Linear Quadratic Optimal Control Based on Dynamic Compensation for Rectangular Descriptor Systems

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Abstract The linear-quadratic optimal control by dynamic compensation for rectangular descriptor system is considered in this paper. First, a dynamic compensator with a proper dynamic order is given such that the closed-loop system is regular, impulse-free, and stable (it is called admissible), and its associated matrix inequality and Lyapunov equation have a solution. Also, the quadratic performance index is expressed in a simple form related to the solution and the initial value of the closed-loop system. In order to solve the optimal control problem for the system, the proposed Lyapunov equation is transformed into a bilinear matrix inequality (BMI), and a corresponding path-following algorithm to minimize the quadratic performance index is proposed in which an optimal dynamic compensator can be obtained. Finally, a numerical example is provided to demonstrate the effectiveness and feasibility of the proposed approach.

Key words Rectangular descriptor system, dynamic compensator, optimal control, path-following algorithm, bilinear matrix inequality (BMI)

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The rectangular descriptor systems have been investigated by many researchers for a few years[1−8]. Among them, [1−2] have considered the issues of admissibility of initial conditions and controls, and [3] has derived the necessary and sufficient condition for a given definition of the impulsive-mode controllability. Since rectangular descriptor systems cannot be turned into square systems by static feedback (state feedback or static output feedback), the closed-loop systems based on the static feedback cannot meet the requirement of regularity, impulse-freeness, and stability. To solve this problem, a new approach called dynamic compensation is adopted. For the rectangular descriptor systems, dynamic compensator′s design was solved in [2, 4−6]. In [2], with regard to dynamic compensation, some new notions on regularizability, controllability, and observability have been proposed and a necessary and sufficient condition for the existence of a dynamic compensator has also been derived. Reference [4] has studied the problems of regularization, impulse-freeness, stability, and pole-placement of the rectangular descriptor systems. Reference [5] has designed a new feedback structure, dynamic compensator plus state feedback, while [6] has considered the constrained regulation problem based on dynamic compensation. In addition to dynamic compensation, proportional integral observers[7] and filters[8] have been designed for the rectangular descriptor systems. For the normal systems, [9] has given a survey of static output feedback control and presented that the dynamic compensator can be brought back to the static output feedback case, and provided some useful methods for the research of the rectangular descriptor systems.

The optimal control problems for rectangular descriptor systems are not yet studied well. The topics of normal systems[10−15] and square descriptor systems[16−19] have been paid considerable attention for many years. Most of the exiting results are focused on state feedback control[10] and output feedback control[10−15]. For the square descriptor systems, [16] has given the optimal state regulators such that the closed-loop systems have robustness properties, and [17] has designed the optimal state feedback in terms of nonsingular transformation, and neural networks[18] and genetic programming[19] have also been used to solve the optimal control problems. The singular linear quadratic (LQ) problem for rectangular descriptor systems is investigated in [20], and it has been transformed to non-singular LQ problem for standard state space systems by using elementary linear algebra and the equivalence principle. The authors of [21] have investigated the LQ suboptimal control problem with disturbance rejection by means of restricted equivalent transformation.

As we know, the LQ optimal control problem can be transformed into a problem of solving Riccati equations/inequalities, and there have been some algorithms to solve these nonlinear matrix equations/inequalities. Reference [11] has given an algorithm, named Levine-Athans algorithm, for solving algebraic Riccati equation. Reference [22] has introduced some iterative algorithms for solving matrix equations. Reference [23] has proposed a simple matrix transformation for turning nonlinear matrix inequality (NLMI) to linear matrix inequality (LMI). But these algorithms cannot be applied directly to solve the linear-quadratic optimal problem based on dynamic compensation. References [24−26] have investigated how to convert bilinear matrix inequality (BMI) problem into an LMI iteratively solving algorithm. Among these algorithms, path-following algorithm[26] is tested to be much more effective since it linearizes the BMI using a first-order perturbation approximation and then iteratively compute a perturbation that "slightly" improves the controller performance by solving a semidefinite program (SDP).

Through the above analysis, dynamic compensation plays an irreplaceable role especially for rectangular descriptor system. However, the optimal control problem for the rectangular descriptor system based on dynamic compensation which represents fruitful research areas is not yet considered. So in this paper, we will present a dynamic compensator with a proper dynamic order such that the closed-loop system is regular, impulse-free, and stable (it is called admissible) and its associated matrix inequality and Lyapunov equation have a solution. Then, the given quadratic performance index can be described in a simple form related to the solution and the initial value of the closed-loop system. In order to solve this problem numerically, an algorithm is proposed in the light of the path-following algorithm[26]. By applying this algorithm, an optimal dynamic compensator and the minimum value of the quadratic performance index can be obtained. Finally, a

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numerical example is provided to demonstrate the effectiveness and feasibility of the proposed approach.

1 Preliminaries and problem formulation

Consider the following linear time-invariant (LTI) rectangular descriptor systems:

$$\begin{cases} E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t) \\ E\boldsymbol{x}(0_-) = E\boldsymbol{x}_0 \end{cases} \quad (1)$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is the state vector, $\boldsymbol{u}(t) \in \mathbf{R}^q$ is the input vector, $\boldsymbol{y}(t) \in \mathbf{R}^p$ is the output vector, $E, A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{m\times q}$, $C \in \mathbf{R}^{p\times n}$ are constant matrices. $E\boldsymbol{x}(0_-) = E\boldsymbol{x}_0$ with $\boldsymbol{x}_0 \in \mathbf{R}^n$ stands for the initial condition. The rank of matrix E is generally assumed as r, i.e., $\operatorname{rank}(E) = r$, with the range of $0 \le r \le \min\{m, n\}$. If $m = n$, system (1) is said to be square, and more, if $\det(sE - A)$ is not identically zero, system (1) is said to be regular, otherwise singular. If $m \ne n$, system (1) is said to be rectangular.

In this paper, we assume that $\boldsymbol{u}(t)$ and $E\boldsymbol{x}_0$ are admissible. Then, the following equation is satisfied:

$$\operatorname{rank}\begin{bmatrix} sE - A & B & E\boldsymbol{x}_0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} sE - A & B \end{bmatrix} \quad (2)$$

Consider the following performance index with the linear quadratic form:

$$J = \frac{1}{2}\int_0^\infty [\boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t)]\mathrm{d}t \quad (3)$$

where $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{q\times q}$ are weight matrices which are symmetric and positive-definite.

In practical applications, it is desired that the state of the system has only one smooth solution, i.e., the system should be regular and impulse-free. But it is clear that any static feedback control for rectangular descriptor system does not meet this requirement. To solve the problem, the dynamic compensators for system (1) are given with following form:

$$\begin{cases} E_c\dot{\boldsymbol{x}}_c(t) = A_c\boldsymbol{x}_c(t) + B_c\boldsymbol{y}(t) \\ \boldsymbol{u}(t) = C_c\boldsymbol{x}_c(t) + D_c\boldsymbol{y}(t) \\ E_c\boldsymbol{x}_c(0_-) = E_c\boldsymbol{x}_{c0} \end{cases} \quad (4)$$

where $\boldsymbol{x}_c(t) \in \mathbf{R}^{n_c}$ is the state vector of dynamic compensator. $E_c, A_c \in \mathbf{R}^{m_c\times n_c}$, $B_c \in \mathbf{R}^{m_c\times p}$, $C_c \in \mathbf{R}^{q\times n_c}$, $D_c \in \mathbf{R}^{q\times p}$ are matrices of dynamic compensator which are to be solved. Let $\operatorname{rank}(E_c) = r_c$, $0 \le r_c \le \min\{m_c, n_c\}$.

Then, the resultant closed-loop system from system (1) and its compensator (4) has the following form:

$$\begin{cases} \begin{bmatrix} E & 0 \\ 0 & E_c \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}}_c \end{bmatrix} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} \\ \boldsymbol{y}(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} \end{cases} \quad (5)$$

Let $\bar{E} = \begin{bmatrix} E & 0 \\ 0 & E_c \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$, and $\boldsymbol{\xi}(t) = \begin{bmatrix} \boldsymbol{x}^{\mathrm{T}}(t) & \boldsymbol{x}_c{}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, then the closed-loop system (5) can be rewritten as

$$\begin{cases} \bar{E}\dot{\boldsymbol{\xi}}(t) = \bar{A}\boldsymbol{\xi}(t) \\ \boldsymbol{y}(t) = \bar{C}\boldsymbol{\xi}(t) \\ \bar{E}\boldsymbol{\xi}(0_-) = \bar{E}\boldsymbol{\xi}_0 \end{cases} \quad (6)$$

It is desirable that system (6) should be a square descriptor system, hence the dimension $m_c \times n_c$ of the compensator (4) is assumed to satisfy

$$n + n_c = m + m_c \quad (7)$$

The following definitions and lemmas are useful for the development of the results in this paper.

Definition 1[2]**.** 1) System (1) is said to be regularizable if there exists a dynamic compensator (4) such that the closed-loop system (6) is regular.

2) System (1) is said to be strongly stabilizable and strongly detectable if there exists adynamic compensator (4) such that the closed-loop system (6) is stable.

Definition 2[27]**.** System (6) is admissible if it is regular, impulse-free, and stable.

Lemma 1[2]**.** There exists a dynamic compensator (4) such that the closed-loop system (6) is admissible, if and only if system (1) is regularizable, strongly stabilizable, and strongly detectable.

In order to be more intuitive, we change the above Lemma 1 to the expression with restriction of rank.

Lemma 2[4]**.** There exists a dynamic compensator (4) such that the closed-loop system (6) is admissible, if and only if

1) $\operatorname{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = m$, $\operatorname{rank}\begin{bmatrix} sE^{\mathrm{T}} - A^{\mathrm{T}} & C^{\mathrm{T}} \end{bmatrix} = n$, for any $s \in \mathbf{C}^+$;

2) $\operatorname{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \end{bmatrix} = m + r$ and $\operatorname{rank}\begin{bmatrix} 0 & E \\ E & A \\ 0 & C \end{bmatrix} = n + r$.

Moreover, the order r_c of the compensator (4) satisfies $f_B + f_C + r_c \ge n + m + r + 1$, where $f_B = \operatorname{rank}\begin{bmatrix} E & 0 \\ A & B \end{bmatrix}$, $f_C = \operatorname{rank}\begin{bmatrix} E & A \\ 0 & C \end{bmatrix}$.

Lemma 3[28]**.** The pair $(\bar{E}, \bar{A})$ is admissible if and only if there exists a matrix P such that

$$\bar{E}^{\mathrm{T}}P = P^{\mathrm{T}}\bar{E} \ge 0 \quad (8)$$

$$\bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A} < 0 \quad (9)$$

The aim of this paper is to design dynamic compensator (4) with proper dynamic order r_c for system (1) such that the closed-loop system (6) is admissible and the linear quadratic performance index (3) is minimized.

2 Main results

2.1 Optimal control based on dynamic compensation

Consider the closed-loop system (6) if we let $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$, $\hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$, $K = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$, and $\bar{A} = \hat{A} + \hat{B}K\hat{C}$, then the case where a dynamic compensator with order r_c is brought back to the static output feedback controller case.

Here, the quadratic performance index (3) is described by

$$J = \frac{1}{2}\int_0^{\infty}[\boldsymbol{\xi}^{\mathrm{T}}(t)\bar{Q}\boldsymbol{\xi}(t)]\mathrm{d}t \tag{10}$$

where $\bar{Q} = \hat{Q} + \hat{C}^{\mathrm{T}}K^{\mathrm{T}}\hat{R}K\hat{C}$, $\hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$, and $\hat{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$.

Remark 1. Applying $\hat{Q}$, $\hat{R}$, K, and $\hat{C}$ to $\bar{Q}$, we obtain

$$\bar{Q} = \begin{bmatrix} Q + C^{\mathrm{T}}D_c^{\mathrm{T}}RD_cC & C^{\mathrm{T}}D_c^{\mathrm{T}}RC_c \\ C_c^{\mathrm{T}}RD_cC & C_c^{\mathrm{T}}RC_c \end{bmatrix}$$

Since Q and R are symmetric and positive-definite, we assume that for controller gain K, $\bar{Q}$ is symmetric and positive-definite.

Theorem 1. Assume that the LTI rectangular descriptor system (1) is regularizable, strongly stabilizable, and strongly detectable. If there exists a dynamic compensator (4) with dynamic order r_c such that the closed-loop system (6) is admissible and $\bar{Q} > 0$, then (8) and the following Lyapunov equation

$$\bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A} + \bar{Q} = 0 \tag{11}$$

have the solution P, and the performance index $J = \frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(0)\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}(0)$ or $J = \frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(0)P^{\mathrm{T}}\bar{E}\boldsymbol{\xi}(0)$.

Proof. From Lemma 1, we know that if the rectangular descriptor systems (1) is regularizable, strongly stabilizable, and strongly detectable, then there exists a dynamic compensator (4) with order r_c which can make the closed-loop system (6) admissible. Then there exists a matrix P such that (8), (9), and

$$\bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A} = -\bar{Q} < 0 \tag{12}$$

hold. Therefore, we choose a Lyapunov function as

$$V(\boldsymbol{\xi}, t) = \boldsymbol{\xi}^{\mathrm{T}}(t)\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}(t) \geq 0 \tag{13}$$

Then, the time-derivative of $V(\boldsymbol{\xi}, t)$ along the solution of (6) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\xi}^{\mathrm{T}}(t)\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}(t)] = \boldsymbol{\xi}^{\mathrm{T}}(t)(\bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A})\boldsymbol{\xi}(t) = -\boldsymbol{\xi}^{\mathrm{T}}(t)\bar{Q}\boldsymbol{\xi}(t) < 0 \tag{14}$$

Substituting (14) into (10)

$$J = \frac{1}{2}\int_0^{\infty}\boldsymbol{\xi}^{\mathrm{T}}(t)\bar{Q}\boldsymbol{\xi}(t)\mathrm{d}t = -\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(t)\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}(t)\Big|_0^{\infty} = -\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(\infty)\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}(\infty) + \frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(0)\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}(0) \tag{15}$$

Because the finite poles of the closed system are in the open left-half-plane, and $\boldsymbol{\xi}(\infty) \to 0$, the following equation can be obtained:

$$J = \frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(0)\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}(0) \tag{16}$$

The matrix P is the solution of the Lyapunov equation (12), i.e., (11). □

In order to obtain the optimal performance index, we should solve the minimization problem described by

$$\min J = \frac{1}{2}\boldsymbol{\xi}_0^{\mathrm{T}}\bar{E}^{\mathrm{T}}P\boldsymbol{\xi}_0$$
$$\text{s.t.}\begin{cases} \bar{E}^{\mathrm{T}}P = P^{\mathrm{T}}\bar{E} \geq 0 \\ \bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A} + \bar{Q} = 0 \\ \bar{Q} > 0 \end{cases} \tag{17}$$

Up to now, no effective algorithm for solving problem (17) is found, and the existing algorithms for solving the optimal control problem of the static output feedback seem unfit to solve this problem. In the next subsection, we will give a new algorithm based on the path-following method to solve the problem.

2.2 Solving the minimization problem

In order to solve Lyapunov equation (11) conveniently, we should add a small positive slack factor $\epsilon_1 > 0$ to transform (11) into the following inequality:

$$|\bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A} + \bar{Q}| < \epsilon_1 I \tag{18}$$

where I is an identity matrix, which has the same dimension as that of $\bar{A}$, and $|X| < \epsilon_1 I$ is defined as $-\epsilon_1 I < X < \epsilon_1 I$.

Equation (18) represents

$$\bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A} + \bar{Q} = M$$

where $|M| < \epsilon_1 I$. That is

$$\bar{A}^{\mathrm{T}}P + P^{\mathrm{T}}\bar{A} + \bar{Q} - M = 0$$

Set $\bar{Q}_{\mathrm{approx}} = \bar{Q} - M$, then $\bar{Q} = \bar{Q}_{\mathrm{approx}} + M$. Therefore, the corresponding performance index is

$$J = \int_0^{\infty}\boldsymbol{\xi}^{\mathrm{T}}(t)(\bar{Q}_{\mathrm{approx}} + M)\boldsymbol{\xi}(t)\mathrm{d}t = \boldsymbol{\xi}^{\mathrm{T}}(0)(P_{\mathrm{approx}} + M)\boldsymbol{\xi}(0) \approx \boldsymbol{\xi}^{\mathrm{T}}(0)P_{\mathrm{approx}}\boldsymbol{\xi}(0)$$

So, by adding a small positive slack factor $\epsilon_1 > 0$ to transform equation (11) into inequality (18), it is clear that the performance index is approximately obtained. If ϵ_1 is small enough, then the approximation is the performance index to be expected.

In order to solve the matrix P in (8), we introduce some new variables to obtain the parameterized expression of P. Select a matrix $U \in \mathbf{R}^{(n+n_c)\times l}$ such that $\bar{E}^{\mathrm{T}}U = 0$ and $\mathrm{rank}(U) = l = (n + n_c) - (r + r_c)$. Then,

$$P = (\bar{E}^{\mathrm{T}}X + YU^{\mathrm{T}})^{\mathrm{T}} \tag{19}$$

where $X \in \mathbf{R}^{(n+n_c)\times(n+n_c)}$, $Y \in \mathbf{R}^{(n+n_c)\times l}$, and X is symmetric and positive-definite. Since

$$\bar{E}^{\mathrm{T}}P = \bar{E}^{\mathrm{T}}(\bar{E}^{\mathrm{T}}X + YU^{\mathrm{T}})^{\mathrm{T}} = \bar{E}^{\mathrm{T}}X\bar{E}$$

$$P^{\mathrm{T}}\bar{E} = (\bar{E}^{\mathrm{T}}X + YU^{\mathrm{T}})\bar{E} = \bar{E}^{\mathrm{T}}X\bar{E}$$

if follows that

$$\bar{E}^{\mathrm{T}}P = P^{\mathrm{T}}\bar{E} = \bar{E}^{\mathrm{T}}X\bar{E} \geq 0$$

Therefore, if P is expressed in (19), then (8) holds.

As K in $\bar{A}$ and P are both unknown matrix variables, the inequality (18) is actually BMI, which cannot be solved directly by LMI. In the following, we present a path-following method for solving BMI (18). In fact, BMI (18) is linearized by using a perturbation approximation, and then it becomes an LMI. The detailed algorithm is as follows.

Algorithm 1.

Step 1. Check if the system (1) meets the conditions 1) and 2) in Lemma 2. If it does not meet one of them, there does not exist adynamic compensator such that the closed-loop system is admissible. Otherwise, determine the order of the dynamic compensator r_c, which satisfies the conditions in Lemma 2, so does E_c. Then, go to Step 2.

Step 2. Let $j=1$. Select an initial feedback gain K_j such that the closed-loop system is admissible and $\bar{Q}>0$.

Step 3. Solve the following LMI problem:

$$\min J_j = \boldsymbol{\xi}_0^{\mathrm{T}}\bar{E}^{\mathrm{T}}X_j\bar{E}\boldsymbol{\xi}_0$$
$$\text{s.t.}\begin{cases} |(\hat{A}+\hat{B}K_j\hat{C})^{\mathrm{T}}(\bar{E}^{\mathrm{T}}X_j+Y_jU^{\mathrm{T}})^{\mathrm{T}}+ \\ (\bar{E}^{\mathrm{T}}X_j+Y_jU^{\mathrm{T}})(\hat{A}+\hat{B}K_j\hat{C})+ \\ \hat{Q}+\hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{R}K_j\hat{C}|<\epsilon_1 I \\ X_j>0\end{cases} \tag{20}$$

We obtain X_j, Y_j and P_j which can be computed by (19). If this LMI optimal problem has a solution, go to Step 4. Otherwise, go to Step 2.

Step 4. Substituting $P_j=P_j+\delta P$, $K_j=K_j+\delta K$ into (20). It is reasonable to assume that δP and δK are small and therefore by neglecting the second order terms, we can obtain the following optimization problem:

$$\begin{aligned}&|\hat{A}^{\mathrm{T}}P_j+P_j^{\mathrm{T}}\hat{A}+P_j^{\mathrm{T}}\hat{B}K_j\hat{C}+\\&\quad\hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{B}^{\mathrm{T}}P_j+P_j\hat{B}\delta K\hat{C}+\\&\quad\hat{C}^{\mathrm{T}}\delta K^{\mathrm{T}}\hat{B}^{\mathrm{T}}P_j^{\mathrm{T}}+\delta P^{\mathrm{T}}\hat{A}+\hat{A}^{\mathrm{T}}\delta P+\\&\quad\delta P^{\mathrm{T}}\hat{B}K_j\hat{C}+\hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{B}^{\mathrm{T}}\delta P+\hat{Q}+\\&\quad\hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{R}K_j\hat{C}+\hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{R}\delta K\hat{C}+\\&\quad\hat{C}^{\mathrm{T}}\delta K^{\mathrm{T}}\hat{R}K_j\hat{C}|<\epsilon_2 I\end{aligned} \tag{21}$$

Note that the constraints of δP and δK are

$$|\delta P|<I,\quad |\delta K|<I \tag{22}$$

Suppose that ϵ_2 is a small positive scalar, then we obtain δP and δK. If this LMI problem has a solution, go to Step 5. Otherwise, go to Step 2.

Step 5. Let $j=j+1$, $P_j=P_{j-1}+\delta P$, $K_j=K_{j-1}+\delta K$, compute $J_j=\frac{1}{2}\boldsymbol{\xi}_0^{\mathrm{T}}\bar{E}^{\mathrm{T}}P^{(j)}\boldsymbol{\xi}_0$.

If $J_j<J_{j-1}$, $J_{j-1}-J_j>\epsilon_3$, and $j<N$ (ϵ_3 is a given small positive scalar, N is the upper bound for the iteration number), then go back to Step 3. Otherwise, stop. Then, the optimal performance index is obtained.

This algorithm ends until a desired performance is achieved, or the performance cannot be improved further. Although there is no convergence analysis to guarantee an acceptable solution, the choice of initial values of K is important for convergence to an acceptable solution[26]. As long as K_j is close enough to the optimal values, we conclude that K_j can be adjusted iteratively using the free variable δK. Then, the optimal performance index J can be obtained accordingly.

The algorithm above can also be applied to solve the problem of optimal control based on static output feedback by solving BMI. Compared with the existing literatures, the proposed method is more general. A numerical example in next section demonstrates the effectiveness of the proposed approach.

3 Numerical example

Since the performance index is related to the initial value of the closed-loop system, we set $E_c\boldsymbol{x}_{c0}=0$ in the following example so as to obtain the minimal index for convenient comparisons.

Example 1. Consider the rectangular descriptor system (1) and its quadratic performance index (3) with the parameters as follows: $E=\begin{bmatrix}1 & 0\end{bmatrix}$, $A=\begin{bmatrix}-1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $Q=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $R=1$ and the admissible initial value of the state vectors $E\boldsymbol{x}_0=1$.

From the given parameters above, we get $m=1$, $n=2$, $q=1$, and $p=2$. According to Lemma 2, there exists a dynamic compensator such that the closed-loop system is admissible, and the dynamic order $r_c\geq 0$.

1) Set $r_c=1$, then $m_c=2$, $n_c=1$ based on (7).

Let $E_c=\begin{bmatrix}0\\1\end{bmatrix}$ and $U=\begin{bmatrix}0\\1\\0\end{bmatrix}$. Take $K_1=\begin{bmatrix}-2 & 0 & -1\\2 & -1 & 1\\2 & -2 & 1\end{bmatrix}$.

Now, the trajectories of the state vector are shown in Fig. 1.

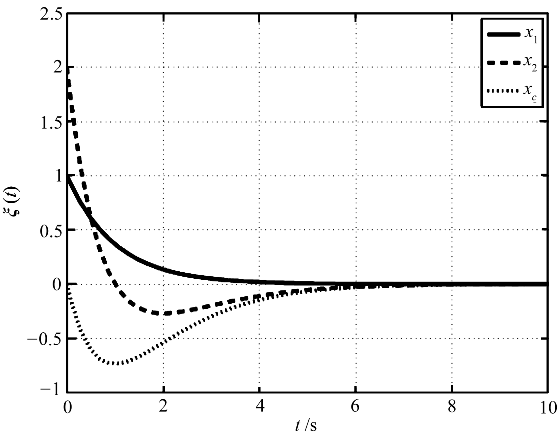

Fig. 1 State curves of the closed-loop system

Using Algorithm 1, we can obtain the optimal controller gain $K=\begin{bmatrix}-2.2663 & -0.1846 & -1.1819\\3.5680 & -1.7486 & 2.0306\\0.9945 & -1.8985 & 0.4228\end{bmatrix}$ and minimal performance index $J^*=1.0806$. Now the trajectories of the state vector are shown in Fig. 2.

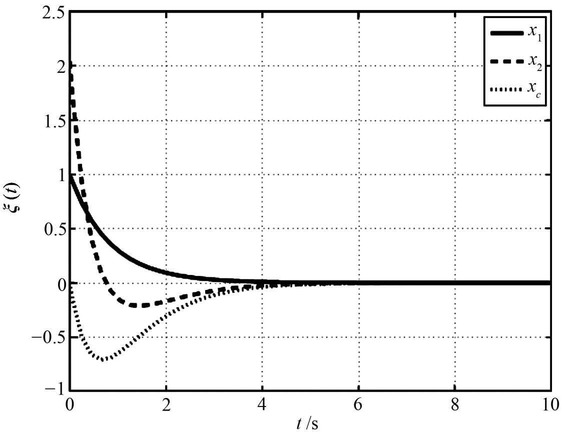

Fig. 2 Optimal state curves of the closed-loop system

By observing these figures and results, we can find that the closed-loop system by optimization can achieve better trajectory performance (the settling time is 4 s which is shorter than that without optimization).

2) Set $r_c=2$, then $m_c=3$, $n_c=2$ based on (7).

Let $E_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Take $K_1 = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -0.5 & -5 & -2 \\ 0 & 2 & -0.2 & -1 \\ 0 & 1 & -10 & 8 \end{bmatrix}$. Now, the trajectories of the state vector are shown in Fig. 3.

Using Algorithm 1, we can obtain the optimal controller gain $K = \begin{bmatrix} -1.0674 & -1.5258 & 0.7321 & -1.0471 \\ -0.2457 & -4.9039 & -3.1287 & 0.0140 \\ -0.1607 & 1.6523 & -2.6331 & -7.5731 \\ 0.0791 & 2.4347 & -10.0928 & 7.2891 \end{bmatrix}$ and minimal performance index $J^* = 0.2375$. Now the trajectories of the state vector are shown in Fig. 4.

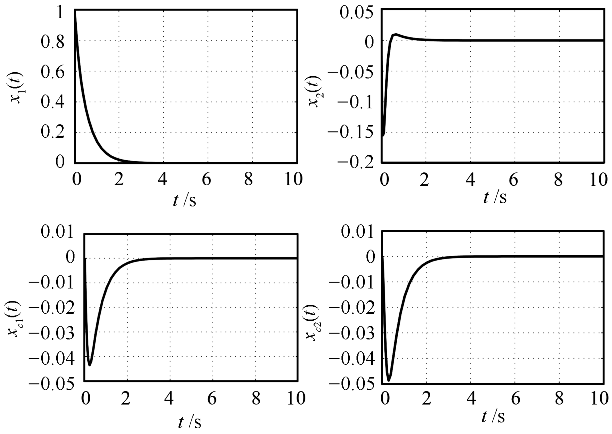

Fig. 3 State curves of the closed-loop system

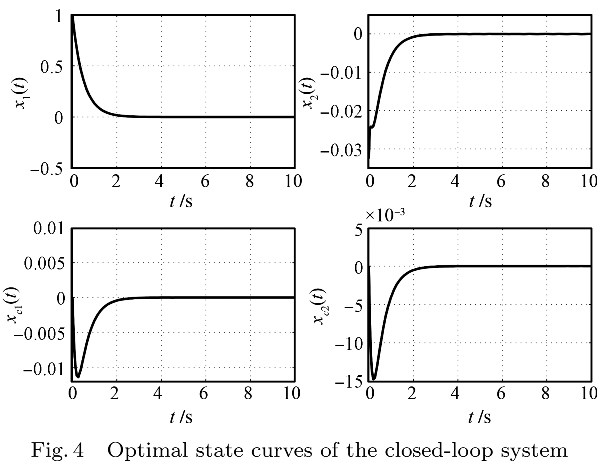

Fig. 4 Optimal state curves of the closed-loop system

From these figures and results, we can see that the performance of closed-loop system can become better with the increase of the dynamic order r_c.

4 Conclusion

In this paper, we have considered the optimal control problem for rectangular descriptor system based on dynamic compensation. It is shown that if there exists a dynamic compensator with proper dynamic order such that the closed-loop system is admissible, then associated matrix inequality and Lyapunov equation have a solution and the given quadratic performance index can be expressed in a simple form. In order to acquire the optimal index for the system, we have transformed this Lyapunov equation into a BMI and a new algorithm has been proposed in terms of path-following algorithm. Then, the optimal dynamic compensator and the minimum value of the quadratic performance index have been obtained by Matlab LMI Toolbox. Finally, a numerical example is given to show that the optimized closed-loop system has better transient characteristics and a comparison is given by the help of curves of with and without optimization. Simulation results also show that dynamic compensators with higher order r_c can achieve better performance.

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