1). From the definition of the homomorphism q it follows immediately that for all $A \in \mathrm{Gp}(2n,\mathbb{C})$,

$$\det A = q(A)^n\,. \tag{2.2}$$

Let X be an irreducible smooth complex projective curve of genus g, with $g \geq 3$. A symplectic bundle on X of rank $2n$ with values in a holomorphic line bundle L is a pair (E,φ), where E is a holomorphic vector bundle of rank $2n$ and

$$\varphi \,:\, E\bigwedge E \longrightarrow L$$

is a homomorphism of coherent sheaves which is fiberwise nondegenerate. The line bundle $\det(E)$ is canonically a direct summand of $(E\bigwedge E)^{\otimes n}$, and the composition

$$\det(E) \hookrightarrow (E\bigwedge E)^{\otimes n} \xrightarrow{\varphi^{\otimes n}} L^{\otimes n}$$

is an isomorphism. Giving a symplectic bundle is equivalent to giving a principal $\mathrm{Gp}(2n,\mathbb{C})$-bundle.

Let (E,φ) be a symplectic bundle. A holomorphic subbundle F of E is called *isotropic* if $\varphi(F\bigwedge F) = 0$.

A symplectic bundle (E,φ) is called *stable* (respectively, *semistable*) if, for all isotropic proper subbundles $E' \subset E$ of positive rank,

$$\frac{\deg E'}{\mathrm{rk}\, E'} < \frac{\deg E}{\mathrm{rk}\, E} \quad \left(\text{respectively, } \frac{\deg E'}{\mathrm{rk}\, E'} \leq \frac{\deg E}{\mathrm{rk}\, E}\right)$$

See [BG] for more on symplectic bundles.

We denote by $M_{\mathrm{Sp}}(L)$ the moduli space of stable symplectic bundles with values in a fixed line bundle L.

Lemma 2.1. *Assume that* $\deg L \leq 2(g-1)$. *Then* $H^0(X,E) = 0$ *for a general stable bundle* $(E,\varphi) \in M_{\mathrm{Sp}}(L)$.

Proof. Using Riemann–Roch, $\dim M_{\mathrm{Sp}}(L) = n(2n+1)(g-1)$. By semicontinuity,

$$\left\{(E,\varphi) \in M_{\mathrm{Sp}}(L) \mid H^0(X,E) \neq 0\right\} \subset M_{\mathrm{Sp}}(L)$$

is a Zariski closed subset. The lemma will be proved by showing that the codimension of this subset is positive.

Take a pair $((E,\varphi)\,,s)$ such that $(E,\varphi) \in M_{\mathrm{Sp}}(L)$ and $s \in H^0(X,E)\setminus\{0\}$. It defines a short exact sequence

$$0 \longrightarrow M = \mathcal{O}_X(D) \xrightarrow{s} E \longrightarrow Q \longrightarrow 0\,, \tag{2.3}$$

where D is the effective divisor defined by s. Let K be the kernel of the composition

$$E \xrightarrow{\varphi} E^\vee \otimes L \longrightarrow M^\vee \otimes L\,.$$

Define $Q := E/M$. Since $\varphi(M \otimes K) = 0$, it follows that φ defines a pairing

$$Q \otimes K \longrightarrow L\,.$$

This pairing is perfect because φ is pointwise nondegenerate. In particular, $Q \cong K^\vee \otimes L$. We have a diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & F & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & M^\vee \otimes L & = & M^\vee \otimes L & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & , \end{array} \tag{2.4}$$

and there is a symplectic form $\omega_F : F \otimes F \longrightarrow L$ induced by φ (recall that $\varphi(M \otimes K) = 0$).

Note that under the homomorphism

$$\mathrm{Ext}^1(Q, M) \longrightarrow \mathrm{Ext}^1(F, M) = \mathrm{Ext}^1(M^\vee \otimes L, F)\,,$$

the class $\xi_1 \in \mathrm{Ext}^1(Q, M)$ for the bottom exact sequence in (2.4) maps to the class $\xi_2 \in \mathrm{Ext}^1(M^\vee \otimes L, F)$ for the vertical exact sequence in the right of (2.4). For a general $(E', \varphi') \in M_{\mathrm{Sp}}(L)$, the underlying vector bundle E' is stable. If E is a stable vector bundle, then $\mathrm{Hom}(Q, M) = 0$, because any nonzero homomorphism from Q to M produces a nilpotent endomorphism of E.

Let $\deg M = \ell$ and $\deg E = n \cdot \deg L = d$. If $\mathrm{Hom}(Q, M) = 0$, then

$$\dim \mathrm{Ext}^1(Q, M) = -\deg(Q^\vee \otimes M) + (2n-1)(g-1) = d - 2n\ell + (2n-1)(g-1)\,. \tag{2.5}$$

Now let us see that the symplectic form $\omega_F : F \otimes F \longrightarrow L$ determines the symplectic form on E. First, ω_F extends uniquely to a homomorphism $F \otimes K \longrightarrow L$, which extends naturally to a homomorphism $Q \otimes K \longrightarrow L$; both these extensions are consequences of the fact that $\varphi(M \otimes K) = 0$. Any two extensions of the pairing $F \otimes K \longrightarrow L$ to a pairing $Q \otimes K \longrightarrow L$ differ by a section contained in $\mathrm{Hom}((Q/F) \otimes K, L) = \mathrm{Hom}((M^\vee \otimes L) \otimes K, L) = \mathrm{Hom}(K, M)$.

We will show that

$$\mathrm{Hom}(K, M) = 0\,. \tag{2.6}$$

First, if a homomorphism $K \longrightarrow M$ composed with $M \longrightarrow K$ is non-zero, then it produces a splitting of the short exact sequence

$$0 \longrightarrow M \longrightarrow K \longrightarrow F \longrightarrow 0\,.$$

So the extension $F \longrightarrow Q \longrightarrow M^\vee \otimes L$ is split. Therefore there are maps $Q \longrightarrow F$ and $F \longrightarrow K$, which composed with $K \longrightarrow E$ splits the diagram $Q \longrightarrow E$, but this is not possible since E is stable. So the homomorphism $K \longrightarrow M$ composed with $M \longrightarrow K$ is the zero homomorphism. But then the homomorphism $K \longrightarrow M$ descends to a homomorphism $F \longrightarrow M$. Let S_1 (respectively, S_2) be the kernel of $K \longrightarrow M$ (respectively, of $F \longrightarrow M$). Then there is an exact sequence

$$0 \longrightarrow M \longrightarrow S_1 \longrightarrow S_2 \longrightarrow 0\,.$$

So it follows that $\deg F \leq \deg S_1$. As $S_1 \subset K \subset E$, and E is a stable bundle, then $\mu(S_1) < \mu(E)$. So

$$\frac{\deg L}{2} = \mu(F) \leq \mu(S_1) < \mu(E) = \frac{\deg L}{2}\,,$$

which is a contradiction. Therefore, (2.6) is proved.

Now the homomorphism $Q \otimes K \longrightarrow L$ extends uniquely to a map $E \otimes K \longrightarrow L$. This again extends to the map $\omega_E : E \otimes E \longrightarrow L$, up to an indeterminacy contained in $\mathrm{Hom}(E \otimes (E/K), L) = \mathrm{Hom}(E \otimes (M^\vee \otimes L), L)$, which actually lives in the subspace $\mathrm{Hom}((E/M) \otimes (M^\vee \otimes L), L) = \mathrm{Hom}(Q, M) = 0$.

Then the dimension of the family of bundles parametrizing (2.3) is

$$\begin{aligned}&(2n-1)(n-1)(g-1) + d - 2n\ell + (2n-1)(g-1) - 1\\ &\leq (2n+1)n(g-1) + d - 2n(g-1) - 1 < (2n+1)n(g-1),\end{aligned}$$

for $d \leq 2n(g-1)$. This completes the proof of the lemma. $\square$

For a symplectic bundle (E, φ), let

$$\mathrm{End}_{\mathrm{Sp}}(E) \;:=\; \mathrm{Sym}^2(E) \otimes L^\vee \;\subset\; \mathrm{End}(E) \;=\; E \otimes E \otimes L^\vee$$

be the set consisting of symmetric symplectic endomorphisms of E.

Lemma 2.2. *Let D be an effective divisor of degree ℓ, with $g \geq \max\{2\ell, \ell+2\}$. Then $H^0(\mathrm{End}_{\mathrm{Sp}}\, E(D)) = 0$ for a general stable symplectic bundle $(E, \varphi) \in M_{\mathrm{Sp}}(L)$.*

Proof. By tensoring with a suitable line bundle, we may assume that L has degree $\epsilon \in \{0, 1\}$. This makes the slope of any symplectic bundle to be $\frac{\epsilon}{2} < g - 1$.

Moreover, we may assume that L is generic in the sense that

$$H^0(L(D)) = 0 \quad \text{and} \quad H^0(L^*(D)) = 0\,,$$

for any D effective divisor of degree $\ell \leq g - 1 - \epsilon$. Let $X^{(\ell)} = \mathrm{Sym}^\ell(X)$ be the set of effective divisors of degree ℓ.

For $r = 1$, consider any stable vector bundle F of rank two with determinant L. It has a symplectic structure:

$$\omega : F \otimes F \longrightarrow \wedge^2 F = L\,.$$

The symplectic bundle (F, ω) is automatically stable. However, we are going to construct an specific bundle F_0 for later use. Consider an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow F_0 \longrightarrow L \longrightarrow 0\,. \tag{2.7}$$

These extensions are parametrized by elements in $H^1(L^*)$, and $\dim H^1(L^*) = g - 1 + \epsilon$. Consider an effective divisor $D \in X^{(\ell)}$. Then the exact sequence

$$0 \longrightarrow L^* \longrightarrow L^*(D) \longrightarrow L^*(D)|_D \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow L^*(D)|_D \longrightarrow H^1(L^*) \longrightarrow H^1(L^*(D)) \longrightarrow 0\,,$$

so we get a subspace $V_D := L^*(D)|_D \subset H^1(L^*)$ of dimension ℓ. Moving D over $X^{(\ell)}$, we see that if $g - 1 + \epsilon > 2\ell$, then there is an extension (2.7) whose class $\xi \in H^1(L^*)$ goes to a non-zero element under the homomorphism $H^1(L^*) \longrightarrow H^1(L^*(D))$ for any $D \in X^{(\ell)}$. Now the connecting homomorphism $H^0(\mathcal{O}(D)) \longrightarrow H^1(L^*(D))$ for the dual sequence

$$0 \longrightarrow L^*(D) \longrightarrow F_0^*(D) \longrightarrow \mathcal{O}(D) \longrightarrow 0 \tag{2.8}$$

of (2.7), which is multiplication by ξ, is injective; indeed, if a section $s \in H^0(\mathcal{O}(D))$, defining a divisor D', maps to zero, then the extension class ξ goes to zero under the homomorphism $H^1(L^*) \longrightarrow H^1(L^*(D'))$, but this is not the case by construction. This implies that

$$H^0(L^*(D)) = H^0(L^* \otimes F_0(D)),$$

which is zero by assumption. Also the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}(D)) \longrightarrow H^0(F_0(D)) \longrightarrow H^0(L(D)) = 0$$

implies that $H^0(F_0(D)) = H^0(\mathcal{O}(D))$. And finally, the exact sequence

$$\mathrm{Hom}(L, F_0(D)) = 0 \longrightarrow \mathrm{Hom}(F_0, F_0(D)) \longrightarrow \mathrm{Hom}(\mathcal{O}, F_0(D)) = H^0(\mathcal{O}(D))$$

gives that $H^0(\mathrm{End}\, F_0(D)) = H^0(\mathcal{O}(D))$. Hence $H^0(\mathrm{End}_0\, F_0(D)) = 0$, where End_0 denotes the space of trace-free endomorphisms. Note that $\mathrm{End}_{\mathrm{Sp}}\, F_0 = \mathrm{End}_0\, F_0$, so $H^0(\mathrm{End}_{\mathrm{Sp}}\, F_0(D)) = 0$.

Now for $r > 1$, consider a general symplectic bundle (F_1, ω_1) of rank $2r-2$. By induction hypothesis, $H^0(\mathrm{End}_{\mathrm{Sp}}\, F_1(D)) = 0$, for any effective divisor D of degree ℓ. Consider the symplectic bundle $E = F_0 \oplus F_1$. This is a symplectic semistable bundle of rank $2r$. Let us see that

$$H^0(\mathrm{End}_{\mathrm{Sp}}\, E(D)) = 0 \,. \tag{2.9}$$

This would imply that also for a general stable bundle $\widetilde{E}$, we have that

$$H^0(\mathrm{End}_{\mathrm{Sp}}\, \widetilde{E}(D)) = 0$$

for all $D \in X^{(\ell)}$ (note that $X^{(\ell)}$ is a complete variety).

The vector space $H^0(\mathrm{End}_{\mathrm{Sp}}\, E(D))$ has four components:

- $H^0(\mathrm{End}_{\mathrm{Sp}}\, F_0(D)) = 0$, by construction.
- $H^0(\mathrm{End}_{\mathrm{Sp}}\, F_1(D)) = 0$, by induction hypothesis.
- $H^0(\mathrm{Hom}_{\mathrm{Sp}}(F_0, F_1(D))) = 0$. A homomorphism $\varphi : F_0 \longrightarrow F_1(D)$ can be restricted to $\mathcal{O} \subset F_0$, so it defines a section in $H^0(F_1(D))$. By Lemma 2.1, this is zero (as $\mu(F_1(D)) = \frac{\epsilon}{2} + \ell < g - 1$). So φ defines a section of the quotient $L \longrightarrow F_1(D)$, i.e., a section of $H^0(L^* \otimes F_1(D))$, which is also zero (L is fixed, so can take both F_1 and $F_1 \otimes L^*$ to be simultaneously generic).
- $H^0(\mathrm{Hom}_{\mathrm{Sp}}(F_1, F_0(D))) = 0$. A homomorphism $\varphi : F_1 \longrightarrow F_0(D)$ gives a homomorphism $F_0^\vee = F_0 \otimes L^{-1} \longrightarrow F_1^\vee(D) = F_1 \otimes L^{-1}(D)$, i.e., a symplectic map $F_0 \to F_1(D)$, which is zero as above.

This completes the proof of the lemma. □

3. Hitchin discriminant

Let us recall the definition of the Hitchin map (see [Hi, Section 5.10]). A symplectic Higgs bundle is a triple (E, ω, θ), where (E, ω) is a symplectic bundle and $\theta : E \longrightarrow E \otimes K_X$ is a symmetric map with respect to ω:

$$\omega(u, \theta(v)) = -\omega(\theta(u), v)$$

for $u, v \in E_x$, $x \in X$.

Let $\mathcal{M}_{\mathrm{Sp}}(L)$ be the moduli space of symplectic Higgs bundles of rank $2n$.

As before, $M_{\mathrm{Sp}}(L)$ is the moduli space of symplectic bundles. The cotangent bundle $T^*M_{\mathrm{Sp}}(L) \subset \mathcal{M}_{\mathrm{Sp}}(L)$ is an open subset. Consider the affine space:

$$W = H^0(X, K_X^2) \oplus \ldots \oplus H^0(X, K_X^{2n}),$$

and the *Hitchin map on* $T^*M_{\mathrm{Sp}}(L)$

$$h : T^*M_{\mathrm{Sp}}(L) \longrightarrow W\,,$$

defined by $h(\theta) = (s_2(\theta), \ldots, s_{2n}(\theta))$, $s_i(\theta) = \mathrm{tr}(\wedge^i\theta)$, for

$$\theta \in T^*_E M_{\mathrm{Sp}}(L) = H^0(X, \mathrm{End}_{\mathrm{Sp}}(E) \otimes K_X).$$

This extends to the *Hitchin map on* $\mathcal{M}_{\mathrm{Sp}}(L)$, the moduli space of semistable symplectic Higgs bundles,

$$H : \mathcal{M}_{\mathrm{Sp}}(L) \longrightarrow W\,.$$

For an element $s = (s_2, \ldots, s_{2n}) \in W$, the *spectral curve* X_s associated to s is the curve in the total space $\mathbb{V}(K_X)$ of K_X defined by the equation

$$y^{2n} + s_2(x)y^{2n-2} + \ldots + s_{2n-2}(x)y^2 + s_{2n}(x) = 0 \tag{3.1}$$

(x is a coordinate for X, and y is the tautological coordinate dx along the fibers of the projection $\mathbb{V}(K_X) \longrightarrow X$).

Consider the compactification

$$S := \mathbb{P}(\mathcal{O}_X \oplus K_X) \,\supset\, \mathbb{V}(K_X)\,.$$

Let $p : S \longrightarrow X$ be the projection. Giving a Higgs bundle $(E, \theta : E \to E \otimes K_X)$ is equivalent to giving a coherent sheaf A of rank one supported on some spectral curve $S \subset \mathbb{V}(K_X)$. Indeed, $E = p_*A$, and the Higgs bundle θ corresponds to multiplication with the tautological coordinate of $\mathbb{V}(K_X)$ on the $\mathcal{O}_{\mathbb{V}(K_X)}$-module structure of A. The support of A is given by the equation (3.1). For more details, see [Hi], [BNR] and [Si].

The symplectic bundle structure $\omega : E \otimes E \longrightarrow L$ corresponds to an isomorphism

$$\sigma^*A \xrightarrow{\cong} Ext^1(A, K_S \otimes p^*K_X) \otimes p^*L$$

where $\sigma : S \longrightarrow S$ is the involution $y \longmapsto -y$ (note that the spectral curve is invariant under this involution because all the exponents of y in (3.1) are even integers). Indeed, applying p_* to this isomorphism we obtain the symplectic structure:

$$p_*\sigma^*A = E \longrightarrow p_*(Ext^1(A, K_S \otimes p^*K_X^{-1})) \otimes L = E^\vee \otimes L\,.$$

The second equality is proved in two steps. There is a spectral sequence

$$R^ip_*Ext^j(\cdot,\cdot) \Rightarrow Ext_p^{i+j}(\cdot,\cdot)\,,$$

and since A has support of dimension 1, we obtain

$$p_*(Ext^1(A, K_S \otimes p^*K_X^{-1})) = Ext^1_p(A, K_S \otimes p^*K_X^{-1})\,,$$

and the relative Serre duality for the projective morphism p gives

$$Ext^1_p(A, K_S \otimes p^*K_X^{-1}) = p_*(A)^\vee = E^\vee\,.$$

We can think of A as a sheaf on the spectral curve X_s. If this is integral, then A is torsionfree as a sheaf on X_s, and then

$$Ext^1(A, K_S \otimes p^*K_X) \,=\, A^\vee \otimes K_{X_s} \otimes \pi^*K_X^{-1}\,,$$

where $\pi : X_s \longrightarrow X$ is the projection. For an arbitrary coherent sheaf A on S supported on X_s, $A^\vee$ is defined to be $Ext^1(A, K_S) \otimes Ext^1(\mathcal{O}_{X_s}, K_S)^\vee$. If A is locally free on X_s, then this is the usual dual line bundle on X_s.

Fix once and for all a square root $R = (K_{X_s} \otimes \pi^* K_X^{-1} \otimes \pi^* L)^{1/2}$. If we denote $U = A \otimes R$, then $\sigma^* U \cong U^\vee$. In other words, U is an element of the Prym subvariety of the compactified Jacobian $\overline{J}(X_s)$

$$\mathrm{Prym}(X_s, \sigma) = \left\{ U \in \overline{J}(X_s) : \sigma^* U \cong U^\vee \right\},$$

and, conversely, an element of this Prym produces a symplectic Higgs bundle whose spectral curve is X_s. Therefore, the fiber of H over $s \in W$ is isomorphic to $\mathrm{Prym}(X_s, \sigma)$, and the isomorphism depends only on the choice of square root R. The dimension of this Prym variety is

$$g(X_s) - g(X_s/\sigma) = n(2n+1)(g(X)-1) = \dim \mathrm{Sp}(2n)(g-1)\,.$$

Let Y be an integral curve whose only singularity is one simple node at a point y. Let

$$\pi_Y : \widetilde{Y} \longrightarrow Y$$

be the normalization, and let x and z be the preimages of y. The compactified Jacobian $\overline{J}(Y)$, parametrizing torsionfree sheaves of rank 1 and degree 0 on Y, is birational to a $\mathbb{P}^1$-fibration P over $J(\widetilde{Y})$, whose fiber over $L \in J(\widetilde{Y})$ is $\mathbb{P}^1(L_x \oplus L_z)$. The morphism $P \longrightarrow \overline{J}(Y)$ is constructed as follows. A point of P corresponds to a line bundle L on $\widetilde{Y}$ and a one dimensional quotient $q : L_x \oplus L_z \twoheadrightarrow \mathbb{C}$ (up to scalar multiple). This is sent to the torsionfree sheaf L' on Y defined as

$$0 \longrightarrow L' \longrightarrow (\pi_Y)_* L \xrightarrow{q} \mathbb{C}_y \longrightarrow 0\,.$$

For the proof, see [Bh, Theorem 4].

Assume that Y has an involution σ. It lifts to an involution $\widetilde{\sigma}$ of $\widetilde{Y}$. This induces an involution in P. Indeed, if (L, q) is a point, and $q : L_x \oplus L_z \twoheadrightarrow \mathbb{C}$ is represented by $[a : b]$, then this point is sent to $(\widetilde{\sigma}^* L^\vee, q^\vee := [b : a])$. Note that the definition of $q^\vee$ makes sense: if $[a : b] \in \mathbb{P}(L_x \oplus L_z)$, then

$$[b : a] \in \mathbb{P}(L_z \oplus L_x) = \mathbb{P}(L_x^\vee \otimes L_z^\vee \otimes (L_z \oplus L_x)) = \mathbb{P}(L_x^\vee \oplus L_z^\vee)\,.$$

The involution on P induces an involution in $\overline{J}(Y)$, which restricts to $A \longmapsto \sigma^* A^\vee$ on the open subset $J(Y) \subset \overline{J}(Y)$ corresponding to line bundles. The fixed point variety of this involution is the Prym variety

$$\mathrm{Prym}(Y, \sigma) \subset \overline{J}(Y)\,.$$

It is a uniruled variety, because it has a surjective morphism from the $\mathbb{P}^1$ fibration $P|_{\mathrm{Prym}}$ defined by the pullback

$$\begin{tikzcd} P|_{\mathrm{Prym}} \arrow[r] \arrow[d] & P \arrow[d] \\ \mathrm{Prym}(\widetilde{Y}, \widetilde{\sigma}) \arrow[r] & J(\widetilde{Y}) \end{tikzcd}$$

Analogously, if Y is an integral curve with two simple nodes, and $\widetilde{Y}$ is the normalization, then $\overline{J}(Y)$ is birational to a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle P on $J(\widetilde{Y})$. If the nodes are called y_1 and y_2, then the two $\mathbb{P}^1$-factors in the Cartesian product correspond to one dimensional quotients $q_1 : L_{x_1} \oplus L_{z_1} \twoheadrightarrow \mathbb{C}$ and $q_2 : L_{x_2} \oplus L_{z_2} \twoheadrightarrow \mathbb{C}$.

Let σ be an involution of Y interchanging both nodes. It lifts to an involution of $\widetilde{Y}$, and also to an involution of P, sending (L, q_1, q_2) to $(\widetilde{\sigma}^* L^\vee, q_2^\vee, q_1^\vee)$, and this induces an involution of $\overline{J}(Y)$. A fixed point in P of this involution has $L \cong \widetilde{\sigma}^* L^\vee$

and $q_2 = q_1^\vee$, hence it is a $\mathbb{P}^1$-fibration on $\mathrm{Prym}(\widetilde{Y})$, and the image of this map is the fixed point locus on $\overline{J}(Y)$, which is denoted by $\mathrm{Prym}(Y,\sigma)$. We again obtain that this Prym is a uniruled variety.

Consider

$$\mathcal{D} \subset W$$

the divisor consisting of characteristic polynomials with singular spectral curves. This has two components

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \,,$$

where $\mathcal{D}_1$ consists of those curves for which s_{2n} has a double root (then (3.1) has a node at the horizontal axis), and $\mathcal{D}_2$ consists of curves with two symmetrical nodes (i.e., $y^n + s_2(x)y^{n-1} + \ldots + s_{2n-2}(x)y + s_{2n}(x) = 0$ has a node). Let $\mathcal{D}_i^o \subset \mathcal{D}_i$, $i = 1, 2$, be the locus of all those curves that do not contain extra singularities. Finally let $\mathcal{D}^* = \mathcal{D} - (\mathcal{D}_1^o \cup \mathcal{D}_2^o)$.

Proposition 3.1. *As before, $h : T^*M_{\mathrm{Sp}}(L) \longrightarrow W$ is the Hitchin map. The following statements hold:*

(1) *For $w \in W - \mathcal{D}$, the fiber $h^{-1}(w)$ is an open subset of an abelian variety (actually a Prym variety).*
(2) *For $w \in \mathcal{D}_1^o$, the fiber $h^{-1}(w)$ is an open subset of the uniruled variety $\mathrm{Prym}(X_w, \sigma)$.*
(3) *For $w \in \mathcal{D}_2^o$, the fiber $h^{-1}(w)$ is an open subset of the uniruled variety $\mathrm{Prym}(X_w, \sigma)$.*

The complement of the open subsets in each of the cases is of codimension at least 2 *(at least for generic w in the corresponding set).*

Proof. The map $H : \mathcal{M}_{\mathrm{Sp}}(L) \longrightarrow W$ is proper. By [Hi], $H^{-1}(w)$ is an abelian variety for $w \in W - \mathcal{D}$. The complement

$$\mathcal{M}_{\mathrm{Sp}}(L) - T^*M_{\mathrm{Sp}}$$

is of codimension ≥ 3 (the assumption that $g \geq 3$ is used here). In [Fa, Theorem II.6 (iii)] it is proved that the complement has codimension ≥ 2 under a weaker assumption, but if we assume $g \geq 3$, then the same proof gives that the codimension is ≥ 3.

Therefore, $(\mathcal{M}_{\mathrm{Sp}}(L) - T^*M_{\mathrm{Sp}}(L)) \cap \mathcal{D}_i$ is of codimension at least 2 in $\mathcal{D}_i$, so for generic $w \in \mathcal{D}_i^o$,

$$H^{-1}(w) - h^{-1}(w) \ \subset \ H^{-1}(w)$$

is of codimension at least 2.

The computations of $H^{-1}(w)$ for $w \in \mathcal{D}_i^o$ were done in the arguments above. □

Proposition 3.2. *The hypersurfaces $h^{-1}(\mathcal{D}_i)$ are irreducible.*

Proof. We need to see that $h^{-1}(\mathcal{D}^*)$ is of codimension at least two in $T^*M_{\mathrm{Sp}}(L)$. This follows easily from Theorem II.5 of [Fa], which says that the fibers of the Hitchin map $H : \mathcal{M}_{\mathrm{Sp}}(L) \longrightarrow W$ are Lagrangian (hence of half-dimension). So the fibers of H are equidimensional, and in particular the codimension of $h^{-1}(\mathcal{D}^*)$ is that of $\mathcal{D}^* \subset W$, which is at least two. □

The inverse image $h^{-1}(\mathcal{D})$ is called the *Hitchin discriminant.*

Theorem 3.3. *The Hitchin discriminant $h^{-1}(\mathcal{D})$ is the closure of the union of the (complete) rational curves in $T^*M_{\mathrm{Sp}}(L)$.*

Proof. Let $l \cong \mathbb{P}^1 \subset h^{-1}(\mathcal{D})$. Then $h(l) \subset W$. As it is a complete curve, it should be a point. So l is included in a fiber. By Proposition 3.1, it cannot be contained in a fiber over $w \in W - \mathcal{D}$.

Now let $w \in \mathcal{D}^o$. Then Proposition 3.1 again shows that there is a family of $\mathbb{P}^1$ covering these fibers. Now using Proposition 3.2, we get that the closure is the entire $h^{-1}(\mathcal{D})$. □

4. Torelli theorem

This section is devoted to a *Torelli type theorem* for the moduli space $M_{\mathrm{Sp}}(L)$, i.e., to prove that the moduli space determines the curve X up to isomorphism.

Lemma 4.1. *The global algebraic functions $\Gamma(T^*M_{\mathrm{Sp}}(L))$ produce a map*

$$\widetilde{h} : T^*M_{\mathrm{Sp}}(L) \longrightarrow \mathrm{Spec}(\Gamma(T^*M_{\mathrm{Sp}}(L))) \cong W \cong \mathbb{C}^N$$

which is the Hitchin map up to an automorphism of $\mathbb{C}^N$, where $N = \dim M_{\mathrm{Sp}}(L)$.

Moreover, consider the standard dilation action of $\mathbb{C}^*$ on the fibers of $T^*M_{\mathrm{Sp}}(L)$. Then there is a unique $\mathbb{C}^*$-action "$\cdot$" on W such that $\widetilde{h}$ is $\mathbb{C}^*$-equivariant, meaning $\widetilde{h}(E, \lambda\theta) = \lambda \cdot \widetilde{h}(E, \theta)$.

Proof. This holds for the Hitchin map H on the moduli of semistable symplectic Higgs bundles $\mathcal{M}_{\mathrm{Sp}}(L)$ (cf. [Hi]). On the other hand, the generic fiber of H is smooth and the codimension of $T^*M_{\mathrm{Sp}}(L) \subset \mathcal{M}_{\mathrm{Sp}}(L)$ on these fibers is at least two (cf. [Fa, Theorem II.6 (i)]. and note that $T^*M_{\mathrm{Sp}}(L)$ is a subset of the moduli $\mathcal{M}^0_{\mathrm{Sp}}(L)$ of stable Higgs bundles). Therefore, it follows that the lemma also holds for the restriction of the Hitchin map to the cotangent bundle $T^*M_{\mathrm{Sp}}(L)$. □

We decompose W into eigenspaces for the action of $\mathbb{C}^*$,

$$W = \bigoplus_{k=0}^{n} W_{2k}\,,$$

and note that $W_{2k} \cong H^0(K_X^{2k})$. Each of these pieces is intrinsically defined. To see this, note that Lemma 4.1 allows us to recover the base W of the Hitchin fibration as an algebraic manifold. But we also have the $\mathbb{C}^*$-action on W. Therefore we have the origin (as the only fixed point of the action). Linearizing the action at each point, we recover the tangent spaces to the subvarieties $W_{2k} + y \subset W$, for any $1 \leq k \leq n$ and $y \in W$. Hence we recover the subvarieties $W_{2k} \subset W$ themselves, and their translates. This easily produces the addition of vectors of W_{2k} and $W_{2k'}$, $k \neq k$. Finally, there is only one way to give a vector space structure to W_{2k} so that the $\mathbb{C}^*$-action is linear on it. The conclusion is that the vector space structure of W and the decomposition $W = \bigoplus W_{2k}$ is recovered from the Hitchin fibration.

Proposition 4.2. *Let $\mathcal{C}$ be the intersection of $W_{2n} = H^0(K_X^{2n}) \subset W$ with $\mathcal{D}_1 \cup \mathcal{D}_2$. This is irreducible. Moreover $\mathbb{P}(\mathcal{C}) \subset \mathbb{P}(W_{2n})$ is the dual variety of $X \subset \mathbb{P}(W_{2n}^*)$ for the embedding given by the linear series $|K_X^{2n}|$.*

Proof. A spectral curve corresponding to a point of $s_{2n} \in H^0(K_X^{2n})$ has equation $y^{2n} + s_{2n}(x) = 0$, and this curve is singular at the points with coordinates $(x, 0)$ such that x is a zero of s_{2n} of order at least two. Clearly $\mathcal{C} = \mathcal{D}_1 \cap W_{2n}$. On the other hand, $\mathcal{D}_2 \cap W_{2n} \subset \mathcal{C}$, since it consists of singular curves. Therefore, the first statement follows.

The elements $b \in W_{2n}$ correspond to spectral curves of the form $y^{2n} + b(x) = 0$. We have $b \in \mathcal{C}$ if and only if there is some x_0 such that $b(x_0) = 0$ and $b'(x_0) = 0$ simultaneously, therefore $b \in H^0(K_X^{2n}(-2x_0)) \subset H^0(K_X^{2n})$. From this the second statement follows, taking into account that the linear system $|K_X^{2n}|$ is very ample, so X is embedded. □

Denote

$$\mathcal{C}_x = H^0(K_X^{2n}(-2x)) \subset W_{2n}$$

Then

$$\mathcal{C} = \bigcup_{x \in X} \mathcal{C}_x \,,$$

and taking the bundle of tangent hyperplanes to $X \subset \mathbb{P}(W_{2n}^*)$, we have

$$\begin{array}{ccc} \widetilde{\mathcal{C}} = \bigsqcup \mathcal{C}_x & \stackrel{F}{\longrightarrow} & \mathcal{C} \\ \downarrow & & \\ X & & \end{array}$$

We shall also need to consider the bundle of hyperplanes through a given point of x, i.e.,

$$\widetilde{\mathcal{H}} = \bigsqcup \mathcal{H}_x \longrightarrow X \,,$$

where $\mathcal{H}_x = \mathbb{P}(K_X^{2n}(-x)) \subset W_{2n}$. This is intrinsically defined once we have obtained X.

The following theorem is proved in [BH] by a different method.

Theorem 4.3 (Torelli)**.** *Let X, X' be two smooth projective curves of genus $g \geq 3$, and consider $M_{\mathrm{Sp}}(L)$, $M'_{\mathrm{Sp}}(L')$ two moduli spaces of symplectic bundles over both curves. If the moduli spaces are isomorphic, then $X \cong X'$.*

Proof. Suppose $\Phi : M_{\mathrm{Sp}}(L) \longrightarrow M'_{\mathrm{Sp}}(L')$ is an isomorphism. Then there is an isomorphism $d\Phi : T^*M_{\mathrm{Sp}}(L) \longrightarrow T^*M'_{\mathrm{Sp}}(L')$. By Lemma 4.1, there is a commutative diagram

$$\begin{array}{ccc} T^*M_{\mathrm{Sp}}(L) & \stackrel{d\Phi}{\longrightarrow} & T^*M'_{\mathrm{Sp}}(L') \\ \downarrow & & \downarrow \\ W & \stackrel{f}{\longrightarrow} & W' \end{array}$$

for some isomorphism $f : W \longrightarrow W'$. The $\mathbb{C}^*$-action by dilations on the fibers of $T^*M_{\mathrm{Sp}}(L)$ and $T^*M'_{\mathrm{Sp}}(L')$ induce $\mathbb{C}^*$-actions on W and W', and f should be $\mathbb{C}^*$-equivariant (as $d\Phi$ is $\mathbb{C}^*$-equivariant). Therefore $f : W_{2n} \longrightarrow W'_{2n}$, and it is linear.

We have seen in Proposition 3.1 that the Hitchin discriminant $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \subset W$ is an intrinsically defined subset, and therefore it is preserved by f. So f preserves $\mathcal{C} = \mathcal{D} \cap W_{2n}$. This induces an isomorphism of the corresponding dual varieties, hence by Proposition 4.2, an isomorphism $\sigma : X \longrightarrow X'$ is obtained. □

5. Nilpotent cone and flag variety

We need some results on linear algebra about the space of symplectic endomorphisms. Let (V,ω) be a symplectic vector space of dimension $2n$. In this section, we want to study the *symplectic nilpotent cone*[1]

$$\mathcal{N} = \{A \in \mathrm{End}_{\mathrm{Sp}} V \,|\, A^{2n} = 0\}.$$

Lemma 5.1. *The following statements hold:*

(1) *$\mathcal{N}$ is a $2n^2$-dimensional algebraic variety.*
(2) *$A \in \mathcal{N}$ if and only if $\mathrm{tr}(A^{2r}) = 0$, for all $r = 1, \dots, n$.*
(3) *$A \in \mathcal{N}^{sm}$ (the smooth locus of $\mathcal{N}$) if and only if $\mathrm{rk}\, A = 2n - 1$.*
(4) *Let $Fl(V,\omega)$ be the set of full isotropic flags on $\mathbb{C}^{2n}$. Then there is a fibration $\pi : \mathcal{N}^{sm} \longrightarrow Fl(V,\omega)$ with fibers isomorphic to $(\mathbb{C}^*)^n \times \mathbb{C}^{n^2-n}$.*

Proof. Statement (1) is clear, since $\mathcal{N}$ is defined by the equations $q_2(A) = \ldots = q_{2n}(A) = 0$, where $p_A(t) = t^{2n} + q_2(A)t^{2n-2} + \ldots + q_{2n}(A)$ is the characteristic polynomial of $A \in \mathrm{End}_{\mathrm{Sp}} V$. As $\dim \mathrm{End}_{\mathrm{Sp}} V = n(2n+1)$, and we have n equations, it follows that $\dim \mathcal{N} = 2n^2$.

To prove statement (2), note that if $\pm\lambda_1, \dots, \pm\lambda_n$ are the eigenvalues of A, then $\mathrm{tr}(A^r) = 0$ for r odd, and $\mathrm{tr}(A^r) = 2\sum \lambda_i^r$ for r even. Then the equations $\mathrm{tr}(A^r) = 0$, $r = 2, 4, \dots, 2n$, are equivalent to $\lambda_1 = \ldots = \lambda_n = 0$, i.e., $A^{2n} = 0$.

Now we will prove statement (3). Let $B \in \mathrm{End}_{\mathrm{Sp}} V$. Considering $\frac{d}{d\epsilon}\big|_{\epsilon=0}\mathrm{tr}((A + \epsilon B)^{2r}) = 0$, $r = 1, \dots, n$, we see that

$$T_A\mathcal{N} = \{B \,|\, \mathrm{tr}(A^{2r-1}B) = 0,\, r = 1, \dots, n\}\,.$$

This has codimension $< n$ when $A^{2n-1} = 0$. So $\mathrm{rk}\, A = 2n - 1$ at a smooth point. For the converse, if $\mathrm{rk}\, A = 2n - 1$, then the matrices $I, A, A^2, \dots, A^{2n-1}$ are linearly independent, and $A^{2k-1} \in \mathrm{End}_{\mathrm{Sp}} V$, $k = 1, \dots, n$. Therefore, the n equations $\mathrm{tr}(A^{2r-1}B) = 0$, $r = 1, \dots, n$, for $B \in \mathrm{End}_{\mathrm{Sp}} V$ are linearly independent, and $\mathrm{codim}\, T_A\mathcal{N} = n$. Hence $A \in \mathcal{N}^{sm}$, as required.

Finally we prove statement (4). Note that if $A \in \mathcal{N}^{sm}$, then $\mathrm{rk}\, A = 2n - 1$. This determines a well-defined full flag

$$0 \subset \ker A \subset \ker A^2 \subset \ldots \subset \ker A^{2n-1} \subset \mathbb{C}^{2n} \tag{5.1}$$

Let us see that $\ker A^i$ is dual (with respect ω) to $\ker A^{2n-i}$. For this, note that $\ker A^{2n-i} = \mathrm{im}\, A^i$. If $u = A^i u_0$, and $v \in \ker A^i$, then

$$\omega(u, v) = \omega(A^i u_0, v) = (-1)^i \omega(u_0, A^i v) = 0\,.$$

This means that the flag in (5.1) is isotropic (in particular, $\ker A^n$ is Lagrangian).

The fiber over a point of the flag variety is given as follows. Fix a symplectic basis $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$, such that the flag is

$$\langle e_n \rangle \subset \langle e_{n-1}, e_n \rangle \subset \cdots \subset \langle e_1, \dots, e_n \rangle \subset \langle e_1, \dots, e_n, e_{n+1} \rangle \subset \cdots \subset \langle e_1, \dots, e_{2n} \rangle\,. \tag{5.2}$$

[1] We remark that this is not the same nilpotent cone defined in [La]. Our nilpotent cone is the fiber over the identity of Laumon's nilpotent cone.

Then the matrices in the fiber are of the form

$$\begin{pmatrix} A & B \\ 0 & -A^T \end{pmatrix}, \text{ where } A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & a_{32} & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & 0 \end{pmatrix} \tag{5.3}$$

with $a_{i+1,i} \neq 0$, and $B = (b_{ij})$ is symmetric with $b_{11} \neq 0$. So the fiber of

$$\pi : \mathcal{N}^{sm} \longrightarrow Fl(V, \omega)$$

is $(\mathbb{C}^*)^n \times \mathbb{C}^{n^2-n}$. □

We will now prove that $\mathcal{N}^{sm}$ determines $Fl(V, \omega)$. Consider the fibration π in Lemma 5.1 (4). We shall show that the fibers of π are intrinsically defined. Take any $F \in Fl(V, \omega)$. Let

$$U_F \subset \mathcal{N} \subset \mathrm{End}_{\mathrm{Sp}}(V)$$

be the space of symmetric endomorphisms respecting the flag F. It is a linear subspace of $\mathrm{End}_{\mathrm{Sp}}(V)$ of dimension n^2 contained in the nilpotent cone. By Lemma 5.2 below, all subspaces of $\mathrm{End}_{\mathrm{Sp}}(V)$ of dimension n^2 contained in the nilpotent cone are of the form U_F for some F. Moreover, $U_F \cap \mathcal{N}^{sm}$ is the fiber of π over F. Therefore π is uniquely defined, up to automorphism of the base.

Lemma 5.2. *Let $L \subset \mathcal{N} \subset \mathrm{End}_{\mathrm{Sp}}(V)$ be a linear subspace of dimension n^2 such that $L \cap \mathcal{N}^{sm} \neq \emptyset$. Then there exists a (unique) flag F such that $L = U_F$.*

Proof. We shall divide the proof in several steps.
Step 1. $\mathrm{tr}(A^i B) = 0$, for any $A, B \in L$, $i \geq 0$. If i is even, then $A^i B$ is an anti-symmetric endomorphism, hence of zero trace. For i odd, note that $\mathrm{tr}(C^{2j}) = 0$ for any $C \in \mathcal{N}$, $j \geq 0$. Then considering that $C_\lambda = A + \lambda B \in L \subset \mathcal{N}$, we have that $\mathrm{tr}(C_\lambda^{i+1}) = \mathrm{tr}((A + \lambda B)^{i+1}) = 0$. Take the coefficient of λ to get $\mathrm{tr}(A^i B) = 0$.

Step 2. Let $A \in \mathcal{N}$. Then $A \in L$ if and only if $\mathrm{tr}(AB) = 0$, for all $B \in L$.

The "only if" part follows from Step 1. To prove the "if" part, suppose that $\mathrm{tr}(AB) = 0$, for all $B \in L$. As $A \in \mathcal{N}$, we can choose a flag so that $A \in U$ (this is unique if $A \in \mathcal{N}^{sm}$). Taking an appropriate symplectic basis, the flag is (5.2), and there is a well-defined space U of (symplectic symmetric) matrices (5.3) preserving the flag.

Denote $U^T = \{B^T \mid B \in U\} \subset \mathrm{End}_{\mathrm{Sp}}(V)$. Consider also the space $D \subset \mathrm{End}_{\mathrm{Sp}}(V)$ consisting of (symmetric symplectic) diagonal matrices. Therefore

$$\mathrm{End}_{\mathrm{Sp}}(V) = U \oplus D \oplus U^T .$$

The bilinear map $q(B_1, B_2) = \mathrm{tr}(B_1 B_2)$ is symmetric and non-degenerate. The q-dual of U is $U + D$. As $\mathrm{End}_{\mathrm{Sp}}(V)/(U + D) = U^T$, there is an induced perfect pairing $q : U \times U^T \longrightarrow \mathbb{C}$. Let $p = p_{U^T} : \mathrm{End}_{\mathrm{Sp}}(V) \longrightarrow U^T$ be the projection. Now $L \cap (U + D) = L \cap U$, since all elements of L are nilpotent. Let us see that $L \cap U$ is q-dual to $p(L)$. Clearly, $q(B, C_1) = \mathrm{tr}(BC_1) = \mathrm{tr}(BC) = 0$, for $B \in L \cap U$, $C_1 \in p(L)$, $C = C_1 + C_2 \in L$, where $C_2 \in U + D$. On the other hand,

$$\dim(L \cap U) + \dim p(L) = \dim L = \dim U .$$

Therefore, $p(L) \subset U^T$ and $L \cap U \subset U$ are q-orthogonal complements.

The conclusion is that given $B \in U$, it is $B \in L \cap U$ if and only if $q(B, C_1) = 0$ for any $C_1 \in p(L)$. In short, $B \in L \cap U$ if and only if $\mathrm{tr}(BC) = 0$, for any $C \in L$.

Step 3. If $A \in L$ then $A^{2i-1} \in L$ for any $i \geq 1$. Without loss of generality, we can suppose $A \in \mathcal{N}^{sm}$. Therefore A determines a flag F, and a space $U = U_F$ of endomorphisms preserving the flag. By Step 1, $\mathrm{tr}(A^{2i-1}B) = 0$, for any $B \in L$. By Step 2, $A^{2i-1} \in L$.

Step 4. Let $A, B \in L$, then $A^2B + BA^2 + ABA \in L$. Let $C_\lambda = A + \lambda B \in L$. Then $C_\lambda^3 = (A + \lambda B)^3 \in L$, by Step 3. Take the coefficient of λ, to conclude the statement.

Step 5. Let $A \in L \cap \mathcal{N}^{sm}$. Let F be the (isotropic) flag determined by A, and let $U = U_F$. Denote as $0 \subset V_1 \subset V_2 \subset \cdots \subset V_{2n} = V$ the flag F. For $B \in L$, let $r(B) \in \mathbb{Z}$ be the minimum integer such that $B(V_i) \subset V_{i+r}$. (Note that if $r(B) < 0$ is equivalent to $B \in U$.) We claim that either $r(B) < 0$ or $r(B)$ is odd.

Let $r = r(B)$. If $r = 0$ then $B \in U + D$. As B is nilpotent, it must be $B \in U$, meaning that $r < 0$. Now we work by induction on r. Suppose that $r > 0$ and it is even. Write $B = (b_{ij})$ and note that $b_{ij} = 0$ for $i - j > r$. Let $b_i = b_{r+i,i}$, $i = 1, \ldots, 2n - r$. By Step 4, $C = A^2B + BA^2 + ABA \in L$. It is easy to see that $r(C) \leq r - 2$, and that it has coefficients

$$\begin{aligned} &c_1 = b_1, c_2 = b_1 + b_2, c_3 = b_1 + b_2 + b_3, \ldots, c_i = b_{i-2} + b_{i-1} + b_i, \\ &\ldots, c_{2n-r+1} = b_{2n-r-1} + b_{2n-r}, c_{2n-r+2} = b_{2n-r} \,. \end{aligned}$$

By induction hypothesis, $r(C) \neq r - 2$, so $r(C) < r - 2$ and all $c_i = 0$. From here, $b_i = 0$ for all i, and so $r(B) < r$.

Step 6. With the notation as in Step 5, $r = r(B) > 1$. Suppose that $r = 1$. Let $\{v_i\}$ be a basis adapted to the flag, i.e. $V_i = \langle v_1, \ldots, v_i \rangle$, for all $i = 0, \ldots, 2n$, and consider the coefficients $b_i := b_{i+1,i}$, $i = 1, \ldots, 2n - 1$, not all equal to zero.

Suppose first that all $b_i \neq 0$. Then B has rank $2n - 1$, so $\ker B$ is 1-dimensional. Actually, if v spans $\ker B$, then $v \notin V_{2n-1}$. So choose the basis so that $v_{2n} = v$ and $v_k = A(v_{k+1})$, $k = 1, \ldots, 2n-1$. Therefore A has standard form and $b_{j,2n} = 0$, all j. So $\det(B + \lambda A) = b_{2n-1}\lambda \det(B' + \lambda A')$, where A', B' are $(2n-2) \times (2n-2)$-matrices obtained from A, B by removing the last two columns and rows. By induction, this determinant is non-zero. Therefore $A + \lambda B$ is not nilpotent for generic value of λ. This is a contradiction, since $A + \lambda B \in L \subset \mathcal{N}$.

Now suppose that $b_{i_0} = 0$, $b_{i_0+2k} = 0$, but $b_{i_0+1}, \ldots, b_{i_0+2k-1} \neq 0$. Then take the blocks formed by rows and columns $i_0 + 1, \ldots, i_0 + 2k$. This produces matrices A', B' of even size, such that $A' + \lambda B'$ is nilpotent for all λ. But $\det(A' + \lambda B') \neq 0$, which is proved as before.

The next case is that $b_{i_0} = 0$, $b_{i_0+2k+1} = 0$, but $b_{i_0+1}, \ldots, b_{i_0+2k} \neq 0$. Choose one such possibility with the smallest possible value of k. Let $W \subset \mathbb{C}^{2n-1}$ be the subspace parametrizing vectors (b_i) arising from matrices $B \in L$ with $r(B) = 1$. And let $W_{i_0,2k+1}$ be the subspace of those vectors $(b_{i_0+1}, \ldots, b_{i_0+2k})$ where $(b_i) \in W$, $b_{i_0} = 0$, $b_{i_0+2k+1} = 0$. If this has dimension ≥ 2, then there is a vector with some coordinate zero. Therefore, there is a smaller k. So $\dim W_{i_0,2k+1} = 1$. Step 4 implies that if $(b_{i_0+1}, \ldots, b_{i_0+2k}) \in W_{i_0,2k+1}$ then

$$\begin{aligned} &(b_{i_0+1}(b_{i_0+1} + b_{i_0+2}), b_{i_0+2}(b_{i_0+1} + b_{i_0+2} + b_{i_0+3}), \ldots, \\ &\qquad b_{i_0+2k-1}(b_{i_0+2k-2} + b_{i_0+2k-1} + b_{i_0+2k}), b_{i_0+2k}(b_{i_0+2k-1} + b_{i_0+2k})) \in W_{i_0,2k+1} \,. \end{aligned}$$

As this vector is a multiple of the previous one, it must be

$$b_{i_0+1} + b_{i_0+2} = b_{i_0+1} + b_{i_0+2} + b_{i_0+3} = \ldots = b_{i_0+2k-1} + b_{i_0+2k}.$$

This implies the vanishing of all b_i unless $k = 1$. And moreover, if $k = 1$, then taking the 3×3-matrix with rows and columns i_0+1, i_0+2, i_0+3, we get that $b_{i_0+1} = \alpha$, $b_{i_0+2} = -\alpha$, for some $\alpha \in \mathbb{C}$, by using that $B' + \lambda A'$ should be nilpotent.

So the elements of W are of the form $(\ldots, 0, \alpha_1, -\alpha_1, 0, \ldots, 0, \alpha_2, -\alpha_2, 0, \ldots)$.

Now consider the space $H^T := \{B \in U^T | r(B) \leq 2\}$. The dual of H^T under q is denoted $Z \subset U$, and let $H := U/Z$, with projection $p_H : U \longrightarrow H$. So there is a perfect pairing $q : H^T \times H \longrightarrow \mathbb{C}$. Recall that Step 2 says that $U \cap L$ is q-dual to $p_{U^T}(L)$. Therefore $p_{U^T}(L) \cap H^T$ and $p_H(U \cap L)$ are q-dual. Now consider $B_2 \in p_{U^T}(L) \cap W^T$. Then $B = B_1 + B_2 \in L$, where $B_2 \in U + C$, and $r(B) \leq 2$. By Step 5, $r(B) \leq 1$. By the previous discussion, if $r(B) = 1$, then the entries $(b_i) \in W$ have the form given above.

So $p_H(U \cap L)$ contains matrices with any values in the second diagonal, and with values in the first diagonal of the form $(\ldots, *, \beta_1, \beta_1, *, \ldots, *, \beta_2, \beta_2, *, \ldots)$.

Now just consider a matrix $C \in U \cap L$ with all zeroes on the first diagonal, and just one 1 in the second diagonal, in a position (i_0, i_0+2), so that the 3×3-block coming from $B + \lambda A + C \in L$ has a 1 in the right-top corner. This matrix is not nilpotent. This is a contradiction.

Step 7. $L = U$. Fix some $A \in L \cap \mathcal{N}^{sm}$, and the corresponding subspace U. Let $B \in L$, and let $r = r(B)$. We only have to see that $r < 0$. If $r \geq 0$, then r is odd from Step 5. By Step 6, it cannot be $r = 1$. The same argument as in Step 5, proves that it cannot be $r > 1$ and odd. So $B \in U$. Hence $L \subset U$, so they are equal by dimensionality. □

Remark 5.3. Lemma 5.2 also follows from the main theorem in [DKK]. The above proof, which uses only elementary methods, is included in order to be self-contained.

6. Automorphisms of the moduli space

In this section we will compute the automorphism group of a moduli spaces of symplectic bundles on X. As before, we assume that $g \geq 3$.

Proposition 6.1. *Fix a generic stable bundle $E \in M_{\mathrm{Sp}}(L)$, and consider the map*

$$h_{2n} : H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \longrightarrow W_{2n}\,,$$

given as composition of the Hitchin map on $H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) = T^*_E M_{\mathrm{Sp}} \subset T^* M_{\mathrm{Sp}}$, followed by projection $W \to W_{2n}$. Then

$$H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x_0))$$
$$= \left\{ \psi \in H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \,|\, h_{2n}(\psi + \phi) \in \mathcal{H}_{x_0}, \forall \phi \in h_{2n}^{-1}(\mathcal{H}_{x_0}) \right\}.$$

Proof. First, note that the sequence

$$0 \longrightarrow H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x_0)) \longrightarrow H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \longrightarrow \mathrm{End}_{\mathrm{Sp}} E \otimes K_X|_{x_0} \longrightarrow 0$$

is exact, since $H^1(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x_0)) = H^0(\mathrm{End}_{\mathrm{Sp}} E(x_0))^* = 0$, for a generic bundle, by Lemma 2.2. So the map

$$H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \longrightarrow \mathrm{End}_{\mathrm{Sp}} E \otimes K_X|_{x_0}\,,$$

given by $\phi \longmapsto \phi(x_0)$, is surjective.

Note that $h_{2n}(\phi) = \det(\phi) \in W_{2n} = H^0(K_X^{2n})$. So

$$h_{2n}(\phi) \in \mathcal{H}_{x_0} \iff \det(\phi(x_0)) = 0 \,.$$

The result follows from this easy linear algebra fact: if (V, ω) is a symplectic vector space, and $A \in \mathrm{End}_{\mathrm{Sp}}(V)$ satisfies that $\det(A + C) = 0$ for any $C \in \mathrm{End}_{\mathrm{Sp}}(V)$ with $\det(C) = 0$, then $A = 0$. □

Proposition 6.1 allows to construct the bundle

$$\mathcal{E} \longrightarrow X$$

whose fiber over $x \in X$ is $\mathcal{E}_x = H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x))$. This is a subbundle of the trivial bundle

$$H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \otimes \mathcal{O}_X \longrightarrow X \,,$$

and there is an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \otimes \mathcal{O}_X \xrightarrow{\pi} \mathrm{End}_{\mathrm{Sp}} E \otimes K_X \longrightarrow 0 \,. \tag{6.1}$$

This is exact on the right by Lemma 2.2. So we recover the bundle

$$\mathrm{End}_{\mathrm{Sp}} E \otimes K_X \longrightarrow X \,.$$

Our next step is to recover the nilpotent cone bundle

$$\mathcal{N}_E \longrightarrow X \,,$$

whose fibers are the symplectic nilpotent cone spaces

$$\mathcal{N}_{E,x} = \{A \in \mathrm{End}_{\mathrm{Sp}} E \otimes K_X|_x \mid A^{2n} = 0\} \subset \mathrm{End}_{\mathrm{Sp}} E \otimes K_X|_x, \; x \in X \,.$$

Note that this nilpotent cone bundle sits as $\mathcal{N}_E \subset \mathrm{End}_{\mathrm{Sp}} E \otimes K_X$.

Lemma 6.2. *Consider the map*

$$h_{2k} : H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \longrightarrow W_{2k} \,,$$

given as composition of the Hitchin map on $H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) = T^*_E M_{\mathrm{Sp}} \subset T^* M_{\mathrm{Sp}}$, followed by projection $W \to W_{2k}$. Then the vector subspace generated by the image $h_{2k}(H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x)))$ is $H^0(K_X^{2k}(-2k\,x)) \subset W_{2k} = H^0(K_X^{2k})$.

Proof. The map h_{2k} sends $(E, \phi) \mapsto \mathrm{tr}(\wedge^{2k}\phi)$. Therefore,

$$h_{2k}(H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x))) \subset H^0(K_X^{2k}(-2k\,x)) \,.$$

Now the vector space generated by $h_{2k}(H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x)))$ equals to the image of

$$\wedge^{2k} : H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x))^{\otimes 2k} \longrightarrow H^0(\mathrm{End}_{\mathrm{ant}} E \otimes K_X^{2k}(-2k\,x)) \,,$$

followed by the trace map $H^0(\mathrm{End}_{\mathrm{ant}} E \otimes K_X^{2k}(-2k\,x)) \longrightarrow H^0(K_X^{2k}(-2k\,x))$, where $\mathrm{End}_{\mathrm{ant}} E$ consists of the anti-symmetric symplectic endomorphisms of E (i.e., those φ such that $\omega(\varphi\, u, v) = \omega(u, \varphi\, v)$). Note that the multiples of the identity are in $\mathrm{End}_{\mathrm{ant}} E$.

There is a bundle map

$$(\mathrm{End}_{\mathrm{Sp}} E)^{\otimes 2k} \longrightarrow \mathrm{End}_{\mathrm{ant}} E \longrightarrow \mathcal{O}_X$$

(first map is composition of endomorphisms, second map is the trace). This is split, therefore the map

$$H^0((\mathrm{End}_{\mathrm{Sp}} E)^{\otimes 2k} \otimes K_X^{2k}(-2k\,x)) \longrightarrow H^0(K_X^{2k}(-2k\,x))$$

is surjective, as required. □

Consider the vector subspace generated by the image $h_{2k}(H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x)))$ which is $H^0(K_X^{2k}(-2k\,x)) \subset W_{2k} = H^0(K_X^{2k})$, for varying $x \in X$. This gives a fiber bundle over X, which has co-rank $2k$. The curve X is embedded into $\mathbb{P}(W_{2k})$ via the linear system $|K_X^{2k}|$, and the osculating $2k$-space at x is given as

$$Osc_{2k}(x) = \mathbb{P}(V_x) \subset \mathbb{P}(W_{2k}), \qquad V_x := \ker(H^0(K_X^{2k})^* \to H^0(K_X^{2k}(-2k\,x)^*))\,.$$

The embedding of the curve X in $\mathbb{P}(W_{2k})$ is recovered from the osculating $2k$-spaces. More specifically, if $g: X \to \mathbb{P}^N$ and $\tilde{g}: X \to \mathrm{Gr}(2k+1, N+1)$ is the map giving the osculating $2k$-spaces, then $\tilde{g}$ determines g. This is proved as follows: the pull-back of the universal bundle through $\tilde{g}$ is the bundle

$$\mathbb{P}(\mathcal{O}_X \oplus TX \oplus (TX)^{\otimes 2} \oplus \ldots \oplus (TX)^{\otimes 2k}) \longrightarrow X\,,$$

and g is determined by a section of this bundle. As TX is of negative degree, this only has one section.

So we recover the embeddings $X \hookrightarrow \mathbb{P}(W_{2k}) = \mathbb{P}(H^0(K_X^{2k}))$, and hence the hyperplanes

$$H_x^{(2k)} := H^0(K_X^{2k}(-x)) \subset W_{2k}\,.$$

Finally, consider

$$\{\phi \in H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \,|\, h_{2k}(\phi) \in H_x^{(2k)}, \forall k = 1, \ldots, n\}.$$

This is the preimage of the nilpotent cone under the surjective map $H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X) \longrightarrow \mathrm{End}_{\mathrm{Sp}} E \otimes K_X|_x$. Take its image to get the bundle

$$\mathcal{N}_E \longrightarrow X\,.$$

Now we are ready to prove the main result of the paper. First, as explained in the introduction, line bundles of order two and automorphisms of X produce automorphisms of moduli spaces of symplectic bundles.

Theorem 6.3. *Let $M_{\mathrm{Sp}}(L)$ be the moduli space of symplectic bundles. Let*

$$\Phi: M_{\mathrm{Sp}}(L) \longrightarrow M_{\mathrm{Sp}}(L)$$

be an automorphism. Then Φ is induced by an automorphism of X and a line bundle of order two.

Proof. As in the proof of Theorem 4.3, the automorphism Φ yields an isomorphism $\sigma: X \longrightarrow X$ and commutative diagrams

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \longrightarrow & \widetilde{\mathcal{C}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widetilde{\mathcal{H}} & \longrightarrow & \widetilde{\mathcal{H}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X \end{array}$$

Let M be a line bundle such that $M^{\otimes 2} \cong L \otimes (\sigma^* L)^\vee$ Composing with the automorphism given by σ^{-1} and M^{-1} (see the introduction), we may assume that $\sigma = Id$. Take a *generic* bundle E, and let E' be its image by Φ. Then we have

$$\begin{array}{ccc} T_E^* M_{\mathrm{Sp}}(L) & \xrightarrow{d\Phi} & T_{E'}^* M_{\mathrm{Sp}}(L) \\ h\big\downarrow & & h\big\downarrow \\ W & \xrightarrow{\cong} & W \end{array}$$

where the lower horizontal map is a linear isomorphism with commutes with the $\mathbb{C}^*$-action.

There is a bundle $\mathcal{E} \longrightarrow X$ whose fiber over any $x \in X$ is the subspace $H^0(\mathrm{End}_{\mathrm{Sp}} E \otimes K_X(-x)) \subset T^*_E M_{\mathrm{Sp}}(L)$. Analogously, there is a bundle $\mathcal{E}' \longrightarrow X$ whose fiber over x is the subspace $H^0(\mathrm{End}_{\mathrm{Sp}} E' \otimes K_X(-x)) \subset T^*_{E'} M_{\mathrm{Sp}}(L)$. By Proposition 6.1, the map $d\Phi : T^*_E M_{\mathrm{Sp}}(L) \longrightarrow T^*_{E'} M_{\mathrm{Sp}}(L)$ gives an isomorphism $\mathcal{E} \longrightarrow \mathcal{E}'$. Going to the quotient bundle (6.1), we have a bundle isomorphism

$$\mathrm{End}_{\mathrm{Sp}} E \otimes K_X \longrightarrow \mathrm{End}_{\mathrm{Sp}} E' \otimes K_X \,.$$

By Lemma 6.2 and the discussion following it, this produces an isomorphism

$$\begin{array}{ccc} \mathcal{N}_E & \longrightarrow & \mathcal{N}_{E'} \\ \big\downarrow & & \big\downarrow \\ X & = & X \end{array}$$

where $\mathcal{N}_E, \mathcal{N}_{E'}$ are the corresponding symplectic nilpotent cone bundles. By Lemma 5.2, we get an isomorphism

$$\begin{array}{ccc} Fl(E,\omega) & \longrightarrow & Fl(E',\omega') \\ \big\downarrow & & \big\downarrow \\ X & = & X \end{array}$$

of the corresponding isotropic flag varieties bundles. Going to global vertical fields, we have a (Lie algebra) bundle isomorphism

$$\begin{array}{ccc} ad_{\mathrm{Sp}} E & \longrightarrow & ad_{\mathrm{Sp}} E' \\ \big\downarrow & & \big\downarrow \\ X & = & X \end{array}$$

Using Lemma 6.4 below, it follows that $E' \cong E \otimes M$, for some line bundle M with $M^2 \cong \mathcal{O}_X$.

As this holds for a generic E, it holds for all E. □

Lemma 6.4. *Let $(E, E \otimes E \to L)$ and $(E', E' \otimes E' \to L')$ be two symplectic vector bundles such that $ad_{\mathrm{Sp}} E$ and $ad_{\mathrm{Sp}} E'$ are isomorphic as Lie algebra bundles. Then there is a line bundle M such that $E' \cong E \otimes M$.*

Furthermore, if we assume $L \cong L'$, then $M^{\otimes 2} \cong \mathcal{O}_X$.

Proof. Giving a vector bundle $ad_{\mathrm{Sp}} E$ with its Lie algebra structure is equivalent to giving a principal $\mathrm{Aut}(\mathfrak{sp}_{2n})$-bundle $P_{\mathrm{Aut}(\mathfrak{sp}_{2n})}$ which admits a reduction to a principal Gp_{2n}-bundle $P_{\mathrm{Gp}_{2n}}$, corresponding to $(E, E \otimes E \to L)$.

Since $\mathfrak{sp}_{2n}$ does not have outer automorphisms, all automorphisms are inner, and $\mathrm{Aut}(\mathfrak{sp}_{2n})$ is connected. Therefore, we have

$$\mathrm{Gp}_{2n} \twoheadrightarrow \mathrm{PGp}_{2n} = \mathrm{Inn}(\mathfrak{sl}_{2n}) = \mathrm{Aut}(\mathfrak{sl}_{2n})$$

Consider the short exact sequence of groups

$$e \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Gp}_{2n} \longrightarrow \mathrm{PGp}_{2n} \longrightarrow e$$

Hence, the set of reductions of a PGp_{2n}-bundle to Gp_{2n} is a torsor for the group $H^1(X, \mathcal{O}^*_X)$. Therefore, if $(E, E \otimes E \to L)$ is a symplectic bundle corresponding to a reduction, the other reductions are of the form

$$\big(E \otimes M, (E \otimes M) \otimes (E \otimes M) \longrightarrow L \otimes M^{\otimes 2}\big)$$

for any line bundle M.

Finally, it follows from this expression that, if $L \cong L'$, then $M^{\otimes 2} \cong \mathcal{O}_X$. □

Let $\mathrm{Aut}(X)$ and $\mathrm{Aut}(M_{\mathrm{Sp}}(L))$ be the automorphisms of X and $M_{\mathrm{Sp}}(L)$ respectively. Let $J(X)_2$ be the group of line bundles on X of order two.

Proposition 6.5. *There is a natural short exact sequence of groups*

$$e \longrightarrow J(X)_2 \longrightarrow \mathrm{Aut}(M_{\mathrm{Sp}}(L)) \longrightarrow \mathrm{Aut}(X) \longrightarrow e\,.$$

Proof. From the proof of Theorem 6.3, if follows that we have a surjective homomorphism

$$\rho\,:\,\mathrm{Aut}(M_{\mathrm{Sp}}(L)) \longrightarrow \mathrm{Aut}(X)\,.$$

The homomorphism sends Φ to σ (see the proof of Theorem 6.3). The kernel of ρ is a quotient of $J(X)_2$. Therefore, to prove the proposition it suffices to show that the action of $J(X)_2$ on $M_{\mathrm{Sp}}(L)$ is effective.

Let $M_{\mathrm{Sp}}(2, L)$ (respectively, $M_{\mathrm{Sp}}(2n-2, L)$) be the moduli space of symplectic bundles of rank 2 (respectively, $2n-2$) such that the symplectic form takes values in L. There is a natural embedding

$$M_{\mathrm{Sp}}(2, L) \times M_{\mathrm{Sp}}(2n-2, L) \;\longrightarrow\; M_{\mathrm{Sp}}(L)$$

defined by $((E_1, \varphi_1)\,, (E_2, \varphi_2)) \longmapsto (E_1 \oplus E_2, \varphi_1 \oplus \varphi_2)$. To prove that the action of $J(X)_2$ on $M_{\mathrm{Sp}}(L)$ is effective it is enough to show that the action of $J(X)_2$ on $M_{\mathrm{Sp}}(2, L)$ is effective.

First assume that $\deg L = 2\delta$, where δ is an integer. Then for a general line bundle $M \in J^\delta(X)$, the symplectic bundle $M \oplus (L \otimes M^*) \in M_{\mathrm{Sp}}(2, L)$ is moved by the action of every nontrivial element of $J(X)_2$. Therefore, the action of $J(X)_2$ on $M_{\mathrm{Sp}}(2, L)$ is effective.

Now assume that $\deg L = 2\delta + 1$. Fix a nontrivial line bundle $\xi \in J(X)_2$. Take a pair (E, θ), where E is a stable vector bundle of rank two with $\bigwedge^2 E = L$, and

$$\theta\,:\,E \;\longrightarrow\; E \otimes \xi$$

is an isomorphism. Therefore, E is a fixed point for the action of ξ on $M_{\mathrm{Sp}}(2, L)$.

The line bundle ξ defines a nontrivial étale covering

$$f\,:\,Y \;\longrightarrow\; X$$

of degree two, and E produces a line bundle $\eta \;\longrightarrow\; Y$ such that $f_*\eta = E$ (see [BNR], [Hi]). Therefore, η lies in the Prym subvariety of $J^{2\delta+1}(Y)$ associated to the covering f. The dimension of the Prym variety is $g-1$. On the other hand, the dimension of $M_{\mathrm{Sp}}(2, L)$ is $3g-3$. Since $3g-3 > g-1$, we conclude that the action of ξ on $M_{\mathrm{Sp}}(2, L)$ is effective. This completes the proof of the proposition. ☐

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