A NOTE ON DYNAMICAL SYSTEMS DEFINING JACOBI'S ϑ-CONSTANTS

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Abstract. We propose a system of ordinary differential equations which defines Jacobi's theta-constant series. The relations of this system to the classical Darboux–Halphen equations and equations introduced by Jacobi are studied. The systems admit a Hamiltonian formulation with a rich structure. We explicitly construct a pencil of nonlinear Poisson brackets and a complete set of involutive integrals of motion.

Contents

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1. Introduction and motivation

In this work we propose the ordinary differential equations (ODEs) defining the classical ϑ-constants and the Weierstrass η-function. By a certain variable change these equations are transformed into a remarkable Jacoby system [14] which does not seem to have received mention in the modern literature. Some other known differential systems, having an extensive literature can be derived from this system by rational transformations. The most known of such ODEs include the Darboux–Halphen system [9, 13], some its varieties [18, 2], and the famous Chazy equation [8]. For equations defining the ϑ, η-series we work out the Hamiltonian formulation and show that the systems admit a pencil of (compatible) Poisson structures, in the sense of Magri [16].

The three Jacobi theta-constants are defined by the classical series

$$\vartheta_2(\tau) = \mathrm{e}^{\frac{1}{4}\pi\mathrm{i}\tau} \sum_{k=-\infty}^{\infty} \mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}, \qquad \vartheta_3(\tau) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{k^2\pi\mathrm{i}\tau}, \qquad \vartheta_4(\tau) = \sum_{k=-\infty}^{\infty} (-1)^k \mathrm{e}^{k^2\pi\mathrm{i}\tau}$$

and the Weierstrass η-function is defined by the series

$$\eta(\tau) = 2\pi^2 \left\{ \frac{1}{24} - \sum_{k=1}^{\infty} \frac{\mathrm{e}^{2k\pi\mathrm{i}\tau}}{(1-\mathrm{e}^{2k\pi\mathrm{i}\tau})^2} \right\}.$$

Here, the 'time' τ is considered to be a complex variable belonging to the upper half-plane $\mathbb{H}^+$: $\Im(\tau) > 0$. These series appear in various problems of mathematical and theoretical physics because of their numerous differential properties [2, 8]. Let us mention some of them.

Three ϑ-constant series satisfy the following differential identities for logarithmic derivatives of their ratios:

$$\frac{d}{d\tau} \ln \frac{\vartheta_2}{\vartheta_3} = \frac{\pi}{4}\mathrm{i}\vartheta_4^4, \qquad \frac{d}{d\tau} \ln \frac{\vartheta_3}{\vartheta_4} = \frac{\pi}{4}\mathrm{i}\vartheta_2^4, \qquad \frac{d}{d\tau} \ln \frac{\vartheta_2}{\vartheta_4} = \frac{\pi}{4}\mathrm{i}\vartheta_3^4.$$

If we introduce the following notation for logarithmic derivatives

$$(X, Y, Z) := 2\left(\frac{\dot\vartheta_2}{\vartheta_2}, \frac{\dot\vartheta_3}{\vartheta_3}, \frac{\dot\vartheta_4}{\vartheta_4} \right)$$

then the quantities (X, Y, Z) satisfy the 3rd order differential system

$$\dot X = (Y+Z)X - YZ, \qquad \dot Y = (X+Z)Y - XZ, \qquad \dot Z = (X+Y)Z - XY, \tag{1}$$

which is widely known as the famous Darboux–Halphen system [9, p. 149], [13, **I**: p. 330–331]. Its physical applications were initiated in the 1990's by M. Ablowitz, J. Chakravarty et all [6, 1] in connection with reductions of self-dual Yang–Mills equations. They usually provide the main motivation for studying both the η, ϑ-series and allied objects (modular forms). However, applications go beyond the Yang–Mills theory. In succeeding years the system appeared in the vacuum cosmological Bianchi–IX model [8, p. 143, 147], [2, p. 577], [1], theory of monopoles [4], and many other areas of mathematical physics. The system

(1) has also varieties; not a lesser-known one is the Weierstrass–Halphen dynamical system on Weierstrass' invariants g_2, g_3, and η-series:

$$\frac{dg_2}{d\tau}=\frac{\mathrm{i}}{\pi}\big(8g_2\eta-12g_3\big),\qquad \frac{dg_3}{d\tau}=\frac{\mathrm{i}}{\pi}\left(12g_3\eta-\frac{2}{3}g_2^2\right),\qquad \frac{d\eta}{d\tau}=\frac{\mathrm{i}}{\pi}\left(2\eta^2-\frac{1}{6}g_2\right). \tag{2}$$

It is known that invariants g_2, g_3 are related to the ϑ-series by the polynomial formulae

$$\begin{aligned} g_2(\tau)&=\frac{\pi^4}{24}\big\{\vartheta_2^8(\tau)+\vartheta_3^8(\tau)+\vartheta_4^8(\tau)\big\},\\ g_3(\tau)&=\frac{\pi^6}{432}\big\{\vartheta_2^4(\tau)+\vartheta_3^4(\tau)\big\}\big\{\vartheta_3^4(\tau)+\vartheta_4^4(\tau)\big\}\big\{\vartheta_4^4(\tau)-\vartheta_2^4(\tau)\big\} \end{aligned} \tag{3}$$

and the series themselves satisfy the famous Jacobi identity

$$\vartheta_3^4(\tau)=\vartheta_2^4(\tau)+\vartheta_4^4(\tau). \tag{4}$$

In different notation and definition for series the system (2) is known as the Ramanujan system of differential equations [7] for modular forms

$$E_2(\tau)\sim\eta(\tau),\qquad E_4(\tau)\sim g_2(\tau),\qquad E_6(\tau)\sim g_3(\tau), \tag{5}$$

where $E_{2k}(\tau):=\sum'(m\tau+n)^{-2k}$. Ramanujan's system is sometimes referred as the Eisenstein system of differential equations [7], though Eisenstein himself did not derive it [11]. Number theoretic and differential treatment of equations satisfied by the series $E_{2k}(\tau)$ was given by Ramanujan [18]. Additional discussion (and references) of the systems mentioned above can be found in the cited works [1, 2, 3, 6, 8, 12].

Dynamical variables for all the systems above are determined usually rationally through the ϑ-variables. Therefore inverse transformations will always involve inversions by multi-valued functions, as further examples show. Yet another example is more nontrivial and comes from equations for modular forms on group $\mathbf{\Gamma}_0(2)$. They define the Ramamani system and were studied recently in work [3] (see also [15, 12]). In this case relation between dynamical variables and the ϑ,η-variables is not obvious because it is given by a duplication of the τ-argument in forms (5) (see formula (3.3) in [3]). If we make use, however, the duplication rules[1]

$$\eta(2\tau)=\frac{1}{2}\eta(\tau)+\frac{\pi^2}{48}\big\{\vartheta_3^4(\tau)+\vartheta_4^4(\tau)\big\},\quad g_2(2\tau)=\frac{1}{8}g_2(\tau)+\frac{\pi^4}{192}\big\{14\vartheta_4^4(\tau)\vartheta_3^4(\tau)-\vartheta_2^8(\tau)\big\}$$

we arrive again at substitutions of a rational type.

For the reasons given above it is essential to have an exhaustive description to differential properties of the ϑ,η-series as such. In particular, we display here a version of differential relations between the objects because we were not able to find it in closed and explicit form in so comprehensive literature on theta-functions.

[1]Though these rules have not appeared in the standard texts known to us, these identities may be established by standard techniques.

Theorem 1. *The canonical Jacobi's ϑ-constant series satisfy the closed differential identities upon adjoining the Weierstrass η-series*:

$$\begin{aligned} \frac{d\vartheta_2}{d\tau} &= \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\vartheta_2, & \frac{d\vartheta_4}{d\tau} &= \frac{\mathrm{i}}{\pi}\Big\{\eta - \frac{\pi^2}{12}\big(\vartheta_2^4+\vartheta_3^4\big)\Big\}\vartheta_4, \\ \frac{d\vartheta_3}{d\tau} &= \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}\big(\vartheta_2^4-\vartheta_4^4\big)\Big\}\vartheta_3, & \frac{d\eta}{d\tau} &= \frac{\mathrm{i}}{\pi}\Big\{2\eta^2 - \frac{\pi^4}{12^2}\big(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8\big)\Big\}. \end{aligned} \tag{6}$$

The identities (6), once considered as dynamical system, have the only algebraic integral being the rational function of ϑ's:

$$U\cdot\vartheta_2^4\vartheta_3^4\vartheta_4^4 = \big(\vartheta_3^4-\vartheta_2^4-\vartheta_4^4\big)^3. \tag{7}$$

This integral generalizes Jacobi's identity (4) if $U \neq 0$.

Proof is given by a straightforward computation with use of Eqs. (2), (3), and (4). ■

Varieties of the system (6) will be the main subject of our consideration. In particular, it is of interest to propose a Hamiltonian formulation for the system and consider some related topics. This would provide fresh insight into properties of the theta-constants. This circle of questions, as applied to an equivalent of the system (2), is addressed in the work [7] by D. Chudnovsky & G. Chudnovsky and, to the best of our knowledge, this is the only paper[2] where the question on Hamiltonian treatment for dynamical systems of modular type has been raised. These authors proposed a 4th order differential system [7, p. 111] and its reduction to the 3rd order equations. Although their Hamilton function is ingenious enough and correct, proposed reduction to the system (2) is not preserved by the constraint[3] $\lambda = 1$ defining the reduction itself. In other words, this reduction is satisfied only by a trivial solution.

1.1. **A Jacobi dynamical system.** Since the late 1850's C. Borchardt, being the editor-in-chief of Crelle's Journal, began to edit and publish the manuscript material kept after Jacobi's death in 1851. In particular, in 1857 he published [14, p. 383–394] calculations where Jacobi constructed power series developments for his $\theta(z|\tau)$-functions. The power θ-series are of interest in their own rights but not a lesser remarkable fact is that they produce the nice dynamical systems integrable in terms of ϑ-constants.

Using conventional notation for classical objects of Legendre's 'elliptic theory'

$$K(k) = \int\limits_0^1 \frac{d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}, \qquad K'(k) = \int\limits_k^1 \frac{d\lambda}{\sqrt{(1-\lambda^2)(\lambda^2-k^2)}}, \tag{8}$$

$$E(k) = \int\limits_0^1 \sqrt{\frac{1-k^2\lambda^2}{1-\lambda^2}}\,d\lambda, \qquad E'(k) = \int\limits_0^1 \sqrt{\frac{1-(1-k^2)\lambda^2}{1-\lambda^2}}\,d\lambda, \tag{9}$$

[2] See also important comments on pp. 5709–5710 in work [17] concerning the system (1) and its relation to Euler's equations and the Lotka–Volterra system.

[3] Notation as on p. 111 of [7].

Jacobi introduces the four variables (we keep completely to Jacobi's notation in [14, p. 386])

$$A=\frac{2K}{\pi},\qquad B=\frac{2E}{\pi}-k'^2\frac{2K}{\pi},\qquad a=4(1-2k^2),\qquad b=2k^2k'^2, \tag{10}$$

where $k^2+k'^2=1$, and shows that these satisfy the very elegant dynamical system

$$\left\{\begin{aligned} \frac{\partial A}{\partial h}&=2A^2B, & \frac{\partial a}{\partial h}&=-16bA^2,\\ \frac{\partial B}{\partial h}&=bA^3, & \frac{\partial b}{\partial h}&=abA^2, \end{aligned}\right. \tag{11}$$

where $h=\frac{1}{4}\pi\mathrm{i}\tau$, and the following constraint

$$a^2=16(1-2b) \tag{12}$$

should be imposed. It is of course an equivalent to the Jacobi ϑ-identity (4). Halphen does not mention system (11) and, to all appearances, it has not received mention in the later literature in the context. Jacobi does not restrict his consideration to variables (10) and exhibits what is called presently the canonical transformations, i.e. transformation of dynamical variables preserving the shape of equations. Here are his versions of the transformations [14, p. 387]:

$$\begin{aligned} A&=\frac{2kK}{\pi}, & B&=\frac{1}{k}\cdot\frac{2E}{\pi}, & a&=-\frac{4(1+k'^2)}{k^2}, & b&=-\frac{2k'^2}{k^4},\\ A&=\frac{2k'K}{\pi}, & B&=\frac{1}{k'}\left(\frac{2E}{\pi}-\frac{2K}{\pi}\right), & a&=\frac{4(1+k^2)}{k'^2}, & b&=-\frac{2k^2}{k'^4} \end{aligned}$$

and complete set of differential relations between these and auxiliary variables $\{k,k',K,E\}$ was written down by Jacobi earlier [14, p. 176–177]. As in the previous differential systems (1), (2), dynamical variables $\{A,B,a,b\}$ are expressed through the η,ϑ-constants rationally. For example, a simple computation for version (10) implies that

$$A=\vartheta_3^2,\quad B=\frac{4}{\pi^2\vartheta_3^2}\left\{\eta+\frac{\pi^2}{12}\left(\vartheta_2^4-\vartheta_4^4\right)\right\},\quad a=4-8\frac{\vartheta_2^4}{\vartheta_3^4},\quad b=2\frac{\vartheta_2^4}{\vartheta_3^4}\frac{\vartheta_4^4}{\vartheta_3^4}. \tag{13}$$

We also note that system (11) is notable for its homogenous monomial structure. Jacobi exploits intensively this fact when deriving the power θ-series; the pages 388–391 of his Werke [14] contain a lot of useful formulae along these lines. The system (11) is not the only dynamical system that was derived by Jacobi in connection with θ-functions; see also [14, p. 173–190] and [10]. He does not pose a question about integration of (11) as system of ODEs[4], however earlier, in 1847, he obtained a complete integral for his famous 3rd order differential equation

$$C^4(\ln C^3C_{\tau\tau})_\tau^2=16C^3C_{\tau\tau}-\pi^2, \tag{14}$$

satisfied by any of the ϑ-constants: $C=\vartheta^{-2}$ (Jacobi's notation [14, p. 179]). This equation is of course a consequence of the system (6).

[4]Complete integral to the system (11) and some computational details will be given in appendix.

2. ODEs determining ϑ-constants

2.1. **Remarks concerning symmetrical system (6).** As we mentioned above all the varieties of dynamical systems under question are algebraically related each other. In this respect equations (6) stand out because this system represents ϑ-constants by itself. However point transformation between dynamical variables is not unique and resulting ODEs for ϑ, η-variables may contain parameters. For example, if we drop out the fix constraint (12) and consider (13) just as a point change in (11), we shall not arrive at symmetrical equations (6). We may also insert into the change (13) some parameters and yield the more symmetrical or simpler form to resulting ODEs, say,

$$a = 4 - \boldsymbol{\alpha}\frac{\vartheta_2^4}{\vartheta_3^4}, \qquad b = \boldsymbol{\beta}\frac{\vartheta_2^4}{\vartheta_3^4}\frac{\vartheta_4^4}{\vartheta_3^4}, \tag{15}$$

but we never get the system (6) in this way. Any of such ODEs will be integrable in terms of ϑ, η-series since they were obtained from (11) by coordinate changes of dynamical variables. These are generally algebraic, i.e. multi-valued in both directions. In this respect even Jacobi's system (11) is not a best choice because we would have

$$\text{cumbrous}\quad K\left(\sqrt{\frac{1}{2}-\frac{1}{2}\frac{a}{\sqrt{a^2+32b}}}\right)\quad \text{instead of simple } K(k)$$

(see theorem 7 in appendix). In other words, the choice of a representative for differential system defining Jacobi–Weierstrass' series is not a trivial question and we need to choose, in some sense, 'natural/optimal' version for such a system (call it canonical one) reflecting the principal property of the series, namely, the property of being uniformizing for other algebraic versions like systems (1), (2), or (11).

For this purpose, however, symmetrical form (6) is apt to be not a good candidate because it is not amenable to integration and we failed to find out its complete integral. That such a strong distinction between systems is inherent in the nature of the case (6) will be apparent from the consideration of their algebraic integrals as algebraic curves in homogeneous coordinates $\vartheta_2 : \vartheta_3 : \vartheta_4$. The equation (30), under generalization (15), has genus 9, whereas integral (7) is a curve of genus $g = 19$. The best we have succeed in solution of the system (6) is its partial solution in terms of elliptic functions.

Indeed, let us rewrite integral (7) in the form of elliptic curve[5]

$$2U^2\boldsymbol{xy} = (\boldsymbol{y}-\boldsymbol{x}-2)^3, \qquad \boldsymbol{x} = 2\frac{\vartheta_2^4}{\vartheta_4^4}, \quad \boldsymbol{y} = 2\frac{\vartheta_3^4}{\vartheta_4^4}.$$

Hence it follows that the pair $(\boldsymbol{x}, \boldsymbol{y})$ is parametrized by Weierstrass' $(\wp, \wp')$-functions and this curve can be transformed into the canonical Weierstrassian form

$$\wp'(\mathfrak{u})^2 = 4\wp^3(\mathfrak{u}) - g_2\,\wp(\mathfrak{u}) - g_3\,.$$

[5]The analogous transformation to the fourth powers of ϑ's in integrals like (30), (15) leads to a zero genus curve. Equations are easily integrated and no elliptic functions appear in this case.

The computation is rather simple and we obtain

$$\boldsymbol{x}=\frac{1}{U}\wp'(\mathfrak{u})-\wp(\mathfrak{u})+\frac{U^2}{12}-1,\qquad \boldsymbol{y}=\frac{1}{U}\wp'(\mathfrak{u})+\wp(\mathfrak{u})-\frac{U^2}{12}+1, \tag{16}$$

where invariants g_2, g_3 are expressed through the integral U by the formulae

$$g_2=\frac{U^4}{12}-2U^2,\qquad g_3=-\frac{U^6}{216}+\frac{U^4}{6}-U^2.$$

Therefore expressions (16) substituted into the system (6) must cause this system to become a τ-evolution of uniformizer $\mathfrak{u}=\mathfrak{u}(\tau)$. Indeed, after some algebra we derive that

$$\frac{36U}{\pi\mathrm{i}}\cdot\frac{1}{\vartheta_4^4}\frac{d\mathfrak{u}}{d\tau}=12\wp(\mathfrak{u})-U^2$$

and therefore

$$\int\frac{d\mathfrak{u}}{12\wp(\mathfrak{u})-U^2}=\frac{\pi\mathrm{i}}{36U}\int\vartheta_4^4 d\tau+\mathrm{const}.$$

Left hand side of this equation is easily integrated because

$$\frac{12U}{12\wp(\mathfrak{u})-U^2}=\zeta(\mathfrak{u}-\varkappa)-\zeta(\mathfrak{u}+\varkappa)+2\zeta(\varkappa),$$

where $12\wp(\varkappa)=U^2$, $\wp'(\varkappa)=\pm U$, and $\zeta(\mathfrak{u})$, $\sigma(\mathfrak{u})$ are the standard Weierstrassian functions associated to the basis $\wp(\mathfrak{u}),\wp'(\mathfrak{u})$. We get

$$\frac{3\mathrm{i}}{\pi}\ln\frac{\sigma(\mathfrak{u}+\varkappa)}{\sigma(\mathfrak{u}-\varkappa)}\mathrm{e}^{2\zeta(\varkappa)\mathfrak{u}}=\int\vartheta_4^4 d\tau+\mathrm{const},$$

but integral in the right hand side requires a further integration of the system. This step is unknown.

2.2. **An integrable version for the ϑ,η-constants.** Returning to the question of canonical representative for ODEs defining ϑ,η-series and using some heuristic reasoning[6] we choose the following modification of equations (6):

$$\begin{aligned}\frac{d\vartheta_2}{d\tau}&=\frac{\mathrm{i}}{\pi}\left\{\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\vartheta_2, &\qquad \frac{d\vartheta_4}{d\tau}&=\frac{\mathrm{i}}{\pi}\left\{\eta-\frac{\pi^2}{12}(2\vartheta_3^4-\vartheta_4^4)\right\}\vartheta_4,\\ \frac{d\vartheta_3}{d\tau}&=\frac{\mathrm{i}}{\pi}\left\{\eta+\frac{\pi^2}{12}(\vartheta_3^4-2\vartheta_4^4)\right\}\vartheta_3, &\qquad \frac{d\eta}{d\tau}&=\frac{\mathrm{i}}{\pi}\left\{2\eta^2-\frac{\pi^4}{72}(\vartheta_3^8-\vartheta_3^4\vartheta_4^4+\vartheta_4^8)\right\}.\end{aligned} \tag{17}$$

It is of interest to observe that all the previous dynamical systems reduce in fact to the *squares* of ϑ-constants. For this reason, in the sequel it will be convenient to renormalize variables ϑ,η and adopt the following notation:

$$x=\sqrt{\frac{\pi\mathrm{i}}{6}}\vartheta_2^2,\qquad y=\sqrt{\frac{\pi\mathrm{i}}{6}}\vartheta_3^2,\qquad z=\sqrt{\frac{\pi\mathrm{i}}{6}}\vartheta_4^2,\qquad u=\frac{2\mathrm{i}}{\pi}\eta.$$

[6] An exhaustive explanation as to why the 'determining ϑ,η-equations' should have the form (17) has been detailed in work [5].

The resulting equivalent of the system (17)

$$\begin{aligned} \dot{x} &= (u + y^2 + z^2)x, & \dot{z} &= (u - 2y^2 + z^2)z, \\ \dot{y} &= (u + y^2 - 2z^2)y, & \dot{u} &= u^2 - y^4 + y^2z^2 - z^4 \end{aligned} \tag{18}$$

will be the main subject of further study. Apart from symmetry $y \rightleftarrows \pm z$ and simplicity, there are some additional properties justifying usefulness of canonical system (18).

First of all, the function u, independently of (x, y, z), satisfies the famous Chazy equation

$$\dddot{u} = 6(2u\ddot{u} - 3\dot{u}^2),$$

(proof is a direct calculation) which cannot be said of η-solution to symmetrical version (6). For the latter, the function η (more precisely $\frac{2\mathrm{i}}{\pi}\eta$) solves this equation only if the U-integral (7) is equal to zero. Similarly, functions y and z also satisfy a third (not fourth) order ODE. This is the very Jacobi C-equation (14):

$$C^4(\ln C^3 C_{\tau\tau})^2_\tau = 16C^3C_{\tau\tau} + 36, \qquad C = \frac{1}{z} \text{ or } \frac{1}{y}.$$

Algebraic integral for (18) is easily found because x is absent in three equations (18). Elimination of u from these equations leads to that the function

$$\pi \boldsymbol{I}^2 = \frac{y^2 - z^2}{x^2} \tag{19}$$

is a constant on solutions of (18), that is integral. This integral is much simpler than those we discussed in sect. 2.1. As for solutions to system (18), these have the most simple form as against the other equations we consider. We shall give these solutions in the next theorem. The last fact we should mention here is the point transformation which leads to the system (18). Most simple form of such a transformation is realized through the 'linearizing' systems (34), (35) which can be thought of as intermediate equivalents for Jacobi's one (11) or (18). Explanation and details have been given in appendix. From now on we change Jacobi's h-notation and put

$$\mathrm{T} := \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$$

with normalization $\alpha\delta - \beta\gamma = 1$.

Theorem 2. *The transformations between Jacobi's system* (11) *and canonical dynamical system* (18) *defining* ϑ, η*-constants are given by the substitution*

$$\begin{aligned} A &= \frac{1 - \mathrm{i}}{2\boldsymbol{I}}y, & B &= \frac{1 + \mathrm{i}}{2}\frac{\boldsymbol{I}}{y}(u + y^2 - 2z^2), \\ a &= \frac{12}{\pi\mathrm{i}}\frac{y^2 - z^2}{x^2}\frac{y^2 - 2z^2}{y^2}, & b &= -\frac{18}{\pi^2}\frac{z^2}{x^4y^4}(y^2 - z^2)^3. \end{aligned} \tag{20}$$

The system (18) *has the following general solution:*

$$x = \varepsilon\frac{\vartheta_2^2(\mathrm{T})}{\gamma\tau + \delta}, \qquad y = \sqrt{\frac{\pi\mathrm{i}}{6}}\frac{\vartheta_3^2(\mathrm{T})}{\gamma\tau + \delta}, \qquad z = \sqrt{\frac{\pi\mathrm{i}}{6}}\frac{\vartheta_4^2(\mathrm{T})}{\gamma\tau + \delta}, \qquad u = \frac{2\mathrm{i}}{\pi}\frac{\eta(\mathrm{T})}{(\gamma\tau + \delta)^2} - \frac{\gamma}{\gamma\tau + \delta},$$

where $\varepsilon \neq 0$ is the fourth free constant. Under $\varepsilon = 0$ the solution is degenerated into the two parametric one

$$x = 0, \qquad y = \frac{\pm 1}{\gamma\tau+\delta}, \qquad z = \frac{1}{\gamma\tau+\delta}, \qquad u = -\frac{\gamma^2\tau+\gamma\delta-1}{(\gamma\tau+\delta)^2}.$$

Proof is a straightforward calculation with use of (6). Derivation of the change (20) follows from formulae (42)–(43). ■

In other words, the change (20) is a quite non-obvious correction of the change (13) and integral $\pi \boldsymbol{I}^2 x^2 = y^2 - z^2$ is a correct modification of complicated integral (7). The Jacobi identity (4) is thus a surface of a constant level in the phase-space (x, y, z, u).

3. Integrals and Poisson structures

3.1. **Function integrals.** Lagrangians, Hamiltonians, and Poisson structures for dynamical systems are known to be closely related to integrals of the systems. This being so, we may use results of the previous sections in order to propose corresponding description of both the system (11) and (18). At first, let us tabulate the complete set of integrals to equations (18).

Proposition 1. *The system (18) has the only algebraic integral (19) and the two transcendental multi-valued ones*

$$\boldsymbol{J}_1 = \frac{1}{y}(u-2y^2+z^2)K\Big(\frac{z}{y}\Big)+3yE\Big(\frac{z}{y}\Big), \qquad \boldsymbol{J}_2 = \frac{1}{y}(u+y^2+z^2)K'\Big(\frac{z}{y}\Big)-3yE'\Big(\frac{z}{y}\Big), \tag{21}$$

that is $\dot{\boldsymbol{J}}_1 = \dot{\boldsymbol{J}}_2 \equiv 0$. The integrals satisfy the identity

$$\boldsymbol{J}_1 K'\Big(\frac{z}{y}\Big) - \boldsymbol{J}_2 K\Big(\frac{z}{y}\Big) = \frac{3}{2}\pi y.$$

Proof and derivation use computations outlined in sect. 4.2 of appendix. ■

It should be emphasized here that integrability of any modular systems is always related to the linear ODEs of Fuchsian class. By this means the appearance of transcendentally multi-valued objects like K, K', E, E' is *inevitable* point. Transition between such a 'k-linear' and 'modularly nonlinear' τ-representation has been detailed in sect. 4.1 of appendix.

3.2. **Action and Lagrangians.** Let us denote coordinates of the phase-space $X^{\intercal} := (A, B, a, b)$ for system (11) or $X^{\intercal} := (x, y, z, u)$ for (18). We can use the standard relations between action functional

$$S = \int \mathcal{L}(X, \dot{X})\, d\tau, \tag{22}$$

Legendre transformation from the Lagrange function

$$\mathcal{L}(X, \dot{X}) = \varrho_k(X)\dot{X}^k - \mathcal{H}(X) \tag{23}$$

to Hamiltonian $\mathcal{H}(X)$, and Poisson brackets. Variation of action $\delta S = 0$ entails equations of motion (dynamical system) and their Hamiltonian form $\dot{X} = \Omega\nabla\mathcal{H}$:

$$\dot{X}^n = \Omega^{nk}(X)\frac{\partial \mathcal{H}}{\partial X^k} \quad\Leftrightarrow\quad \boldsymbol{\Omega}_{kn}(X)\dot{X}^n = \frac{\partial \mathcal{H}}{\partial X^k} \quad\Leftrightarrow\quad (11),\ (18),$$

where

$$\boldsymbol{\Omega}_{kn} = \frac{\partial \varrho_n(X)}{\partial X^k} - \frac{\partial \varrho_k(X)}{\partial X^n}$$

and Ω, $\boldsymbol{\Omega}$ are mutually inverse matrices: $\boldsymbol{\Omega} = \Omega^{-1}$. These equations do not depend on choice of the Lagrangian $\mathcal{L}$ but bi-vector Ω^{nk} defining Poisson bracket does, even though the Hamilton function $H = \mathcal{H}(X)$ and coordinates X have been fixed. We thus have to choose, apart from choice of $\mathcal{H}(X)$, any two independent integrals $I_j = I_j(X)$ in invariant form to Lagrangian density (23)

$$\mathcal{L}(X, \dot{X}) = \mathcal{H}(\dot{\mathcal{N}} - 1) + I_1 \dot{I}_2$$

and to find the quantity $N = \mathcal{N}(X)$ with normalization $\dot{\mathcal{N}} \equiv 1$. Clearly, it is a linear function of τ, i.e. $\mathcal{N}(X) = \tau + \mathrm{const}(\mathcal{H}, I_1, I_2)$, and we have built such objects in appendix. A computation, based on (44) followed by use of (41), (34), and (37), shows that

$$\mathcal{N} = -2\frac{K(k)}{AJ_1} \qquad \Rightarrow \qquad \frac{d\mathcal{N}}{dh} \equiv 1 .$$

The quantity $\mathcal{N}(X)$ thus becomes

$$\mathcal{N}(X) = -\frac{K\big(\frac{z}{y}\big)}{y\boldsymbol{J}_1} = \frac{-K\big(\frac{z}{y}\big)}{(u-2y^2+z^2)K\big(\frac{z}{y}\big) + 3y^2 E\big(\frac{z}{y}\big)} \qquad (\dot{\mathcal{N}} \equiv 1).$$

The Lagrangian density is defined up to a perfect τ-derivative and therefore its choice always contains some heuristic arguments (simplicity of Lagrangians, brackets, etc.) The most compact Lagrange function we have found is given by the following statement.

Proposition 2. *The equations (11), (18), and (34) are lagrangian for action (22) with the following expression for Lagrangian density $\mathcal{L}$:*

$$\begin{aligned} \mathcal{L} &= J_1^2(\dot{\mathcal{N}} - 1) + J_2 \dot{I} - 8\frac{d}{d\tau}\left(\frac{B}{A}K^2\right) = \\ &= 4\frac{J_1 K}{A^2}\cdot\dot{A} - 2\left\{kIK^2 + \frac{J_1^2 - 16B^2K^2}{k(k^2-1)IA^2}\right\}\cdot\dot{k} + \left\{J_2 + \frac{2K}{AI}(J_1 - 4BK)\right\}\cdot\dot{I} - J_1^2, \end{aligned} \tag{24}$$

where we omit indication of argument in Legendre's integral $K(k)$ and expressions for $J_{1,2}(A, B, k, I)$ are taken from theorem 7 of appendix. Transitions between variables are given by substitutions (42)–(43) and (20).

3.3. **Poisson structures.** The following property characterizes some non-triviality of dynamical systems under consideration.

Theorem 3. *Whatever the Hamilton function $\mathcal{H}(X)$ may be (single- or multi-valued analytic function), none of the systems (6), (11), or (17), (18) does admit a constant non-degenerated Poisson bracket Ω.*

Proof. Let X denotes a phase-space coordinate vector for any of the systems above: $\dot{X}^k = V^k(X)$. Assuming the availability of the form $\boldsymbol{\Omega}\dot{X} = \nabla\mathcal{H}(X)$ with constant matrix $\boldsymbol{\Omega}$ we apply integrability condition to equations $\nabla\mathcal{H} = \boldsymbol{\Omega}\dot{X}$, once considered as equations for the Hamiltonian $\mathcal{H}$:

$$\nabla_k \mathcal{H} = \boldsymbol{\Omega}_{kj}V^j \quad \Rightarrow \quad \nabla_n(\boldsymbol{\Omega}_{kj}V^j) = \nabla_k(\boldsymbol{\Omega}_{nj}V^j) \quad \Rightarrow \quad \boldsymbol{\Omega}_{kj}\nabla_n V^j = \boldsymbol{\Omega}_{nj}\nabla_k V^j . \tag{25}$$

It follows that $\boldsymbol{\Omega}\cdot\partial_X V$ is a symmetric matrix $\forall X$. Straightforward computations show that this property is compatible with vector fields V's determining the systems (6), (11), and (17)–(18) if and only if $\boldsymbol{\Omega} \equiv 0$. ■

This proof gives in fact a criteria for availability of a canonical symplectic form (given coordinates) and absence of such a bracket suggests to look for non-canonical one. Insomuch as (19) is the only single-valued function integral we have to take it (or function of it) as a Hamiltonian. Again, choosing $\mathcal{L}$ in order that bracket Ω be simplest, we put

$$\mathcal{L} = \mathcal{H}(\dot{\mathcal{N}} - 1) + (\lambda \boldsymbol{J}_1)^{-1}\dot{\boldsymbol{J}}_2, \tag{26}$$

where λ is an arbitrary constant, and derive the bracket by formulae from the previous section. The computational rule thus acquires the following form:

$$\boldsymbol{\Omega} = \boldsymbol{M} - \boldsymbol{M}^{\intercal}, \qquad \boldsymbol{M}_{kn} = \nabla_k\mathcal{H}\cdot\nabla_n\mathcal{N} + \nabla_k I_1\cdot\nabla_n I_2 .$$

We insert here $I_1 = (\lambda\boldsymbol{J}_1)^{-1}$ and $I_2 = \boldsymbol{J}_2$ and use proposition 3 of appendix.

Theorem 4. *Let $X^{\intercal} := (x, y, z, u)$. Dynamical system (18) admits a Hamiltonian form*

$$\dot{X} = \omega\nabla\mathcal{H}, \qquad \mathcal{H} = \frac{1}{2}\frac{y^2 - z^2}{x^2}$$

with the degenerated rational (single-valued) Poisson bracket

$$\omega = \frac{x}{2\mathcal{H}}\begin{pmatrix} 0 & (u+y^2-2z^2)y & (u-2y^2+z^2)z & u^2-y^4+y^2z^2-z^4 \\ -(u+y^2-2z^2)y & 0 & 0 & 0 \\ -(u-2y^2+z^2)z & 0 & 0 & 0 \\ -u^2+y^4-y^2z^2+z^4 & 0 & 0 & 0 \end{pmatrix}.$$

Non-degenerated but transcendental multi-valued extension of ω is given by the bracket $\Omega = \omega + \lambda\tilde{\omega}$ $(\det\Omega = 4\pi^{-2}\lambda^2x^6y^2z^2\boldsymbol{J}_1^4)$, where

$$\tilde{\omega} = \frac{2}{\pi}K^2\begin{pmatrix} 0 & \frac{x}{y}z^2 & xz & xM_1 \\ -\frac{x}{y}z^2 & 0 & \frac{z}{y}(y^2-z^2) & \frac{1}{y}M_2 \\ -xz & \frac{z}{y}(z^2-y^2) & 0 & zM_3 \\ -xM_1 & -\frac{1}{y}M_2 & -zM_3 & 0 \end{pmatrix} \qquad (\det\tilde{\omega} = 0)$$

and

$$M_1 := 3y^2(EK^{-1}-1)^2 - z^2, \qquad M_3 := y^2(3E^2K^{-2}-1) + z^2,$$

$$M_2 := 3y^4(EK^{-1}-1)^2 + y^2z^2(6EK^{-1}-5) + 2z^4.$$

Brackets ω, $\tilde{\omega}$ are self-consistent and therefore have the following Casimir's functions:

$$\omega\nabla\boldsymbol{J}_1 = \omega\nabla\boldsymbol{J}_2 \equiv 0, \qquad \tilde{\omega}\nabla\mathcal{H} = \tilde{\omega}\nabla\mathcal{N} \equiv 0.$$

The system is thus bi-Hamiltonian in the sense of Magri [16].

Incidentally it should be observed that degenerated but well-defined rational bracket ω is obtained from non-degenerated but multi-valued bracket Ω by a passage to the limit

$\lambda \to 0$ in transcendental part of the Ω and this procedure can be interpreted as a formal separability of canonically conjugated pairs $(\mathcal{H}, \mathcal{N})$ and $(\boldsymbol{J}_1, \boldsymbol{J}_2)$ in Lagrangian (26). Their commutation relations (algebra of integrals) are standard:

$$\{\mathcal{H}, \boldsymbol{J}_1\}_\Omega = \{\mathcal{H}, \boldsymbol{J}_2\}_\Omega = 0, \qquad \{\boldsymbol{J}_2, (\lambda \boldsymbol{J}_1)^{-1}\}_\Omega = 1.$$

Remark. An explicit analog of theorem 4 for Jacobi's system (11) is obtained with avail of transformation law for tensor $\Omega(x,y,z,u) \mapsto \widetilde{\Omega}(A,B,a,b)$ under the coordinate change $X^\intercal := (x,y,z,u) \mapsto (A,B,a,b) =: Y^\intercal$. The explicit form of the transformations reads

$$\widetilde{\Omega}^{jp}(Y) = \frac{\partial Y^j}{\partial X^n}\frac{\partial Y^p}{\partial X^m}\Omega^{nm}(X) \qquad \Rightarrow \qquad \widetilde{\Omega} = \boldsymbol{T}\Omega\boldsymbol{T}^\intercal, \quad \boldsymbol{T}_{kn} := \frac{\partial Y^k}{\partial X^n}$$

and implies equations

$$\dot{Y}^j = \widetilde{\Omega}^{jp}(Y)\frac{\partial \mathcal{H}}{\partial Y^p} \qquad \Leftrightarrow \qquad (11).$$

We do not display the formulae here since they are not quite compact.

We conclude the section with general remarks concerning other non-constant brackets. All of them are obtainable by re-normalization of the objects

$$N \mapsto N + F_1(H, \boldsymbol{J}_1, \boldsymbol{J}_2),\ H \mapsto F_2(H, \boldsymbol{J}_1, \boldsymbol{J}_2),\ \boldsymbol{J}_2 \mapsto F_3(H, \boldsymbol{J}_1, \boldsymbol{J}_2),\ \boldsymbol{J}_2 \mapsto F_4(H, \boldsymbol{J}_1, \boldsymbol{J}_2) \quad (27)$$

entering into Lagrangian (26). This defines a function freedom of the three variables $(\alpha, \beta, \gamma) = (H, \boldsymbol{J}_1, \boldsymbol{J}_2)$. On the other hand, all the dependencies $\Omega(X)$, including possible change of the Hamilton function $\mathcal{H}$, are determined by the following modification of the line (25):

$$\nabla_n(\boldsymbol{\Omega}_{kj} V^j) = \nabla_k(\boldsymbol{\Omega}_{nj} V^j) \quad \Rightarrow \quad (\nabla_n \boldsymbol{\Omega}_{kj} - \nabla_k \boldsymbol{\Omega}_{nj}) V^j = \boldsymbol{\Omega}_{nj} W^j_k - \boldsymbol{\Omega}_{kj} W^j_n, \qquad (28)$$

where tensor field

$$W^j_k := \frac{\partial V^j}{\partial X^k}$$

can be thought of as given. Equations (28) are a set of partial DE's for $\Omega(X)$'s but, thanks to function freedom mentioned above, we may pass from old set of variables, say (x,y,z,u), to the new one $(N, \alpha, \beta, \gamma)$ and thereby turn these equations into ordinary differential equations in variable N.

Theorem 5. *Denote $\mathit{\Omega}(N; \alpha, \beta, \gamma) := \Omega(x,y,z,u)$ and matrix $W = W(N; \alpha, \beta, \gamma)$:*

$$W_{jk} := \left.\frac{\partial V^j}{\partial X^k}\right|_{X = X(N;\alpha,\beta,\gamma)},$$

where (α, β, γ) are seen as parameters. Then all the brackets $\Omega(X) = \mathit{\Omega}(N)$ satisfy the following dynamical system

$$\frac{d\mathit{\Omega}}{dN} = W\mathit{\Omega} + \mathit{\Omega} W^\intercal \qquad (29)$$

supplemented with arbitrary initial condition (bracket) $\mathit{\Omega}(0) = \Lambda(\alpha, \beta, \gamma)$.

Proof. With use of antisymmetry $\boldsymbol{\Omega}_{kj} = -\boldsymbol{\Omega}_{jk}$ and Jacobi's identity $\nabla_n \boldsymbol{\Omega}_{kj} + \nabla_k \boldsymbol{\Omega}_{jn} + \nabla_j \boldsymbol{\Omega}_{nk} = 0$ equations (28) can be rewritten as $V^j \nabla_j \boldsymbol{\Omega}_{nk} = \boldsymbol{\Omega}_{kj} W^j_n - \boldsymbol{\Omega}_{nj} W^j_k$. Hence $\dot{\boldsymbol{\Omega}} = -\boldsymbol{\Omega} W - W^\intercal \boldsymbol{\Omega}$ and, subsequently, (29) since $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$ and $\boldsymbol{\Omega}\dot{\mathit{\Omega}} = -\dot{\boldsymbol{\Omega}}\mathit{\Omega}$. ■

To put it differently, function freedom (27) with three functions of three variables $\alpha = \mathcal{H}(X)$, $\beta = \boldsymbol{J}_1(X)$, $\gamma = \boldsymbol{J}_2(X)$ is converted into coefficients of dynamical system (29) (matrix W) and its initial condition $\Lambda(\alpha,\beta,\gamma)$. As the latter one may take any particular bracket, say, the bracket Ω of theorem 4.

4. Appendix: computational details

4.1. **Solutions.** The general integral of system (11) is found without difficulties because all the objects of the theory are completely at hand. From (11) it follows that $a\partial a = -16\partial b$ and this yields an algebraic integral generalizing Jacobi's restriction (12)

$$I^2 = a^2 + 32b \qquad \Rightarrow \qquad \dot{I} \equiv 0. \tag{30}$$

Therefore b is expressed through the variable a and a itself satisfies a simple differential consequence of (11), namely the 3rd order equation

$$\frac{a_{hhh}}{a_h^3} - \frac{3}{2}\frac{a_{hh}^2}{a_h^4} = -\frac{1}{2}\frac{a^2+3I^2}{(a^2-I^2)^2}.$$

This is a variety of the standard differential equation for Legendre's modulus $\lambda := k^2(\tau)$:

$$\frac{\lambda_{\tau\tau\tau}}{\lambda_\tau^3} - \frac{3}{2}\frac{\lambda_{\tau\tau}^2}{\lambda_\tau^4} = -\frac{1}{2}\frac{\lambda^2-\lambda+1}{\lambda^2(\lambda-1)^2}. \tag{31}$$

It immediately follows that there is bound to be a linear fractional change between a and λ transforming these equations into each other. This simple computation gives that

$$\lambda = \frac{I-a}{2I}$$

and with use of well-known ϑ-constant representation to the function λ we get

$$a = I - 2I\frac{\vartheta_2^4}{\vartheta_3^4}\Big(\frac{\alpha h+\beta}{\gamma h+\delta}\Big), \tag{32}$$

where, as usual, $\{\alpha,\beta,\gamma,\delta\}$ are free constants and $\alpha\delta \neq \beta\gamma$. The further integration for variables A and B can be continued in two ways. Having rules for differential computations (17) of the ϑ-series we can find, by differentiating formulae for $a(h)$ and $b(h)$, expressions for $A(h)$, $B(h)$. The second way is to linearize the system because any Schwarz's equation is known to be related to a certain linear ODE. We shall give solutions both in h- and k-representations.

4.1.1. *The k-representation.* Using (32) we have the obvious transformations between pairs (a,b) and (k,I):

$$a = I - 2Ik^2, \qquad 8b = I^2k^2(1-k^2). \tag{33}$$

and therefore the following variety of the system (11):

$$\dot{A} = 2A^2B, \qquad \dot{B} = \frac{1}{8}I^2k^2(1-k^2)A^3, \qquad \dot{k} = \frac{1}{2}Ik(1-k^2)A^2, \qquad \dot{I} = 0, \tag{34}$$

where we let the dot above a symbol denote an h-derivative. We shall use this system as an intermediate equivalent of Jacobi's one (11) because of its relation to linear ODEs is elementary. Indeed, as it follows from (34), the quantities A and B, as functions of k, satisfy the two linear equations

$$\frac{dA}{dk} = \frac{4}{I}\frac{1}{(1-k^2)k}B, \qquad \frac{dB}{dk} = \frac{I}{4}kA \tag{35}$$

and their consequences

$$k(k^2-1)A_{kk} + (3k^2-1)A_k + kA = 0, \qquad k(k^2-1)B_{kk} - (k^2-1)B_k + kB = 0. \tag{36}$$

Since k is Legendre's modulus, it is naturally to expect that these ODEs are integrable in terms of integrals (8)–(9).

Proposition 3. *Canonical Legendre's elliptic integrals (8)–(9) are differentially closed:*

$$\begin{aligned} \frac{dK}{dk} &= -\frac{K}{k} - \frac{E}{(k^2-1)k}, & \frac{dK'}{dk} &= \frac{kK'}{1-k^2} + \frac{E'}{(k^2-1)k}, \\ \frac{dE}{dk} &= -\frac{K}{k} + \frac{E}{k}, & \frac{dE'}{dk} &= \frac{kK'}{1-k^2} + \frac{kE'}{k^2-1}. \end{aligned} \tag{37}$$

This system, once considered as a dynamical one, has the following general solution

$$\begin{aligned} K &= \boldsymbol{\alpha}K(k) - \boldsymbol{\beta}K'(k), & K' &= \boldsymbol{\gamma}K(k) + \boldsymbol{\delta}K'(k), \\ E &= \boldsymbol{\alpha}E(k) + \boldsymbol{\beta}\big[E'(k) - K'(k)\big], & E' &= \boldsymbol{\delta}E'(k) + \boldsymbol{\gamma}\big[K(k) - E(k)\big], \end{aligned}$$

where $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\}$ *are free constants.*

Of course, one should bear in mind that canonical functions (8)–(9) themselves are not independent but satisfy the Legendre identity

$$K(k)E'(k) + K'(k)E(k) - K(k)K'(k) = \frac{\pi}{2} \quad \forall k.$$

The second order differential consequences of system (37) are as follows. The quantities K, K' satisfy the one equation

$$k(k^2-1)\frac{d^2\Psi}{dk^2} + (3k^2-1)\frac{d\Psi}{dk} + k\Psi = 0 \qquad \Rightarrow \qquad \Psi = \big\{K(k), K'(k)\big\}$$

and common equation for E and E' reads as

$$k(k^2-1)\frac{d^2\Psi}{dk^2} + (k^2-1)\frac{d\Psi}{dk} - k\Psi = 0 \qquad \Rightarrow \qquad \Psi = \big\{E(k), E'(k)\big\}.$$

The search for solution to the system (35) is thus just a technical task. Indeed, in addition to solution pair (33), we derive that

$$\begin{aligned} A &= 4\boldsymbol{\alpha}K(k) + 4\boldsymbol{\gamma}K'(k), \\ IB &= \boldsymbol{\alpha}\big[E(k) + (k^2-1)K(k)\big] - \boldsymbol{\gamma}\big[E'(k) - k^2K'(k)\big] \end{aligned} \tag{38}$$

with some free constants $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$. We can now combine the 'k-formulae' (38) and h-time dynamics to obtain a complete integral of system (11).

4.1.2. *The modular h-representation.* Let us denote

$$\mathrm{T} := \frac{\alpha h+\beta}{\gamma h+\delta} \tag{39}$$

and therefore, by virtue of (31),

$$\frac{\alpha h+\beta}{\gamma h+\delta} = \mathrm{i}\frac{K'(k)}{K(k)} \qquad \Leftrightarrow \qquad k = \frac{\vartheta_2^2(\mathrm{T})}{\vartheta_3^2(\mathrm{T})}. \tag{40}$$

Make use of representation to integrals (8)–(9) through Jacobi's η, ϑ-constants. Canonical formulae for K and K' are known:

$$K(k) = \frac{\pi}{2}\vartheta_3^2(h), \qquad K'(k) = \frac{\pi}{2\mathrm{i}}h\vartheta_3^2(h), \qquad k = \frac{\vartheta_2^2(h)}{\vartheta_3^2(h)}.$$

One can also show that second pair $\{E, E'\}$ has the following modular h-representation:

$$E(k) = \frac{2}{\pi}\frac{1}{\vartheta_3^2(h)}\left\{ \quad \eta(h) + \frac{\pi^2}{12}\big[\vartheta_3^4(h) + \vartheta_4^4(h)\big]\right\},$$

$$E'(k) = \frac{2}{\pi}\frac{\mathrm{i}}{\vartheta_3^2(h)}\left\{h\eta(h) - \frac{\pi^2}{12}\big[\vartheta_2^4(h) + \vartheta_3^4(h)\big]h - \frac{\pi}{2}\mathrm{i}\right\}.$$

Modifying these formulae for the ratio (39) we obtain

$$K(k) = \frac{\pi}{2}\vartheta_3^2(\mathrm{T}) \qquad \Rightarrow \qquad \alpha K(k) - \mathrm{i}\gamma K'(k) = \frac{K(k)}{\gamma h+\delta} = \frac{\pi}{2}\frac{\vartheta_3^2(\mathrm{T})}{\gamma h+\delta}.$$

Adjust free integration constants in (38) with those of (39) and (40). We then may write

$$A = \pm\sqrt{\frac{4\mathrm{i}}{\pi I}}\,\{\alpha K(k) - \mathrm{i}\gamma K'(k)\} \tag{41}$$

and therefore

$$B = \pm\sqrt{\frac{\mathrm{i}}{4\pi I^3}}\Big\{\alpha\big[E(k) + (k^2-1)K(k)\big] - \mathrm{i}\gamma\big[E'(k) - k^2K'(k)\big]\Big\}.$$

Passing to the ϑ, η-representation we arrive at a final form of the sought solution.

Theorem 6. *General solution to the dynamical system of Jacobi* (11) *has the form:*

$$a = I - 2I\frac{\vartheta_2^4(\mathrm{T})}{\vartheta_3^4(\mathrm{T})}, \qquad b = \frac{I^2}{8}\frac{\vartheta_2^4(\mathrm{T})\vartheta_4^4(\mathrm{T})}{\vartheta_3^8(\mathrm{T})}, \qquad A = \pm\sqrt{\frac{\pi\mathrm{i}}{I}}\frac{\vartheta_3^2(\mathrm{T})}{\gamma h+\delta},$$

$$B = \pm\sqrt{\frac{\mathrm{i}I}{\pi^3}}\frac{1}{(\gamma h+\delta)\vartheta_3^2(\mathrm{T})}\left\{\frac{\pi^2}{12}\big[\vartheta_2^4(\mathrm{T}) - \vartheta_4^4(\mathrm{T})\big] + \eta(\mathrm{T}) + \frac{\pi}{2}\mathrm{i}\gamma(\gamma h+\delta)\right\},$$

where $\{I, \alpha, \beta, \gamma, \delta\}$ *are free constants supplemented with normalization* $\alpha\delta - \beta\gamma = 1$.

These formulae are in effect the hidden correction of change (13). As a references source, we give here also the explicit transformation $\{A, B, k, \boldsymbol{I}\} \to \{x, y, z, u\}$ ($I := 12\mathrm{i}\boldsymbol{I}^2$):

$$x = \frac{1+\mathrm{i}}{\sqrt{\pi}}kA, \quad y = (1+\mathrm{i})\boldsymbol{I}A, \quad z^2 = 2\mathrm{i}\boldsymbol{I}^2(1-k^2)A^2, \quad u = 2A\big\{B - \mathrm{i}\boldsymbol{I}^2(2k^2-1)A\big\}. \tag{42}$$

This change turns (18) exactly into the system (34). Inverting this change we obtain

$$A=\frac{1-\mathrm{i}}{2\boldsymbol{I}}\,y,\qquad B=\frac{1+\mathrm{i}}{2}\frac{\boldsymbol{I}}{y}(u+y^2-2z^2),\qquad \boldsymbol{I}^2=\frac{1}{\pi}\frac{y^2-z^2}{x^2},\qquad k^2=1-\frac{z^2}{y^2} \tag{43}$$

and thereby derive theorem 2.

4.2. **Function integrals.** In order to integrate system (11) we made use its first integral (30). Having a complete solution we can find the remaining two function integrals and the fourth 'integral' corresponds to a time shift $h\mapsto h+\varepsilon$. The simplest way to derive the integrals is to use the linear fractional formula (40). Indeed, the h-derivative of this formula produces

$$(\gamma h+\delta)^2=\left(\frac{d}{dh}\frac{\alpha h+\beta}{\gamma h+\delta}\right)^{-1}=\left(\mathrm{i}\frac{d}{dh}\frac{K'(k)}{K(k)}\right)^{-1}=\cdots$$

and therefore expression

$$\cdots=\left(\frac{I}{2}k(1-k^2)A^2\cdot\mathrm{i}\frac{d}{dk}\frac{K'(k)}{K(k)}\right)^{-1}$$

is a perfect square. Upon rooting we get a linear function $\gamma h+\delta$ of dynamical variables $\{A,B,k,I\}\Longleftrightarrow\{A,B,a,b\}$. Its h-derivative yields an h-independent constant γ, i.e. integral $J_1(A,B,a,b)\cong\gamma$. Doing the same for

$$(\alpha h+\beta)^2=\left(\frac{d}{dh}\frac{\gamma h+\delta}{\alpha h+\beta}\right)^{-1}=\left(-\mathrm{i}\frac{d}{dh}\frac{K(k)}{K'(k)}\right)^{-1}=\cdots$$

we get one more integral $J_2(A,B,a,b)\cong\alpha$. Both of these integrals are independent of each other since $\{\alpha,\gamma\}$ are independent constants. All the calculus with objects $K,k,\ldots$ has been described in previous section and computations are somewhat lengthy but routine. We therefore omit them entirely.

Theorem 7. *The Jacobi system (11) has the only algebraic integral $I^2=a^2+32b$ and the two functionally independent transcendental integrals*

$$\begin{aligned}J_1&=4K(k)\cdot B-\big\{E(k)+(k^2-1)K(k)\big\}\cdot AI,\\ J_2&=4K'(k)\cdot B+\big\{E'(k)-k^2K'(k)\big\}\cdot AI,\end{aligned}\tag{44}$$

where $\{I,k\}$, if required, can be expressed via $\{a,b\}$ by the inversion of formulae (33):

$$I=\sqrt{a^2+32b},\qquad k^2=\frac12-\frac12\frac{a}{\sqrt{a^2+32b}}.$$

The integrals J_1, J_2 are multi-valued transcendental functions of dynamical variables $\{a,b\}$ and linear ones of $\{A,B\}$.

Another way of derivation of integrals rests on linear equations (35)–(36) and the well-known Wronskian relation for 2nd order linear ODEs. For example, the A-equation in (36) has $K(k)$ as its particular solution. Therefore

$$\left\{K(k)\cdot\frac{dA}{dk}-\frac{dK(k)}{dk}\cdot A\right\}(k^2-1)k=\mathrm{const}.$$

Replacing here A_k through B by (35) and using rules (37) we arrive again at integral J_1. Choice of $K'(k)$ for particular solution produces the second integral J_2 in (44).

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