COMPUTING THE TOPOLOGICAL ENTROPY OF UNIMODAL MAPS

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Abstract. We derive an algorithm to determine recursively the lap number (minimal number of monotone pieces) of the iterates of unimodal maps of an interval with free end-points. The algorithm is obtained by the sign analysis of the itineraries of the critical point and of the boundary points of the interval map. We apply this algorithm to the estimation of the growth number and the topological entropy of maps with direct and reverse bifurcations.

1. Introduction

Topological entropy, introduced in 1965 by Adler, Konheim and McAndrew, [1], is an invariant of topological conjugacy for self-maps of an interval. Here, using symbolic dynamical techniques, we derive a formula to calculate the lap number of the iterates of the class of maps of an interval with free end-points, leading to a straightforward estimation of the topological entropy of these maps. The lap numbers of this class of maps depends on the symbolic itineraries of the critical point and of the two extreme points.

Let $I = [a, b]$ be a closed interval of the real line $\mathbb{R}$. We consider the class $\mathcal{F}$ of $C^2(I)$ maps $f : I \to I$, with a critical point at $x_c \in (a, b)$, and such that,

(I): $f''(x_c) < 0$,
(II): $f'(x) > 0$ for $x < x_c$, and $f'(x) < 0$ for $x > x_c$.

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In this paper, the single-humped maps in $\mathcal{F}$ are called *unimodal maps*. The maps in the class $\mathcal{F}$ are not necessarily symmetric around the critical point, and the iterates of the end-points of the interval I need not to converge to the same limit set.

The chain rule of differentiation applied to the nth iterate of f, written f^n (f^0 is the identity map), shows trivially that along with f, all iterates f^n are also twice differentiable on I. In particular,

$$f^{n\prime}(x) = f'(f^{n-1}(x))f'(f^{n-2}(x))\cdots f'(x) \tag{1}$$

implies that x_c is a critical point of f^n for every $n \geq 1$. From (1) and condition (II) we have:

Lemma 1.1. *If $f \in \mathcal{F}$, then the critical points of f^n, $n > 1$, are the points $x \in (a,b)$ such that $f^i(x) = x_c$, for some $0 < i \leq n-1$. Moreover, all the critical points of f^n, with $n \geq 1$, are maxima or minima, but not inflection points.*

It follows from Lemma 1.1, that the critical points of f^n with $n \geq 1$, are the pre-images of the critical point x_c up to order $n-1$. If $f(x_c) > x_c$, $f(a) < x_c$ and $f(b) < x_c$, then the iterated maps f^n, with $n \geq 2$, have more than one critical point. On the other hand, if $f(x_c) \leq x_c$, or $f(x_c) > x_c$, $f(a) > x_c$ and $f(b) > x_c$, then the only critical point of f^n, with $n \geq 2$, is x_c.

A well-known example of a family of maps in $\mathcal{F}$ is provided by the one-parameter quadratic (or logistic) family $f_\mu(x) = 4\mu x(1-x)$, where $I = [0,1]$ and the parameter $\mu \in (0,1]$. For this family of maps, the critical point $x_c = 1/2$ is μ-independent. If $\mu \in (0,1/2]$, then the only critical point of f_μ^n, with $n \geq 1$, is $x = x_c = 1/2$.

Two self-maps f_1 and f_2 of the intervals I_1 and I_2, respectively, are topologically conjugate if there exists an homeomorphism $h : I_1 \to I_2$ such that $f_2 = h \circ f_1 \circ h^{-1}$. The topological entropy of a piecewise continuous interval map f is an invariant of topological conjugacy, [1], and its topological entropy is calculated through the minimal number of monotone pieces or laps of the iterates f^n, or by the number of fixed points of f^n, [1, 14, 15]. To be more precise, denoting by ℓ_n the minimal number of monotone pieces of f^n, and by $N(f^n)$ the number of fixed points of f^n, then the topological entropy of f is given by, [15],

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log \ell_n = \limsup_{n\to\infty} \frac{1}{n} \log N(f^n) \tag{2}$$

The number $s = \limsup_{n\to\infty} \ell_n^{1/n}$ is called the growth number of f. In the case of the logistic family of maps $f_\mu(x) = 4\mu x(1-x)$, we have $h(f_1) = \log 2$ and $h(f_\mu) = 0$, for $\mu \in [0,1/2]$.

An important characteristic of a unimodal map f with positive topological entropy is the existence of a semiconjugacy to the tent map with slopes $\pm e^{h(f)}$. This is the Milnor and Thurston classification theorem, [14, 11]. The variation of the topological entropy as a parameter of a family of maps is changed, implies that the number of periodic points of the family of maps also changes, showing the existence of bifurcations in the dynamics described by interval maps. This is in general associated with a change in complexity in the sense of Li and Yorke's chaos scenario [12]. The topological entropy for families of interval maps is a way to analyze bifurcations and is also a quantitative evaluation of the complexity of the dynamics.

Several authors have proposed approximating algorithms to calculate the topological entropy of unimodal maps with fixed boundary points. In the case of the logistic map, techniques based on Markov processes have been introduced, [2, 10]. In this approach the spectral properties of a transfer matrix with increasing dimension are analyzed. This transfer matrix is defined by the images by the logistic map f_μ, [2, 10]. Another technique consists in approximating unimodal maps by piecewise monotone maps and then calculating the topological entropy of the piecewise approximation to the map, [3]. All these methods converge to the topological entropy, however they are indirect and do not give information about the topological characteristics of the map, as, for example, the lap numbers and the number of critical points.

A direct method of estimating and calculating the topological entropy of an interval map is with the lap numbers ℓ_n of the iterates of a map. In [5, 6] a recursive formula to calculate ℓ_n was proposed for the class of unimodal maps with fixed boundary points. The proof for unimodal maps with a local minimum and fixed boundary points was given in [7]. Here, we generalize this result for the larger class of maps $\mathcal{F}$. The main result of this paper is Theorem 4.2, where we derive a general formula to calculate recursively the lap number ℓ_n of the iterates of $f \in \mathcal{F}$. This formula depends on the orbits (itineraries) of the critical point and of the boundary points of the map. On the other hand, this formula can also be efficiently used to estimate the topological entropy and the growth number for unimodal maps obtained by time series analysis.

The proof of Theorem 4.2 is based on the kneading calculus of Milnor-Thurston [14] and on a symbolic sequence introduced in [5] which is derived from the signed itinerary of the critical point of a unimodal map. This new symbolic sequence will be called the Min-Max sequence (MMS) of the map. The MMS gives a procedure to determine on the interval I the sequence of maxima and minima of any iterated map f^n, with $n \geq 1$.

To fix the notation, next, we describe the symbolic approach due to Metropolis *et al*, [13], and recall some basics related to the kneading calculus. The Min-Max sequence of a unimodal map and its most important properties will be analyzed in the next section.

Consider a map $f \in \mathcal{F}$. Following Metropolis *et al*, [13], we assign the symbols I_0 and I_1 to the intervals $[a, x_c)$ and $(x_c, b]$, respectively, and the symbol C to the critical point x_c. We then associate to every $(f, x) \in \mathcal{F} \times I$, a sequence $\Theta_f(x) = (\Theta_{f,n}(x))_{n \geq 0}$ with entries $\Theta_{f,n}(x)$ belonging to the alphabet $\mathcal{A} = \{I_0, C, I_1\}$. The sequence $\Theta_f(x)$, called the *itinerary* of x under f, is defined as follows:

$$\Theta_{f,n}(x) = \begin{cases} I_0 & \text{if } \ f^n(x) < x_c, \\ C & \text{if } \ f^n(x) = x_c, \\ I_1 & \text{if } \ f^n(x) > x_c. \end{cases}$$

The space of all the itineraries will be denoted by $\mathcal{A}^{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, ...\}$. Since,

$$\Theta_{f,n}(f(x)) = \Theta_{f,n+1}(x), \tag{3}$$

the action of f on I translates into a left shift on $\mathcal{A}^{\mathbb{N}_0}$. The sequence $\gamma_f = (\gamma_{f,n})_{n \in \mathbb{N}}$, defined as,

$$\gamma_{f,n} = \Theta_{f,n}(x_c) = \Theta_{f,n-1}(f(x_c)), \quad n \geq 1, \tag{4}$$

is called the *kneading sequence* (KS for short) of f, [14]. Hence, γ_f is the itinerary of the critical value $f(x_c)$.

The *signed itinerary* of $x \in I$ under the iteration of f, is the sequence $\Theta^\varepsilon_f(x) = (\Theta^\varepsilon_{f,n}(x))_{n \geq 0}$ on the extended alphabet $\mathcal{A}_\varepsilon \in \{-I_1, -C, -I_0, 0, I_0, C, I_1\}$, where,

$$\Theta^\varepsilon_{f,0}(x) = \Theta_{f,0}(x), \ \Theta^\varepsilon_{f,1}(x) = \Theta_{f,1}(x), \tag{5}$$

$$\Theta^\varepsilon_{f,n}(x) = \varepsilon(\Theta_{f,1}(x)) ... \varepsilon(\Theta_{f,n-1}(x)) \Theta_{f,n}(x) \ \text{ for } \ n \geq 2, \tag{6}$$

and,

$$\varepsilon(I_0) = 1, \varepsilon(C) = 0, \text{ and } \varepsilon(I_1) = -1.$$

Note that, $\varepsilon(I_0)$, $\varepsilon(C)$, and $\varepsilon(I_1)$ coincide with the signs of f' on I_0, C, and I_1, respectively.

We say that a sequence $\alpha \in \mathcal{A}_\varepsilon^{\mathbb{N}_0}$ is *admissible* if there exists $f \in \mathcal{F}$ such that $\alpha = \Theta^\varepsilon(f(x_c))$. From now on, we shall always consider admissible sequences without explicitly stating it.

In the following, we write $\Theta(x)$ instead of $\Theta_f(x)$ whenever the map f is understood from the context, and the same applies to the kneading sequence and signed itineraries.

2. Geometry of the signed itineraries: The Min-Max sequence

We now relate the signed itineraries of the critical point of a unimodal map $f \in \mathcal{F}$ with the structure of maxima and minima of the iterates of f.

Suppose that f^n, $n \geq 1$, has a local maximum (resp. minimum) at some point $x \in I = [a, b]$. In order to simplify the notation and the proofs, in the following, we say that $f^n(x)$ is a "positive" maximum (resp. minimum), if $f^n(x) - x_c > 0$. If, otherwise, $f^n(x) - x_c < 0$, then we say that $f^n(x)$ is a "negative" maximum (resp. minimum). In the remaining case, $f^n(x) - x_c = 0$, and we say that $f^n(x)$ is a "zero" maximum (resp. minimum).

Lemma 2.1 ([5]). *Let $f \in \mathcal{F}$, and $k \geq 1$. Then:*

- **(a):** *If $f^k(x) = x_c$ and $f(x_c) > x_c$, then $f^{k+1}(x)$ is a positive maximum.*
- **(b):** *If $f^k(x)$ is a negative [resp. positive] minimum, then $f^{k+1}(x)$ is a minimum [resp. maximum].*
- **(c):** *If $f^k(x)$ is a negative [resp. positive] maximum, then $f^{k+1}(x)$ is a maximum [resp. minimum].*
- **(d):** *If $f(x_c) \leq x_c$, then $f^k(x)$ has only one critical point, a negative or a zero maximum.*

Proof. For $k \geq 1$ we have,

$$f^{k+1\prime}(x) = (f \circ f^k)'(x) = f'(f^k(x))f^{k\prime}(x) \tag{7}$$

and,

$$f^{k+1\prime\prime}(x) = f''(f^k(x))(f^{k\prime}(x))^2 + f'(f^k(x))f^{k\prime\prime}(x). \tag{8}$$

(a) If $f^k(x) = x_c$, by (II), $f^{k+1\prime}(x) = f'(x_c)f^{k\prime}(x) = 0$, and by (I), $f^{k+1\prime\prime}(x) = f''(x_c)(f^{k\prime}(x))^2 < 0$. As $f(x_c) > x_c$, $f^{k+1}(x) = f(x_c) > x_c$.

(b) If $f^k(x)$ is a minimum, then $f^{k+1\prime}(x) = f'(f^k(x))f^{k\prime}(x) = 0$, and,

$$f^{k+1\prime\prime}(x) = f'(f^k(x))f^{k\prime\prime}(x),$$

where $f^{k\prime\prime}(x) > 0$. Hence, $f^{k+1\prime\prime}(x)$ has the same sign as $f'(f^k(x))$. If $f^k(x) < x_c$, then, by (II), $f^{k+1\prime\prime}(x) > 0$ and $f^{k+1}(x)$ is a minimum. Likewise, if $f^k(x) > x_c$, then, by (II), $f^{k+1\prime\prime}(x) < 0$ and $f^{k+1}(x)$ is a maximum.

The proof of (c) is similar and (d) follows from Lemma 1.1. □

Let $f \in \mathcal{F}$ and set,

$$\mathcal{S}^k = \{x \in I : x \text{ is a critical point for } f^k\}. \tag{9}$$

In particular, $\mathcal{S}^1 = \{x_c\}$. According to Lemma 1.1, for $k \geq 1$, $\mathcal{S}^k$ contains x_c and its preimages by the iteration of f up to order $k-1$. This same lemma or Eq. (7) imply that if $x \in (a,b)$ is a critical point of f^k, $k \geq 1$, then x is also a critical point of f^n for $n \geq k$. Hence, $\mathcal{S}^k \subset \mathcal{S}^{k+1}$. These critical points can be maxima or minima, and the corresponding critical values can be greater than, equal to, or smaller than the critical point x_c.

In order to distinguish all these possibilities, we introduce the new alphabet

$$\mathcal{M} = \{m^-, m^0, m^+, M^-, M^0, M^+\}\,, \tag{10}$$

where "m" stands for minimum and "M" stands for maximum. The superscript signs attached to m and M specify additionally whether the extreme in question is positive, negative or zero in the sense explained above in the beginning of this section.

Now, we define the sequence $\omega_f = (\omega_{f,n})_{n\geq 1} \in \mathcal{M}^{\mathbb{N}}$ as follows:

$$\omega_{f,n} = \begin{cases} m^- & \text{if } f^n(x_c) \text{ is a "negative" minimum,} \\ m^+ & \text{if } f^n(x_c) \text{ is a "positive" minimum,} \\ m^0 & \text{if } f^n(x_c) \text{ is a "zero" minimum,} \\ M^0 & \text{if } f^n(x_c) \text{ is a "zero" maximum,} \\ M^- & \text{if } f^n(x_c) \text{ is a "negative" maximum,} \\ M^+ & \text{if } f^n(x_c) \text{ is a "positive" maximum.} \end{cases}$$

The sequence ω_f is called the *Min-Max sequence* of $f \in \mathcal{F}$, or MMS for short, [5, 7]. For example, if $f^n(x)$ is a positive maximum, then we say that the extremum $f^n(x)$ is of type M^+. As usual, we say that ω_i and ω_j have opposite signs if the product of their signs is strictly negative.

The geometric meaning of the MMS of f is clear: The nth iterate of f, f^n, has a critical point of type $\omega_n \in \mathcal{M}$ at $x = x_c$.

Furthermore, if $f \in \mathcal{F}$, according to (I),

$$\text{sign } f''(x_c) = -1. \tag{11}$$

If $n \geq 2$ and $f^k(x_c) \neq x_c$ for $1 \leq k \leq n-1$, then, by (1) and (II) we have

$$f^{n\prime\prime}(x_c) = f'(f^{n-1}(x_c)) \cdots f'(f(x_c)) f''(x_c) \neq 0,$$

and hence

$$\text{sign } f^{n\prime\prime}(x_c) = -\varepsilon(\gamma_{f,1}) \cdots \varepsilon(\gamma_{f,n-1}), \quad (n \geq 2). \tag{12}$$

Comparison of (11)-(12) with (5)-(6) leads to the conclusion that the symbols $\Theta^{\varepsilon}_{f,n}(x_c)$ can be identified with the symbols $\omega_{f,n}$ as in Table 1, provided $\Theta^{\varepsilon}_{f,n}(x_c) \neq 0$, for $n \geq 1$. More specifically, by the rules of Lemma 2.1, the symbols I_0, C and I_1 in $\Theta^{\varepsilon}_{f,n}(x_c)$ translate into the superscript signs $-$, 0 and $+$, in $\omega_{f,n}$, respectively, while the signs $+$

and $-$ in $\Theta^{\varepsilon}_{f,n}(x_c)$ translate into the symbols M and m in $\omega_{f,n}$. If, for some $k \geq 2$, $\gamma_{f,k} = C$ (thus $\varepsilon(\gamma_{f,k}) = 0$ and $\Theta^{\varepsilon}_{f,k+1}(x_c) = 0$), then the construction of the MMS for $n \geq k+1$ follows the rules of Lemma 2.1. These rules are summarized in Table 2.

$\Theta^{\varepsilon}_{f,n}(x_c)$		$\omega_{f,n}$
I_1	$\longleftrightarrow$	M^+
$-I_1$	$\longleftrightarrow$	m^+
I_0	$\longleftrightarrow$	M^-
$-I_0$	$\longleftrightarrow$	m^-
C	$\longleftrightarrow$	M^0
$-C$	$\longleftrightarrow$	m^0

TABLE 1. Correspondence between the symbols of the signed itinerary of x_c and the *Min-Max sequence* (MMS).

Moreover, Lemma 2.1(a-c) also implies that consecutive symbols in the MMS obey the transition diagram in Table 2.

$\omega_{f,n}$		$\omega_{f,n+1}$
m^0, M^0	$\longrightarrow$	M^+
m^+, M^-	$\longrightarrow$	$M^{-,0,+}$
m^-, M^+	$\longrightarrow$	$m^{-,0,+}$

TABLE 2. Possible transitions between consecutive symbols of the MMS.

In the following, we use the shorthand notation $(\alpha_0...\alpha_n)^\infty$ for any symbolic sequence that repeats indefinitely the block $(\alpha_0...\alpha_n)$. A symbolic sequence of the form $(\alpha_0 \dots \alpha_n(\alpha_{n+1} \dots \alpha_{n+k})^\infty)$ is called eventually periodic.

Lemma 2.2. *Let γ and ω be the KS and MMS of $f \in \mathcal{F}$, respectively. If γ is periodic of period k, then ω is also periodic and has period k or $2k$. If γ is eventually periodic, then ω is also eventually periodic.*

Proof. Suppose that the KS γ is periodic with period k, $\gamma = (\gamma_1...\gamma_k)^\infty$. Without loss of generality assume that $f(x_c) > x_c$, otherwise all the KS have one of the forms $\gamma = (I_0)^\infty$ or $\gamma = (C)^\infty$. Thus, $\gamma_{k+1} = \gamma_1 = I_1$, and $\Theta^{\varepsilon}_{k+1}(x_c) = \pm\gamma_{k+1} = \pm I_1$. From Table 1, it follows that $\omega_{k+1} = M^+$ or $\omega_{k+1} = m^+$. Since $\omega_1 = M^+$, in the first case, it follows that ω has period k. In the second case, since $\gamma_{k+j} = \gamma_j$ for $j \geq 1$, we deduce that

$f'(f^{k+j}(x_c))$ and $f'(f^j(x_c))$ have the same sign for $j = 1, ..., k$, hence, by (I),

$$f^{2k+1\prime\prime}(x_c) = f'(f^{2k}(x_c)) \cdots f'(f^{k+1}(x_c))f'(f^k(x_c)) \cdots f'(f(x_c))f''(x_c) < 0$$

and $f^{2k+1}(x_c)$ is a maximum. By the periodicity of the KS, $\gamma_{2k+1} = \gamma_{k+1} = I_1$, which means that $f^{2k+1}(x_c) > x_c$, thus $\omega_{2k+1} = M^+$.

The second part of the lemma follows readily. □

Example 2.3. We consider unimodal maps $f \in \mathcal{F} \cap C^3(I)$ with negative Schwarzian derivative,

$$Sf(x) \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0\,.$$

Unimodal maps with negative Schwarzian derivative have some special properties, like possessing at most one stable periodic orbit, [16].

It can be shown that

$$(I_1 I_0 I_1 I_1)^\infty,\ I_1 I_0 (I_1)^\infty,\ (I_1 I_0 I_1)^\infty \text{ and } I_1(I_0)^\infty$$

are admissible KS for maps of this class, [9]. Using the rules (5)-(6) to construct the respective signed KS and the rules in Table 1, the corresponding MMS are:

$$(M^+m^-m^+M^+m^+M^-M^+m^+)^\infty,\ M^+m^-(m^+M^+)^\infty,\ (M^+m^-m^+)^\infty \text{ and } M^+(m^-)^\infty.$$

Let $f \in \mathcal{F}$, $f : [a, b] \to [a, b]$, and let $\mathcal{S}^k$ be the set of critical points of f^k (see (9)). Let x_k [resp. y_k] be the leftmost [resp. rightmost] critical point of f^k, i.e.,

$$x_k = \min \mathcal{S}^k, \quad y_k = \max \mathcal{S}^k, \tag{13}$$

for $k \geq 1$. The relation between the critical points of f^k and f^{k+1} in the interval (x_k, y_k) is described by the following lemma.

Lemma 2.4. *Let $f \in \mathcal{F}$ and $z_{k,1} < z_{k,2}$ be two consecutive critical points for f^k, with $k \geq 2$. Then,*

- **(a):** *If $f^k(z_{k,1})$ and $f^k(z_{k,2})$ have opposite signs, then there exists one and only one $z_{k+1} \in (z_{k,1}, z_{k,2})$ such that f^{k+1} has a positive maximum at z_{k+1}. Furthermore, $f^{k+1}(z_{k+1}) = f(x_c)$.*
- **(b):** *Otherwise, there is no critical point of f^{k+1} on $(z_{k,1}, z_{k,2})$.*

Proof. We may assume that $f(x_c) > x_c$ (otherwise, by Lemma 2.1d), f^k has only one critical point). (a) By assumption, f^k is monotone on $[z_{k,1}, z_{k,2}]$. Suppose $f^k(z_{k,1}) < x_c < f^k(z_{k,2})$. By the Mean Value Theorem, there exists one and only one $z \in (z_{k,1}, z_{k,2})$ such that $f^k(z) =$

x_c. Set $z = z_{k+1}$. Hence, by (I) and as $f(x_c) > x_c$, $f^{k+1}(z) = f(x_c) > x_c$,

$$f^{k+1\prime}(z) = f'(f^k(z))f^{k\prime}(z) = f'(x_c)f^{k\prime}(z) = 0,$$

and, $f^{k+1\prime\prime}(z) = f''(x_c)(f^{k\prime}(z))^2 < 0$. Therefore, f^{k+1} has a positive maximum at z_{k+1}. The case, $f^k(z_{k,1}) > x_c > f^k(z_{k,2})$ is dealt similarly.

(b) This assertion is straightforward. □

3. Counting laps

Before analyzing the general case $f \in \mathcal{F}$ without any particular boundary conditions, we consider provisionally the following additional boundary condition:

(III): $f^k(a) < x_c$ and $f^k(b) < x_c$, for every $k \geq 1$

If $f \in \mathcal{F}$ and $f(x_c) \leq x_c$, then the boundary conditions (III) are trivially fulfilled and, as mentioned in the Introduction, x_c is the only critical point of f^k for all $k \geq 1$. This means that $x_k = y_k = x_c$ for all $k \geq 1$, see (13).

Interesting dynamics is possible if $f(x_c) > x_c$. In this case, the boundary conditions (III) imply that the orbits of the endpoints of the interval I, a and b, are confined within the interval $[a, x_c)$. This happens, in particular, for maps anchored at the boundary points of the interval $[a, b]$, with $f(a) = f(b) = a$, as is the case of the logistic map.

Lemma 3.1. *Let $f \in \mathcal{F}$ with $f(x_c) > x_c$, and suppose further that condition (III) is fulfilled. Let $\mathcal{S}^k$ be the set of critical points of f^k, and let x_k and y_k, $k \geq 1$, be the leftmost and rightmost critical points of f^k, respectively. Then,*

(a): *f^k has a positive maximum at x_k with (i) $x_k < x_{k-1}$, for $k \geq 2$, and (ii) $f^k(x_k) = f(x_c)$.*

(b): *f^k has a positive maximum at y_k with (i) $y_k > y_{k-1}$, for $k \geq 2$, and (ii) $f^k(y_k) = f(x_c)$.*

Proof. As f is unimodal, $x_1 = y_1 = x_c$.

(a) Let $k = 2$. Then, by (I) and (II), $f^{2\prime\prime}(x_c) = f'(f(x_c))f''(x_c) > 0$, so $f^2(x_c)$ is a minimum. Furthermore, the equation $f^{2\prime}(x) = f'(f(x))f'(x) = 0$ has one and only one solution on $(a, x_c) = (a, x_1)$. In fact, by (I) and (II), $f'(x) > 0$ for $x \in (a, x_c)$, and $f'(f(x)) = 0$ implies that $f(x) = x_c$. By (III) and the condition $f(x_c) > x_c$, we have $f(a) < x_c < f(x_c)$, and the equation $f(x) = x_c$ has a solution on (a, x_c). This solution is unique due to the monotonicity of f on the interval (a, x_c). Hence, the unique critical point of f^2 on (a, x_c) is x_2, and $x_2 = \min \mathcal{S}^2$.

We claim now that $f^2(x_2)$ is a positive maximum. Indeed, as $f(x_c) > x_c$,

$$f(x_2) = x_c \;\Rightarrow\; f^2(x_2) = f(x_c) > x_c\,,$$

and, by (I),

$$f^{2\prime\prime}(x_2) = f''(x_c)(f'(x_2))^2 < 0.$$

We proceed now by induction, supposing that (a) is true for $k = 2, 3, ..., n-1$. As by hypothesis, $f^{n-1}(x_{n-1})$ is a positive maximum, then $f^{n-1}(x_{n-1}) > x_c$ and $f^{n-1\prime\prime}(x_{n-1}) < 0$. Hence,

$$f^{n\prime\prime}(x_{n-1}) = f'(f^{n-1}(x_{n-1}))f^{n-1\prime\prime}(x_{n-1}) > 0\,.$$

So, it follows that $f^n(x_{n-1})$ is a minimum. Furthermore, the equation $f^{n\prime}(x) = f'(f^{n-1}(x))f^{n-1\prime}(x) = 0$ has one and only one solution on the interval (a, x_{n-1}). Indeed, (i) $f^{n-1\prime}(x) > 0$ on (a, x_{n-1}) because x_{n-1} is the leftmost critical point of f^{n-1}, and $f^{n-1}(x_{n-1})$ is a maximum; (ii) by (III), the induction hypothesis and the monotonicity of f^{n-1} on (a, x_{n-1}), $f^{n-1}(x) = x_c$ has a unique solution on (a, x_{n-1}). We call x_n the unique critical point of f^n on (a, x_{n-1}). Finally, to prove that $f^n(x_n)$ is a positive maximum, by (I) and as $f(x_c) > x_c$,

$$f^{n-1}(x_n) = x_c \;\Rightarrow\; f^n(x_n) = f(x_c) > x_c,$$

and,

$$f^{n\prime\prime}(x_n) = f''(x_c)(f^{n-1\prime}(x_n))^2 < 0.$$

The proof of (b) is similar. □

Given the KS of a map $f \in \mathcal{F}$, it is possible to sketch symbolically and qualitatively the graph of f^k, for any $k \geq 1$. Suppose first that f obeys the boundary condition (III). We proceed as follows:

- **(A):** Fix $k \geq 1$ and from the KS of $f \in \mathcal{F}$, construct the first k terms of the MMS, $\omega = (\omega_1 = M^+, \omega_2, ..., \omega_k)$.
- **(B):** Draw two perpendicular coordinate axes and divide the vertical axis into k rows, corresponding, top to bottom, to the iterates f^i, $1 \leq i \leq k$, Table 3. The horizontal axis represents the interval $[a, x_c]$. Divide the horizontal axis into $k+1$ columns, corresponding, right to left, to the critical points $x_c = x_1 > x_2 > ... > x_k$ of Lemma 3.1 and to the left endpoint a. On the leftmost column, above a, enter on each row i the sign of $f^i(a) - x_c$, for $1 \leq i \leq k$ (i.e., "+" if $f^i(a) > x_c$, "0" if $f^i(a) = x_c$, and "−" if $f^i(a) > x_c$). On the rightmost (or "first") column, above $x_c = x_1 = y_1$, enter on each row i the element ω_i of the MMS, for $i = 1, ..., k$. Due to the boundary conditions (III) as assumed in Lemma 3.1, the leftmost column of Table 3 contains only minus signs. However, dropping boundary condition (III),

the left most column entries can have all the three signs. This construction is shown in Table 3.

(C): On the second column of Table 3, above x_2, enter ω_{i-1} on row i, for $2 \leq i \leq k$. Continue filling out the remaining columns above x_j. In the step j, enter ω_{i-j+1} on row i, for $j \leq i \leq k$. In this way, we get a triangular matrix, with the symbol M^+ along the secondary diagonal, and each column being the down shift by one entry of the next column to the right. This construction shown in Table 3, is based on the geometrical meaning of the MMS and on Lemma 3.1(a). In view of Lemma 3.1(b), we can extend the horizontal axis to include the interval $(x_c, b]$, but, due to the boundary condition (III), we do not get additional information. Up to this point, we know the structure of the local extrema of f^i at the special critical points $x_1, ..., x_k$ (and $y_1, ..., y_k$).

-					ω_1	1
-				ω_1	ω_2	2
-			ω_1	ω_2	ω_3	3
-			$\vdots$	$\vdots$	$\vdots$	$\vdots$
-	ω_1	$\cdots$	ω_{k-2}	ω_{k-1}	ω_k	k
a	x_k	$\cdots$	x_3	x_2	$x_c = x_1 = y_1$	

TABLE 3. Ordering of some of the critical points of the iterates f^i on the interval $[a, x_c]$. This ordering is derived from Lemma 3.1, and the construction (A)-(C). On the leftmost column, we show the signs of $f^i(a) - x_c$, determined in this case by condition (III). Note that the iterates of the leftmost and rightmost critical points of f^i have the same signed itinerary as the critical point of the unimodal map f. However, the localization of the maxima and minima of the iterated map f^i is not complete. To finish it, we must incorporate the results of Lemma 2.4, as explained in step (D).

(D): In order to determine the location of the remaining critical points of f^2,..., f^k on the interval (a, x_c), we analyze the symbols of the MMS in row 2 of Table 3. Consider the (only) two consecutive elements of the MMS on row 2, namely ω_1 and ω_2. If they have opposite sign, then by Lemma 2.4(a), f^3 has a positive maximum at $z_3 \in (x_2, x_1)$. Thus, from row 3 downwards, a new column is inserted between the x_2- and the x_1-columns. This new column contains the symbols, $\omega_1, \omega_2, ..., \omega_{k-2}$ (see Example 3.2 below). If the sign of the two consecutive elements on this row are not opposite, we are in the conditions of Lemma 2.4(b) and no new column is created. Next, go to rows 3, 4, etc. and compare row-wise all pairs of consecutive elements. If two consecutive elements on the same row have opposite signs, then we proceed as before and, under application of Lemma 2.4(a), we insert a new column in-between, beginning on the next row with the entry ω_1, followed by $\omega_2, \omega_3, ...$ on the remaining entries of this new column. The resulting diagram displays the qualitative structure of local maxima and minima of the iterates f^i, with $1 \leq i \leq k$. According to Lemma 1.1, all critical points of f^i are pre-images of x_c up to order $i-1$.

We call the "MM-table or the Min-Max table of f" a table like the one in Table 4, extended to the whole interval $[a, b]$.

Example 3.2. Let $f \in \mathcal{F}$ obeying the boundary condition (III), and suppose that the KS of f is $\gamma = (I_1 I_1 C)^\infty$. By the rules of Table 2, the corresponding MMS is $\omega = (M^+ m^- m^0)^\infty$. In order to know the qualitative shape of, say, f^4, we follow the steps (A)-(D) above. The symbolic table of maxima and minima is represented in Table 4. Row 4 of Table 4 provides the necessary information to draw qualitatively the graph of f^4. From this construction, we can calculate directly the lap numbers of f^k for $1 \leq k \leq 4$. From Table 4, and counting also laps on the right hand side of the critical point x_c, we obtain, $\ell_1 = 2$, $\ell_2 = 4$, $\ell_3 = 8$ and $\ell_4 = 14$.

The construction just done provides the tools in order to prove Theorem 4.2 of the next section, where we drop the boundary condition (III). The existence of a simple zero of the equation $f^n(x) - x_c = 0$ on (a, x_{n-1}) [resp. on (y_{n-1}, b)] depends on the signs of $f^n(a) - x_c$ and $f^n(x_{n-1}) - x_c$ [resp. $f^n(y_{n-1}) - x_c$ and $f^n(b) - x_c$]. If the signs of, say, $f^n(a) - x_c$ and $f^n(x_{n-1}) - x_c$ are opposite, then, as in the proof of Lemma 3.1, it follows that there exists a unique $x_n \in (a, x_{n-1})$ such that $f^n(x_n) = x_c$. Due to monotonicity, x_n is a simple zero of $f^n(x) - x_c$. Otherwise, there is no such solution and so $x_n = x_{n-1}$, since x_{n-1} is the leftmost critical point of f^n. This translates *mutatis mutandis* to the half interval (x_c, b). In the following, we consider the general case, without assuming the boundary conditions (III).

Example 3.3. We consider the map $f_{\alpha,\beta}(x) = e^{-\alpha^2 x^2} + \beta \in \mathcal{F}$, with $\alpha = 2.8$ and $\beta = -0.1$, which will be analyzed in detail in section 5. This map has a critical point at $x = 0$. We suppose further that $f_{\alpha,\beta}(x) : [-(1+\beta), (1+\beta)] \to [-(1+\beta), (1+\beta)]$, and $a = -b = -(1+\beta)$. The

-							M^+	1
-				M^+			m^-	2
-		M^+		m^-		M^+	m^0	3
-	M^+	m^-	M^+	m^0	M^+	m^-	M^+	4
a	x_4	x_3		x_2			$x_c = x_1 = y_1$	

TABLE 4. Structure of the maxima and minima on the interval $[a, x_c]$ of the first four iterates of a unimodal map f obeying the boundary condition (III), with KS $\gamma = (I_1 I_1 C)^\infty$ and MMS $\omega = (M^+ m^- m^0)^\infty$. After having constructed the table for the first row ($k = 1$), if two consecutive symbols of the row number 1 have opposite signs, counting with the boundary conditions, then there is a new column beginning in the second row with the first symbol of the MMS sequence. From this symbolic representation we can count the number of laps of each iterated map. Due to the symmetry of the boundary condition (III), this diagrams extends symmetrically with respect to the critical point $x_c = x_1 = y_1$ to the interval $[x_c, b]$, and the distribution of maxima and minima is mirrored.

first elements of the MMS of $f_{\alpha,\beta}$ are $(M^+ m^- m^+ M^- M^+ \ldots)$. The signs of the iterates $f^n_{\alpha,\beta}(a)$ and $f^n_{\alpha,\beta}(b)$, with $n \geq 1$, are $(-,+,-,+,-,\cdots)$. With the construction done above in the steps (A)-(D), the MM-table of the first five iterates of $f_{\alpha,\beta}$ is represented in Table 5. The difference from this case to the one presented in Example 3.2 is on the boundary points. Observe that $x_3 = x_2$, since $f^2_{\alpha,\beta}$ has a positive maximum at x_2 and $f^2_{\alpha,\beta}(a) > x_c$; $x_4 = x_3$ because $f^3_{\alpha,\beta}$ has a negative minimum at x_3 and $f^3_{\alpha,\beta}(a) < x_c$. The equality $x_5 = x_4$ follows from a similar argument.

4. Main results

Consider a unimodal map $f \in \mathcal{F}$, let ℓ_n denote the *minimal number of laps of* f^n, and let e_n be the *number of local extrema of* f^n, with $n \geq 1$. Since f^n is continuous and piecewise monotone, the laps are separated by critical points, and the relation,

$$\ell_n = e_n + 1 \tag{14}$$

holds.

Furthermore, let s_n stand for the *number of interior simple zeros of* $f^n(x) - x_c$, $n \geq 1$, i.e., solutions of $f^n(x) = x_c$, with $x \in (a, b)$,

-					M^+	1
+	M^+				m^-	2
-	m^-			M^+	m^+	3
+	m^+	M^+		m^-	M^-	4
-	M^-	m^-	M^+	m^+	M^+	5
a	$x_5 = \ldots = x_2$				$x_c = x_1 = y_1$	

TABLE 5. Structure of the maxima and minima on the interval $[a, x_c]$ of the first five iterates of the unimodal map $f_{\alpha,\beta}$ of Example 3.3. The first elements of the MMS are $(M^+m^-m^*M^-M^+\ldots)$. Due to symmetry of $f_{\alpha,\beta}$ with respect to $x_c = 0$, the distribution of maxima and minima of $f^i_{\alpha,\beta}$ on $[x_c, b] = [0, 0.9]$ is the mirrored image of the maxima and minima of $f^i_{\alpha,\beta}$ on the interval $[a, x_c] = [-0.9, 0]$.

$f^i(x) \neq x_c$ for $0 \leq i \leq n-1$, and $f^{n\prime}(x) \neq 0$. In particular, for unimodal maps $f \in \mathcal{F}$, we have $\ell_1 = 2$, $e_1 = 1$ and $s_1 \in \{0, 1, 2\}$. If in addition the boundary condition (III) is fulfilled, then $s_1 = 2$.

Lemma 4.1 ([5]). *Let $f \in \mathcal{F}$. Let e_n be the number of critical points of f^n, and let s_n be the number of interior simple zeros of $f^n(x) - x_c$. Then, for $n \geq 2$,*

$$e_n = e_{n-1} + s_{n-1}. \tag{15}$$

Proof. Indeed, e_n equals the number of sign changes of $f^{n\prime}$. Then, Eq. (15) follows from the relation $f^{n\prime}(x) = f'(f^{n-1}(x))f^{n-1\prime}(x)$. □

From (14) and (15), we have,

$$\ell_n - \ell_{n-1} = e_n - e_{n-1} = s_{n-1} \tag{16}$$

for $n \geq 2$. Moreover, induction on n with (15) yields,

$$e_n = e_1 + \sum_{i=1}^{n-1} s_i = 1 + \sum_{i=1}^{n-1} s_i. \tag{17}$$

For each iterate of a unimodal map, maxima and minima alternate along the x direction. This is easily seen in Table 4. In the simplest case, the row k of the MM-table of $f \in \mathcal{F}$ contains only the symbols M^+ and m^-. Therefore, the laps between two consecutive extrema on the graph of f^k always cross the line $y = x_c$. Whether this also happens in the intervals (a, x_k) and (y_k, b) depends, of course, on the signs of $f^k(a) - x_c$ and $f^k(x_k) - x_c$ on the left side of the interval I, and of $f^k(b) - x_c$ and $f^k(y_k) - x_c$ on the right side of I. All the four possibilities are encapsulated in the relation,

$$s_k = e_k + 1 - \alpha_k - \beta_k, \tag{18}$$

where,

$$\alpha_k = \begin{cases} 0 & \text{if} \;\; \mathrm{sign}(f^k(a) - x_c) \cdot \mathrm{sign}(f^k(x_k) - x_c) < 0 \\ 1 & \text{if} \;\; \mathrm{sign}(f^k(a) - x_c) \cdot \mathrm{sign}(f^k(x_k) - x_c) \geq 0 \end{cases}$$
$$\beta_k = \begin{cases} 0 & \text{if} \;\; \mathrm{sign}(f^k(b) - x_c) \cdot \mathrm{sign}(f^k(y_k) - x_c) < 0 \\ 1 & \text{if} \;\; \mathrm{sign}(f^k(b) - x_c) \cdot \mathrm{sign}(f^k(y_k) - x_c) \geq 0\,. \end{cases} \tag{19}$$

In the construction of the MM-table of f of Example 3.3, we have set $x_i = x_{i-1}$ ($i \geq 2$) every time $f^{i-1}(a) - x_c$ and $f^{i-1}(x_{i-1}) - x_c$ have no opposite signs (see Table 5). The first time this occurs, the type of the leftmost extreme of f^i is not ω_1 but ω_2; if $f^i(a) - x_c$ and $f^i(x_i) - x_c$ have not opposite signs again, the type of the leftmost extremum of f^{i+1} is ω_3. In general, if $f^{i-1}(a) - x_c$ and $f^{i-1}(x_{i-1}) - x_c$ have not opposite signs j times, for $1 \leq i \leq k-1$, then the leftmost extremum $f^k(x_k)$ is of type ω_{1+j} (see Table 5). Therefore, $\mathrm{sign}(f^k(x_k) - x_c) = \mathrm{sign}(\omega_{1+j})$. A similar discussion holds for the rightmost extremum $f^k(y_k)$.

In general, the row k contains any symbol from the whole alphabet $\mathcal{M} = \{m^-, m^0, m^+, M^-, M^0, M^+\}$. In this case, if some symbol ω_i on the row k belongs to the set,

$$\mathcal{M}_b = \{m^+, M^-, m^0, M^0\}$$

then, none of the two entries adjacent to ω_i (eventually including $f^k(a) - x_c$ or $f^k(b) - x_c$) can have a sign opposite to the sign of ω_i. Every time this happens, there are two laps, one on each side of the extremum correponding to the symbol $\omega_i \in \mathcal{M}_b$, which do not cross the line $y = x_c$. If all such symbols belonging to $\mathcal{M}_b$ correspond to extrema other than the leftmost and rightmost extremun (so these are of types M^+ or m^-), one gets from (18),

$$s_k = e_k + 1 - \alpha_k - \beta_k - 2b_k \tag{20}$$

where b_k is the number of symbols from $\mathcal{M}_b$ on row k of the MM-table of f (see rows $k = 3, 4$ in Table 4 for an example on the half-interval $[a, x_c]$). If, otherwise, $f^k(x_k)$ (resp. $f^k(y_k)$) is of a type included in $\mathcal{M}_b$, then we have to set $\alpha_k = 0$ (resp. $\beta_k = 0$) in (20) since the term $2b_k$ already accounts for the absence of a simple zero of $f^k(x) - x_c$ in the interval (a, x_k) (resp. (y_k, b)).

From the above discussion we conclude that Eq. (20) holds without restrictions, provided that,

(21)

$$\alpha_k = \begin{cases} 0 & \text{if } \operatorname{sign}(f^k(a) - x_c) \cdot \operatorname{sign}(\omega_{1+\lambda(k-1)}) < 0 \\ 0 & \text{if } \operatorname{sign}(f^k(a) - x_c) \cdot \operatorname{sign}(\omega_{1+\lambda(k-1)}) \geq 0 \text{ and } \omega_{1+\lambda(k-1)} \in \mathcal{M}_b \\ 1 & \text{if } \operatorname{sign}(f^k(a) - x_c) \cdot \operatorname{sign}(\omega_{1+\lambda(k-1)}) \geq 0 \text{ and } \omega_{1+\lambda(k-1)} \notin \mathcal{M}_b \end{cases}$$

$$\beta_k = \begin{cases} 0 & \text{if } \operatorname{sign}(f^k(b) - x_c) \cdot \operatorname{sign}(\omega_{1+\rho(k-1)}) < 0 \\ 0 & \text{if } \operatorname{sign}(f^k(b) - x_c) \cdot \operatorname{sign}(\omega_{1+\rho(k-1)}) \geq 0 \text{ and } \omega_{1+\rho(k-1)} \in \mathcal{M}_b \\ 1 & \text{if } \operatorname{sign}(f^k(b) - x_c) \cdot \operatorname{sign}(\omega_{1+\rho(k-1)}) \geq 0 \text{ and } \omega_{1+\rho(k-1)} \notin \mathcal{M}_b \end{cases}$$

where $\operatorname{sign}(\omega_i) = -1, 0, 1$ depending on whether $\omega_i \in \{m^-, M^-\}$, $\{m^0, M^0\}$, or $\{m^+, M^+\}$, respectively. The functions $\lambda(k)$ and $\rho(k)$ are recursively calculated as follows: $\lambda(0) = \rho(0) = 0$, and for $k \geq 1$,

(22)

$$\lambda(k) = \begin{cases} 0 & \text{if } \operatorname{sign}(f^k(a) - x_c) \cdot \operatorname{sign}(\omega_{1+\lambda(k-1)}) < 0 \\ \lambda(k-1) + 1 & \text{if } \operatorname{sign}(f^k(a) - x_c) \cdot \operatorname{sign}(\omega_{1+\lambda(k-1)}) \geq 0 \end{cases}$$

$$\rho(k) = \begin{cases} 0 & \text{if } \operatorname{sign}(f^k(b) - x_c) \cdot \operatorname{sign}(\omega_{1+\rho(k-1)}) < 0 \\ \rho(k-1) + 1 & \text{if } \operatorname{sign}(f^k(b) - x_c) \cdot \operatorname{sign}(\omega_{1+\rho(k-1)}) \geq 0\,. \end{cases}$$

We can now derive the main result of this paper.

Theorem 4.2. *Let $\omega = (\omega_k)_{k\geq 1}$ be the MMS of $f \in \mathcal{F}$, and set,*

$$J_n = \{1 \leq j \leq n : \omega_j \in \mathcal{M}_b\}$$

where $\mathcal{M}_b = \{m^+, M^-, m^0, M^0\}$. Let $(\alpha_k)_{k\geq 1}$ and $(\beta_k)_{k\geq 1}$ be the sequences of integers calculated recursively from (21) and (22), then

$$\ell_{n+1} = 2\ell_n - 2\sum_{j\in J_n} (\ell_{n+1-j} - \ell_{n-j}) - \alpha_n - \beta_n, \tag{23}$$

where $\ell_0 = 1$ and $n \geq 1$.

Proof. Let N_p be the number of columns that begin at row p on the MM-table of f, and, as before, let b_n be the number of symbols from the set $\{m^+, M^-, m^0, M^0\}$ on row n of the MM-table of f. Then,

$$b_n = \sum_{j\in J_n} N_{n-j+1}. \tag{24}$$

By Lemma 2.4(a), $N_p = s_{p-1}$. Upon substitution of this equality into (24) and (20), we obtain,

$$s_n = e_n + 1 - 2\sum_{j\in J_n} s_{n-j} - \alpha_n - \beta_n$$

From (17) it follows now that,

$$s_n = 2 + \sum_{i=1}^{n-1} s_i - 2 \sum_{j \in J_n} s_{n-j} - \alpha_n - \beta_n. \tag{25}$$

Finally, with the condition $\ell_1 = 2$ and substitution of (25) into (16), we obtain (23). □

For example, we can check the formulas (20)-(23) with the geometric information contained in Table 5 (extended to the half-interval $[x_c, b]$), Example 3.3. Here, $\alpha_k = \beta_k$ because the map $f_{2.8,-0.1}$ of Example 3.3 is symmetric around the critical point $x_c = 0$, $J_1 = J_2 = \emptyset$, $J_3 = \{3\}$ and $J_4 = \{3, 4\}$. So, by Theorem 4.2, we obtain,

k	$\alpha_k = \beta_k$	b_k	e_k	s_k	ℓ_k
1	0	0	1	2	2
2	1	0	3	2	4
3	1	1	5	2	6
4	0	3	7	2	8
5	0	4	9	2	10

where b_k is the number of symbols of the set $\mathcal{M}_b$ of the row number k.

In the case of maps with the boundary condition (III), then $\alpha_k = \beta_k = 0$ for every $k \geq 1$ (see the proof of Lemma 3.1 and Table 4). So, from (23), we conclude:

Corollary 4.3 ([5, 7])**.** *Let $\omega = (\omega_k)_{k\geq 1}$ be the MMS of $f \in \mathcal{F}$ fulfilling the boundary condition (III), and set,*

$$J_n = \{1 \leq j \leq n : \omega_j \in \mathcal{M}_b\}.$$

where $\mathcal{M}_b = \{m^+, M^-, m^0, M^0\}$. Then,

$$\ell_{n+1} = 2\ell_n - 2 \sum_{j \in J_n} (\ell_{n+1-j} - \ell_{n-j}), \tag{26}$$

where $\ell_0 = 1$ and $n \geq 1$.

The theorem 4.2 and the corollary 4.3 provide fast algorithms to calculate lap numbers, thus allowing to estimate the topological entropy of smooth unimodal maps in a simple and efficient way. Note that these formulas involve the MMS of the map and the itineraries of the endpoints of the interval I.

From the technical point of view, the conclusions of theorem 4.2 remain true if we change the smoth conditions (I) a (II), by continuity and monotinicity conditions on the left and right sides of the critical point.

5. Computing the topological entropy

Theorem 4.2 and Corollary 4.3 are the basic results in order to estimate the topological entropy of interval maps.

The first example we consider here is the one parameter family of logistic maps, $x_{n+1} = f_\mu(x_n) = 4\mu x_n(1-x_n)$, defined on the interval $I = [0,1]$ and with $\mu \in (0,1]$. Clearly, $f_\mu \in \mathcal{F}$, for all the values of $\mu \in (0,1]$, and the critical point of the family is $x_c = 1/2$, independently of μ.

n	$\mu = 0.875$ $(I_1I_0I_1I_1)^\infty$ $(M^+m^-m^+M^+m^+M^-M^+m^+)^\infty$	$\mu = 0.9196433776$ $I_1I_0(I_1)^\infty$ $M^+m^-(m^+M^+)^\infty$	$\mu = 0.957$ $(I_1I_0I_1)^\infty$ $(M^+m^-m^+)^\infty$	$\mu = 1.0$ $I_1(I_0)^\infty$ $M^+(m^-)^\infty$
1	2	2	2	2
2	4	4	4	4
3	8	8	8	8
4	14	14	14	16
5	24	24	24	32
6	38	38	40	64
7	58	60	66	128

Table 6. Lap number for several parameter values of the logistic map $f_\mu(x) = 4\mu x(1-x)$. For each parameter value, we show the corresponding KS and MMS.

To calculate the lap numbers for several values of the parameter μ, we choose $\mu = 0.875$, $\mu = 0.9196433776$, $\mu = 0.957$ and $\mu = 1$. For these parameter values, the logistic maps have the KS and the MMS of Example 2.3 and, in Table 6, we show the first lap numbers calculated from corollary 4.3.

To calculate the growth numbers and the topological entropies for this family of maps, we can use theorem 4.2 and corollary 4.3 to calculate numerically the lap numbers and then estimating $\limsup \ell_n^{1/n}$. As the rate of convergence of $\limsup \ell_n^{1/n}$ to its limit s is not known, in order to have an estimate of the numerical errors $|s - \ell_n^{1/n}|$, we use the following lemma adapted from [7].

Lemma 5.1 ([7]). *Let $\omega = (\omega_k)_{k\geq 1}$ be the MMS of $f \in \mathcal{F}$ fulfilling the boundary condition (III). Suppose in addition that the MMS has least period $k > 1$, $\omega = (\omega_1\omega_2\dots\omega_k)^\infty$, and $f^k(x_c) \neq x_c$. Then, the growth number (s) of f is the largest real root of the polynomial,*

$$P(x) = x^{k-1} - \sum_{i=1}^{k-1} \varepsilon_i x^{k-1-i}$$

where,

$$\varepsilon_i = \begin{cases} 1 & \text{if } \varepsilon_i = m^- \text{ or } M^+ \\ -1 & \text{if } \varepsilon_i = m^+ \text{ or } M^- . \end{cases}$$

For the proof, See the Appendix A.

To test the convergence of $(\ell_n)^{1/n}$ to the growth numbers of the maps f_μ, we take the MMS in Table 6. For the first and third MMS, we use lemma 5.1, and we obtain, $s = 1$ and $s = (1+\sqrt{5})/2$, respectively. The growth numbers of the second and forth MMS listed in Table 6 are, $s = \sqrt{2}$ and $s = 2$, [5]. In the four cases, the growth numbers and the topological entropies are,

$$\begin{array}{cclcl} (I_1 I_0 I_1 I_1)^\infty & : & s = 1 & , & h = 0 \\ I_1 I_0 (I_1)^\infty & : & s = \sqrt{2} & , & h = \frac{1}{2}\log 2 \\ (I_1 I_0 I_1)^\infty & : & s = \frac{1+\sqrt{5}}{2} & , & h = \log \frac{1+\sqrt{5}}{2} \\ I_1 (I_0)^\infty & : & s = 2 & , & h = \log 2 . \end{array} \tag{27}$$

In Figure 1, for the same parameter values in Table 6, we compare the behavior of $(\ell_n)^{(1/n)}$ as a function of n with the growth numbers (27). In Table 7, we show the error between the exact and the numerical values of the growth number and of the topological entropy for the maps of Table 6. In most cases, the error for the estimation of the topological entropy is acceptable for $n \simeq 512$. The worst estimate in Table 7, correspond to the MMS of a dynamics on the Feigenbaum period doubling bifurcation cascade, prior to the appearance of fixed points with odd periods ([5]).

	$\Delta s = \lvert s - \ell_n^{1/n} \rvert$	$\Delta h = \lvert h - (\log \ell_n)/n \rvert$
$(I_1 I_0 I_1 I_1)^\infty$	0.034	0.033
$I_1 I_0 (I_1)^\infty$	0.005	0.004
$(I_1 I_0 I_1)^\infty$	0.003	0.002
$I_1 (I_0)^\infty$	0	0

TABLE 7. Error between the exact and the numerical values of the growth number and of the topological entropy for the maps of Table 6, estimated with the choice $n = 512$.

In Figure 2, we show the topological entropy of the logistic map as a function of the parameter μ and calculated from (2). We have calculated the KS and the MMS up to iterate number $n = 512$, and we have plotted $\log(\ell_n)/n$ ($\simeq h$), for $n = 32$, $n = 128$ and $n = 512$, as a function of the parameter μ. The lap number ℓ_n has been calculated from Corollary 4.3. For $\mu \leq \mu_\infty = 0.892486416\ldots$ (the Feigenbaum

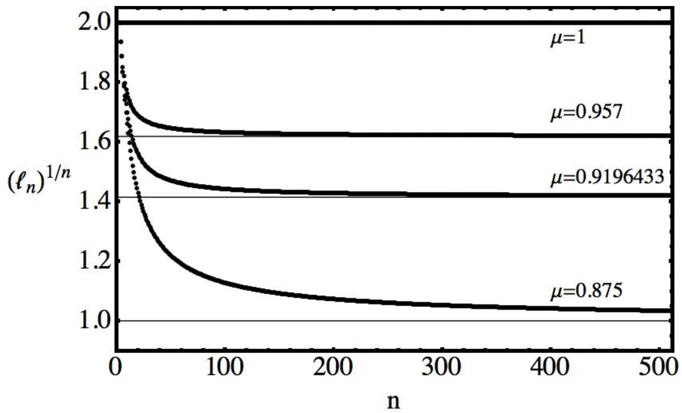

FIGURE 1. Behavior of the growth number $(\ell_n)^{(1/n)}$ as a function of n, for several parameter values of the logistic map $f_\mu(x) = 4\mu x(1-x)$. The parameter values are the same as in Table 6. The thin lines show the exact values of the growth numbers, calculated in (27). For $n = 512$, the errors in the growth number estimates are below $\Delta s = 0.005$, except for the first case of table Table 6, where $\Delta s = 0.034$.

point), the topological entropy of the logistic map is zero ([6, 7]), and, for the case $n = 512$, this condition has been introduced by hand in the plot of Figure 2. Near μ_∞ the convergence of $\log(\ell_n)/n$ to the topological entropy is very slow, Table 7. This can be seen in the approximations to the topological entropy with $n = 32$ and $n = 128$. For these cases, we have not imposed the condition $h(f_\mu) = 0$, for $\mu \leq \mu_\infty$.

The monotonicity of the topological entropy for the logistic map, [8] and [17], shows the existence of bifurcations as the parameter μ is varied, and it measures the complexity of the dynamics of the family of maps. In the chaotic region, where the topological entropy is positive, these bifurcations are associated with bifurcations of fixed points with large periods, which are difficult to detect numerically. The variation of the lap number and of the topological entropy as the parameter μ changes, shows the existence of bifurcations. In [6] and [7], it has been shown that for the logistic family of maps the topological entropy in the chaotic region is well approximated by the function,

$$\tilde{h}(\mu) = \begin{cases} a\,(\mu-\mu_\infty)^b & \text{if} \quad \mu \in [\mu_\infty, \eta_2] \\ \log\left(2 - \frac{2}{c(1-\mu)^{-d}-2}\right) & \text{if} \quad \mu \in [\eta_2, 1] \end{cases} \tag{28}$$

where $a = 1.82957$, $b = \log 2/\log\delta = 0.4498069$, $c = \pi/2 = 1.534780$, $d = 1/2$, $\mu_\infty = 0.892486416$, $\eta_2 = 0.919643377$ and $\delta = 4.6692\ldots$ is the Feigenbaum constant. In Figure 2, we have also plotted the approximation (28) to the topological entropy.

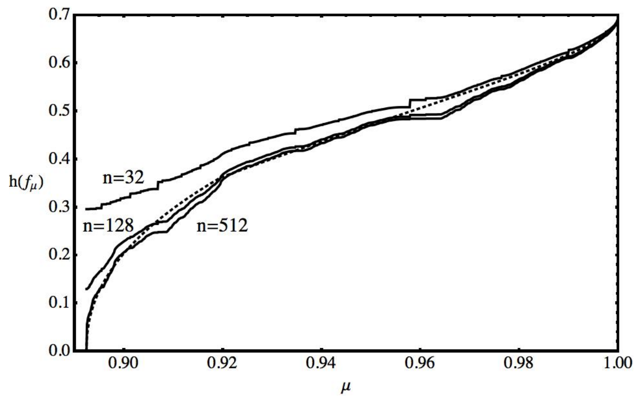

FIGURE 2. Topological entropy of the logistic map as a function of the parameter μ, calculated from (2), for $n = 32$, $n = 128$ and $n = 512$. For $\mu \leq \mu_\infty = 0.892486416\ldots$, the topological entropy of the logistic family is zero, [7]. For $n = 512$, we have imposed the condition $h(f_\mu) = 0$, for $\mu \leq \mu_\infty$. The dotted line is the approximation (28) to the topological entropy.

We consider now the interval map $x_{n+1} = f_{\alpha,\beta}(x_n)$, where,

$$f_{\alpha,\beta}(x) = e^{-\alpha^2 x^2} + \beta, \tag{29}$$

and $\alpha \neq 0$ and β are real parameters. This two-parameter family of interval maps qualitatively describes the dynamics of an electrical circuit with a nonlinear diode showing chaotic behavior. The map (29) shows direct and reverse bifurcations when the parameters are monotonically changed, [4].

The map (29) has a critical point at $x_c = 0$. As $f_{\alpha,\beta} \in C^\infty(\mathbb{R})$, and $f''_{\alpha,\beta}(0) = -2\alpha^2 < 0$, the map obeys condition (I). Condition (II) is trivially fulfilled, and without loss of generality, we take $\alpha > 0$. To choose an interval I such that $f_{\alpha,\beta}$ maps I to itself, we take the non-trivial case $\beta \in (-1, 0]$. As $f_{\alpha,\beta}(0) = 1 + \beta$, and,

$$f^2_{\alpha,\beta}(0) = \beta + e^{-\alpha^2(1+\beta)^2} < 1 + \beta \quad (> 0)\,,$$

we can choose $I = [-(1+\beta), (1+\beta)]$. Therefore, all the maps $f_{\alpha,\beta} : [-(1+\beta), (1+\beta)] \to [-(1+\beta), (1+\beta)]$, with $\alpha > 0$ and $\beta > -1$, are unimodal. As the iterates of the boundary points $a = -(1+\beta)$ and

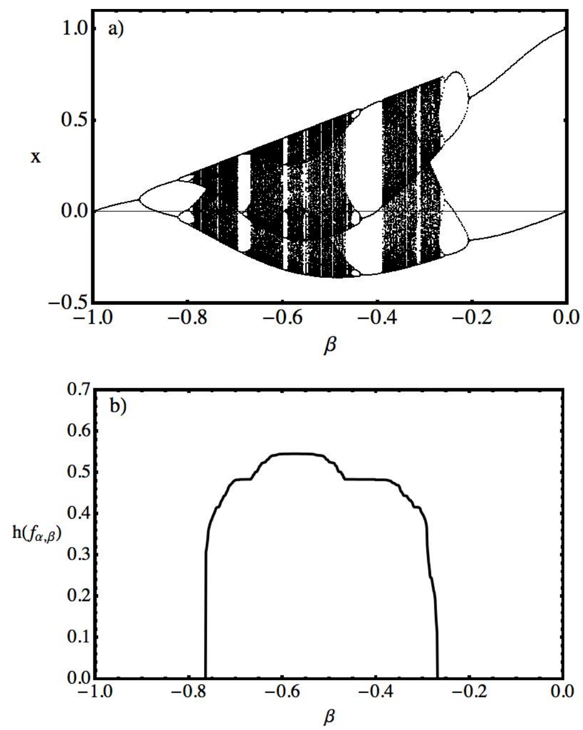

FIGURE 3. a) Bifurcation diagram of the map (29) as a function of the parameter β in the interval $[-1, 0]$ and for $\alpha = 2.8$. b) Topological entropy of the map (29), with $\alpha = 2.8$ and $\beta \in (-1, 0]$. The topological entropy has been calculated by (2) with $n = 512$, and the lap number has been calculated by Theorem 4.2. The parameter value where topological entropy changes from a increasing to a decreasing function of β corresponds to the reversion of bifurcations.

$b = (1 + \beta)$ can be on both sides of the critical point $x_c = 0$, we are in the conditions of Theorem 4.2. Therefore, in order to estimate the topological entropy of the map, we have to calculate numerically the iterates of the critical point and the iterates of the boundary points of the interval I.

To better understand the relation between the topological entropy and the dynamics generated by the family of maps (29), we have calculated an estimate for the topological entropy through the exact value of the lap number obtained from Theorem 4.2, and we have also computed the bifurcation diagram of the dynamical system $x_{n+1} = f_{\alpha,\beta}(x_n)$. In Figure 3, we show the bifurcation diagram and the topological entropy of the maps (29), for the parameter value $\alpha = 2.8$ and $\beta \in [-1, 0]$. As it is seen in Figure 3, the region where the topological entropy is increasing corresponds to an increase in the complexity of the dynamics of the maps. This is due to the emergence of several periodic points. When the reversal of bifurcations occur, the topological entropy decreases.

6. **Conclusion**

We have derived an algorithm to efficiently calculate the lap number and the topological entropy of the class of single-humped unimodal maps with free end-points. We have shown that the lap number is determined by the itinerary of the critical point and by the itinearies of the boundary points (Theorem 4.2). The algorithm is based on the geometric interpretation of the kneading sequence of a map, leading us to a new symbolic sequence $(\omega_{f,n})_{n\in\mathbb{N}}$ — the Min-Max sequence — that contains qualitative information about the sequence of maxima and minima of the iterates $f^n(x)$.

Appendix A. Proof of Lemma 5.1

Proof of Lemma 5.1. As $f \in \mathcal{F}$, fulfilling the boundary condition (III), we have $\alpha_k = \beta_k = 0$, and by (25), we obtain,

$$\begin{aligned} s_{n+k} - s_n &= \sum_{i=1}^{n+k-1} s_i - 2\sum_{j\in J_{n+k}} s_{n+k-j} - \sum_{i=1}^{n-1} s_i + 2\sum_{j\in J_n} s_{n-j} \\ &= \sum_{i=n}^{n+k-1} s_i - 2\sum_{j\in J_k} s_{n+k-j}\,. \end{aligned} \tag{30}$$

As ω has least period k, $\omega_1 = \omega_{k+1} = M^+$ or $\omega_1 = \omega_{k+1} = M^-$, and therefore, as in Table 2, $\omega_k \in \mathcal{M}_b$. Introducing this condition in (30) and as $f^k(x_c) \neq x_c$, we get,

$$\begin{aligned} s_{n+k} &= \sum_{i=n+1}^{n+k-1} s_i - 2\sum_{j\in J_{k-1}} s_{n+k-j} \\ &= \sum_{j=1}^{k-1} \varepsilon_j s_{n+k-j} \end{aligned} \tag{31}$$

where,

$$\varepsilon_i = \begin{cases} 1 & \text{if } \varepsilon_i = m^- \text{ or } M^+ \\ -1 & \text{if } \varepsilon_i = m^+ \text{ or } M^-\,. \end{cases}$$

The recurrence relation (31), can be written in the matrix form,

$$\begin{pmatrix} s_{n+2} \\ \vdots \\ s_{n+k} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 \\ \varepsilon_{k-1} & \varepsilon_{k-2} & \varepsilon_{k-3} & \cdots & \varepsilon_1 \end{pmatrix} \begin{pmatrix} s_{n+1} \\ \vdots \\ s_{n+k-1} \end{pmatrix} \tag{32}$$

and $P(x)$ is the characteristic polynomial of the matrix in (32). The solution of the recurrence relation (32) has the form,

$$s_n = \sum_{i=1}^{m} c_i p_i(n) \lambda_i^n \tag{33}$$

where, λ_i are the eigenvalues of the characteristic polynomial $P(x)$, c_i are constants and $p_i(n)$ are polynomials in n of degree $m_i - 1$, where m_i is the multiplicity of λ_i. By (14) and (15), we have,

$$\ell_n = 2 + \sum_{i=1}^{n-1} s_i \, . \tag{34}$$

Substitution of (33) into (34), gives the result of the lemma.

□

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