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Schur function expansions of KP tau functions associated to algebraic curves[1]

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Abstract

The Schur function expansion of Sato-Segal-Wilson KP τ-functions is reviewed. The case of τ-functions related to algebraic curves of arbitrary genus is studied in detail. Explicit expressions for the Plücker coordinate coefficients appearing in the expansion are obtained in terms of directional derivatives of the Riemann theta function or Klein sigma function along the KP flow directions. Using the fundamental bi-differential, it is shown how the coefficients can be expressed as polynomials in terms of Klein's higher genus generalizations of Weierstrass' ζ and $\wp$ functions. The cases of genus two hyperelliptic and genus three trigonal curves are detailed as illustrations of the approach developed here.

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Contents

1 Introduction

In the mid-1970s Novikov and Dubrovin ([Nov74], [DN74], [Dub75]), Its and Matveev ([IM75], [IM76]) and others ([Ma74], [La75], [MvM75], [DT76]) applied the Lax pair (isospectral deformation) approach to Hill's operator for periodic potentials with finite band spectrum and thereby determined finite gap periodic solutions to the KdV equation

$$\mathcal{U}_t = 6\,\mathcal{U}\mathcal{U}_x - \mathcal{U}_{xxx}. \tag{1.1}$$

Its and Matveev discovered a remarkable formula that provides periodic and, more generally, quasi-periodic solutions of the KdV equation explicitly as a second logarithmic derivative of the Riemann theta-function:

$$\mathcal{U}(x,t) = -\frac{\partial^2}{\partial x^2}\ln\theta(\mathbf{U}x + \mathbf{V}t + \mathbf{W}) + C \tag{1.2}$$

with $\mathbf{U},\mathbf{V},\mathbf{W} = \text{const} \in \mathbb{C}^g$ and $C \in \mathbb{C}$. The theta-function appearing here is determined by the period lattice of a hyperelliptic curve X of arbitrary genus g, and the "winding vectors" $\mathbf{U},\mathbf{V}$ are periods of abelian differentials of the second kind. This formed an important part of the general theory of algebro-geometric solutions of the KdV equation (see e.g. ([DMN76] for a review). Krichever [Kri77] extended these considerations more generally to quasi-periodic solutions of the KP hierarchy. This led to a general method of integration of such partial differential equations determined by the specification of an algebraic curve, and certain additional data on it. (See([Dub81]) for an overview of this approach and further applications.) These results had a great influence on subsequent developments in the theory of integrable nonlinear hierarchies, and their applications in various domains of mathematics and physics.

The phenomenon of algebro-geometric integrability has been considered from different viewpoints. In this paper we discuss the theory of tau-functions as initiated by Sato [SM80, SM81, SS82] and developed in the works Sato, Date, Jimbo, Kashiwara, Miwa and others (see e.g. [DJKM83]). We also make use of the geometrical formulation of Segal and Wilson [SW85]. In this approach the tau-function

$$\tau = \tau_w(\mathbf{t}), \quad \tau_w(\mathbf{0}) \neq 0 \tag{1.3}$$

is understood to depend on two sets of variables, an infinite dimensional vector $\mathbf{t} = (t_1, t_2, \dots) \in \mathbb{C}^\infty$ and an element $w \in Gr_{\mathcal{H}_+}(\mathcal{H})$ of an infinite dimensional Grassmannian consisting of subspaces of a polarized Hilbert space (the direct sum of two mutually orthogonal subspaces of essentially equal size)

$$\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_- \tag{1.4}$$

that are "commensurable" with the fixed subspace $\mathcal{H}_+$ (i.e., orthogonal projection to $\mathcal{H}_+$ is "large" - a Fredholm operator - while projection to $\mathcal{H}_-$ is, e.g., of Hilbert-Schmidt class). The Plücker relations, defining the embedding of $Gr_{\mathcal{H}_+}(\mathcal{H})$ as a subvariety of the projectivized infinite exterior space $\mathbf{P}(\Lambda\mathcal{H})$ are equivalent to an infinite set of bilinear differential relations in the $\mathbf{t}$ variables, the Hirota equations which, in turn imply the equations of the KP hierarchy.

In the special case to be considered here, w is determined by certain algebro-geometric data, the Krichever-Novikov-Dubrovin (KND) data, consisting of an algebraic curve X of genus g, and a non-special positive divisor of degree g:

$$\mathcal{D} = \sum_{i=1}^{g} p_i, \qquad p_i \in X, \tag{1.5}$$

a point p_∞ identified as "infinity" and a local parameter $\xi = \frac{1}{z}$ with $\xi(p_\infty) = 0$.

We further make a choice of a homology basis $\{\mathfrak{a}_1, \dots, \mathfrak{a}_g; \mathfrak{b}_1, \dots, \mathfrak{b}_g\}$ for X consisting of $\mathfrak{a}$ and $\mathfrak{b}$ cycles satisfying the intersection relations

$$\mathfrak{a}_i \circ \mathfrak{a}_j = 0, \quad \mathfrak{b}_i \circ \mathfrak{b}_j = 0, \quad \mathfrak{a}_i \circ \mathfrak{b}_j = \delta_{ij}, \tag{1.6}$$

a normalized basis $\{u_1, \dots, u_g\}$ of holomorphic abelian differentials, and a canonical polygonization of X obtained by cutting along the $\mathfrak{a}$ and $\mathfrak{b}$-cycles. Selecting an arbitrary base point p_0 the Abel map,

$$\mathcal{A} : (\mathcal{S}^g X) \to \mathcal{J}(X) = \mathbb{C}^g/\Gamma \tag{1.7}$$

from the gth symmetric power of X to its Jacobian variety (the quotient of $\mathbb{C}^g$ by the lattice of periods) is defined by integration of these differentials from the base point to the points in question. The KND data determine a vector $\mathbf{e} = (e_1, \dots, e_g)^T \in \mathbb{C}^g$ as the image of the divisor $\mathcal{D}$ under the Abel map (within the polygonization)

$$\mathbf{e} = \mathcal{A}(\mathcal{D}) - \mathcal{A}(p_\infty) + \mathbf{K} \tag{1.8}$$

translated by the Riemann constant $\mathbf{K}$ corresponding to the polygonization. We may then denote, in short, the corresponding τ-function as $\tau(\mathbf{e}, \mathbf{t})$.

The tau function can be represented as a Taylor series in $\mathbf{t}$ and then re-expressed as an expansion in a basis consisting of Schur function $s_\lambda(\mathbf{t})$, labelled by partitions, $\lambda = (\lambda_1, \lambda_2, \dots \lambda_{\ell(\lambda)})$, (where the λ_i's form a weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots$, with the last nonzero term $\lambda_{\ell(n)}$, where $\ell(\lambda)$ is the *length* of λ). The flow parameters $(t_1, t_2, \dots)$ may be identified with monomial sums

$$t_i = \frac{1}{i} \sum_{a=1}^{N} x_a^i \tag{1.9}$$

in terms of N auxiliary variables for any $N \geq \ell(\lambda)$, by taking the (stable) limit $N \to \infty$.

The Cauchy-Littlewood identity ([Mac90]) (or equivalently, the abelian group property of the KP flow flows) permits us to express this expansion in the form

$$\tau(\mathbf{e}, \mathbf{t}) = \sum_\lambda \left[s_\lambda \left(\frac{\partial}{\partial t_1}, \dots, \frac{1}{k} \frac{\partial}{\partial t_k}, \dots \right) \tau(\mathbf{e}, \mathbf{t}) \right] \Big|_{\mathbf{t}=\mathbf{0}} s_\lambda(\mathbf{t}), \tag{1.10}$$

where the summation runs over all partitions λ. The important point is that the coefficients in this expansion

$$\pi_\lambda(w) := s_\lambda \left(\frac{\partial}{\partial t_1}, \dots, \frac{1}{k} \frac{\partial}{\partial t_k}, \dots \right) \tau(\mathbf{e}, \mathbf{t})|_{\mathbf{t}=\mathbf{0}} \tag{1.11}$$

are just the Plücker coordinates of the element $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ under the Plücker embedding

$$\mathfrak{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbf{P}(\mathcal{F}) \tag{1.12}$$

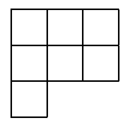
into the projectivization of the exterior space

$$\mathcal{F} = \Lambda\mathcal{H}, \tag{1.13}$$

which is a completion of the space of sums over a basis consisting of semi-infinite wedge products of basis elements of $\mathcal{H}$ (the Fermionic Fock space). In this setting, the Hirota bilinear relations of the KP hierarchy are simply equivalent to the Plücker relations satisfied by the coefficients $\{\pi_\lambda(w)\}$.

To each partition $\lambda = (\lambda_1, \lambda_2 \dots,)$ we associate, as usual, a Young diagram with λ_1 boxes in the first row, λ_2 in the second, etc. For example,

$$\lambda = (3,3,1) \quad \Leftrightarrow \tag{1.14}$$



Partitions of the form $(k, \underbrace{1, 1, \dots, 1}_{j}) \equiv (k, 1^j)$ are called *hooks*. Partitions may equivalently be expressed in Frobenius notation as

$$\lambda \sim (a_1, \dots, a_k | b_1, \dots, b_k), \tag{1.15}$$

where k is the number of diagonal boxes in the Young diagram, called the rank of the partition and the a_i's and b_i's are, respectively, the number of boxes to the right of and below the diagonal ones. The Giambelli formula

$$s_{(a_1,\dots,a_k|b_1,\dots,b_k)} = \det\left(s_{(a_i|b_j)}\right) \tag{1.16}$$

expresses the Schur function corresponding to an arbitrary partition as a determinant whose entries are the Schur functions corresponding to hook diagrams only. Sato also used Giambelli's formula in the decomposition of the coefficients of the expansion (1.10), since the same determinantal relations are valid, when expressed for the Plücker coordinates of the image of any element w of the Grassmannian (i.e. for any completely decomposable element of $\Lambda\mathcal{H}$), expressing them in terms those for hook partitions (see eq. (2.23). This amounts to an explicit solution of the Plücker relations, valid on the affine neighborhood corresponding to the "big cell" Using eq. (1.11), the resulting relations appear as partial differential equations, for which the tau-function plays the role of generating function.

Such expansions are valid for any tau-function $\tau_w(\mathbf{t})$, but here we will mainly consider τ-functions $\tau(\mathbf{e}, \mathbf{t})$ associated to the KND data on an algebraic curve X of genus g, and compute the coefficients in the Schur function expansion for this case.

Let the curve X be equipped with a canonical homology bases $(\mathfrak{a}_1, \ldots, \mathfrak{a}_g; \mathfrak{b}_1, \ldots, \mathfrak{b}_g)$, and corresponding polygonization, and choose a basis of holomorphic differentials $\mathbf{u} = (u_1, \ldots, u_g)^T$ ordered according to the degree of their vanishing at the the Weierstrass point p_∞ at infinity, $n_g + 1, \ldots, n_1 + 1$, where $\mathfrak{W} = (n_1, \ldots, n_g)$ is the Weierstrass gap sequence at p_∞ (see section 3.2).

Denote by

$$\mathfrak{A} = \left(\oint_{\mathfrak{a}_j} u_i \right)_{i,j=1,\ldots,g}, \quad \mathfrak{B} = \left(\oint_{\mathfrak{b}_j} u_i \right)_{i,j=1,\ldots,g} \tag{1.17}$$

the matrices of periods. The Jacobian of the curve is then $\mathrm{Jac}(X) = \mathbb{C}^g / \mathfrak{A} \oplus \mathfrak{B}$.

We will refer to $(\mathfrak{A}, \mathfrak{B})$ as the first period matrices or Riemann period matrices. The second period matrices, $(\mathfrak{S}, \mathfrak{T})$ are similarly formed from the periods of meromorphic differentials $\mathbf{r} = (r_1, \ldots, r_g)^T$ with poles only at the Weierstrass point p_∞, of orders $n_g + 1, \ldots, n_1 + 1$, respectively.

$$\mathfrak{S} = -\left(\oint_{\mathfrak{a}_j} r_i \right)_{i,j=1,\ldots,g}, \quad \mathfrak{T} = -\left(\oint_{\mathfrak{b}_j} r_i \right)_{i,j=1,\ldots,g} \tag{1.18}$$

normalized by the condition (generalized Legendre equation)

$$\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{S} & \mathfrak{T} \end{pmatrix} \mathbf{J} \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{S} & \mathfrak{T} \end{pmatrix}^T = -2\imath\pi \mathbf{J}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{0}_g & -\mathbf{1}_g \\ \mathbf{1}_g & \mathbf{0}_g \end{pmatrix}. \tag{1.19}$$

The matrix

$$\varkappa := \mathfrak{A}^{-1} \mathfrak{S} \tag{1.20}$$

is necessarily symmetric, $\varkappa^T = \varkappa$, and

$$\mathfrak{S} = \varkappa \mathfrak{A}, \qquad \mathfrak{T} = \varkappa \mathfrak{B} - \frac{\imath\pi}{2} \mathfrak{A}^{-1T}. \tag{1.21}$$

Remark 1.1 In the basis $(\mathbf{u}, \mathbf{r})$ the meromorphic differentials $\mathbf{r}$ are not uniquely defined, only up to the addition of a holomorphic differential. Therefore the matrix $\varkappa$ is only defined up to the addition of an arbitrary symmetric matrix. Nevertheless for our purposes, it is sufficient to choose a specific representative of the class of differentials $\mathbf{r}$, and this will be specified in each case treated in detail in the examples.

Following Baker [Bak07], we associate to the curve X the fundamental bi-differential $\Omega(p, q)$, which is the unique symmetric meromorphic 2-form on $X \times X$, with a second order pole on the diagonal $p = q$, and elsewhere holomorphic in each variable. Locally expressed, this has the form

$$\Omega(p,q) = \frac{\mathrm{d}\xi(p)\mathrm{d}\xi(q)}{(\xi(p)-\xi(q))^2} + \sum_{i,j=0}^{\infty} \mu_{ij}(p_0)\xi(p)^i\xi(q)^j\mathrm{d}\xi(p)\mathrm{d}\xi(q), \tag{1.22}$$

where and $\xi(p), \xi(q)$ are local coordinates in the vicinity of a base point p_0, $\xi(p_0) = 0$ and the coefficients $\mu_{ij}(p_0)$ are symmetric in the i,j-indices, $\mu_{ij}(p_0) = \mu_{ji}(p_0)$. The bi-form $\Omega(p,q)$ is normalized by the conditions

$$\oint_{\mathfrak{a}_j} \Omega(p,q) = 0, \quad j = 1,\dots,g. \tag{1.23}$$

Usually $\Omega(p,q)$ is realized as the second logarithmic derivative of the prime-form or theta-function [Fay73]. But in our development we use an alternative representation of $\Omega(p,q)$ in an *algebraic form* that goes back to Weierstrass and Klein, as described by Baker in [Bak97]

$$\Omega(p,q) = \frac{\mathcal{F}(p,q)}{P_y(p)P_w(q)(x-z)^2}\mathrm{d}x\mathrm{d}z + 2\mathbf{u}(p)^T\varkappa\mathbf{u}(q), \tag{1.24}$$

where $p = (x,y)$, $q = (z,w)$, the function $\mathcal{F}(p,q) = \mathcal{F}((x,y),(z,w))$ is a polynomial in its arguments, with coefficients depending on the parameters defining the curve X explicitly, as a planar model given by the polynomial equation

$$P(x,y) = 0, \tag{1.25}$$

$\mathbf{u}$ is the g-component vector whose entries are the holomorphic differentials and $\varkappa$ is a symmetric matrix (1.20) thereby providing the normalization (1.23) of $\Omega(p,q)$. We will refer to the first term on the right of (1.24), which involves the polynomial $\mathcal{F}(p,q)$, as the algebraic part[4], denoting it as $\Omega^{\mathrm{alg}}(p,q)$.[5] In the vicinity of the base point p_0, $\xi(p_0) = 0$ the form $\Omega^{\mathrm{alg}}(p,q)$ is expanded in a power series as

$$\Omega^{\mathrm{alg}}(p,q) = \frac{\mathrm{d}\xi(p)\mathrm{d}\xi(q)}{(\xi(p)-\xi(q))^2} + \sum_{i,j=0}^{\infty} \mu_{ij}^{\mathrm{alg}}(p_0)\xi(p)^i\xi(q)^j\mathrm{d}\xi(p)\mathrm{d}\xi(q), \tag{1.26}$$

[4]The factor 2 in the normalization makes the case $g = 1$ agree with the usual one in the Weierstrass theory of elliptic functions

[5]The representation (1.24) was further developed by Buchstaber, Leykin and one of the authors [BEL97] and more recently by Nakayashiki [Nak08].

where the quantities $\mu_{ij}^{\rm alg}(p_0)$ are algebraic functions of $\xi(p_0)$ and the coefficients of the curve. The transcendental part $\Omega^{\rm trans}(p,q)$ of $\Omega(p,q)$ is holomorphic and its expansion in the vicinity of the base point p_0 has the form

$$\begin{aligned}\Omega^{\rm trans}(p,q) &\equiv 2\mathbf{u}(p)^T\varkappa\mathbf{u}(q)\\ &= \sum_{i,j=0}^{\infty}\mu_{ij}^{\rm trans}(p_0)\xi(p)^i\xi(q)^j\mathrm{d}\xi(p)\mathrm{d}\xi(q).\end{aligned}\tag{1.27}$$

Therefore

$$\Omega(p,q)=\Omega^{\rm alg}(p,q)+\Omega^{\rm trans}(p,q)\tag{1.28}$$

and

$$\mu_{ij}(p_0)=\mu_{ij}^{\rm alg}(p_0)+\mu_{ij}^{\rm trans}(p_0)\tag{1.29}$$

for all p_0 and $i,j\in\mathbb{Z}$.

The algebraic representation of the fundamental differential, as described above, lies behind the definition of the multivariate sigma function in terms of the mutivariate (Riemann) theta-functions θ. It differs from θ by an exponential factor and a modular factor C:

$$\sigma(\mathbf{v})=C\exp\left\{\frac{1}{2}\mathbf{v}^T\varkappa\mathbf{v}\right\}\theta\left(\mathfrak{A}^{-1}\mathbf{v};\tau\right),\tag{1.30}$$

where $\varkappa$ and $\mathfrak{A}$ are defined in (1.20) and (1.17). These modifications make $\sigma(\boldsymbol{v})$ invariant with respect to the action of the symplectic group, so that for any $\gamma\in\mathrm{S}p(2g,\mathbb{Z})$, we have:

$$\sigma(\mathbf{v};\gamma\circ\mathfrak{M})=\sigma(\mathbf{v};\mathfrak{M}),\tag{1.31}$$

where $\mathfrak{M}$ is the set of periods of the curve X. The multivariate sigma-function is the natural generalization of the Weierstrass sigma function to algebraic curves of higher genera.

Remark 1.2 In his lectures, [Wei93], Weierstrass defined the sigma-function in terms of a series with coefficients given recursively, a key point of the Weierstrass theory of elliptic functions. A generalization of this result to genus two curves was begun by Baker [Bak07] and recently completed by Buchstaber and Leykin [BL05], who obtained recurrence relations between coefficients of the sigma-series in closed form. Buchstaber and Leykin also recently found an operator algebra that annihilates the sigma-function of higher genera (n,s)-curves [BL08]. Finding a suitable recursive definition of the higher genera sigma-functions along the lines of [BL08] remains a challenging problem.

In this paper we study the relation between the multivariate sigma function and the Sato τ-function [SM80, SM81, SS82] for the case of quasi-periodic solutions associated to KND data on an algebraic curve. Thus, we mainly consider this class of "algebro-geometric τ-function" solutions. These are essentially the same as those studied by Fay [Fay83, Fay89] in terms of θ-functions. In terms of σ functions such τ-functions may be expressed (see Proposition 3.1)[6]

$$\frac{\tau(\mathbf{e},\mathbf{t})}{\tau(\mathbf{e},\mathbf{0})} = \frac{\sigma\left(\sum_{k=1}^{\infty} \mathfrak{A}\mathbf{U}_k(p_\infty)t_k + \mathfrak{A}\mathbf{e}\right)}{\sigma(\mathfrak{A}\mathbf{e})} \exp\left\{\frac{1}{2}\sum_{k,l=0}^{\infty} \mu_{kl}^{\mathrm{alg}}(p_\infty)t_k t_l\right\}. \tag{1.32}$$

Here, as above, $\mathfrak{A}$ is the period matrix of holomorphic differentials, $\mathbf{U}_k(p_\infty), k = 1, 2, \ldots$ are "winding vectors" i.e. $\mathfrak{b}$ -periods of normalized second kind differentials with poles of order $k+1$ at the point p_∞, and $\mathbf{e}$ is an arbitrary point of the Jacobian variety $\mathrm{Jac}(X)$.

The paper is organized as follows. In section 2, we review the geometric formulation of the Sato-Segal-Wilson τ-function in terms of Hilbert space Grassmannians. The interpretation of the coefficients of the Schur function expansion as Plücker coordinates is derived (Proposition 2.2), as well as their expression in terms of the affine coordinates on the big cell, and hook partitions (Corollary 2.3).

In section 3, we restrict to the special case of τ-functions associated to KND data on an algebraic curve. The explicit formula for such τ-functions in terms of Riemann θ-functions is given in eq. (3.37). The affine coordinates determining the Schur function expansion are computed explicitly (eq. (3.31)) in terms of directional derivatives of the Riemann θ-function along the flow directions. In subsection 3.2, we review the Weierstrass gap theorem and introduce the normalized symmetric bi-differential $\Omega(p, q)$. This allows us to make explicit the splitting into "algebraic" and "transcendental" parts (Corollary 3.1) of the infinite quadratic form Q appearing in the exponential term in eq. (3.37) and the formula eq.(3.31) determining the affine coordinates. The equivalent expression for the τ-function in terms of the σ-function, which displays more clearly its modular transformation properties is derived in Proposition 3.1, eq. (3.77). From this, it is shown how to compute the affine coordinate matrix elements explicitly as polynomials in terms of the Kleinian functions $\{\zeta_i, \wp_{ij}\}$, that generalize the Weierstrass ζ- and $\wp$-functions to curves of arbitrary genus.

In section 4, we consider several examples for which the affine coordinates are explicitly calculated and use the Plücker relations to derive identities relating Kleinian ζ and $\wp$

[6]Recently A. Nakayashiki [Nak09] has independently suggested a similar expression for the algebro-geometric tau functions in terms of multivariate σ-functions and studied properties of the sigma-series.

functions of different degrees. The case of hyperelliptic curves is treated in detail, with explicit formulae computed for the case of genus $g = 2$. In Proposition 4.1 we derive algebraic relations between the Kleinian $\wp$-functions of different orders for this case. A further example based on a planar model of a class of trigonal curves is considered, and explicit expressions for the lowest affine coordinate matrix elements are computed in terms of the Kleinian ζ and $\wp$-functions for this case, as well as identities relating $\wp$-functions of different orders that follow again from the Plücker relations.

2 Background. Sato-Segal-Wilson τ-function

The following is a brief summary of the Sato [SS82] and Segal and Wilson [SW85] approach to KP τ-functions. The first is essentially algebraic in nature, the second functional analytic, but we combine elements of both. For more precise definitions of the main ingredients (Hilbert space Grassmannians, infinite transformation groups, determinant line bundles, Fermionic Fock space, the Plücker map, etc.), the reader is referred to these two sources, which agree in the geometric framework but not in analytic details. The introductory summary given here is intended to be as simple and self-contained as possible, and applicable to the cases at hand. Functional analytic details of the general formulation are either omitted or referred to [SW85], and the ingredients are made as much as possible to appear like the finite dimensional case.

2.1 Hilbert space Grassmannian and Plücker coordinates

Following [SW85], we start with the Hilbert space

$$\mathcal{H} := L^2(S^1) = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{2.1}$$

of square integrable complex valued functions f on the unit circle $\{z = e^{i\phi}\}$ in the complex plane, which may be split as a sum

$$f = f_+ + f_-, \quad f_\pm \in \mathcal{H}_\pm, \tag{2.2}$$

where $f_\pm \in \mathcal{H}_\pm$ are the positive and negative parts of the Fourier series.

Equivalently, $\mathcal{H}_+$ may be interpreted as the space of holomorphic functions on the interior unit disk and $\mathcal{H}_-$ as the space of holomorhic functions on the exterior, vanishing at $z = \infty$, with orthonormal bases provided by the monomials in z

$$\mathcal{H}_+ = \mathrm{span}\{e_j := z^{-j-1}\}_{j=-1,-2\ldots,}, \qquad \mathcal{H}_- = \mathrm{span}\{e_j := z^{-j-1}\}_{j=0,1,2\ldots}. \tag{2.3}$$

Remark 2.1 The convention of labeling the basis vectors e_j so $\mathcal{H}_+$ is the span of those having negative indexes j is chosen so that under the Plücker map (see below) $\mathcal{H}_+$ is taken into $|0\rangle = e_{-1} \wedge e_{-2} \wedge \ldots$ which is the "Dirac sea", in which all negative "energy" stated are occupied.

We denote by $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ the Hilbert space Grassmannian whose points are closed subspaces $w \subset \mathcal{H}$ that are commensurable with $\mathcal{H}_+$ in the sense that orthogonal projection $\pi^\perp : w \rightarrow \mathcal{H}_+$ to $\mathcal{H}_+$ along $\mathcal{H}_-$ is a Fredholm map with index zero and orthogonal projection $\pi^\perp : w \rightarrow \mathcal{H}_-$ to $\mathcal{H}_-$ along $\mathcal{H}_+$ is Hilbert-Schmidt. (In [SW85] this is called the *zero virtual dimension* component of the full Hilbert space Grassmannian.)

Let

$$\begin{aligned} w &= \mathrm{span}\{w_j\}_{j\in\mathbf{N}} \\ w_j &= \sum_{i\in\mathbf{Z}} w_{ij} e_i. \end{aligned} \tag{2.4}$$

Relative to the monomial basis $\{e_j\}$, the frame $\{w_0, w_1, \ldots\}$ may be represented as an infinite matrix W with components $\{w_{ij}\}_{i\in\mathbf{Z}, j\in\mathbf{N}}$ whose jth column vector W_j has components $\{w_{ij}\}_{i\in\mathbf{Z}}$

$$W = (W_0, W_1, \ldots) = \begin{array}{cc} \left(\begin{array}{c} \\ W_+ \\ \\ \\ \\ W_- \\ \\ \end{array}\right) & \begin{array}{c} -\infty \\ \vdots \\ -1 \\ \\ 0 \\ \vdots \\ +\infty \end{array} \end{array} \tag{2.5}$$

where the rows are labeled by the integers, increasing downward with the 0-th row at the top of the W_- block, and the columns are labeled by the non-negative integers, starting with 0 on the left and increasing to the right.

Remark 2.2 Note that the column labeling corresponds to the monomials degrees in $\mathcal{H}_+$ whereas the row labeling corresponds to the basis elements $\{e_j\}$

Here W_- may be viewed as representing a map $w_- : \mathcal{H}_+ \rightarrow \mathcal{H}_-$ and W_+ a map $w_+ : \mathcal{H}_+ \rightarrow \mathcal{H}_+$ such that w is the image of their sum

$$w = (w_- + w_+)(\mathcal{H}_+). \tag{2.6}$$

The Grassmannian $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ may be interpreted as the quotient of the bundle $\mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})$ of admissible frames by the right action of the group $\mathfrak{Gl}(\mathcal{H}_+)$ of linear changes of basis. (See [SW85] for the precise definition, which requires elements of $\mathfrak{Gl}(\mathcal{H}_+)$ to have a non-vanishing, finite determinant). Thus, $\mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})$ may be viewed, similarly to the finite dimensional case, as a principal $\mathfrak{Gl}(\mathcal{H}_+)$ bundle over $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$), to which we may associate, through the determinant representation, a line bundle $\mathrm{Det}\rightarrow\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ and its dual $\mathrm{Det}^*\rightarrow\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$. A holomorphic section of the latter is determined by each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0)$, $\lambda_i \in \mathbf{N}$, where $\ell(\lambda)$ denotes the length, in a way that mimics the finite dimensional case. Extending the set of parts $\{\lambda_j\}_{j=1,\ldots\ell(\lambda)}$ in the usual way [Mac90] to an infinite sequence $\{\lambda_j\}_{j=1,\ldots\infty}$ by defining,

$$\lambda_j = 0 \quad \text{if} \quad j > \ell(\lambda), \tag{2.7}$$

the determinant $\det(W_\lambda)$ of the submatrix W_λ of W consisting of the rows $\{\lambda_i - i\}_{i\in\mathbf{N}^+}$ defines a holomorphic section $\sigma : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})\rightarrow\mathrm{Det}^*$ of the bundle $\mathrm{Det}^*\rightarrow\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ and these span the space of admissible sections. (Again, for the full analytic details required to define the class of admissible sections, see [SW85].)

As in the finite dimensional case, the space of such holomorphic sections may be identified with a certain subspace $\mathcal{F}_0 \subset \mathcal{F}$ of the exterior space $\mathcal{F} := \Lambda\mathcal{H}$, the "zero charge sector" of $\mathcal{F}$, with the latter interpreted as the full "Fermionic Fock space". Let $\mathcal{F}_0$ be the span of the exterior elements

$$|\lambda\rangle := e_{l_1} \wedge e_{l_2} \wedge e_{l_3} \wedge \cdots\ , \tag{2.8}$$

where, for any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, the l_i's are the "particle coordinates",

$$l_i = \lambda_i - i. \tag{2.9}$$

These form an orthonormal basis with respect to the inner product induced on $\mathcal{F}_0 \subset \Lambda\mathcal{H}$ by the one on $\mathcal{H}$.

For each partition λ, we may also define $\mathcal{H}_\lambda \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ as

$$\mathcal{H}_\lambda = \mathrm{span}\{e_{l_i}\}_{i\in\mathbf{N}^+} \tag{2.10}$$

In particular, the element $\mathcal{H}_0$ corresponding to the trivial partition $\lambda = 0$ is $\mathcal{H}_+$. We define the Plücker map $\hat{\mathfrak{P}} : \mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})\rightarrow\mathcal{F}_0$ in the natural way

$$\hat{\mathfrak{P}} : \mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})\rightarrow\mathcal{F}_0 \tag{2.11}$$

$$\hat{\mathfrak{P}} : \{w_0, w_1, \ldots\}\mapsto w_0 \wedge w_1 \wedge w_2 \wedge \cdots \tag{2.12}$$

Changing the frame $\{w_0, w_1, \dots\}$ spanning the subspace $w \subset \mathcal{H}$ by application (on the right) of an element $g \in \mathfrak{Gl}(\mathcal{H}_+)$ just changes the image under $\hat{\mathfrak{P}}$ by the nonzero multiplicative factor $\det(g)$. Therefore the Plücker map (2.12) on the frame bundle projects to a map embedding $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ into the projectivization $\mathbf{P}(\mathcal{F}_0)$ of $\mathcal{F}_0$

$$\begin{aligned} \mathfrak{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) &\rightarrow \mathbf{P}(\mathcal{F}_0) \\ \mathfrak{P} : \{w_0, w_1, \dots\} &\mapsto [w_0 \wedge w_1 \wedge w_2 \wedge \cdots], \end{aligned} \tag{2.13}$$

where $[|v\rangle]$ denotes the projective equivalence class of $|v\rangle \in \mathcal{F}_0$. In particular,

$$\mathfrak{P}(\mathcal{H}_\lambda) = [|\lambda\rangle]. \tag{2.14}$$

The image of $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ under $\mathfrak{P}$ is the orbit of $\mathcal{H}_+$ under the identity component of the general linear group $\mathfrak{Gl}(\mathcal{H})$ (again, suitably defined, as in [SS82] or [SW85]).

The Plücker coordinates $\{ \pi_\lambda(|v\rangle)\}$ of an element $|v\rangle \in \mathcal{F}_0$ are just its components relative to the orthonormal basis $\{|\lambda\rangle\}$:

$$\begin{aligned} \pi_\lambda(|v\rangle) &= \langle\lambda|v\rangle \\ |v\rangle &= \sum_\lambda \pi_\lambda |\lambda\rangle. \end{aligned} \tag{2.15}$$

Applying the Plücker map $\hat{\mathfrak{P}}$ to an element $\{w_0, w_1, \dots\} \in \mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})$ spanning $w \in \mathrm{Gr}_{\mathcal{H}_+(\mathcal{H})}$, the Plücker coordinates of its image are the homogeneous coordinates of $\mathfrak{P}(w)$ under the Plücker map (2.13)

$$\pi_\lambda(w) := \pi_\lambda(\mathfrak{P}(w)). \tag{2.16}$$

It follows from the above that

$$\pi_\lambda(w) = \det(W_\lambda), \tag{2.17}$$

and hence the basis of holomorphic sections $H^0(\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}), \mathrm{Det}^*)$ of Det^* defined by the partitions λ correspond precisely to the Plücker coordinates.

The Plücker relations are the infinite set of quadratic relations that determine the image of $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ under the Plücker map. They follow from the fact that for any $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$, this image $\mathfrak{P}(w)$ is a decomposable element of $\mathcal{F}_0$. The Plücker coordinates of $\pi_\lambda(w)$ are therefore not independent; it is possible to express them on an open dense affine subvariety as finite determinants in terms of a much smaller subset consisting, e.g., of those Plücker coordinates corresponding to hook partitions. It is convenient to use Frobenius notation $(a_1 a_2 \cdots a_r | b_1 b_2 \cdots b_r)$ for a partition, [Mac90], where (a_i, b_i) are the number of elements to the right and below the ith diagonal element of the Young diagram, for $i = 1 \dots r$. A hook partition $\lambda = (a+1, 1^b)$ is one for which $r = 1$, and hence in Frobenius notation is expressed $(a \mid b)$.

To see how to express the Plücker coordinate $\pi_{(a_1 a_2 \cdots a_r | b_1 b_2 \cdots b_r)}(w)$ corresponding to an arbitrary partition in terms of those for the hook partitions $\{(a_i \mid b_j)\}_{1 \le i,j \le r}$, it is easiest to assume that w is in the "big cell", in which the map $w_+ : \mathcal{H}_+ \to \mathcal{H}_+$ is invertible. The infinite matrix W_+ in (2.5) is therefore also invertible, and we may define affine coordinates as the matrix entries of

$$A := W_- W_+^{-1}. \tag{2.18}$$

By convention, we will let the indices (a, b) range over the non-negative integers, and therefore the componentwise interpretation of (2.18) is

$$A_{ab} := (W_- W_+^{-1})_{ab}, \qquad a, b \in \mathbb{N}. \tag{2.19}$$

The homogeneous coordinates in this basis have the form

$$W W_+^{-1} = \begin{pmatrix} \mathbf{I} \\ A \end{pmatrix}, \tag{2.20}$$

where $\mathbf{I}$ is the semi-infinite identity matrix, $I_{ij} = \delta_{-i-1,j}$ which, in view of the labeling convention has $\{ij\}$-th entry $i \le -1,\ j \ge 0$ and the labeling convention for the matrix A in the lower block is that the pair of indices (ab) start with $(0,0)$ in the upper left-hand corner, and increase downward and to the right. This allows us to express all the Plücker coordinates as finite determinants in terms of the affine coordinates on the big cell.

Proposition 2.1 *The Plücker coordinate $\pi_{(a_1 a_2 \cdots a_r | b_1 b_2 \cdots b_r)}(w)$ corresponding to the partition $(a_1 a_2 \cdots a_r | b_1 b_2 \cdots b_r)$ is*

$$\begin{aligned} \pi_{(a_1 a_2 \cdots a_r | b_1 b_2 \cdots b_r)}(w) &= \det(W_{(a_1 a_2 \cdots a_r | b_1 b_2 \cdots b_r)}) \\ &= (-1)^{\sum_{k=1}^r b_k} \det\left(A_{a_i, b_j}|_{1 \le i,j \le r} \right) \det(W_+)\ . \end{aligned} \tag{2.21}$$

In particular, we may consider the case of hook partitions, which, according to (2.21), coincide within a sign with the components of the affine coordinate matrix A, allowing all other Plücker coordinates to be expressed as finite determinants in terms of these.

Corollary 2.1 *The Plücker coordinate corresponding to a hook partition $(a|b)$ is, within a sign, the (a, b) affine coordinate,*

$$\pi_{(a|b)}(w) = (-1)^b A_{ab}, \tag{2.22}$$

and hence

$$\pi_{(a_1 a_2 \cdots a_r | b_1 b_2 \cdots b_r)}(w) = \det\left(\pi_{(a_i | b_j)}(w)\Big|_{1 \le i,j \le r} \right). \tag{2.23}$$

2.2 Abelian flow group Γ_+ and the KP τ-function

We now introduce the abelian subgroup $\Gamma_+ \subset \mathfrak{Gl}(\mathcal{H})$ consisting of nonvanishing elements of $\mathcal{H}_+$ normalized to equal 1 at the origin $z = 0$

$$\begin{aligned} \Gamma_+ &:= \{\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i z^i}\} \\ \mathbf{t} &:= (t_1, t_2, \ldots\}, \end{aligned} \tag{2.24}$$

acting on $\mathcal{H}$ by multiplication

$$\begin{aligned} \Gamma_+ \times \mathcal{H} &\to \mathcal{H} \\ (\gamma, f) &\mapsto \gamma f. \end{aligned} \tag{2.25}$$

This induces an action on the Grassmannian

$$\begin{aligned} \Gamma_+ \times Gr_{\mathcal{H}_+}(\mathcal{H}) &\to Gr_{\mathcal{H}_+}(\mathcal{H}) \\ (\gamma(\mathbf{t}), w) &\mapsto \gamma(\mathbf{t}) w \end{aligned} \tag{2.26}$$

which lifts to one on the bundle $\mathrm{Det}^* \to \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$, and determines an action on the space of holomorphic sections $H^0(\mathrm{Det}^*, \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}))$

$$\begin{aligned} \Gamma_+ \times H^0(\mathrm{Det}^*, \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) &\to H^0(\mathrm{Det}^*, \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \\ (\gamma(\mathbf{t}), \sigma) &\mapsto \tilde{\gamma}(\mathbf{t})\sigma := \sigma \circ \gamma^{-1}(\mathbf{t}) \\ \tilde{\gamma}(\mathbf{t})\sigma(w) &:= \tilde{\gamma}(\mathbf{t})\sigma(\gamma^{-1}(\mathbf{t})w) \end{aligned} \tag{2.27}$$

The latter coincides, within normalization, with the induced action on $\mathcal{F}_0 \subset \Lambda\mathcal{H}$. Let

$$w(\mathbf{t}) = \gamma(\mathbf{t})(w) \tag{2.28}$$

be the image of $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ under the action of the group element $\gamma(\mathbf{t})$ and let $W(\mathbf{t})$ be its matrix of homogeneous coordinates relative to the standard basis of monomials $\{e_j\}_{j \in \mathbf{Z}}$. Then

$$\pi_\lambda(w(\mathbf{t})) = \det(W_\lambda(\mathbf{t})) \tag{2.29}$$

is the Plücker coordinate of $w(\mathbf{t})$ corresponding to the partition λ. The KP τ-function is defined to be the Plücker coordinate corresponding to the trivial partition $\lambda = 0$

$$\tau_w(\mathbf{t}) := \pi_0(w(\mathbf{t})) = \det(W_+(\mathbf{t})). \tag{2.30}$$

Since, as will be seen in the next subsection, all other Plücker coordinates of $w(\mathbf{t})$ may be determined from $\tau_w(\mathbf{t})$ by applying constant coefficient differential operators in the $\mathbf{t}$ variables, defined in terms of the Schur functions, the Plücker relations for $\mathfrak{P}(w(\mathbf{t}))$ may be expressed as an infinite system of bilinear differential relations satisfied by $\tau_w(\mathbf{t})$, the Hirota relations [SS82], which are equivalent to the hierarchy of KP flow equations.

2.3 Schur function expansions

Recall that if we view the flow parameters $\mathbf{t} = (t_1, t_2, \ldots)$ as power sums in terms of a set $\{x_1, \ldots, x_N\}$ of N auxiliary variables

$$t_i = \frac{1}{i} \sum_{a=1}^{N} x_a^i, \tag{2.31}$$

the Schur function s_λ, which is the irreducible character of the tensor representation of $\mathfrak{Gl}(N)$ with symmetry type corresponding to partition λ, is given by the Jacobi-Trudi formula [Mac90]

$$s_\lambda(\mathbf{t}) = \det(h_{\lambda_i - i + j})|_{1 \le i,j \le n} \tag{2.32}$$

for any $n \geq \ell(\lambda)$, where $\{h_j(\mathbf{t})\}_{j=1,\ldots\infty}$ are the complete symmetric functions defined by the generating function formula

$$e^{\sum_{i=1}^{\infty} t_i z^i} = \sum_{j=0}^{\infty} h_j(\mathbf{t}) z^j. \tag{2.33}$$

We have $h_0(\mathbf{t}) = 1$ and it is understood in (2.32) that $h_j(\mathbf{t}) := 0$ for $j < 0$.

Given any function $f(\mathbf{t})$ that admits a Taylor expansion in the flow variables about the origin $\mathbf{0} := (0, 0, \ldots)$,

$$f(\mathbf{t}) = (e^{\sum_{i=1}^{\infty} t_i \frac{\partial}{\partial \tilde{t}_i}} f(\tilde{\mathbf{t}}))|_{\tilde{\mathbf{t}}=\mathbf{0}}, \tag{2.34}$$

where $\tilde{\mathbf{t}} = (\tilde{t}_1, \tilde{t}_2, \ldots)$, we may use the Schur functions as a basis and express the series as

$$f(\mathbf{t}) = \sum_{\lambda} f_\lambda s_\lambda(\mathbf{t}). \tag{2.35}$$

(This determines the "Bose-Fermi equivalence", which associates an element $\sum_\lambda f_\lambda |\lambda\rangle$ of the Fermi Fock space $\mathcal{F}_0$ to the Bose-Fock space element f, viewed as a symmetric function of an underlying infinite series of parameters $\{x_a\}_{a=1,\ldots\infty}$ related by (2.31), in the inductive limit, to the flow parameters $\mathbf{t}$.) Using the Cauchy-Littlewood identity [Mac90]

$$e^{\sum_{i=1}^{\infty} i t_i \tilde{t}_i} = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}}) \tag{2.36}$$

in the form

$$e^{\sum_{i=1}^{\infty} t_i \frac{\partial}{\partial t_i}} = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\partial_{\mathbf{t}}), \tag{2.37}$$

where

$$\partial_{\mathbf{t}} := \left\{ \frac{1}{i} \frac{\partial}{\partial t_i} \right\}_{i=1,2,\ldots}, \tag{2.38}$$

we obtain

$$f_\lambda = s_\lambda(\partial_{\mathbf{t}})(f(\mathbf{t}))|_{\mathbf{t}=\mathbf{0}}. \tag{2.39}$$

For the tau function $\tau_w(\mathbf{t})$, the coefficients in this expansion coincide with the Plücker coordinates $\pi_\lambda(w)$ of the initial point $w \in Gr_{\mathcal{H}_+}(\mathcal{H})$.

Proposition 2.2 (Sato [SS82]) *The Schur function expansion of $\tau_w(\mathbf{t})$ is*

$$\tau_w(\mathbf{t}) = \sum_\lambda \pi_\lambda(w) s_\lambda(\mathbf{t}). \tag{2.40}$$

The Plücker coordinates are therefore given by

$$\pi_\lambda(w) = s_\lambda(\partial_{\mathbf{t}})\left(\tau_w(\mathbf{t})\right)|_{\mathbf{t}=\mathbf{0}}. \tag{2.41}$$

Proof. Using formula (2.33), it is easy to see that the matrix representation of the action (2.25) is given by:

$$W(\mathbf{t}) = \begin{pmatrix} H_{++}(\mathbf{t}) & H_{+-}(\mathbf{t}) \\ 0 & H_{--}^T(\mathbf{t}) \end{pmatrix} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}, \tag{2.42}$$

where

$$H_{++}(\mathbf{t}) := \begin{pmatrix} \ddots & \ddots & \vdots & \vdots \\ \ddots & 1 & h_1 & h_2 \\ \cdots & 0 & 1 & h_1 \\ \cdots & 0 & 0 & 1 \end{pmatrix}$$

$$H_{+-}(\mathbf{t}) := \begin{pmatrix} \vdots & & & \\ h_3 & \vdots & & \\ h_2 & h_3 & \vdots & \\ h_1 & h_2 & h_3 & \cdots \end{pmatrix}, \qquad H_{--}(\mathbf{t}) := \begin{pmatrix} 1 & h_1 & h_2 & h_3 & \cdots \\ 0 & 1 & h_1 & h_2 & \cdots \\ 0 & 0 & 1 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{2.43}$$

Letting

$$H(\mathbf{t}) := \left(H_{++}(\mathbf{t}) \quad H_{+-}(\mathbf{t}) \right), \tag{2.44}$$

we have

$$\tau_w(\mathbf{t}) = \det(H(\mathbf{t})W) = \sum_\lambda \det(H_\lambda(\mathbf{t}))\det(W_\lambda) = \sum_\lambda \pi_\lambda(H^T(\mathbf{t}))\pi_\lambda(w), \tag{2.45}$$

where the second equality is the Cauchy-Binet identity and

$$\pi_\lambda(H(\mathbf{t})) := \det(h_{\lambda_i-i+j}(\mathbf{t}))|_{1\le i,j\le\infty} = s_\lambda(\mathbf{t}) \tag{2.46}$$

by (2.32). We thus obtain the Schur function expansion (2.40) of the τ-function.

On the "big cell", each $\pi_\lambda(w)$ is determined through (2.21) as a finite determinant in terms of the affine coordinates $\{A_{ab}\}$ of w which, by (2.22), coincide within sign with the hook partition Plücker coordinates. Therefore, we need only apply (2.41) to obtain these

$$(-1)^b A_{ab} = \pi_{(a|b)}(w) = s_{(a|b)}(\partial_{\mathbf{t}})\left(\tau_w(\mathbf{t})\right)|_{\mathbf{t}=\mathbf{0}}. \tag{2.47}$$

Substituting the expression (2.21) for the Plücker coordinates $\pi_{(a_1,\dots,a_r|b_1,\dots,b_r)}$ in (2.40) we obtain

Corollary 2.2

$$\frac{\tau_w(\mathbf{t})}{\tau_w(\mathbf{0})} = \sum_\lambda (-1)^{\sum_{k=1}^r b_k}\det\left(A_{a_i,b_j}\big|_{1\le i,j\le r}\right) s_\lambda(\mathbf{t}). \tag{2.48}$$

A further simplification may be made using the following identity, for which a simple proof follows from the above definitions.

Lemma 2.1

$$s_{(a|b)}(\mathbf{t}) = (-1)^b \sum_{j=1}^{b+1} h_{b-j+1}(-\mathbf{t}) h_{a+j}(\mathbf{t}). \tag{2.49}$$

Proof. By the Jacobi-Trudi formula (2.32),

$$s_{(a|b)}(\mathbf{t}) = \det\begin{pmatrix} \mathbf{h}^T & h \\ \mathbf{H} & \mathbf{k}\end{pmatrix}, \tag{2.50}$$

where

$$\mathbf{f}^T = (h_{a+1}(\mathbf{t}), h_{a+2}(\mathbf{t}) \cdots h_{a+b}(\mathbf{t})), \qquad h := h_{a+b+1}(\mathbf{t}), \tag{2.51}$$

$$\mathbf{k} := \begin{pmatrix} h_b(\mathbf{t}) \\ h_{b-1}(\mathbf{t}) \\ \vdots \\ h_1(\mathbf{t})\end{pmatrix}, \quad \mathbf{H} := \begin{pmatrix} 1 & h_1(\mathbf{t}) & h_2(\mathbf{t}) & \cdots & \cdots & h_{b-1}(\mathbf{t}) \\ 0 & 1 & h_1(\mathbf{t}) & h_2(\mathbf{t}) & \cdots & h_{b-2}(\mathbf{t}) \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & & \cdots & 1 \end{pmatrix} \tag{2.52}$$

It follows from the generating function formula (2.33) that the inverse $\mathbf{H}^{-1}$ is given by

$$\mathbf{H}^{-1} := \begin{pmatrix} 1 & h_1(-\mathbf{t}) & h_2(-\mathbf{t}) & \cdots & \cdots & h_{b-1}(-\mathbf{t}) \\ 0 & 1 & h_1(-\mathbf{t}) & h_2(-\mathbf{t}) & \cdots & h_{b-2}(-\mathbf{t}) \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & & \cdots & 1 \end{pmatrix} \tag{2.53}$$

and

$$\sum_{j=-a}^{b} h_{a+j}(-\mathbf{t}) h_{b-j}(\mathbf{t}) = \delta_{ab}. \tag{2.54}$$

The matrix

$$\begin{pmatrix} \mathbf{0}^T & \mathbf{H}^{-1} \\ 1 & -h^{-1}\mathbf{h}^T\mathbf{H}^{-1} \end{pmatrix} \tag{2.55}$$

has determinant $(-1)^b$, and its product with the matrix in (2.50) is

$$\begin{pmatrix} \mathbf{h}^T & h \\ \mathbf{H} & \mathbf{k} \end{pmatrix} \begin{pmatrix} \mathbf{0}^T & \mathbf{H}^{-1} \\ 1 & -h^{-1}\mathbf{h}^T\mathbf{H}^{-1} \end{pmatrix} = \begin{pmatrix} h & \mathbf{0}^T \\ \mathbf{k} & \mathbf{I} - h^{-1}\mathbf{k}\mathbf{h}^T\mathbf{H}^{-1} \end{pmatrix}. \tag{2.56}$$

Therefore

$$s_{(a|b)}(\mathbf{t}) = (-1)^b h \det(\mathbf{I} - h^{-1}\mathbf{k}\mathbf{h}^T\mathbf{H}^{-1}) = (-1)^b (h - \mathbf{H}^{-1}\mathbf{k}\mathbf{h}^T). \tag{2.57}$$

But it follows from (2.53) and (2.54) that

$$\mathbf{H}^{-1}\mathbf{k} = -\begin{pmatrix} h_b(-\mathbf{t}) \\ h_{b-1}(-\mathbf{t}) \\ \vdots \\ h_1(-\mathbf{t}) \end{pmatrix}, \tag{2.58}$$

from which (2.49) follows, in view of the definition (2.51) of h. QED.

Substituting the identity (2.49) into (2.47) thus gives

Corollary 2.3

$$A_{ab} = -\sum_{j=0}^{b} h_{a+j-1}(-\partial_{\mathbf{t}}) h_{b-j}(\partial_{\mathbf{t}}) \left(\tau_w(\mathbf{t})\right)|_{\mathbf{t}=0}. \tag{2.59}$$

The relations (2.21), (2.41) (2.47) determining the Plücker coordinates of $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ on the "big cell" are equivalent to the fact that the formal Baker-Akhiezer function [SM80],[SM81], [SW85], defined by the Sato formula [SS82]

$$\psi_w(z, \mathbf{t}) = e^{\sum_{i=1}^{\infty} t_i z^i} \frac{\tau_w(\mathbf{t} - [z^{-1}])}{\tau_w(\mathbf{t})}, \tag{2.60}$$

where

$$[z^{-1}] := \left(\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \dots\right), \tag{2.61}$$

takes its values in $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ for all values of $\mathbf{t}$.

Following Sato, we may also introduce the dual Baker function:

$$\Psi_w^*(z, \mathbf{t}) = -\frac{\tau_w(\mathbf{t} + [z^{-1}])}{\tau(\mathbf{t})} \exp\left\{-\sum_{i=1}^{\infty} t_i z^i\right\}. \tag{2.62}$$

Then, as shown in [SM80, SM81, SS82], the KP hierarchy equations may all be expressed in the form of Hirota bilinear equations for the τ-function.

Theorem 2.1 (Hirota bilinear relation [SM80, SM81, SS82])

$$\mathrm{Res}_{z=0}\, \Psi_z^*(z, \mathbf{t}) \Psi_z^*(z, \widetilde{\mathbf{t}}) \equiv 0, \tag{2.63}$$

where the $\mathrm{Res}_{z=0}$ *just means the coefficient of* z^{-1} *in the formal Laurent expansion about* $z = 0$ *and the relation is satisfied identically in the infinite set of KP flow variables* $\mathbf{t} = (t_1, t_2, \dots)$, $\widetilde{\mathbf{t}} = (\widetilde{t}_1, \widetilde{t}_2, \dots)$.

Remark 2.3 In view of (2.62) , eq.(2.63) is equivalently written as

$$\mathrm{Res}_{z=0} \exp\left\{\sum_{i=1}^{\infty} t_i z^i\right\} \exp\left\{-\sum_{i=1}^{\infty} \widetilde{t}_i z^i\right\} \tau(\mathbf{t} - [z^{-1}]) \tau(\widetilde{\mathbf{t}} + [z^{-1}]) \equiv 0 \tag{2.64}$$

identically in $\mathbf{t}$ and $\widetilde{\mathbf{t}}$.

Remark 2.4 It is also shown in [SM80, SM81, SS82] that (2.60) is simply an expression of the infinite set of Plücker relations satisfied by the coefficients $\pi_\lambda(w)$ appearing in the Schur function expansion (2.40)

3 Algebraic curves

3.1 Baker-Akhiezer function and tau function for algebraic curves

A particularly important class of tau functions consists of those associated to algebraic curves [Dub81, Kri77, DJKM83]. The relevant data needed to define these are: an algebraic curve X of genus g, a positive nonspecial divisor of degree g

$$\mathcal{D} := \sum_{i=1}^{g} p_i, \quad p_i \in X, \tag{3.1}$$

(or, equivalently, a degree g line positive bundle $\mathcal{L}{\rightarrow}X$ in general position satisfying suitable generic stability conditions), a point "at infinity" $p_\infty \in X$ and a local parameter $\xi = \frac{1}{z}$ defined on a disc

$$D_\infty := \{p(\zeta), \quad |\zeta| \leq 1, \quad p(0) = p_\infty\} \tag{3.2}$$

centered at p_∞. The points p_i are assumed to lie in the complement $D_0 := X - D_\infty$. Identifying S^1 with ∂D_∞, the associated element $w := w(X, \mathcal{D}, p_\infty, \zeta) \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is the closure of the space of functions $f \in L^2(S^1)$ admitting a meromorphic extension to $\bar{D}_0$ with pole divisor subordinate to $\mathcal{D}$.

To realize the construction we use canonical theta-functions of g variables, $g \geq 1$

$$\theta(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\left\{\imath\pi\mathbf{m}^T\mathbf{T}\mathbf{m} + 2\imath\pi\mathbf{m}^T\mathbf{z}\right\}, \quad \mathbf{z} \in \mathbb{C}^g, \tag{3.3}$$

where $\mathbf{T}$ is a complex symmetric $g \times g$-matrix with positive definite imaginary part. The space of such matrices (the Siegel upper half-space) will be denoted $\mathcal{S}^g$. The theta-function is holomorphic on $\mathbb{C}^g \times \mathcal{S}^g$ and satisfies

$$\theta(\mathbf{z} + \mathbf{n}) = \theta(\mathbf{z}), \quad \theta(\mathbf{z} + \mathbf{T}\mathbf{n}) = \exp\{-\imath\pi(\mathbf{n}^T\mathbf{T}\mathbf{n} + 2\mathbf{z}^T\mathbf{n})\}\theta(\mathbf{z}). \tag{3.4}$$

In the considered case of θ-functions associated to an algebraic curve

$$\mathbf{T} := \mathfrak{A}^{-1}\mathfrak{B}, \tag{3.5}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are the period matrices of a basis of holomorphic differentials.

As shown in [Dub81, Kri77, DJKM83], the corresponding Baker-Akhiezer function may be chosen as the restriction to ∂D_+ of a meromorphic function on $X - p_\infty$ with pole divisor $\mathcal{D}$ and having an essential singularity at p_∞ of the form

$$\psi_w(p(\zeta), \mathbf{t}) \sim e^{\sum_{i=1}^\infty t_i z^i}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right). \tag{3.6}$$

The Riemann-Roch theorem implies that there is just a one dimensional space of such functions. They may be expressed, within a $\mathbf{t}$-dependent normalization, as

$$\tilde{\psi}_w(p,\mathbf{t}) = e^{\int_{p_0}^{p}\Omega(\mathbf{t})}\frac{\theta(\mathcal{A}(p)-\mathcal{A}(\mathcal{D})+\sum_{i=1}^{\infty}\mathbf{U}_i t_i-\mathbf{K})}{\theta(\mathcal{A}(p)-\mathcal{A}(\mathcal{D})-\mathbf{K})}, \tag{3.7}$$

where θ is the theta-function with T equal to the Riemann matrix of periods defined below relative to a suitable polygonization obtained by cutting along a canonical homology basis $(\mathfrak{a}_1,\ldots,\mathfrak{a}_g,\mathfrak{b}_1,\ldots,\mathfrak{b}_g)$ with intersection matrix

$$\mathfrak{a}_i\circ\mathfrak{a}_j = \mathfrak{b}_i\circ\mathfrak{b}_j = 0, \quad \mathfrak{a}_i\circ\mathfrak{b}_j = \delta_{ij}, \tag{3.8}$$

p_0 is an arbitrarily chosen base point,

$$\begin{aligned}\mathcal{A}:\quad & \mathcal{S}^g(X)\rightarrow\mathbb{C}^g\\ \mathcal{A}:\ = \quad & \sum_{j=1}^{g}p_j\mapsto\sum_{j=1}^{g}\int_{p_0}^{p_j}\boldsymbol{\omega}, \quad \boldsymbol{\omega} = \begin{pmatrix}\omega_1\\ \vdots\\ \omega_g\end{pmatrix}\end{aligned} \tag{3.9}$$

is the Abel map with ith component

$$\mathcal{A}_i := \sum_{j=1}^{g}\int_{p_0}^{p_j}\omega_i, \quad i=1,\ldots,g, \tag{3.10}$$

where $(\omega_1,\ldots,\omega_g)$ is a canonically normalized basis of the space $H^0(K)$ of holomorphic abelian differentials

$$\oint_{a_i}\omega_j = \delta_{ij}, \quad \oint_{b_i}\omega_j = T_{ij}, \tag{3.11}$$

and $\mathbf{K}\in\mathbf{C}^g$ is the Riemann constant, chosen so that $\theta(\mathcal{A}(p)-\mathcal{A}(\mathcal{D})-\mathbf{K})$ vanishes at the g points $p=p_i$ in the divisor $\mathcal{D}$.

Define the linear family of abelian differentials of the second type

$$\Omega(\mathbf{t}) = \sum_{j=1}^{\infty}\Omega_j t_j, \tag{3.12}$$

where Ω_i is the unique normalized abelian differential of the second kind with pole divisor of degree $j+1$ at p_∞ having local form

$$\Omega_j \sim d(z^j) + \text{holomorphic} \tag{3.13}$$

near p_∞, with vanishing $a-$cycles

$$\oint_{\mathfrak{a}_i} \Omega_j = 0, \quad i = 1, \dots, g. \tag{3.14}$$

Then $2\pi i \mathbf{U}_j \in \mathbb{C}^g$ is defined to be its vector of b-cycles with components

$$\oint_{\mathfrak{b}_k} \Omega_j =: 2\pi i (\mathbf{U}_j)_k, \quad k = 1, \dots, g. \tag{3.15}$$

In order to compare with the formal Baker function $\psi_a(z, \mathbf{t})$ appearing in the Sato formula (2.60), we must interpret $p = p(z)$ within the punctured disc $D_\infty - p_\infty$ and on its boundary, and normalize $\tilde{\psi}(p, \mathbf{t})$ in formula (3.7) so as to obtain the correct local expansion (3.6) near $\xi = 0$

$$\psi_w(p(\xi), \mathbf{t}) = \frac{\tilde{\psi}_w(p(\xi), \mathbf{t})}{a_0(\mathbf{t})}, \tag{3.16}$$

where

$$\tilde{\psi}_w(p(\xi), \mathbf{t}) \sim e^{\int_{p_0}^{p(\xi)} \Omega(\mathbf{t})} \left(a_0(\mathbf{t}) + a_1(\mathbf{t})\xi + \cdots\right). \tag{3.17}$$

Since $\int_{p_0}^{p} \Omega_i$ has the local expansion

$$\int_{p_0}^{p} \Omega_i = \xi^{-i} + \sum_{j=1}^{\infty} \frac{1}{j} Q_{ij} \ \xi^j + q_j, \tag{3.18}$$

where

$$Q_{ij} = Q_{ji}, \quad 1 \le i, j \le \infty, \tag{3.19}$$

and $\mathcal{A}(p(z))$ has the expansion [Dub81], [Dic03]

$$\mathcal{A}(p(z)) = \mathcal{A}(p_\infty) - \sum_{j=1}^{\infty} \frac{1}{j} \mathbf{U}_j z^{-j}, \tag{3.20}$$

this gives the formula

$$\psi_w(p(z), \mathbf{t}) = e^{\sum_{i=1}^\infty t_i \left(z^i + \sum_{j=1}^\infty \frac{1}{j} Q_{ji} z^{-j}\right)} \ \frac{\theta(\mathbf{e} + \sum_{i=1}^\infty \mathbf{U}_i (t_i - \frac{1}{i} z^{-i}))\theta(\mathbf{e})}{\theta(\mathbf{e} - \sum_{i=1}^\infty \frac{1}{i} \mathbf{U}_i z^{-i})\theta(\mathbf{e} + \sum_{i=1}^\infty \mathbf{U}_i t_i)} \tag{3.21}$$

near $z = \infty$ where

$$\mathbf{e} := \mathcal{A}(p_\infty) - \mathcal{A}(\mathcal{D}) - \mathbf{K}. \tag{3.22}$$

(Note that the assumptions behind formula (3.7) imply that $\mathbf{e}$ is not on the theta divisor; $\theta(\mathbf{e}) \neq 0$.) The last ratio of theta function factors in (3.21),

$$\frac{\theta(\mathbf{e})}{\theta\left(\mathbf{e}+\sum_{i=1}^{\infty}\mathbf{U}_i t_i\right)}$$

does not depend on z, and hence the space spanned by the values of $\psi_w(p(z),\mathbf{t})$ is the same as that spanned by

$$\check{\psi}_w(p(z),\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i\left(z^i+\sum_{j=1}^{\infty}\frac{1}{j}Q_{ji}z^{-j}\right)}\ \frac{\theta(\mathbf{e}+\sum_{i=1}^{\infty}\mathbf{U}_i(t_i-\frac{1}{i}z^{-i}))}{\theta(\mathbf{e}-\sum_{i=1}^{\infty}\frac{1}{i}\mathbf{U}_i z^{-i})}. \tag{3.23}$$

We may expand the remaining ratio of theta function terms as a power series in z^{-1}

$$\begin{aligned}\theta(\mathbf{e}+\sum_{i=1}^{\infty}\mathbf{U}_i(t_i-\ \tfrac{1}{i}z^{-i}))e^{\sum_{i=1}^{\infty}-\frac{1}{iz^i}\frac{\partial}{\partial t_i}}\left(\theta(\mathbf{e}+\sum_{i=1}^{\infty}\mathbf{U}_i t_i)\right)&\\ =\sum_{j=0}^{\infty}z^{-j}h_j(-\nabla_{\mathbf{U}})\theta(\mathbf{e}+\sum_{i=1}^{\infty}\mathbf{U}_i t_i)&\end{aligned} \tag{3.24}$$

and

$$\theta(\mathbf{e}-\sum_{i=1}^{\infty}\ \tfrac{1}{i}\mathbf{U}_i z^{-i}) = \sum_{j=0}^{\infty} z^{-j}h_j(-\nabla_{\mathbf{U}})\theta(\mathbf{e}), \tag{3.25}$$

where

$$\nabla_{\mathbf{U}} := (\nabla_{\mathbf{U}_1}, \frac{1}{2}\nabla_{\mathbf{U}_2}, \frac{1}{3}\nabla_{\mathbf{U}_3}, \dots) \tag{3.26}$$

and $\nabla_{\mathbf{U}_i}$ is the directional derivative in $\mathbf{C}^g$ along U_i.

Now define a basis $\{w_0, w_1, \dots\}$ for w as

$$w_0(z) := \check{\psi}(z,\mathbf{t})_{\mathbf{t}=0} = 1, \quad w_j(z) := \left(\frac{\partial\check{\psi}(z,\mathbf{t})}{\partial t_j}\right)_{\mathbf{t}=0}, \quad j\geq 1. \tag{3.27}$$

Then

$$w_j(z) = z^j + P_{0j} + \sum_{i=1}^{\infty}(\ \tfrac{1}{i}Q_{ij}+P_{ij})z^{-i}, \quad j=1,2,\dots \tag{3.28}$$

where

$$\sum_{i=0}^{\infty}P_{ij}z^{-i} := \frac{\sum_{i=0}^{\infty}M_{ij}z^{-i}}{\sum_{i=0}^{\infty}N_i z^{-i}} \tag{3.29}$$

and

$$M_{ij} := \nabla_{\mathbf{U}_j}h_i(-\nabla_{\mathbf{U}})\theta(\mathbf{e}), \quad N_i := h_i(-\nabla_{\mathbf{U}})\theta(\mathbf{e}). \tag{3.30}$$

It follows that the affine coordinates A_{ij} of the element w are

$$\begin{aligned} &A_{ij} := \tfrac{1}{i+1} Q_{i+1,j} + P_{i+1,j}, \quad i = 0, 1, \ldots, \quad j = 1, 2, \ldots \\ &A_{i0} = 0, \quad i = 0, 1, 2 \ldots \end{aligned} \tag{3.31}$$

By Corollary 2.1 this determines the Plücker coordinates for all hook partitions and hence, by (2.23), for all partitions.

Comparing formula (3.21) for the Baker function with the Sato formula (2.60), we see that the tau function for $w = w(X, \mathcal{D}, p_\infty, \zeta)$ is given by (cf. [Dic03])

$$\tau_w(\mathbf{t}) = e^{\sum_{i=1}^\infty \lambda_i t_i} e^{-\frac{1}{2}\sum_{i,j=1}^\infty Q_{ij} t_i t_j} \theta(\mathbf{e} + \sum_{i=1}^\infty t_i \mathbf{U}_i), \tag{3.32}$$

where

$$\lambda_i := \mu_i + i \sum_{k=0}^{i-1} \frac{Q_{k,i-k}}{2k(i-k)} \tag{3.33}$$

with the μ_i's defined by

$$\theta(\mathbf{e}) e^{\sum_{i=1}^\infty \frac{\mu_i}{iz^i}} := \theta(\mathbf{e} - \sum_{i=1}^\infty \frac{\mathbf{U}_i}{iz^i}) = \sum_{i=0}^\infty N_i z^{-i}. \tag{3.34}$$

From the viewpoint of the KP hierarchy, however, the linear exponential factor $e^{\sum_{i=1}^\infty \lambda_i t_i}$ in (3.32) may be removed, since this just corresponds to a gauge transformation of the Baker-Akhiezer function that is constant in the $\mathbf{t}$ variables

$$\psi_w(z, \mathbf{t}) \rightarrow k(z) \psi_w(z, \mathbf{t}), \quad k(z) := e^{\sum_{i=1}^\infty \frac{\lambda_i}{iz^i}}, \tag{3.35}$$

which leaves the solutions to the KP flow equations invariant. Equivalently, this means replacing $w \in Gr_{\mathcal{H}_+}(\mathcal{H})$ by

$$w_k := \mathrm{span}\{kv, v \in w\}. \tag{3.36}$$

Therefore the tau function may be chosen in the simpler gauge equivalent form (cf. [DJKM83, Fay83, Dic03])

$$\tau_w(\mathbf{t}) = e^{-\frac{1}{2}\sum_{i,j=1}^\infty Q_{ij} t_i t_j} \theta(\mathbf{e} + \sum_{i=1}^\infty t_i \mathbf{U}_i). \tag{3.37}$$

Henceforth, we denote this τ-function as $\tau(\mathbf{e}, \mathbf{t}) = \tau_w(\mathbf{t})$.

Applying formula (2.59) directly to $\tau_w(\mathbf{t})$ as defined in (3.37) provides an alternative way to compute the affine coordinates A_{ab} that is equivalent, within such a gauge transformation, to (3.31).

The Hirota bilinear relations (2.64) may in this case equivalently be written

$$\begin{aligned}\mathrm{Res}_{\zeta=0}\frac{1}{\xi^2}\Bigg[&\exp\left\{\sum_{n=1}^{\infty}t_n\xi^{-n}\right\}\exp\left\{-\sum_{n=1}^{\infty}\frac{\xi^n}{n}\frac{\partial}{\partial t_n}\right\}\tau(\mathbf{e};\mathbf{t})\\ &\times\exp\left\{-\sum_{n=1}^{\infty}\widetilde{t}_n\xi^{-n}\right\}\exp\left\{\sum_{n=1}^{\infty}\frac{\xi^n}{n}\frac{\partial}{\partial\widetilde{t}_n}\right\}\tau(\mathbf{e};\widetilde{\mathbf{t}})\Bigg]=0,\end{aligned} \tag{3.38}$$

where $\xi = \xi(q)$ is the local coordinate of the point q near p, $\xi(p) = 0$ defining the KP hierarchy.

3.2 Weierstrass gaps, bases, and the fundamental bi-differential

The Weierstrass gap theorem [FK80] says:

Theorem 3.1 (Lückensatz) *For any point $p \in X$ on a nonsingular algebraic curve X of genus g there exist precisely g distinct non-negative integers $n_1, \ldots, n_g$, satisfying the inequalities*

$$n_1 = 1 < n_2 < \ldots < n_g < 2g, \tag{3.39}$$

such that no meromorphic function on X may have pole divisors solely at p of degrees $(n_1, \ldots, n_g)$.

The set of integers $(n_1, \ldots, n_g)$ is called the Weierstrass gap numbers, and will be denoted $\mathfrak{W}(p)$. For a point in "general position" the gap numbers are $1, 2, \ldots, g$. A point $p \in X$ that admits a meromorphic function that has a pole of order smaller then $g + 1$ is called a Weierstrass point.

Given the point p_∞, and local parameter $\xi(p)$, the $2g$ dimensional space $\mathrm{H}_1^*(X,\mathbb{Z})$ dual to the homology group $\mathrm{H}_1(X,\mathbb{Z})$ may be identified with a space of meromorphic differentials having poles at p_∞ only, with vanishing residues, the pairing being given by integration over cycles. A basis $\{u_1, \ldots, u_g, \Omega_{n_1}, \ldots \Omega_{n_g}\}$ for this space consists of the g elements $\{u_1, \ldots, u_g\}$ providing a basis for the subspace $\mathrm{H}^0(X, K)$ of holomorphic differentials, defined so that u_j vanishes to order $n_j - 1$ at p_∞

$$u_k = -\left(\xi(p)^{n_k-1} + \text{higher order terms}\right)\mathrm{d}\xi(p), \quad k = 1, \ldots, g, \quad n_k \in \mathfrak{W}(p), \tag{3.40}$$

with higher order terms $\xi(p)^k$ consisting only of powers k for which $k+1 \notin \mathfrak{W}(p)$.

The remaining basis elements $\{\Omega_{n_1}, \dots, \Omega_{n_g}$ are the normalized differentials of second kind where the Ω_k's are as defined in (3.13), (3.14). These are mutually dual under the pairing

$$\frac{1}{2\pi i}\int\int_X u_j \wedge \Omega_{n_k} = \operatorname*{Res}_{p=p_\infty}\left(\left(\int_{p_0}^{p} u_j\right)\Omega_{n_k}(p)\right) = \delta_{jk}. \tag{3.41}$$

Denote the non-vanishing period matrices around the $\mathfrak{a}$ and $\mathfrak{b}$ cycles

$$\oint_{\mathfrak{a}_j} u_i =: \mathfrak{A}_{ij}, \quad \oint_{\mathfrak{b}_j} u_i =: \mathfrak{B}_{ij}, \tag{3.42}$$

$$\frac{1}{2\pi i}\oint_{\mathfrak{b}_j} \Omega_{n_i} =: C_{ij} = (U_{n_i})_j, \qquad i,j = 1,\dots,g. \tag{3.43}$$

The columns of the $g \times g$ matrix $\mathbf{C}$ are thus given by the vectors $\mathbf{U}_{n_i}$ corresponding to the Weierstrass gaps

$$\mathbf{C} = (\mathbf{U}_{n_1}, \mathbf{U}_{n_2}, \dots, \mathbf{U}_{n_g}), \quad n_j \in \mathfrak{W}(p_\infty). \tag{3.44}$$

The Riemann bilinear relations, obtained by applying Stokes theorem to the 2-forms $u_j \wedge u_k$ and $u_j \wedge \Omega_k$ on the canonical polygonization of the curve, then imply

$$\begin{aligned} \mathfrak{A}\mathfrak{B}^T &= \mathfrak{B}\mathfrak{A}^T, \\ \mathfrak{A}\mathbf{C}^T &= \mathbf{1}_g. \end{aligned} \tag{3.45}$$

The relation to the basis $\{\omega_1, \dots \omega_1, \Omega_{n_1}, \dots, \Omega_{n_g}\}$ of normalized differentials is thus

$$\omega_i = \sum_{j=1}^{g} \mathfrak{A}^{-1}_{ij} u_j = \sum_{j=1}^{g} C_{ji} u_j, \tag{3.46}$$

and the normalized Riemann period matrix $\mathbf{T}$ is

$$\mathbf{T} := \mathfrak{A}^{-1}\mathfrak{B} = \mathbf{C}^T\mathfrak{B}. \tag{3.47}$$

Defining the vectors

$$\mathbf{R}_j = \mathfrak{A}\mathbf{U}_j, \quad j = 1, 2, \dots, \tag{3.48}$$

it follows from (3.20) that the differentials u_i have the local expansion

$$u_i(p(\xi)) = -\sum_{j=n_i}^{\infty} (\mathbf{R}_j)_i \xi(p)^{j-1} d\xi(p). \tag{3.49}$$

It is also convenient to introduce the normalized symmetric bi-differential $\Omega(p,q)$ on $X \times X$, [Kle86],[Kle88], [Fay73] defined by the conditions

- $\Omega(p,q)$ has a second order pole on the diagonal $p=q$ where its local form, expressed in terms of the parameters $\xi(p), \xi(q)$ is

$$\Omega(p,q) = \left(\frac{1}{(\xi(q)-\xi(p)^2} + \sum_{i,j=0}^{\infty} \mu_{ij}\xi(p)^i \xi(q)^j \right) \mathrm{d}\xi(p)\mathrm{d}\xi(q) \tag{3.50}$$

with

$$\mu_{ij} = \mu_{ji}. \tag{3.51}$$

- $\Omega(p,q)$ is holomorphic elsewhere in both p and q.
- The a-cycles all vanish

$$\oint_{p\in\mathfrak{a}_j} \Omega(p,q) = \oint_{q\in\mathfrak{a}_j} \Omega(p,q) = 0. \tag{3.52}$$

These conditions uniquely determine $\Omega(q,p)$, which may be given the explicit representation

$$\Omega(p,q) = \mathrm{d}_p \mathrm{d}_q \ln\theta(\mathcal{A}(p) - \mathcal{A}(q) + \boldsymbol{\delta}), \tag{3.53}$$

where $\boldsymbol{\delta}$ is non-singular odd half-period.

The following further properties also follow from the definitions.

- The b-cycles are given by the normalized holomorphic differentials ω_j, $j=1,\dots g$

$$\oint_{p\in\mathfrak{b}_j} \Omega(p,q) = 2\pi i \omega_j(q), \quad \oint_{q\in\mathfrak{b}_j} \Omega(p,q) = 2\pi i \omega_j(p). \tag{3.54}$$

- The residues of the locally defined bi-differentials $\xi(p)^{-j}\Omega(p,q)$, $\xi(q)^{-j}\Omega(p,q)$ at p_∞ are given by the normalized second type differentials Ω_j,

$$\operatorname*{Res}_{p=p_\infty} \xi(p)^{-j}\Omega(p,q) = -\Omega_j(q), \quad \operatorname*{Res}_{q=p_\infty} \xi(q)^{-j}\Omega(p,q) = -\Omega_j(p). \tag{3.55}$$

- The coefficients μ_{ij} in the expansion (3.50) are related those in the expansion (3.18) as follows

$$\mu_{ij} = -Q_{i+1,j+1}. \tag{3.56}$$

3.3 Planar model of the curve

Henceforth, we assume the algebraic curve X of geometric genus $g \geq 1$ is given by the equation

$$X: \; P(x,y) = 0, \quad x, y \in \mathbf{C}, \tag{3.57}$$

with P a polynomial in x and y

$$P(x,y) = y^n + a_1(x) y^{n-1} + \ldots + a_n(x), \tag{3.58}$$

where $a_k(x), k = 0 \ldots, n$ are polynomials in x and $n > 1$. Suppose that the curve X has a Weierstrass point at infinity, p_∞ where coordinates x, y are locally expressed as

$$x = \frac{1}{\xi^n} + \ldots, \quad y = \frac{1}{\xi^s} + \ldots \tag{3.59}$$

and the order of any monomial term the polynomial $P(x,y)$ is the order of its pole at p_∞. Now suppose that the curve X can be written in the form

$$y^n - x^s + \quad \text{lower order terms} \; = 0, \tag{3.60}$$

where $n, s > 2$ are positive integers .

In what follows the planar coordinates of a point p are denoted $(x(p), y(p))$. When considering local expansions near to a reference point p_∞ with local parameter $\xi(p)$ we also use $p(\xi)$ to denote the point, with $p(0) = p_\infty$. It is then possible to include the principal part of Ω in an explicit algebraic expression in terms of the coefficients of the curve X.

Theorem 3.2 ([Bak97]) *The fundamental bi-differential may be expressed in the form*

$$\Omega(p,q) = \frac{\mathcal{F}(p,q)}{(x(p)-x(q))^2} \frac{\mathrm{d}x(p)\mathrm{d}x(q)}{P_y(x(p),y(p))P_y(x(q),y(q))} + \sum_{i=1}^{g}\sum_{j=1}^{g} \omega_i(p)\gamma_{ij}\omega_j(q) \tag{3.61}$$

$$= \frac{\mathcal{F}(p,q)}{(x(p)-x(q))^2} \frac{\mathrm{d}x(p)\mathrm{d}x(q)}{P_y(x(p),y(p))P_y(x(q),y(q))} + \sum_{i=1}^{g}\sum_{j=1}^{g} u_i(p)\varkappa_{ij}u_j(q) \tag{3.62}$$

where $\mathcal{F}(p,q)$ is a polynomial function of the coordinates $(x(p), y(p), x(q), y(q))$ which is a linear form in the coefficients of the polynomials $\{a_0(x), \ldots a_n(x)\}$ defining the planar model of the curve (3.58), and the symmetric $g \times g$ matrices $\varkappa$, γ have elements $\varkappa_{ij}$, γ_{ij} given by

$$\gamma_{ij} = (\mathfrak{A}^T \varkappa \mathfrak{A})_{ij} = -\oint_{p\in\mathfrak{a}_i}\oint_{q\in\mathfrak{a}_j} \frac{\mathcal{F}(p,q)}{(x(p)-x(q))^2} \frac{\mathrm{d}x(p)\mathrm{d}x(q)}{P_y(x(p),y(p))P_y(x(q),y(q))}. \tag{3.63}$$

Remark 3.1 The representation (3.62) of the fundamental bi-differential $\Omega(p,q)$ in the hyperelliptic case is classical and may be found in the books [Bak97] and [Bak07]. It is summarized in [BEL97] and extended to non-hyperelliptic curves in [EEL00], [BEL00].

Remark 3.2 As already mentioned above in Remark 1.1, the matrix $\varkappa$ appearing in the definition of the multi-variable σ-function is defined only up to the addition of an arbitrary symmetric matrix, say $\boldsymbol{\chi}$. The change $\varkappa \to \varkappa + \boldsymbol{\chi}$, however, does not affect the higher Klein formula nor the algebraic and differential relations between the $\wp$-functions that follow from this formula. In the examples to follow, explicit expressions will be given for the polynomial $\mathcal{F}(p,q)$ defining the fundamental bi-differential $\Omega(p,q)$.

The following is a consequence of (3.62).

Corollary 3.1 *The coefficients μ_{ij} in the expansion (3.50) can be decomposed as a sum*

$$\mu_{ij} = -Q_{i+1,j+1} = \mu_{ij}^{\mathrm{alg}} + \mu_{ij}^{\mathrm{trans}}, \tag{3.64}$$

where μ_{ij}^{alg} is defined by the expansion

$$\begin{aligned} \Omega^{\mathrm{alg}}(p,q) &= \frac{\mathcal{F}(p,q)}{(x(p)-x(q))^2}\frac{\mathrm{d}x(p)\mathrm{d}x(q)}{P_y(x(p),y(p))P_y(x(q),y(q))} \\ &= \left(\frac{1}{(\xi(q)-\xi(p))^2} + \sum_{i,j=0}^{\infty} \mu_{ij}^{\mathrm{alg}}\xi(p)^i\xi(q)^j\right)\mathrm{d}\xi(q)\mathrm{d}\xi(p), \end{aligned} \tag{3.65}$$

$\xi(p)$ and $\xi(q)$ are local coordinates of the points p,q and $\mu_{ij}^{\mathrm{trans}}$ is given by

$$\mu_{ij}^{\mathrm{trans}} = \mathbf{R}_{i+1}^T \varkappa \mathbf{R}_{j+1}. \tag{3.66}$$

Proof: The coefficients μ_{ij} in the expansion of the normalized symmetric bi-differential $\Omega(p,q)$ given in (3.50) (projective connection) near the diagonal $p=q$ can be expressed as a sum $\mu_{ij} = \mu_{ij}^{\mathrm{alg}} + \mu_{ij}^{\mathrm{trans}}$. The first term is obtained by expansion of the l.h.s of (3.65) as a rational bi-form on $X \times X$. The second term follows from the local expansion in powers of ξ of the holomorphic differentials u_i in the second term of the r.h.s of (3.62). It is transcendental and given by (3.66). The term μ_{ij}^{alg} defines the holomorphic part of the expansion of the rational 2-form appearing in the first term in (3.61), (3.62).

Remark 3.3 Note that the double integral in (3.63) can be decomposed into periods of basic holomorphic and meromorphic second kind differentials. Following the Baker construction [Bak97],[Bak98], we represent the integrand $\Omega^{\mathrm{alg}}(p,q)$ in the form

$$\begin{aligned} \Omega^{\mathrm{alg}}(p,q) &= \frac{\mathcal{F}((x,y),(z,w))}{(x-z)^2}\frac{\mathrm{d}x\mathrm{d}z}{P_y(x,y)P_z(z,w)} \\ &= \frac{\partial}{\partial z}\Pi_{(z,w)}^{(z',w')}(x,y)dz + \mathbf{u}^T(x,y)\mathbf{r}(z,w). \end{aligned} \tag{3.67}$$

Here $\Pi_{(z,w)}^{(z',w')}(x,y)$ is a third kind differential with first order poles in points (z,w) and (z',w') and corresponding residues ± 1, $\mathbf{u} = (u_1,\dots,u_g)^T$ is the vector of holomorphic differentials normalized as in eq. (3.40), $\mathbf{r} = (r_1,\dots,r_g)^T$ is the vector of meromorphic differentials with poles of orders $n_1+1,\dots,n_g+1$ at p_∞. (The pole location (z',w') can be taken arbitrary and does not affect the construction.) The differentials $\mathbf{r}$ are then chosen to satisfy the symmetry condition

$$\Omega^{\mathrm{alg}}(p,q) = \Omega^{\mathrm{alg}}(q,p). \tag{3.68}$$

The explicit algebraic construction of the differentials $\mathbf{r}$ is described in [Bak97] and further developed in [BEL97]. In particular, in the case of a hyperelliptic curve

$$P(x,y) = y^2 - \mathcal{P}_{2g+1}(x), \tag{3.69}$$

where $\mathcal{P}_{2g+1}(x)$ is a polynomial in x of degree $2g+1$,

$$\Pi_{(z,w)}^{(z',w')}(x,y) = \frac{1}{2y}\left\{\frac{y+w}{x-z} - \frac{y+w'}{x-z'}\right\}\mathrm{d}x \tag{3.70}$$

The second term in (3.70) can be taken as an arbitrary finite point of the curve.

The matrices of $\mathfrak{a}$ and $\mathfrak{b}$-periods of the differentials $\mathbf{r}$ as given in the introduction by eqs. (1.21) and (1.20) enter in the definition of the multi-variable σ-function (3.72) below.

3.4 σ-functions and algebro-geometric formulae for $\pi_\lambda(w)$

Let $\mathcal{D} = p_1 + \ldots + p_g$ be a positive non-special divisor of degree g and

$$\mathbf{v} := \sum_{i=1}^{g}\int_{p_\infty}^{q_i}\mathbf{u} + \mathfrak{A}\mathbf{K} = -\mathfrak{A}\mathbf{e}, \tag{3.71}$$

where $\mathfrak{A}$ is defined in (3.43), $\mathbf{K}$ is the vector of Riemann constants, with the base point at p_∞ and $\mathbf{u} = (u_1,\dots,u_g)^T$.

The multivariable σ-function $\sigma(\mathbf{v})$ is defined by the formula

$$\sigma(\mathbf{v}) = C\theta(\mathbf{e})\exp\left\{\frac{1}{2}\mathbf{v}^T\varkappa\mathbf{v}\right\}, \quad \mathbf{e} = -\mathfrak{A}^{-1}\mathbf{v} \tag{3.72}$$

and C is a constant depending on the moduli of the curve, whose explicit form is not needed here. This definition naturally generalizes the Weierstrass σ-function from the theory of elliptic functions to higher genera.

The multi-variable $\boldsymbol{\zeta}$-functions, $\boldsymbol{\zeta} = (\zeta_1,\dots,\zeta_g)$ are defined as

$$\zeta_k(\mathbf{v}) = \frac{\partial}{\partial v_k}\ln\sigma(\mathbf{v}), \quad k = 1,\dots,g. \tag{3.73}$$

The Kleinian $\wp$-functions are the second logarithmic derivatives of the sigma function

$$\wp_{i,k}(\mathbf{v}) = -\frac{\partial^2}{\partial v_i \partial v_k} \ln \sigma(\mathbf{v}), \quad i,k = 1,\dots,g \tag{3.74}$$

More generally, we denote higher order logarithmic derivatives as

$$\wp_{\underbrace{i_1,\dots,i_1}_{m_1},\dots,\underbrace{i_k,\dots,i_k}_{m_k}}(\mathbf{v}) = -\frac{\partial^{m_1 i_1+\ldots+m_k i_k}}{\partial v_{i_1}^{m_1}\cdots\partial v_{i_k}^{m_k}} \ln \sigma(\mathbf{v}), \quad i_1,\dots,i_k \in \{1,\dots,g\}.$$

In the classical theory the following theorem provides the basic means to derive algebraic and differential relations between multi-variable $\wp$-functions. (See e.g. [Bak97] and the more recent exposition [BEL97]).

Theorem 3.3 (Klein formula) *Let the planar curve X of genus g be defined by the polynomial equation $P(x,y) = 0$. Choose a set of independent holomorphic differentials in the form*

$$u_k = \frac{\phi_k(x,y)}{f_y(x,y)}dx, \quad k = 1,\dots,g, \tag{3.75}$$

where $\phi_k(x,y)$ are polynomials in x and y. Let $p = (x,y)$ be an arbitrary point of X and $\mathcal{D} = p_1+p_2+\ldots+p_g$ a positive non-special divisor on X, $p_k = (x_k,y_k)$ and $p = (x,y) \in X$ - arbitrary point. Let $\mathbf{v}$ be the shifted image of $\mathcal{D}$ under the Abel map as in (3.71). Then

$$\sum_{j,k=1}^{g} \wp_{j,k}\left(\int_{p_0}^{p} \mathbf{u} - \mathbf{v}\right) \phi_k(x,y)\phi_j(x_r,y_r) = \frac{\mathcal{F}(p,p_r)}{(x-x_r)^2}, \quad r = 1,\dots,g, \tag{3.76}$$

where the polynomial $\mathcal{F}(p,p_r) = \mathcal{F}((x,y);(x_r,y_r))$ defines the fundamental bi-differential $\Omega(p,p_r)$.

Remark 3.4 This formula was first given for hyperelliptic curves in [Kle86], [Kle88]. But the proof, which is based on the Riemann vanishing theorem and the representation of the fundamental bi-differential in the form (3.62) can easily be extended to non-hyperelliptic curves.

The Weierstrass-Poincaré theorem (see e.g. [Igu82]) says that any $g+1$ Abelian functions on the Jacobian of an algebraic curve of genus g are algebraically dependent. In particular, it follows from this that in the case of a hyperelliptic curve of genus g, among the $(g+1)g/2$ functions $\wp_{ij}$ there are only g that are algebraically independent, so these functions must satisfy $g(g-1)/2$ relations. (Using the Klein formula (3.76) it was shown in [BEL97] that these relations are quartics that represent a Kummer variety. (See the example of a genus 2 curve in Section 4.1 below).)

Another set of relations that follow from the Klein formula describe the Jacobi variety of the curve X and integrable flows of KP type using $\wp$-functions as coordinates. In particular in the case of hyperelliptic curves all products $\wp_{ijk}\wp_{p,q,r}$ are cubic polynomials in $\wp_{ij}$. (or more details see [BEL97].) Here we will show that these results can also be obtained within τ-function theory on the basis of the Sato formula or, equivalently from the bilinear Identity. To do so we represent the Sato τ-function in terms of the multi-variable σ-function of Klein. These methods of derivation of integrable hierarchies of KP type, Jacobi and Kummer varieties are compared in [EEG10].

Proposition 3.1 *The normalized algebro-geometric function $\tau(\mathbf{e},\mathbf{t};)/\tau(\mathbf{e},\mathbf{0};)$ is expressible in terms of the multi-variable σ-function as*

$$\frac{\tau(\mathbf{e},\mathbf{t})}{\tau(\mathbf{e},\mathbf{0})} = \frac{\sigma\left(\sum_{k=1}^{\infty}\mathbf{R}_k t_k+\mathbf{v}\right)}{\sigma(\mathbf{v})}\exp\left\{\sum_{k=1}^{\infty}\Lambda_k(\mathbf{v})t_k\right\}\exp\left\{\frac{1}{2}\sum_{k,l=1}^{\infty}\mu_{kl}^{\mathrm{alg}}t_k t_l\right\}, \tag{3.77}$$

Here $\mathbf{v}=\mathfrak{A}\mathbf{e}$ is the shifted Abelian image (3.71) of the positive non-special divisor $\mathcal{D}$ and μ_{ik}^{alg} are coefficients in the expansion of the algebraic part of the bi-differential $\Omega(p,q)$ near the point p_∞. The coefficients $\Lambda_k(\mathbf{v})$ are given by

$$\Lambda_k(\mathbf{v}) = \mathbf{R}_k^T\varkappa\mathbf{v}\quad k=1,2,\ldots \tag{3.78}$$

.

Proof. The algebro-geometric τ-function in the gauge transformed form (3.37) leads to the relation

$$\frac{\tau(\mathbf{e},\mathbf{t})}{\tau(\mathbf{e},\mathbf{0})} = \exp\left\{\frac{1}{2}\sum_{i,j=0}^{\infty}\mu_{ij}^{\mathrm{alg}}t_i t_j+\frac{1}{2}\sum_{i,j=0}^{\infty}\mu_{ij}^{\mathrm{trans}}t_i t_j\right\}\frac{\theta(\mathbf{e}+\sum_{i=1}^{\infty}\mathbf{U}_i t_i)}{\theta(\mathbf{e})} \tag{3.79}$$

where λ_i is given in (3.33) and the relation (3.56) was used.

Otherwise from the definition of σ-function we get

$$\begin{aligned}\sigma\left(\sum_{k=1}^{\infty}\mathbf{R}_k t_k+\mathbf{v}\right) &= C\theta\left(\sum_{k=1}^{\infty}\mathbf{U}_k t_k+\mathfrak{A}^{-1}\mathbf{v}\right)\\ &\times\ \exp\left\{\frac{1}{2}\sum_{k,l=1}^{\infty}\mathbf{R}_k^T\varkappa\mathbf{R}_l t_k t_l\right\}\\ &\times\ \exp\left\{\sum_{k=1}^{\infty}\mathbf{R}_k^T\varkappa\mathbf{v}\,t_k\right\}\exp\left\{\frac{1}{2}\mathbf{v}^T\varkappa\mathbf{v}\right\}.\end{aligned} \tag{3.80}$$

Here C is the constant given in the definition of the sigma-function (3.72). Taking into account eq. (3.66) we get (3.77) with the coefficients Λ_k given in (3.78).

Since the τ-function is defined only up to a linear exponential factor in $\mathbf{t}$, we omit the linear term in the exponential (3.77), to get the simpler formula

$$\frac{\tau(\mathbf{e},\mathbf{t})}{\tau(\mathbf{e},\mathbf{0})} = \frac{\sigma\left(\sum_{k=1}^{\infty}\mathbf{R}_k t_k + \mathbf{v}\right)}{\sigma(\mathbf{v})}\exp\left\{\frac{1}{2}\sum_{k,l=1}^{\infty}\mu_{k,l}^{\mathrm{alg}} t_k t_l\right\}, \quad \mathbf{v} = \mathfrak{A}\mathbf{e}. \tag{3.81}$$

Remark 3.5 A similar formula for the algebro-geometric τ-functions in terms of the σ-function was given by A.Nakayashiki in [Nak09], where the terms linear in t_k in the exponent are also taken into account.

At first glance the use of the multi-variable sigma-function instead of the Riemann theta-function in the expression for the algebro-geometric τ-function seems a trivial change. But as a result, the quadratic form in the exponential has coefficients that are algebraically expressed in terms of coefficients of the curve. Moreover, these coefficients, μ_{kl}^{alg} are polynomials in the coefficients of the curve (see [Nak08] for details). As shown below, such a representation of the τ-function results in solutions of the integrable KP hierarchy expressed as differential polynomials in the $\wp$-functions with polynomial coefficients determined directly in terms of the coefficients of the polynomials $P(x,y)$ defining the curve.

According to Propositions 2.1 and 2.2 for any curve X of genus g, the associated algebro-geometric τ-function admits the following expansion

$$\frac{\tau(\mathbf{e},\boldsymbol{t})}{\tau(\mathbf{e},\mathbf{0})} = \sum_{\lambda}\pi_\lambda(w)s_\lambda(\mathbf{t}), \tag{3.82}$$

where

$$\pi_\lambda(w) = (-1)^{\sum_{k=1}^{r} b_k}\det(A_{a_i,b_j}|_{\leq i,j\leq r}), \tag{3.83}$$

the sum being over all partitions λ, which in Frobenius notations have the form $(a_1,\ldots,a_r|b_1,\ldots,b_r)$. Here A_{ij} with $i,j=0\ldots,\infty$ are the elements of the affine coordinate matrix A representing the Grassmannian element $w(X,\mathcal{D},p_\infty,\zeta)\in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$.

Corollary 3.2 *The elements A_{ij} are expressible as polynomials in the Kleinian symbols*

$$\zeta_i(\mathbf{v}),\quad \wp_{ij}(\mathbf{v}),\ \ldots,\quad i,j\in\{1,\ldots,g\} \tag{3.84}$$

and the coefficients of the polynomial $P(x,y)$.

Proof. The quantities $\pi_\lambda(w)$ in the τ-expansion (3.82) are expressible in terms of quotients $\sigma_i(\mathbf{v})/\sigma(\mathbf{v})$, $\sigma_{ij}(\mathbf{v})/\sigma(\mathbf{v})$, etc. But for $i,j,k\ldots=1,\ldots g$ we get

$$\begin{aligned}
\frac{\sigma_i(\mathbf{v})}{\sigma(\mathbf{v})} &= \zeta_i(\mathbf{v}), \quad \frac{\sigma_{ij}(\mathbf{v})}{\sigma(\mathbf{v})} = \zeta_i(\mathbf{v})\zeta_j(\mathbf{v}) - \wp_{ij}(\mathbf{v}), &(3.85)\\
\frac{\sigma_{ijk}(\mathbf{v})}{\sigma(\mathbf{v})} &= \zeta_i(\mathbf{v})(\mathbf{v})\wp_{j,k}(\mathbf{v}) + \zeta_j(\mathbf{v})(\mathbf{v})\wp_{i,k}(\mathbf{v}) + \zeta_k(\mathbf{v})(\mathbf{v})\wp_{ij}(\mathbf{v})\\
&\quad -\zeta_i(\mathbf{v})\zeta_j(\mathbf{v})\zeta_k(\mathbf{v}) + \wp_{ijk}(\mathbf{v}). &(3.86)\\
&\vdots
\end{aligned}$$

To each symbol $\wp_{k_1,\ldots,k_g}$ we assign a weight

$$\wp_{\underbrace{1,\ldots,1}_{k_1},\underbrace{2,\ldots,2}_{k_2},\ldots,\underbrace{g,\ldots,g}_{k_g}} \quad \Leftrightarrow \quad \mathcal{W}_{k_1\ldots k_g} = \sum_{j=1}^{g} k_j n_j, \tag{3.87}$$

where $\{n_i\}_{i=1,\ldots,g}$ is the Weierstrass gap sequence at infinity.

To each coefficient a_{kl} of a monomial term $a_{kl}x^k y^l$, $k<s, l<n$ within the polynomial $P(x,y)$ defining the curve (3.58) we assign the weight $\widehat{\mathcal{W}}_{kl}$,

$$a_{kl} \quad \Leftrightarrow \quad \widehat{\mathcal{W}}_{kl} = ns - (nk+ls). \tag{3.88}$$

Finally, assign to a monomial whose factors are $\wp$-symbols and coefficients $a_{k,j}$ the weight $\mathcal{W}$ that is the sum of weights of factors,

$$a_{ij}\cdots a_{kl}\wp_{i_1\ldots i_g}\cdots\wp_{k_1\ldots k_g} \quad \Leftrightarrow \quad \mathcal{W} = \widehat{\mathcal{W}}_{ij} + \cdots + \widehat{\mathcal{W}}_{kl} + \mathcal{W}_{i_1\ldots i_g} + \cdots + \mathcal{W}_{k_1\ldots k_g}. \tag{3.89}$$

Consider the set of (Giambelli-like) relations

$$\pi_{(a_1,\ldots,a_r|b_1,\ldots,b_r)} = (-1)^{\sum_{i=1}^r b_i}\det(A_{a_i,b_j}) \tag{3.90}$$

corresponding to a partition $\lambda=(a_1,\ldots,a_r|b_1,\ldots,b_r)$ of weight $\mathcal{W}$. The above procedure reduce relations (3.90) to corresponding homogeneous polynomial relations of weight $\mathcal{W}$ between $\wp$-functions, $\wp_{i_1,\ldots,i_k}$ with coefficients that are polynomials in the coefficients of the defining curve polynomial $P(x,y)$. Such relations describe KP-type hierarchies in terms of $\wp$-coordinate.

4 Examples and applications of Schur function expansions

4.1 τ-function of a hyperelliptic curve

Let X be a hyperelliptic curve of genus g defined by the equation

$$X: \quad P(x,y) = 0 \tag{4.1}$$

with polynomial $P(x,y)$ given as

$$\begin{aligned} P(x,y) &= y^2 - 4x^{2g+1} + \ldots + \alpha_0 \\ &= y^2 - 4\prod_{j=1}^{2g+1}(x-a_j). \end{aligned} \tag{4.2}$$

As above, denote by $p=(x,y)$ an arbitrary point of X and $p_\infty = (\infty,\infty)$. We choose a canonical basis of cycles $(\mathfrak{a}_1,\ldots,\mathfrak{a}_g;\mathfrak{b}_1,\ldots,\mathfrak{b}_g) \in H_1(X,\mathbb{Z})$ and fix the basic holomorphic differentials $\mathbf{u}=(u_1,\ldots,u_g)^T$ as

$$u_i(p) = \frac{x^{i-1}\mathrm{d}x}{y}, \quad i=1,\ldots,g. \tag{4.3}$$

Denote as above period matrices, $\mathfrak{A}$, $\mathfrak{B}$ and $\mathbf{T}=\mathfrak{A}^{-1}\mathfrak{B}$.

Let $\mathcal{D}=p_1+\ldots+p_g$ be non-special divisor of degree g and let $\mathbf{v}$ be the shifted Abel map given in (3.71). In this case the vector of Riemann constants $\mathbf{K}$ can be given as the image under the Abel map of the divisor $\mathcal{D}_\kappa = (p_1,\ldots,p_g)$ where $p_k=(a_k,0)$ are branch points whose abelian images are odd half-periods [FK80, Sect VII.1.2],

$$\mathbf{K} = -\sum_{j=1}^{g}\int_{p_0}^{p_k}\boldsymbol{\omega} = -\mathfrak{A}^{-1}\sum_{j=1}^{g}\int_{p_0}^{p_k}\mathbf{u} \tag{4.4}$$

Therefore we can write

$$\mathbf{v} = \sum_{k=1}^{g}\int_{(a_k,0)}^{q_k}\mathbf{u}. \tag{4.5}$$

In the classical theory (see e.g. [Bak97, Bak98]) it was shown that the quadratic bi-differential $\Omega(p,q)$ can be chosen in the form

$$\Omega(p,q) = \frac{F(x,z)+2yw}{4(x-z)^2yw}\mathrm{d}x\mathrm{d}z + 2\mathbf{v}^T(p)\varkappa\mathbf{v}(q), \tag{4.6}$$

where the polynomial $F(x,z)$ is the Kleinian 2-polar

$$F(x,z) = \sum_{m=0}^{g} x^m z^m (2\alpha_{2m} + (x+z)\alpha_{2m+1}) \tag{4.7}$$

and the $g\times g$-matrix symmetric matrix $\varkappa$ is given as $\varkappa = \mathfrak{A}^{-1}\mathfrak{S}$, where $\mathfrak{S}$ the matrix of $\mathfrak{a}$-periods, of meromorphic differentials

$$r_j = \sum_{k=j}^{2g+1-j} (k+1-j)\lambda_{k+1+j}\frac{x^k\mathrm{d}x}{4y}, \quad j=1,\ldots,g. \tag{4.8}$$

Theorem 4.1 (Klein formula for hyperelliptic curve) *Let the planar curve X be defined by the polynomial equation (4.2). Let $p=(x,y)$ be an arbitrary point of X and $\mathcal{D} = p_1 + p_2 + \ldots + p_g$ a non-special divisor on X, $p_k = (x_k, y_k)$. Let $\mathbf{v}$ be the shifted Abel map of $\mathcal{D}$ given in (4.5). Then*

$$\sum_{i,k=1}^{g} \wp_{i,k}\left(\mathcal{A}(p) - \mathcal{A}(p_\infty) + \mathbf{v}\right) x^{k-1} x_r^{i-1} = \frac{F(x,x_r) - 2yy_r}{4(x-x_r)^2}, \quad r=1,\ldots,g, \tag{4.9}$$

where the polynomial $\mathcal{F}(p,p_r) = \mathcal{F}((x,y);(x_r,y_r))$ defines the fundamental bi-differential $\Omega(p,p_r)$.

Remark 4.1 Differential relations between the $\wp_{ij}$ describe all possible integrable equations associated with the given curve. Moreover one can show that for arbitrary genus g any even derivative $\wp_{i_1,\ldots,i_{2k}}$, $k>1$ may be written as a polynomial in $\wp_{ik}$ with coefficients expressible in terms of the invariants of the curve [BEL97]. A complete set of differential relations in the particular cases $g=2$ and $g=3$ can be found in [Bak03] [Bak07] and recently in covariant form these relations were obtained in [Ath08].

We now restrict ourself to the case of a genus two curve, whose equation can be taken in the form

$$\begin{aligned} y^2 &= 4x^5 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 && (4.10)\\ &= 4(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5), \quad a_l \neq a_l. && (4.11)\end{aligned}$$

The holomorphic differentials $\mathbf{u} = (u_1, u_2)^T$ are related to $\mathbf{v} = (v_1, v_2)^T$

$$v_1 = \frac{x\mathrm{d}x}{y}, \quad v_2 = \frac{\mathrm{d}x}{y} \tag{4.12}$$

by the transformation

$$T = \begin{pmatrix} 1 & \frac{\alpha_4}{8} \\ 0 & 1 \end{pmatrix} \tag{4.13}$$

According to the definition (3.65), the first few coefficients μ_{ij}^{alg} are

$$\begin{aligned}
\mu_{ij}^{\mathrm{alg}} &= 0, \quad \text{if } i \text{ or } j \text{ or both are even,} \\
\mu_{1,1}^{\mathrm{alg}} &= -\frac{1}{16}\alpha_4, \\
\mu_{1,3}^{\mathrm{alg}} &= \mu_{3,1} = -\frac{1}{16}\alpha_3 + \frac{3}{256}\alpha_4^2, \\
\mu_{1,5}^{\mathrm{alg}} &= \mu_{5,1} = -\frac{1}{16}\alpha_2 + \frac{3}{128}\alpha_3\alpha_4 - \frac{5}{2048}\alpha_4^3, \\
\mu_{3,3}^{\mathrm{alg}} &= -\frac{3}{16}\alpha_2 + \frac{1}{32}\alpha_3\alpha_4 - \frac{3}{1024}\alpha_4^3, \\
&\vdots
\end{aligned} \tag{4.14}$$

Introduce the set of residue vectors $\mathbf{R}_{2k} = 0, k = 1, \ldots$ and

$$\mathbf{R}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{R}_3 = \begin{pmatrix} -\frac{1}{8}\alpha_4 \\ 1 \end{pmatrix}, \quad \mathbf{R}_5 = \begin{pmatrix} -\frac{1}{8}\alpha_3 + \frac{3}{128}\alpha_4^2 \\ -\frac{1}{8}\alpha_4 \end{pmatrix}, \ldots \tag{4.15}$$

It follows from formulae (2.47) and (2.59) that the first few affine matrix components are

$$\begin{aligned}
A_{00}(\mathbf{v}) &= \zeta_1, \\
A_{01}(\mathbf{v}) &= \frac{1}{2}\zeta_1^2 - \frac{1}{2}\wp_{11} - \frac{1}{16}\alpha_4 \\
A_{02}(\mathbf{v}) &= \frac{1}{6}\zeta_1^3 + \frac{1}{3}\zeta_2 - \left(\frac{1}{2}\wp_{11} + \frac{5}{48}\alpha_4\right)\zeta_1 - \frac{1}{6}\wp_{111}, \\
A_{03}(\mathbf{v}) &= \frac{1}{24}\zeta^4 + \frac{1}{3}\zeta_1\zeta_2 - \left(\frac{7}{96}\alpha_4 + \frac{1}{4}\wp_{11}\right) - \frac{1}{6}\wp_{111}\zeta_1 \\
&- \frac{1}{24}\wp_{1111} - \frac{1}{3}\wp_{12} + \frac{1}{8}\wp_{11}^2 + \frac{7}{96}\alpha_4\wp_{11} - \frac{1}{24}\alpha_3 + \frac{5}{512}\alpha_4^2, \\
&\vdots \\
&\vdots \\
A_{10}(\mathbf{v}) &= A_{01}(\mathbf{v}), \\
A_{11}(\mathbf{v}) &= \frac{1}{3}\zeta_1^3 - \frac{1}{3}\zeta_2 - \left(\wp_{11} + \frac{1}{12}\alpha_4\right)\zeta_1 - \frac{1}{3}\wp_{111},
\end{aligned} \tag{4.16}$$

$$
\begin{aligned}
A_{12}(\mathbf{v}) &= \frac{1}{8}\zeta_1^4 - \frac{1}{2}\zeta_1\wp_{111} + \frac{3}{8}\wp_{11}^2 - \frac{1}{8}\wp_{1111} - \left(\frac{3}{4}\wp_{11} + \frac{3}{32}\alpha_4\right)\zeta_1^2 + \frac{3}{32}\alpha_4\wp_{11} + \frac{3}{512}\alpha_4^2,\\
A_{13}(\mathbf{v}) &= \frac{1}{30}\zeta_1^5 + \frac{1}{6}\zeta_1^2\zeta_2 - \frac{1}{3}\wp_{111}\zeta_1^2 - \left(\frac{1}{3}\wp_{11} + \frac{1}{16}\alpha_4\right)\zeta_1^3 - \left(\frac{1}{6}\wp_{11} + \frac{1}{48}\alpha_4\right)\zeta_2\\
&+ \left(-\frac{1}{6}\wp_{1111} - \frac{1}{3}\wp_{12} + \frac{1}{2}\wp_{11}^2 + \frac{3}{16}\alpha_4\wp_{11} - \frac{1}{24}\alpha_3 + \frac{7}{384}\alpha_4^2\right)\zeta_1\\
&+ \frac{1}{3}\wp_{11}\wp_{111} - \frac{1}{6}\wp_{1112} - \frac{1}{30}\wp_{11111} + \frac{1}{16}\alpha_4\wp_{111},\\
&\vdots
\end{aligned}
$$

All Kleinian symbols, ζ_k, $\wp_{k,l}$ etc. in these formulae are evaluated at $\mathbf{v}$.

Proposition 4.1 *The Plücker relations written for the partition $\lambda = (2,2)$ of weight 4 and partitions $\lambda = (3,2), \lambda = (2,2,1)$ of weight 5 are equivalent to the equation*

$$\wp_{1111}(\mathbf{v}) = 6\wp_{11}^2(\mathbf{v}) + 4\wp_{12}(\mathbf{v}) + \alpha_4\wp_{11}(\mathbf{v}) + \frac{1}{2}\alpha_3. \tag{4.17}$$

Proof. The first non-trivial Plücker relation for the partition $\lambda = (2,2)$

$$\pi_{(1,0|1,0)} = \begin{vmatrix} A_{1,1} & A_{1,0} \\ A_{0,1} & A_{0,0} \end{vmatrix} \tag{4.18}$$

is written in detailed form as

$$\tau(\mathbf{0},\mathbf{v})\left[\frac{1}{12}\frac{\partial^4}{\partial t_1^4} + \frac{1}{4}\frac{\partial^2}{\partial t_2^2} - \frac{1}{3}\frac{\partial^2}{\partial t_1\partial t_3}\right]\tau(\mathbf{t},\mathbf{v})\bigg|_{\mathbf{t}=0}$$

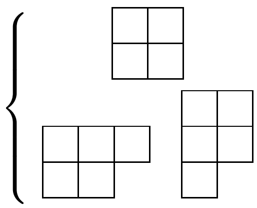
$$\begin{vmatrix} \left[\frac{1}{3}\frac{\partial^3}{\partial t_1^3} - \frac{1}{3}\frac{\partial}{\partial t_3}\right]\tau(\mathbf{t},\mathbf{v})\bigg|_{\mathbf{t}=0} & \left[\frac{1}{2}\frac{\partial}{\partial t_2} + \frac{\partial^2}{\partial t_1^2}\right]\tau(\mathbf{t},\mathbf{v})\bigg|_{\mathbf{t}=0} \\ \left[-\frac{1}{2}\frac{\partial}{\partial t_2} + \frac{1}{2}\frac{\partial^2}{\partial t_1^2}\right]\tau(\mathbf{t},\mathbf{v})\bigg|_{\mathbf{t}=0} & \frac{\partial}{\partial t_1}\tau(\mathbf{t},\mathbf{v})\bigg|_{\mathbf{t}=0} \end{vmatrix} \tag{4.19}$$

Substituting the expression (3.81) for $\tau(\mathbf{t};\mathbf{e})$ into this relation and expressions for μ_{ij} and computing directional derivatives, we obtain equation (4.17). Expressions for the hook diagram coefficients entering in the determinant are given above and

$$
\begin{aligned}
\pi_{(1,0|1,0)} &= -\frac{1}{12}\wp_{1111} + \frac{1}{12}\zeta_1^4 - \left(\frac{1}{2}\wp_{11} + \frac{1}{48}\alpha_4\right)\zeta_1^2 - \frac{1}{3}\zeta_2\zeta_1 - \frac{1}{3}\wp_{111}\zeta_1 \\
&+ \frac{1}{3}\wp_{12} + \frac{1}{4}\wp_{11}^2 + \frac{1}{48}\alpha_4\wp_{11} - \frac{1}{256}\alpha_4^2 + \frac{1}{24}\alpha_3.
\end{aligned} \tag{4.20}
$$

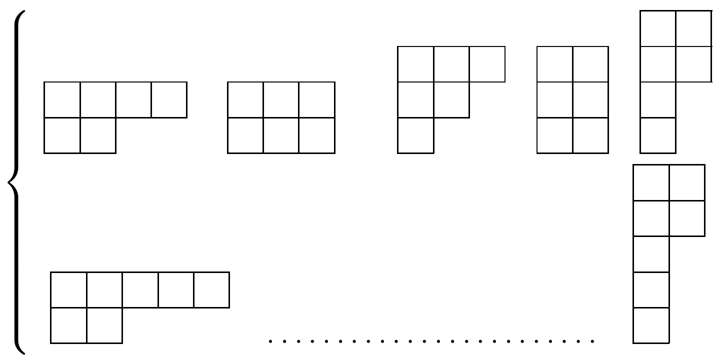
Relation (4.17) follows from these.

The partitions $\lambda = (3, 2)$ and $\lambda = (2, 2, 1)$ of weight 5 give Plücker relations that imply the action of $\zeta_1(\mathbf{v}) + \partial/\partial v_1$ on the above equation. These all yield the single equation.



$$\Leftrightarrow \wp_{1111}(\mathbf{v}) = 6\wp_{11}^2(\mathbf{v}) + 4\wp_{12}(\mathbf{v}) + \alpha_4\wp_{11}(\mathbf{v}) + \frac{1}{2}\alpha_3.$$

Analogous considerations involving weight 6 and 7 partitions lead to the correspondence



$$\Updownarrow$$

$$\wp_{1112}(\mathbf{v}) = 6\wp_{11}\wp_{12}(\mathbf{v}) - 2\wp_{22}(\mathbf{v}) + \alpha_4\wp_{12}(\mathbf{v}) \tag{4.21}$$

$$\wp_{111}^2(\mathbf{v}) = 4\wp_{11}^3(\mathbf{v}) + \wp_{22}(\mathbf{v}) + 4\wp_{12}(\mathbf{v})\wp_{11}(\mathbf{v}) + \alpha_4\wp_{11}^2(\mathbf{v}) + \alpha_3\wp_{11}(\mathbf{v}) \tag{4.22}$$

Using the definition of weight $\mathcal{W}$ for the functions appearing in (4.21) and (4.22) we see that all these equations are homogeneous.

The process described here can be continued. It seems reasonable to conjecture that the whole set of differential relations between multi-index Kleinian symbols can be put into correspondence with the Young diagrams of the partitions $\lambda = (2, 2, i_1, \ldots, i_n)$ in such the way that partitions of weights $2k$ and $2k + 1$ correspond to a set of equations of weight $\mathcal{W} = 2k$.

To complete the interpretation of the basic equations describing Abelian functions in terms of Plücker relations, we find that the Kummer surface arises as the Plücker relation corresponding to the $\lambda = (4,4,4,4)$ diagram with weight 16.

$$\pi_{(3210|3210)} = \begin{vmatrix} A_{3,3} & A_{3,2} & A_{3,1} & A_{3,0} \\ A_{2,3} & A_{2,2} & A_{2,1} & A_{2,0} \\ A_{1,3} & A_{1,2} & A_{1,1} & A_{1,0} \\ A_{0,3} & A_{0,2} & A_{0,1} & A_{0,0} \end{vmatrix}. \tag{4.23}$$

The equation

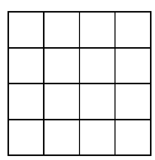
$$\pi_{(3210|3210)} = 0 \tag{4.24}$$

can be written in the form

$$\begin{vmatrix} \alpha_0 & \frac{1}{2}\alpha_1 & -2\wp_{22} & -2\wp_{12} \\ \frac{1}{2}\alpha_1 & \alpha_2 + 4\wp_{22} & \frac{1}{2}\alpha_3 + 2\wp_{12} & -2\wp_{11} \\ -2\wp_{22} & \frac{1}{2}\alpha_3 + 2\wp_{12} & \alpha_4 + 4\wp_{11} & 2 \\ -2\wp_{12} & -2\wp_{11} & 2 & 0 \end{vmatrix} = 0. \tag{4.25}$$

This is the celebrated quartic Kummer surface, $\mathrm{Kum}(X)$, defined as the surface in $\mathbb{C}^3$ with coordinates $x = \wp_{11}, y = \wp_{12}, z = \wp_{22}$. $\mathrm{Kum}(X)$ is the quotient of the Jacobi variety, $\mathrm{Kum}(X) = \mathrm{Jac}(X)/(\mathbf{u} \to -\mathbf{u})$

Therefore we conclude



$\Longleftrightarrow \qquad \mathrm{Kum}(X)$

Remark 4.2 The equation for the Kummer surface in this form was derived by Baker [Bak07] and a generalization to higher genera was given in [BEL97]. Also note that equations written for all 2×2 minors of the matrix $(A_{i,j})_{i,j=0,\ldots,3}$ in (4.23),

$$\pi_{(i,k|j,l)} = \begin{vmatrix} A_{i,j} & A_{i,l} \\ A_{k,j} & A_{k,l} \end{vmatrix}, \quad i \geq j, \quad k \geq l, \quad i,j,k,l \in \{0,1,2,3\} \tag{4.26}$$

give a complete set of algebraic equations describing the Jacobi variety $\mathrm{Jac}(X)$ as an algebraic variety and also the flows of KdV type on $\mathrm{Jac}(X)$. This resembles the matrix realization of the Jacobi and Kummer varieties given by Baker [Bak07], which was generalized to higher genera in [BEL97].

The above considerations lead to the following result.

Theorem 4.2 *Each column vector $\mathbf{A}_k(\mathbf{v})$, $k = 0, \ldots$ in the matrix of affine coordinates of the element of the Grassmanian, whose components $A_{k,l}$ $l = 1, \ldots$ correspond to hook partitions $(k+1, 1^l)$ is a polynomial in a finite set of Kleinian symbols $\zeta_i(\mathbf{v})$, $\wp_{ij}(\mathbf{v})$, $\wp_{ijk}(\mathbf{v})$.*

4.2 τ-function of a trigonal curve

In this section we demonstrate how the above results appear in the case of a trigonal curve. The σ-function theory of trigonal Abelian functions was developed in [BEL00]. Various results in this area were obtained in [EEL00], [BL05], [EEMOP08], [MP08], [EE09], [MP10]. In order to emphasize the main idea and avoid cumbersome formulae we restrict ourselves to the first nontrivial case of the cyclic family of trigonal curves X of genus 3 defined by:

$$P(x,y) = y^3 - (x^4 + \beta_3 x^3 + \beta_6 x^2 + \beta_9 x + \beta_{12}) = 0. \tag{4.27}$$

and fix a canonical basis of cycles $(\mathfrak{a}_1, \ldots, \mathfrak{a}_3; \mathfrak{b}_1, \ldots, \mathfrak{b}_3) \in H_1(X, \mathbb{Z})$ of X.

The explicit calculation of canonical holomorphic differentials and the meromorphic differentials conjugate to them is given in [EEMOP08]. In particular we have for $p = (x,y)$

$$u_1(p) = \frac{\mathrm{d}x}{3y}, \quad u_2(p) = \frac{x\mathrm{d}x}{3y^2}, \quad u_3(p) = \frac{\mathrm{d}x}{3y^2},$$

$$r_1(p) = \frac{x^2\mathrm{d}x}{3y^2}, \qquad r_2(p) = -\frac{2x\mathrm{d}x}{3y}, \qquad r_3(p) = -\frac{(5x^2 + 3\beta_3 x + \beta_6)\mathrm{d}x}{3y}.$$

Denote, as above, the period matrices, $\mathfrak{A}$, $\mathfrak{B}$ and $\boldsymbol{T} = \mathfrak{A}^{-1}\mathfrak{B}$, $\varkappa = \mathfrak{A}^{-1}\mathfrak{S}$.

Let $\mathcal{D} = p_1 + p_2 + p_3$ be a nonspecial divisor of degree 3 and

$$\mathbf{v} = \sum_{i=1}^{3} \int_{p_\infty}^{p_i} \mathbf{u} + \mathfrak{A}\mathbf{K}, \tag{4.28}$$

where $\mathbf{K}$ is the vector of Riemann constants with the base point at p_∞.

The polynomial $\mathcal{F}((x,y);(z,w))$ appearing in the fundamental bi-differential $\Omega(p,q)$ is given by

$$\mathcal{F}((x,y),(z,w)) = 3w^2y^2 + wT(x,z) + yT(z,x), \tag{4.29}$$

where

$$T(x,z) = 3\beta_{12} + (z + 2x)\beta_9 + x(x + 2\,z)\beta_6 + 3\beta_3 x^2 z + x^2 z^2 + 2\,x^3 z. \tag{4.30}$$

Expanding about p_∞ gives the following expressions for the quantities μ_{ij}^{alg}

$$\mu_{0,0}^{\mathrm{alg}} = 0, \tag{4.31}$$

$$\mu_{0,1}^{\mathrm{alg}} = \mu_{1,0}^{\mathrm{alg}} = -\frac{2}{3}\beta_3,$$

$$
\begin{aligned}
\mu^{\rm alg}_{0,4} &= \mu^{\rm alg}_{4,0} = -\frac{2}{3}\beta_6 + \frac{5}{9}\beta_3^2,\\
\mu^{\rm alg}_{1,3} &= \mu^{\rm alg}_{3,1} = -\frac{2}{3}\beta_6 + \frac{4}{9}\beta_3^2,\\
\mu^{\rm alg}_{2,2} &= 0,\\
&\vdots
\end{aligned}
$$

Remark 4.3 $\mu^{\rm alg}_{ij} = 0$ *unless* $(i+j)+2 \equiv 0 \bmod 3$. *This is a consequence of the cyclic symmetry of the curve.*

In this case the Klein formula reads

$$\sum_{i,k=1}^{3} \wp\left(\int_{p_\infty}^{p} \mathbf{u} - \mathbf{v}\right)\phi_i(x,y)\phi_k(x_r,y_r) = \frac{\mathcal{F}(p,p_r)}{(x-x_r)^2}, \quad r = 1,2,3,$$

where $p = (x,y)$, $p_k = (x_k, y_k)$ also $\phi_1(x,y) = y, \phi_2(x,y) = x, \phi_3(x,y) = 1$.

Expanding this relation in the vicinity of p_∞, where local coordinate $x = 1/\xi^3$ is introduced/. Equating principal parts at the poles, we obtain a set of equations involving variables x_x, y_k and $\wp$-symbols. We now show that these relations can be obtained as consequences of Plücker relations via Giambelli-type formula.

The first Young diagrams leading to a nontrivial Plücker relation, as in the hyperelliptic genus 2 case, corresponds to the partition $\lambda = (2,2)$. In this case we obtain, after simplification, the equation

$$\wp_{1111} = 6\,\wp_{11}^2 - 3\,\wp_{22}. \tag{4.32}$$

Differentiating with respect to the coordinate v_1 gives the Boussinesq equation.

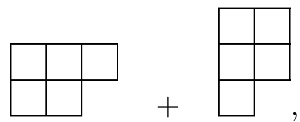
The derivation of (4.32) is based on formula (4.18) for the trigonal curve (4.27). Namely, we have

$$\pi_{(1,0|1,0)} = \frac{1}{4}\wp_{22} + \frac{1}{4}\zeta_2^2 - \frac{1}{12}\wp_{1111} - \frac{1}{3}\wp_{111}\zeta_1 + \frac{1}{4}\wp_{11}^2 - \frac{1}{2}\wp_{11}\zeta_1^2 + \frac{1}{12}\zeta_1^4, \tag{4.33}$$

and also

$$
\begin{aligned}
A_{0,0}(\mathbf{v}) &= \zeta_1(\mathbf{v}),\\
A_{0,1}(\mathbf{v}) &= -\frac{1}{2}\wp_{11}(\mathbf{v}) + \frac{1}{2}\zeta_1^2(\mathbf{v}) - \frac{1}{2}\zeta_2(\mathbf{v}),\\
A_{1,0}(\mathbf{v}) &= \frac{1}{2}\zeta_2(\mathbf{v}) - \frac{1}{2}\wp_{11}(\mathbf{v}) + \frac{1}{2}\zeta_1^2(\mathbf{v}),\\
A_{1,1}(\mathbf{v}) &= -\frac{1}{3}\wp_{111}(\mathbf{v}) - \wp_{11}(\mathbf{v})\zeta_1(\mathbf{v}) + \frac{1}{3}\zeta_1^3(\mathbf{v}).
\end{aligned} \tag{4.34}
$$

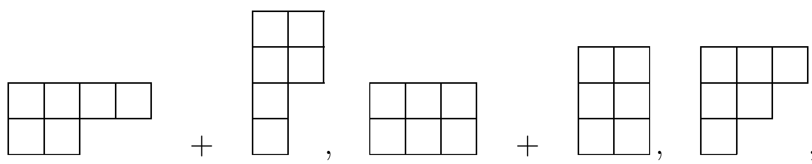
In the trigonal case we no longer have symmetry about the diagonal of the Young diagram that we have in the genus 2 case, but we can restrict ourselves to equations of even degree by taking symmetric combinations of the two diagram related by transposition. In the weight 5 case we have the symmetric combination



which gives the weight 5 trigonal PDE

$$\wp_{1112} = 6\,\wp_{11}\wp_{12} + 3\,\beta_3\wp_{11}.$$

For weight 6 we have the three sets of diagram



which lead to an over-determined set of equations with the unique solution

$$\begin{aligned} \wp_{111}^2 &= 4\wp_{11}^3 + \wp_{12}^2 + 4\,\wp_{13} - 4\,\wp_{11}\wp_{22}, \\ \wp_{1122} &= 4\,\wp_{13} + 4\wp_{12}^2 + 2\,\wp_{11}\wp_{22} + 3\,\beta_3\wp_{12} + 2\,\beta_6. \end{aligned}$$

Continuing in this way, we recover the cyclic trigonal versions of the full set of equations given in [EEMOP08].

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