THE DARBOUX COORDINATES FOR A NEW FAMILY OF HAMILTONIAN OPERATORS AND LINEARIZATION OF ASSOCIATED EVOLUTION EQUATIONS

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Abstract. A. de Sole, V. G. Kac, and M. Wakimoto have recently introduced a new family of compatible Hamiltonian operators of the form $H^{(N,0)} = D^2 \circ ((1/u) \circ D)^{2n} \circ D$, where $N = 2n+3$, $n = 0, 1, 2, \dots$, u is the dependent variable and D is the total derivative with respect to the independent variable. We present a differential substitution that reduces any linear combination of these operators to an operator with constant coefficients and linearizes any evolution equation which is bi-Hamiltonian with respect to a pair of any nontrivial linear combinations of the operators $H^{(N,0)}$. We also give the Darboux coordinates for $H^{(N,0)}$ for any odd $N \geqslant 3$.

1. Introduction

The Hamiltonian evolution equations are well known to play an important role in modern mathematical physics. Indeed, a Hamiltonian operator maps the variational derivatives of the conserved quantities into symmetries; this plays an important role in the theory of integrable systems which often turn out to be *bi*-Hamiltonian, see e.g. [5, 6, 8, 13] and references therein. It is thus no wonder that the study and, in particular, the classification of Hamiltonian operators is a subject of ongoing interest, see for instance [1, 3, 7, 6, 11, 12] and the works cited there.

Recently A. de Sole, V. G. Kac, and M. Wakimoto have made a major advance in this area. Namely, in [4] they gave a conjectural classification of Poisson vertex algebras in one differential variable, i.e., of scalar Hamiltonian operators. *Inter alia*, they have come up with a new infinite family of compatible Hamiltonian operators $H^{(N,0)} = D^2 \circ ((1/u) \circ D)^{2n} \circ D$, where $N = 2n + 3$, $n = 0, 1, 2, \dots$, and D denotes the total derivative with respect to the space variable.

Although it is well known [15, 16] that there is no proper counterpart of the Darboux theorem on canonical forms of finite-dimensional Poisson structures for Hamiltonian operators associated with evolutionary PDEs, it is often possible to find new variables in which a Hamiltonian operator takes a simpler form. One such form is the Gardner operator D; in analogy with the finite-dimensional case the associated new variables are often called the Darboux coordinates, see e.g. [3, 15]. Bringing a Hamiltonian operator into the Gardner form enables one e.g. to render the associated Hamiltonian systems into the canonical Hamiltonian form and construct Lagrangian representations (modulo potentialization) for these systems [16].

In this paper we present the transformations that bring the operators $H^{(N,0)}$ into the Gardner form, see Corollary 1 below. Moreover, in Theorem 1 we give a differential substitution which *simultaneously* turns the operators $H^{(N,0)}$ for all odd $N \geqslant 3$ into the operators with constant coefficients $\widetilde{H}^{(N,0)} = -D^{2n+1}$.

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These results could be employed e.g. for the study of (co)homology of the Poisson complexes associated with the operators $H^{(N,0)}$. The cohomologies in question play an important role e.g. in finding all Hamiltonian operators compatible with a given Hamiltonian operator and the associated multi-Hamiltonian systems, see for example [10, 14, 17] and references therein. Another possible application of the results in question is e.g. the construction of new integrable systems in spirit of [9] in the new variables from Theorem 1 or Corollary 1 with the subsequent pullback to the original variables. Last but not least, the differential substitution from Theorem 1 linearizes any evolution equation which is bi-Hamiltonian with respect to a pair of any nontrivial linear combinations of the operators $H^{(N,0)}$, thus exhibiting a broad class of somewhat unusual (in that they are C-integrable rather than S-integrable) integrable bi-Hamiltonian systems; see Corollary 2 for details.

2. Preliminaries

In what follows we deal with Hamiltonian operators and associated Hamiltonian evolution equations involving a single spatial variable x and a single dependent variable u. An evolution equation of this kind has the form

$$u_t = K[u] = K(x, u, u_x, u_{xx}, \dots),$$

where the square brackets indicate that K is a *differential function* in the sense of [13], meaning that it depends on x, u, and finitely many derivatives of u with respect to the space variable x. Recall (see e.g. [5, 6, 13] for details) that an evolution equation is said to be *Hamiltonian* with respect to the Hamiltonian operator $\mathcal{D}$ if it can be written in the form

$$u_t = \mathcal{D}\delta_u \mathcal{T}[u],$$

where $\mathcal{T} = \int T[u]dx$ is the Hamiltonian functional, and δ_u denotes the variational derivative with respect to u.

Lemma 1 ([11]). *Let L_1 be a Hamiltonian operator in the variables x, u. Under the transformation*

$$x = \varphi(y, v, v_y, \dots, v_m), \quad u = \psi(y, v, v_y, \dots, v_n), \tag{1}$$

where $v_j = D_y^j(v)$, where D_y is the total derivative with respect to y, the operator L_1 goes into the Hamiltonian operator L_2 defined by the formula

$$\overline{L}_1 = (D_y(\varphi))^{-1} K^* \circ L_2 \circ K, \tag{2}$$

where $\overline{L}_1$ is obtained from L_1 under the substitution (1) and upon setting $D_x = (D_y(\varphi))^{-1} D_y$,

$$K = \sum_{i=0}^{\max(m,n)} (-1)^i D_y^i \circ \left(\frac{\partial \psi}{\partial v_i} D_y(\varphi) - \frac{\partial \varphi}{\partial v_i} D_y(\psi) \right),$$

and K^* is the formal adjoint of K.

Remark 1. Note that in general the operator L_2 may contain nonlocal terms unless (1) is a contact transformation, cf. e.g. [1, 4, 11].

3. The main result

Theorem 1. *The transformation $x=v$, $u=1/v_y$ turns the Nth order Hamiltonian operator $H^{(N,0)} = D_x^2\circ((1/u)\circ D_x)^{2n}\circ D_x$, where $N=2n+3$, $n=0,1,2,\dots$, into the Hamiltonian operator* with constant coefficients $\widetilde{H}^{(N,0)} = -D_y^{2n+1}$.

The proof is obtained by a straightforward application of Lemma 1. Even though the transformation from Theorem 1 is not contact, in the particular case under study the transformed operators $\widetilde{H}^{(N,0)}$ happen to be free of nonlocal terms (cf. Remark 1).

Using Lemma 1 we can further amplify the result of Theorem 1 by providing the Darboux coordinates (cf. Introduction) for the operators $H^{(N,0)}$.

Corollary 1. *The transformation $x=(-1)^{\frac{n+1}{2}}w_n$, $u=(-1)^{\frac{n+1}{2}}/w_{n+1}$, where $w_k=D_z^k(w)$, and z is the new independent variable, maps the Hamiltonian operator $H^{(N,0)} = D_x^2\circ((1/u)\circ D_x)^{2n}\circ D_x$ to the first-order Gardner operator D_z for any odd $N\geqslant 3$.*

Remark 2. The inverse of the transformation $x=v$, $u=1/v_y$ is nothing but the extended hodograph transformation in the sense of [2]. Let us stress that, unlike the original transformation, the said inverse is not a differential substitution, i.e., it cannot be written in the form $y=\xi(x,u,u_x,\dots,u_q)$, $v=\chi(x,u,u_x,\dots,u_p)$.

Note that the transformation $x=v$, $u=1/v_y$ can be written as the composition of $x=z$, $u=w_z$ and of the hodograph transformation $z=v$, $w=y$. The first of these is nothing but introduction of the potential w for u. It is readily seen that a bi-Hamiltonian evolution equation

$$u_t = H_1\delta_u\mathfrak{T}_1 = H_2\delta_u\mathfrak{T}_2, \tag{3}$$

where $H_i=\sum_{j=1}^{k_i} c_{ij}H^{(N_{ij},0)}$, $i=1,2$, k_i are arbitrary natural numbers, and c_{ij} are arbitrary constants, is nothing but the pullback of the bi-Hamiltonian equation

$$w_t = \check{H}_1\delta_w\check{\mathfrak{T}}_1 = \check{H}_2\delta_w\check{\mathfrak{T}}_2, \tag{4}$$

where $\check{H}_i=\sum_{j=1}^{k_i} c_{ij}\check{H}^{(N_{ij},0)}$, $i=1,2$,

$$\check{H}^{(N,0)} = -D_z\circ\left(\frac{1}{w_z}D_z\right)^{2n},$$

and $\check{\mathfrak{T}}_i$ are obtained from $\mathfrak{T}_i$ using the substitution $x=z$, $u=w_z$. It is natural to refer to (4) as to the potential form of (3).

Proposition 1. *The transformation $z=v$, $w=y$, where y is the new independent variable, linearizes the potential form (4) of the bi-Hamiltonian evolution equation (3).*

Before proving this let us point out the following important consequence of this result.

Corollary 2. *The differential substitution $x=v$, $u=1/v_y$ relates any equation of the form (3) to a linear evolution equation with constant coefficients.*

Informally, this just means that the inverse of the transformation $x = v$, $u = 1/v_y$ linearizes (3), but this statement should be treated with some care, as this transformation is not uniquely invertible, and the inverse is not a differential subsitution, cf. Remark 2 and [18].

Proof of Proposition 1. The hodograph transformation $z = v$, $w = y$ sends (4) into

$$v_t = \widetilde{H}_1 \delta_v \widetilde{\mathcal{T}}_1 = \widetilde{H}_2 \delta_v \widetilde{\mathcal{T}}_2,$$

where $\widetilde{H}_i = \sum_{j=1}^{k_i} c_{ij} \widetilde{H}^{(N_{ij},0)} = -\sum_{j=1}^{k_i} c_{ij} D_y^{N_{ij}-2}$ are linear differential operators with constant coefficients, and $\widetilde{\mathcal{T}}_i$ are obtained from $\check{\mathcal{T}}_i$ using the transformation in question.

Lemma 2. *Let $v_t = K[v]$ be an nth order evolution equation which is bi-Hamiltonian with respect to a pair of Hamiltonian operators with constant coefficients. Then $v_t = K[v]$ is necessarily a* linear *equation with constant coefficients, i.e., we have $v_t = \sum_{i=0}^{n} c_i v_i$, where $c_i = \text{const}$.*

Proof of the lemma. Denote the Hamiltonian operators in question by $\mathcal{D}_i$, $i = 1, 2$. Since $v_t = K[v]$ is bi-Hamiltonian with respect to these operators by assumption, their ratio $\mathcal{R} = \mathcal{D}_2 \circ \mathcal{D}_1^{-1}$ is a (formal) recursion operator for this equation, that is (see e.g. [13] for details),

$$\mathrm{pr}\ \mathbf{v}_K(\mathcal{R}) - [\mathrm{D}_K, \mathcal{R}] = 0,$$

where $\mathrm{D}_K = \sum_{i=0}^{n} (\partial K/\partial u_i)\, D^i$ is the Fréchet derivative of K and $\mathrm{pr}\ \mathbf{v}_K$ is the prolongation of the evolutionary vector field $\mathbf{v}_K$ with the characteristic K. Since $\mathcal{D}_i$, $i = 1, 2$, have constant coefficients by assumption, we have $\mathrm{pr}\ \mathbf{v}_K(\mathcal{D}_i) = 0$, and therefore $\mathrm{pr}\ \mathbf{v}_K(\mathcal{R}) = 0$, so D_K commutes with $\mathcal{R}$, whence it readily follows that D_K has constant coefficients (recall that K is independent of t by assumption) and therefore K indeed is a linear combination of v_i with constant coefficients. □

The desired result now readily follows from the above lemma. □

Example 1. Consider a bi-Hamiltonian evolution equation

$$u_t = D_x^3 \left(u^{-2}\right) = H^{(3,0)} \delta_u \mathcal{T}_1 = H^{(5,0)} \delta_u \mathcal{T}_2, \tag{5}$$

where $\mathcal{T}_1 = -\int dx/u$ and $\mathcal{T}_2 = \int x^2 u dx$.

The potential form (4) of (5) reads

$$w_t = D_z^2 \left(w_z^{-2}\right) = \check{H}^{(3,0)} \delta_w \check{\mathcal{T}}_1 = \check{H}^{(5,0)} \delta_w \check{\mathcal{T}}_2. \tag{6}$$

Recall that $u = w_z$ and $x = z$; we have $\check{H}^{(3,0)} = -D_z$, $\check{H}^{(5,0)} = -D_z \circ ((1/w_z) D_z)^2$, $\check{\mathcal{T}}_1 = -\int dz/w_z$, and $\check{\mathcal{T}}_2 = \int z^2 w_z dz$. Note that (6) has, up to a rescaling of t, the form (2.31) from [2].

In perfect agreement with Proposition 1 (cf. also Proposition 2.2 in [2]) the hodograph transformation $z = v$, $w = y$ linearizes (6) into a (trivially) bi-Hamiltonian equation

$$v_t = -2v_{yyy} = \widetilde{H}^{(3,0)} \delta_v \widetilde{\mathcal{T}}_1 = \widetilde{H}^{(5,0)} \delta_v \widetilde{\mathcal{T}}_2, \tag{7}$$

where $\widetilde{H}^{(3,0)} = -D_y$, $\widetilde{H}^{(5,0)} = -D_y^3$, $\widetilde{\mathcal{T}}_1 = -\int v_y^2 dy$, and $\widetilde{\mathcal{T}}_2 = \int v^2 dy$.

The transformation $x = v$, $u = 1/v_y$ relates (7) to (5), cf. Corollary 2, so (5) provides an explicit example of a C-integrable (rather than S-integrable) bi-Hamiltonian system, just as discussed in Introduction.

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