# Cosmic alignment of the aether

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In Einstein-aether theory and Horava gravity, a timelike unit vector is coupled to the spacetime metric. It has previously been shown that in an exponentially expanding homogeneous, isotropic background, small perturbations of the vector relax back to the isotropic frame. Here we investigate large deviations from isotropy, maintaining homogeneity. We find that, for generic values of the coupling constants, the aether and metric relax to the isotropic configuration if the initial aether hyperbolic boost angle and its time derivative in units of the cosmological constant are less than something of order unity. For larger angles or angle derivatives, the behavior is strongly dependent on the values of the coupling constants. Generally there is runaway behavior, in which the anisotropy increases with time, and/or singularities occur.

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## I. INTRODUCTION

When the phenomenology of theories with a preferred frame is studied, it is generally assumed that this frame coincides, at least roughly, with the cosmological rest frame defined by the Hubble expansion of the universe. Observations place strong bounds on frame dependent effects which would presumably grow with the relative velocity of the preferred frame and the velocity of the earth (which moves at the "low" speed of $10^{-3}c$ relative to the Hubble frame). In a particular theory with preferred frame effects, the dynamical alignment of the frame (or frames) can be studied to determine stability of cosmic alignment, as well as to characterize the range of initial conditions that could be expected to naturally align.

In this paper we examine this question in the case of Einstein-aether theory[1] and in the IR limit of (extended) Horava gravity[2, 3]. Einstein-aether theory just consists of general relativity coupled, at second derivative order, to a dynamical timelike unit vector field $u^a$, the aether. In Horava gravity, the aether vector is assumed to be hypersurface-orthogonal, i.e. it is the unit normal to level sets of a scalar time function. Various forms of "Horava gravity" have been discussed in the literature. Here we refer exclusively to the one related to Einstein-aether theory as just explained. (This corresponds to the so-called "non-projectable" version, where the lapse function $N$ is an arbitrary function of spacetime, and includes in the Lagrangian a term proportional to the square of the gradient of $\ln N$.) Every hypersurface-orthogonal Einstein-aether solution is a Horava solution. All the solutions to be considered in this paper are of this type.

The alignment of the aether has been studied before in the context of linearized perturbations. The question was first addressed, indirectly, by Lim[4], who found that all perturbations of the aether decay exponentially in a de Sitter background. In particular, this result applies to the homogeneous modes. Subsequent work[5, 6] confirmed this result, but in Ref. [6] it was found that under some circumstances, after inflation, velocity perturbations might grow to be "mildly relativistic" and could still possibly be compatible with observations. In all these analyses, it is assumed that the aether is aligned in a background solution, and the behavior of perturbations is studied.

Kanno and Soda (KS) approached the question from a different point of view. In the Appendix of Ref. [7] they examined homogeneous but anisotropic solutions in the presence of a positive cosmological constant, with three orthogonal principal directions of expansion, and with the aether tilted in one of the principal directions. (This corresponds to Bianchi type I (Kasner-like) symmetry.) They showed that, to linear order in the anisotropy, the system relaxes exponentially to the isotropic, de Sitter solution. Since their analysis was carried out just to linear order in the anisotropy, it is in fact just a special case of the above mentioned perturbative treatments.

In this paper we adopt precisely the setting of the KS analysis but we include the full nonlinear dynamics. We characterize the range of initial data that relax to an isotropic solution. Generically, the aether aligns provided the initial boost angle and its time derivative in units of the cosmological constant are less than something of order unity. The precise stability bounds depend on the values of the coupling parameters in the Lagrangian defining the theory.

## II. BIANCHI TYPE I EINSTEIN-AETHER COSMOLOGY

Einstein-aether theory is general relativity (GR) coupled to a dynamical timelike unit vector field. In terms of the metric $g_{ab}$ of signature $(+---)$ and the unit vector field $u^a$ it is defined by the action

$$S = \frac{-1}{16\pi G}\int d^4x\,\sqrt{-g}\left(R + 2\Lambda + K^{ab}{}_{mn}\nabla_a u^m \nabla_b u^n\right) \tag{1}$$

where $R$ is the Ricci scalar, $\Lambda$ is a cosmological constant, and the tensor $K^{ab}{}_{mn}$ is given by

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m + c_4 u^a u^b g_{mn}, \quad (2)$$

and $c_1, \ldots, c_4$ are dimensionless coupling parameters that define the theory. Since $u^m$ is constrained to be a unit vector, the action need only be stationary under variations orthogonal to the aether, $u_m \delta u^m = 0$. In this paper for simplicity we omit any matter couplings, since the cosmological constant suffices to source the overall expanding solution and it models the conditions that would have pertained in an inflationary early universe. It would be straightforward to add radiation or matter or some form of quintessence to the model.

### A. Bianchi type I symmetry

Following KS, we specialize to Bianchi type I spacetimes, i.e. to metrics that are homogeneous and spatially flat, with three commuting translation symmetries,

$$\begin{aligned} ds^2 &= N^2(t) dt^2 - e^{2\alpha(t)} \big[ e^{-4\sigma_+(t)} dx^2 \\ &\quad + e^{2\sigma_+(t)} (e^{2\sqrt{3}\sigma_-(t)} dy^2 + e^{-2\sqrt{3}\sigma_-(t)} dz^2) \big]. \end{aligned} \quad (3)$$

We also assume that the aether vector is tilted only in the $x$-direction,

$$u = \frac{1}{N(t)} \cosh\theta(t)\, \partial_t + e^{-\alpha(t)+2\sigma_+(t)} \sinh\theta(t)\, \partial_x. \quad (4)$$

The hyperbolic angle $\theta$ measures the boost of the aether relative to the rest frame of the homogeneous, flat spatial sections, i.e. the "homogeneous frame". The metric is determined by four functions. The lapse $N(t)$ specifies the flow rate of proper time with respect to $t$ in the homogeneous frame. Co-moving lengths $L$ in the $x$, $y$, and $z$ directions all have different expansion rates, $\dot{L}/L$. The sum of these is $3\dot{\alpha}$, which is also the fractional rate of change of co-moving volume, $\dot{V}/V$. The quantity $2\sqrt{3}\dot{\sigma}_-$ is the difference between the expansion rates in the two transverse directions $y$ and $z$, while $3\dot{\sigma}_+$ is the difference between the average of these and the rate in the $x$ direction.

The vector field (4) is in effect just two dimensional hence, like all such vector fields, it is hypersurface orthogonal. According to the analysis of Ref. [8], this means that the solutions to the field equations discussed here are also solutions to the field equations of Horava gravity. Hence our results apply to the cosmology of that theory as well. The hypersurface orthogonality also means[9] that the action is unchanged under $c_1 \to c_1 + \delta$, $c_3 \to c_3 - \delta$, and $c_4 \to c_4 - \delta$, so the system depends on these three coupling parameters only through the two invariant combinations $c_1 + c_4$ and $c_1 + c_3$. For notational compactness we shall make use of these quantities, and also drop the subscript 2 on $c_2$:

$$a = c_1 + c_4, \quad b = c_1 + c_3, \quad c = c_2. \quad (5)$$

(These parameters correspond respectively to the parameters $\alpha, \beta, \lambda'$ of the action for Horava gravity, Eq. (5.72) in Ref. [10].)

When the fields have this symmetry structure, the action takes the form (up to a total derivative)

$$S = \frac{1}{16\pi G} \int dt\, e^{3\alpha} \Big( \frac{1}{2N} H_{ij}(\theta) \dot{q}^i \dot{q}^j - 2N\Lambda \Big), \quad (6)$$

where $q^i \leftrightarrow (\theta, \alpha, \sigma_+, \sigma_-)$. Here and below the time dependence of the dynamical variables is implicit, the dot denotes derivative with respect to $t$, and indices $i, j, \ldots$ label the four dynamical variables. The nonzero components of the symmetric array $H_{ij}$ are given by

$$\begin{aligned} H_{\theta\theta} &= 2\big(b + c + (a - b - c)\cosh^2\theta\big) \\ H_{\theta\alpha} &= 2(a - b - 3c)\cosh\theta\,\sinh\theta \\ H_{\theta+} &= 4(-a + b)\cosh\theta\,\sinh\theta \\ H_{\alpha\alpha} &= 2\big(-6 - a + (a - 3b - 9c)\cosh^2\theta\big) \\ H_{\alpha+} &= -4a\sinh^2\theta \\ H_{++} &= 4\big(3 - 2a + (2a - 3b)\cosh^2\theta\big) \\ H_{--} &= 12(1 - b\cosh^2\theta) \end{aligned} \quad (7)$$

The $\theta$ dependence of $H_{ij}(\theta)$ will also be suppressed. The dynamics is symmetric under the inversion $(\theta, \dot{\theta}) \to (-\theta, -\dot{\theta})$.

Key to the general behavior of solutions is the invertibility of $H_{ij}$, whose determinant can be written in the form

$$\begin{aligned} \det H &= -1728a(1-b)^2(2+b+3c) \\ &\quad \times \big(v_0^2 + (1-v_0^2)\cosh^2\theta\big)\big(v_2^2 + (1-v_2^2)\cosh^2\theta\big) \end{aligned} \quad (8)$$

with

$$v_2^2 = \frac{1}{1-b}, \qquad v_0^2 = \frac{(b+c)(2-a)}{a(1-b)(2+b+3c)}. \quad (9)$$

The constants $v_0$ and $v_2$ are, in fact, the speeds of the spin-0 and spin-2 modes linearized around flat space (defined with respect to the background aether frame), in both Einstein-aether theory[1] and Horava gravity. Note that the $v_0$ factor is proportional to $H_{--}$.

It may initially be surprising that the spin-0 and spin-2 perturbations play any role in a homogeneous cosmology. But while the the metric (3) is homogeneous on constant $t$ surfaces, it has $x$ dependence on surfaces orthogonal to the aether. In fact, the form of (8) can be understood from simple kinematic considerations as follows.

The linearized, diagonalized, action for a mode $\psi$ with speed $v$ is proportional to $g^{ab}_{(v)} \partial_a\psi \partial_b\psi$, where $g^{ab}_{(v)} \propto u^a u^b + v^2(g^{ab} - u^a u^b)$ is the effective metric for that mode. Since we consider only homogeneous fields, which depend on $t$ alone, the only component that enters is $g^{tt}_{(v)} \propto \big(v^2 g^{tt} + (1-v^2) u^t u^t\big) \propto \big(v^2 + (1-v^2)\cosh^2\theta\big)$. The symmetry and hypersurface orthogonality of the aether permit only one spin-2 mode and the spin-0 mode. The spin-2 mode is governed by the shear $\sigma_-$, which describes the gravitational wave mode transverse to the tilt and with polarization aligned with the $y$ and $z$ symmetry axes. This explains the form of the determinant (8).

If the coupling constants $(a, b, c)$ are such that one of the mode speeds exceeds unity, then there is a value of $\theta$ for which the determinant of $H_{ij}$ vanishes, corresponding to the condition $g^{tt}_{(v)} = 0$. When the aether reaches this hyperbolic tilt angle, the propagation cone of that mode becomes tangent to the constant $t$ surface, so that becomes a valid constant phase surface for the mode. In other words, the mode propagates instantaneously on the constant $t$ surface. Beyond this aether tilt the kinetic energy of the mode becomes negative, so the system is unstable. In section IV we shall discuss the implications of this phenomenon for the cosmological dynamics.

### B. Equations of motion for $\Lambda = 0$

Although our main interest is in the case of exponential expansion driven by a positive cosmological constant, we begin by looking first at the simpler case of vanishing $\Lambda$. Then variation with respect to the lapse $N$ yields the initial value constraint

$$H_{ij}\dot{q}^i\dot{q}^j = 0. \tag{10}$$

As usual in general relativity, if this constraint is satisfied at one time, then the rest of the equations of motion imply that it remains satisfied for all time. We can choose the nontrivial lapse

$$N = e^{3\alpha} \tag{11}$$

to eliminate the $\alpha$ dependence in the action. Then the dynamics becomes that of affinely parametrized null geodesics on the configuration space $(\theta, \alpha, \sigma_+, \sigma_-)$ with respect to the metric $H_{ij}$ that depends only on $\theta$.

It is convenient to define momenta by

$$p_i = H_{ij}\dot{q}^j, \tag{12}$$

which can be solved for the velocities,

$$\dot{q}^i = H^{ij}p_j, \tag{13}$$

when the inverse $H^{ij}$ of $H_{ij}$ exists. In terms of the momenta, the constraint (10) reads

$$H^{ij}p_ip_j = 0. \tag{14}$$

Since $H_{ij}$ depends only on $\theta$, the momenta $p_\alpha$ and $p_\pm$ are conserved. Moreover the constraint (14) is a quadratic equation in $p_\theta$ that can be solved for $p_\theta(\theta; p_\alpha, p_\pm)$ (there are generically two roots or none). Having solved three of the four evolution equations, as well as the constraint equation, the fourth evolution equation, for $p_\theta$, is now redundant. The dynamics for $\theta$ is thus reduced to a first order differential equation,

$$\dot{\theta} = H^{\theta k}p_k = F(\theta; p_\alpha, p_\pm). \tag{15}$$

Once the evolution of $\theta$ is known, the remaining variables $\alpha$ and $\sigma_\pm$ are determined by integration of the first order equation (13). The character of the evolution of $\theta$ can be seen by inspection of a plot of the graph of the function $F$ defined in (15).

### C. Equations of motion for $\Lambda \neq 0$

For nonvanishing $\Lambda$, the variation with respect to the lapse $N$ yields the initial value constraint,

$$H_{ij}\dot{q}^i\dot{q}^j + 4\Lambda = 0. \tag{16}$$

Because of the $\Lambda$ term in the action, the lapse (11) is no longer the most convenient, and it is simpler to just use

$$N = 1. \tag{17}$$

The Euler-Lagrange equations with this gauge choice are

$$\frac{d}{dt}\left(e^{3\alpha}H_{ij}\dot{q}^j\right) - \partial_i\Big(e^{3\alpha}\big(\tfrac{1}{2}H_{kl}\dot{q}^k\dot{q}^l - 2\Lambda\big)\Big) = 0. \tag{18}$$

The individual components $i = \theta, \alpha, \pm$ read

$$\frac{d}{dt}\left(e^{3\alpha}H_{\theta j}\dot{q}^j\right) = \tfrac{1}{2}e^{3\alpha}H_{ij,\theta}\dot{q}^i\dot{q}^j \tag{19}$$

$$\frac{d}{dt}\left(e^{3\alpha}H_{\alpha j}\dot{q}^j\right) = -12\Lambda e^{3\alpha} \tag{20}$$

$$\frac{d}{dt}\left(e^{3\alpha}H_{\pm j}\dot{q}^j\right) = 0, \tag{21}$$

where in (20) the constraint (16) was used.

Again, it is sometimes convenient to express the field equations in terms of the "momenta" (12). (These are not precisely the conjugate momenta anymore since the factor $e^{3\alpha}$ is not included, but we will nevertheless refer to them as momenta.) Then the constraint (16) takes the form

$$H^{ij}p_ip_j + 4\Lambda = 0, \tag{22}$$

and the equations of motion become

$$\dot{p}_\theta = -3\dot{\alpha}\,p_\theta - \tfrac{1}{2}H^{ij}{}_{,\theta}\,p_ip_j \tag{23}$$

$$\dot{p}_\alpha = -3\dot{\alpha}\,p_\alpha - 12\Lambda \tag{24}$$

$$\dot{p}_\pm = -3\dot{\alpha}\,p_\pm \tag{25}$$

Using (13) we can express $\dot{\alpha}$ in terms of $\theta$ and the momenta,

$$\dot{\alpha} = H^{\alpha k}p_k. \tag{26}$$

Moreover, we can solve the quadratic constraint equation (22) for $p_\alpha$, so that $\alpha$ and $p_\alpha$ can be eliminated completely from the dynamical system.

## III. LIMITING CASES

In this section we discuss various special cases and limits of the theory.

### A. General relativity

If we reduce to the pure GR case, $\theta$ is not present, and $a = b = c = 0$. Then $H_{ij}$ is diagonal and constant, with $H_{++} = H_{--} = -H_{\alpha\alpha} = 12$. The constraint (22) then becomes

$$-p_\alpha^2 + p_+^2 + p_-^2 = -48\Lambda. \tag{27}$$

The only isotropic ($p_\pm = 0$) solutions are Minkowski spacetime (with $\Lambda = 0$) and de Sitter spacetime (with $\dot\alpha = \sqrt{\Lambda/3}$, $\Lambda > 0$). In the anisotropic case, $\Lambda = 0$ yields the Kasner solutions, and $\Lambda \neq 0$ yields a generalization of those.

### B. $\theta = 0$ solutions

Next we characterize the solutions in which $\theta = 0$ for all times, i.e. in which the aether remains orthogonal to the constant $t$ homogeneity surfaces. Then, although the aether has no dynamics, its couplings in the action (1) still contribute to the field equations and we therefore have something different from GR.

There are no terms linear in $\theta$ or $\dot\theta$ alone in the action (6), and terms of quadratic or higher order in these quantities will obviously not contribute to the equations of motion if $\theta = \dot\theta = 0$. If this condition holds initially it is therefore preserved for all time, and for characterizing these solutions it is consistent to simply set $\theta = 0$ in the action. Then $H_{ij}$ is diagonal and constant, the relevant components being

$$\begin{aligned} H_{\alpha\alpha} &= -6(2 + b + 3c) \\ H_{++} &= 12(1 - b) \\ H_{--} &= 12(1 - b). \end{aligned} \tag{28}$$

In the isotropic case $\sigma_\pm = 0$, the system is then equivalent to GR with a rescaled cosmological value of Newton's constant [11, 12], $G_{\rm cosmo} = G/(1 + (b + 3c)/2)$, and with $\Lambda$ replaced by $\Lambda' = \Lambda/(1 + (b + 3c)/2)$. This isotropic solution is the spatially flat slicing of de Sitter spacetime with Hubble constant $H = \sqrt{\Lambda'/3}$. The aether becomes singular because of infinite stretching on the past horizon. In the anisotropic case, interestingly, there is no equally simple relation to GR: *the presence of the isotropic aether induces different rescalings of the kinetic energy associated with expansion and shear.*

### C. Linearized anisotropy

If one drops all terms in the action (6) of higher than quadratic order in the anisotropic coordinates $\theta$ and $\sigma_\pm$, the $H_{ij}$ array reduces to the nonzero elements

$$\begin{aligned} H_{\theta\theta} &= 2a \\ H_{\theta\alpha} &= 2(a - b - 3c)\theta \\ H_{\alpha\alpha} &= -6(2 + b + 3c) + 2(a - 3b - 9c)\theta^2 \\ H_{++} &= 12(1 - b) \\ H_{--} &= 12(1 - b). \end{aligned} \tag{29}$$

Keeping only linear order terms in the anisotropy, the equation of motion for $\theta$ reduces to

$$\ddot\theta + 3\dot\alpha\,\dot\theta + 2\dot\alpha^2\,\theta = 0. \tag{30}$$

To zeroth order in the anisotropy the solution to the constraint (16) is,

$$\dot\alpha^2 = \frac{\Lambda}{3[1 + (b + 3c)/2]} \tag{31}$$

and $\dot\alpha$ in (30) can be replaced by this value. Then (30) is the equation found by KS [7]. They pointed out that the coefficient of $\theta$ is positive provided the effective gravitational coupling is positive, in which case this is the equation of a damped harmonic oscillator. In fact, the oscillator is overdamped, with eigenmode decay rates $\dot\alpha$ and $2\dot\alpha$. This implies that $\theta$ relaxes to zero as the universe expands.

### D. $p_\pm = 0$ solutions

The equation of motion (25) is solved for $p_\pm$ by

$$p_\pm(t) = e^{-3\alpha(t)}\,p_{\pm,0}, \tag{32}$$

where $p_{\pm,0}$ is an integration constant. Using this, the remaining equations of motion (23,24) involve only the variables $(\theta, p_\theta, \alpha, \dot\alpha, p_\alpha)$. Moreover, as mentioned at the end of section II C, Eq. (26) can be used to eliminate $\dot\alpha$, and the constraint can be solved for $p_\alpha$. However this requires that the function $\alpha(t)$ be determined by the previous values of $(\theta, p_\theta, \alpha)$ via $\int^t dt'\,\dot\alpha(t')$, which does not yield an evolution equation that is local in time.

If the co-moving volume is expanding then, according to (32), $p_\pm(t)$ is driven to zero. It is therefore a useful limiting case to set $p_\pm = 0$ from the beginning. Then there is no remaining $\alpha(t)$ dependence, and the system can be reduced to the $\theta$ degree of freedom alone. We now explain in detail how this is achieved.

We assume now that $p_\pm = 0$. Since $H^{-k} = 0$ except for $k = -$, it follows from (13) that $\dot\sigma^- = 0$. Thus the two transverse dimensions must have the same expansion rates. In contrast, it does *not* follow that $\dot\sigma^+ = 0$, since we have in this case

$$\dot\sigma^+ = H^{+\theta}p_\theta + H^{+\alpha}p_\alpha. \tag{33}$$

As might be expected, it is inconsistent to for the metric to be isotropic ($\sigma^\pm = 0$) when the aether is tilted

($\theta \neq 0$). (However, note that $\dot\sigma^+$ is of second order in the $\theta$ anisotropy.) The expansion rate in the tilt direction must generally differ from that in the transverse direction. Eliminating $\dot\sigma^+$ via (33) the system reduces to the variables ($\theta$, $p_\theta$, $\alpha$, $p_\alpha$).

One can write this system in terms of the velocities, i.e. in terms of the variables $(\theta, \dot\theta, \alpha, \dot\alpha)$, by using the constraint (16) to solve for $\dot\sigma^+$ in terms of $(\theta, \dot\theta, \dot\alpha)$, and substituting that into the Euler-Lagrange equation (18). But for the purpose of making a $(\theta, \dot\theta)$ phase portrait of the evolution, it appears more neat to organize the equations as follows.

The idea is to solve for the momenta $(p_\theta, p_\alpha)$ in terms of the velocities $(\dot\theta, \dot\alpha)$, and then to use the equations that were expressed in terms of momenta. To this end, we introduce capital indices $A, B, \ldots$ to refer to the two coordinates $\theta$ and $\alpha$, we define the contravariant tensor $h^{AB}$ to be the restriction of $H^{ij}$,

$$h^{AB} \equiv H^{AB}, \tag{34}$$

and we denote by $h_{AB}$ the inverse of $h^{AB}$. Then from (13) we have

$$\dot q^A = h^{AB} p_B, \tag{35}$$

which can be inverted to yield

$$p_A = h_{AB} \dot q^B. \tag{36}$$

Using (36) the constraint (22) becomes

$$h_{AB} \dot q^A \dot q^B + 4\Lambda = 0, \tag{37}$$

which can be solved as a quadratic equation for $\dot\alpha(\theta, \dot\theta)$, thus eliminating $\dot\alpha$. Explicitly, we have

$$\dot\alpha = \frac{-h_{\theta\alpha}\dot\theta \pm \sqrt{(h_{\theta\alpha}\dot\theta)^2 - h_{\alpha\alpha}(h_{\theta\theta}\dot\theta^2 + 4\Lambda)}}{h_{\alpha\alpha}}. \tag{38}$$

There are generically two solutions or no solutions. If $h_{\alpha\alpha}(h_{\theta\theta}\dot\theta^2 + 4\Lambda) < 0$ there are two solutions, one in which the volume is expanding ($\dot\alpha > 0$) and the other in which it is contracting ($\dot\alpha < 0$). Note that for $\dot\theta = 0$ the solutions are simply $\dot\alpha = \pm\sqrt{-4\Lambda/h_{\alpha\alpha}}$. As $h_{\alpha\alpha} \to 0^-$ this diverges, and no solution exists when $h_{\alpha\alpha} > 0$.

The remaining task is to find an equation for $\ddot\theta$. For this purpose we can use (13) to write $\dot\theta = h^{\theta B} p_B$, hence

$$\ddot\theta = h^{\theta B}{}_{,\theta}\, \dot\theta\, p_B + h^{\theta B} \dot p_B. \tag{39}$$

Then using (36) and the equations of motion (23,24) we find, after some manipulation,

$$\ddot\theta \;=\; -3\dot\alpha\dot\theta - 12\Lambda h^{\theta\alpha} - h^{\theta A} h_{AB,\theta}\, \dot q^B \dot\theta \tag{40}$$

$$+\tfrac{1}{2} h^{\theta\theta} h_{AB,\theta}\, \dot q^A \dot q^B. \tag{41}$$

Together with (38) this yields a dynamics reduced to just the $\theta$ degree of freedom, which can be visualized in a phase portrait.

## IV. GENERIC BEHAVIOR

We have seen that in a small enough neighborhood of $\theta = 0$, the dynamics relaxes to the $\theta = 0$ case, provided the values of $a, b, c$ are such that the effective gravitational coupling constant is positive. On the other hand, once $\theta$ is sufficiently large, the character of the dynamical system can obviously change dramatically because of the growth of the hyperbolic trigonometric functions in the components of $H_{ij}$ (7).

A general feature mentioned earlier is that $\det H$ (8) vanishes if either the spin-0 or spin-2 propagation cone is tangent to the constant $t$ surface. The conditions determining these angles can be expressed as

$$\coth\theta_0 = v_0, \quad \coth\theta_2 = v_2 \tag{42}$$

where $v_{0,2}$ are the mode speeds (9). At either of these angles $H_{ij}$ is not invertible, so the equation of motion (18) cannot be solved for $\ddot q^i$. As such a value of $\theta$ is approached, at least one second derivative component would generally diverge. Hence generically there can be no smooth evolution across the degenerate values of $\theta$. The dynamics may run into a singularity there, or it may "bounce" before reaching such a value of $\theta$.

There is a solution $\theta_*$ to each of the equations in (42) as long as the corresponding mode speed $v$ is greater than unity. The *larger* of the mode speeds defines the *smaller* of the critical angles. The critical angles are of order unity unless $v$ is either very large or very close to 1. In these limits we have

$$\theta_* \approx \begin{cases} 1/v & \text{for} \quad v \gg 1 \\ -\frac{1}{2}\ln(v-1) & \text{for} \quad v - 1 \ll 1 \end{cases} \tag{43}$$

In particular, the degenerate value $\theta_2$ is real only if $0 < b \leq 1$, and is of order unity unless $b$ is very close to either 0 or 1. For instance, for $b = 0.01$, 0.9, or 0.99, we have $\theta_2 \simeq 3$, 0.3, or 0.1 respectively. The degenerate value $\theta_0$ is of order unity for generic values of $a, b, c$ with no large hierarchy amongst them. (If $a, b, c$ are all much smaller than 1, then $v_0 \approx (b+c)/a$, so $\theta_0 \approx \coth^{-1}\big((b+c)/a\big)$.) We infer that exotic behavior, including singularities or runaway solutions, may typically occur for aether boost angles of order unity.

### A. Restriction to physically viable couplings

There are three independent coupling constants that affect the solutions we are studying in either Einstein-aether theory or Horava gravity, but stability and observational constraints restrict the range of physically viable values.

#### 1. *Einstein-aether couplings*

As summarized in Ref. [1], $c_2$ and $c_4$ should be determined by $c_1$ and $c_3$ such that the preferred frame parametrized post-Newtonian (PPN) parameters $\alpha_{1,2}$ vanish (or are very small compared to unity). Moreover, when $\alpha_{1,2}$ vanish, stability and positive energy of linearized modes and the absence of vacuum Cherenkov radiation by ultra high energy cosmic rays require $0 < c_+ < 1$ and $0 < c_- < c_+/(3(1-c_+))$, where

$$c_\pm = c_1 \pm c_3. \qquad (44)$$

(In terms of $a, b, c$ these conditions correspond to $0 < b < 1$, $0 < a < 2b/(4-3b)$, and $c = -(a+b)/3$, or, equivalently, $0 < b < 1$, $b(b-2)/(4-3b) < c < -b/3$ and $a = -b-3c$.) We shall label the examples by their $c_\pm$ values. In particular, the vacuum Cherenkov constraint requires that all the mode speeds be greater than or equal to unity, so except in the case where they are exactly unity, there are values of the tilt angle where the dynamics is degenerate. The degeneracy at $\theta_2$ is relevant to the dynamics only if $\dot\sigma_- \neq 0$.

The remaining observational constraint one might apply is that the radiation damping rate for a binary pulsar system agrees with the rate in GR, which agrees with observations within the present relative uncertainty of about 0.002. The results of Ref. [13] establish that this constraint is satisfied for generic small values $c_i \lesssim 0.001$, and if $c_i \lesssim 0.01 - 0.1$ it is satisfied if $c_- \approx 0.18\, c_+$. For these values the spin-0 mode speed is $v_0 \approx 1.36$ and the critical boost angle is $\theta_0 \approx 0.94$.

#### 2. *Horava gravity couplings*

The constraints on the couplings in Horava gravity are the same as in Einstein-aether theory except for the PPN constraints $\alpha_{1,2} = 0$, which now are equivalent to $a = 2b$ [10]. The other constraints are $0 < b < 1$ and $(b+c)/(b(2+b+3c)) > 1$. Given the first of these, the second is satisfied in two regions: I. $c > (b+b^2)/(1-3b)$, $b < 1/3$, and II. $c < -2/3 - b/3$, $c > (b+b^2)/(1-3b)$ when $b > 1/3$. The radiation damping rate has not yet been calculated in the Horava case.

### B. Phase portraits

As discussed in section III D, in the case when $p_\pm = 0$ one can reduce the dynamics to the $\theta$ degree of freedom. Then the dynamics can be displayed as a phase portrait in the $(\theta, \dot\theta)$ plane, displaying the flow of the vector field $(\dot\theta, \ddot\theta)$. This serves to illustrate the general features of the dynamics discussed above.

Two examples are shown in Figs. 1 and 2. In both of these the parameters $c_2$ and $c_4$ are chosen so as to satisfy the PPN constraints of Einstein-aether theory, and $c_\pm$ satisfy the remaining constraints other than that of gravitational radiation damping. Fig. 1 is qualitatively similar to the phase portrait for the case $c_+ = 1/10$, $c_- = 0.18 c_+$, which is at least close to satisfying the radiation damping constraint. It is also similar to the case

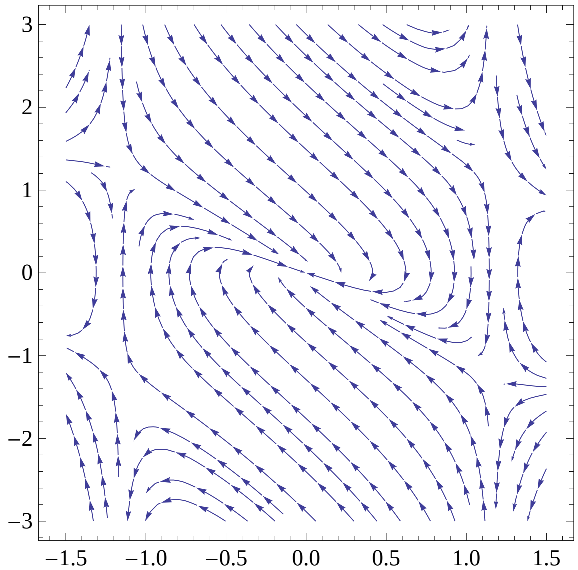


Figure 1: Stream plot of the vector field $(\dot\theta, \ddot\theta)$ on the $(\theta, \dot\theta)$ plane, with $c_+ = 1/10$ and $c_- = 1/40$, and $p_\pm = 0$. In this case the determinant of $H_{ij}$ vanishes at $\theta_0 \simeq 0.86$.

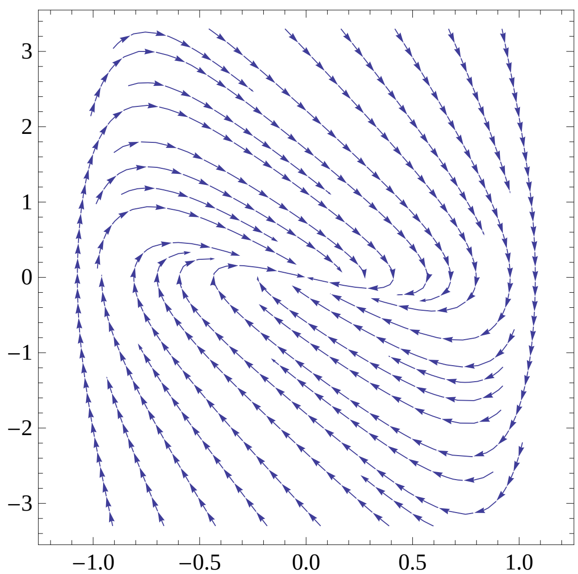


Figure 2: Stream plot with $c_+ = 1/2$ and $c_- = 1/4$, and $p_\pm = 0$. In this case $\theta_0 \simeq 1.3$, but already for $\theta \gtrsim 1.1$ and $\dot\theta = 0$ there is no solution to the constraint equation for $\dot\alpha$.

$(a, b, c) = (2/10, 1/10, 3/10)$ which satisfies the Horava constraints other than the (unknown) radiation damping one.

In the case illustrated by Fig. 1, runaway behavior occurs if $\theta$ or $\dot\theta$ is sufficiently large. Numerical evolution of this example suggests that some of the flow lines end at curvature singularities. The simplest spacetime scalar in this setting is the expansion of the aether, $\nabla_a u^a = 3\dot\alpha\cosh\theta + \dot\theta\sinh\theta$. For example, with initial data $\theta = 0$ and $\dot\theta = 4$, the evolution runs away to large

$\dot{\theta}$. For another example, with initial data $\theta = 1.5$ and $\dot{\theta} \simeq 1.8$, the evolution runs away to large $\dot{\alpha}$.

The singular behavior seen in these solutions might be related to what is seen in some homogeneous anisotropic cosmologies with tilted perfect fluid matter (and vanishing cosmological constant)[14]. It may appear inconsistent with the cosmological "no-hair" theorem proved by Wald[15], which showed that in the presence of a positive cosmological constant $\Lambda$, all expanding Bianchi type cosmologies (except Type-IX) evolve toward the de Sitter solution with time scale $\sqrt{3/\Lambda}$. But that result assumed that the dominant and strong energy conditions hold for the matter stress tensor. These conditions do not generally hold for the stress-tensor associated with the aether part of the action (1).

## V. CONCLUSION

The question driving this investigation was whether it is natural for the aether to be aligned with the isotropic frame of a homogeneous, isotropic cosmology in Einstein-aether theory or Horava gravity? We addressed this question by studying the dynamics of a tilted aether in a homogeneous anisotropic Bianchi type I cosmology with a cosmological constant. We found that generically the aether does align provided its tilt angle and the time derivative of its tilt angle in units of the cosmological constant are smaller than something of order unity. This extends the linearized stability result of KS [7] to a finite "basin of attraction" whose precise shape depends on the coupling parameters of the theory, and in some cases the basin appears to be much broader than order unity. Outside of this basin, the solutions exhibit runaway or singular behavior of one or more of the variables. Some of this behavior occurs when the propagation cone of either the spin-0 or spin-2 mode is tilted enough to meet the homogeneous constant $t$ surface. We do not know whether similar behavior would persist if the homogeneus symmetry condition were dropped.

Our findings show that the fate of a universe with Bianchi I symmetry depends heavily on the initial tilt of the aether. Perhaps the question of the initial tilt could be addressed from the standpoint of quantum cosmology, for example via the "wave function of the universe" or via the distribution of initial conditions for chaotic inflation.

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