

# Tunneling in Fractional Quantum Mechanics

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## Abstract

We study the tunneling through delta and double delta potentials in fractional quantum mechanics. After solving the fractional Schrödinger equation for these potentials, we calculate the corresponding reflection and transmission coefficients. These coefficients have a very interesting behaviour. In particular, we can have zero energy tunneling when the order of the Riesz fractional derivative is different from 2. For both potentials, the zero energy limit of the transmission coefficient is given by  $\mathcal{T}_0 = \cos^2 \pi/\alpha$ , where  $\alpha$  is the order of the derivative ( $1 < \alpha \leq 2$ ).

## 1. Introduction

In recent years the study of fractional integrodifferential equations applied to physics and other areas has grown. Some examples are [1, 2, 3], among many others. More recently, the fractional generalized Langevin equation is proposed to discuss the anomalous diffusive behavior of a harmonic oscillator driven by a two-parameter Mittag-Leffler noise [4].

Fractional Quantum Mechanics (FQM) is the theory of quantum mechanics based on the fractional Schrödinger equation (FSE). In this paper we consider the FSE as introduced by Laskin in [5, 6]. It was obtained in the context of the path integral approach to quantum mechanics. In this approach, path integrals are defined over Lévy flight paths, which is a natural generalization of the Brownian motion [7].

There are some papers in the literature studying solutions of FSE. Some examples are [8, 9, 10]. However, recently Jeng et al. [11] have shown that some claims to solve the FSE have not taken into account the fact that the fractional derivation is a *nonlocal* operation. As a consequence, all those attempts based on local approaches are intrinsically wrong. Jeng et al. pointed out that the only correct one they found is the one [12] involving the delta potential. However, in [12] the FSE with delta potential was

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studied only in the case of *negative* energies. This has been generalized in [13], where we have solved the FSE for the delta and double delta potentials for positive and negative energies.

The objective of this paper is to study the tunneling through delta and double delta potentials in the context of the FSE. As a result, we found some very interesting properties that are not observed in the usual  $\alpha = 2$  quantum mechanics. Probably the most interesting is the presence of tunneling through delta and double delta potentials even at zero energy. Moreover, in the case of the double delta potential, this zero energy tunneling is independent of the relation of the two delta functions.

We organized this paper as follows. Firstly, and for the sake of completeness, we reproduce the solution of the FSE for the delta and double delta potentials, as given in [13], presenting their respective solutions in terms of Fox's  $H$ -function. Some calculations and properties of the Fox's  $H$ -function are given in the Appendixes. Then we study the asymptotic behaviour of those solutions, calculate the reflection and transmission coefficients, and study some of their properties.

## 2. The Fractional Schrödinger Equation

The one-dimensional FSE is

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \psi(x, t) + V(x) \psi(x, t), \quad (1)$$

where  $1 < \alpha \leq 2$ ,  $D_\alpha$  is a constant,  $\Delta = \partial_x^2$  is the Laplacian, and  $(-\hbar^2 \Delta)^{\alpha/2}$  is the Riesz fractional derivative [14], that is,

$$(-\hbar^2 \Delta)^{\alpha/2} \psi(x, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{ipx/\hbar} |p|^\alpha \phi(p, t) dp, \quad (2)$$

where  $\phi(p, t)$  is the Fourier transform of the wave function,

$$\phi(p, t) = \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \psi(x, t) dx, \quad \psi(x, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{ipx/\hbar} \phi(p, t) dp. \quad (3)$$

The time-independent FSE is

$$D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \psi(x) + V(x) \psi(x) = E \psi(x). \quad (4)$$

In the momentum representation, this equation is written as

$$D_\alpha |p|^\alpha \phi(p) + \frac{(W * \phi)(p)}{2\pi\hbar} = E \phi(p), \quad (5)$$

where  $(W * \phi)(p)$  is the convolution

$$(W * \phi)(p) = \int_{-\infty}^{+\infty} W(p - q) \phi(q) dq, \quad (6)$$

and  $W(p) = \mathcal{F}[V(x)]$  is the Fourier transform of the potential  $V(x)$ .

Solutions of the FSE for delta and double delta potentials are given in [13] in the situations of bound and scattering states. Since we need to study the asymptotic behaviour of these solutions in order to find the transmission coefficients, and for the sake of completeness, we will reproduce the calculations of the wave functions in the case of scattering states.

## 2.1. FSE for Delta Potential

Let us consider the case

$$V(x) = V_0\delta(x), \quad (7)$$

where  $\delta(x)$  is the Dirac delta function and  $V_0$  is a constant. Its Fourier transform is  $W(p) = V_0$  and the convolution  $(W * \phi)(p)$  is

$$(W * \phi)(p) = V_0K, \quad (8)$$

where the constant  $K$  is

$$K = \int_{-\infty}^{+\infty} \phi(q) dq. \quad (9)$$

The FSE in the momentum representation (5) is

$$\left( |p|^\alpha - \frac{E}{D_\alpha} \right) \phi(p) = -\gamma K, \quad (10)$$

where

$$\gamma = \frac{V_0}{2\pi\hbar D_\alpha}. \quad (11)$$

Since we are interested in scattering states, we will consider that  $E > 0$  and write

$$\frac{E}{D_\alpha} = \lambda^\alpha, \quad (12)$$

where  $\lambda > 0$ . Since  $f(x)\delta(x) = f(0)\delta(x)$ , the solution of Eq.(10) in this case is

$$\phi(p) = \frac{-\gamma K}{|p|^\alpha - \lambda^\alpha} + 2\pi\hbar C_1 \delta(p - \lambda) + 2\pi\hbar C_2 \delta(p + \lambda), \quad (13)$$

where  $C_1$  and  $C_2$  are arbitrary constants and the constant  $2\pi\hbar$  was introduced for later convenience. Using this in Eq.(9) gives that

$$K = -\gamma K \int_{-\infty}^{+\infty} \frac{dp}{|p|^\alpha - \lambda^\alpha} + 2\pi\hbar C_1 + 2\pi\hbar C_2, \quad (14)$$

where the integral is interpreted in the sense of Cauchy principal value, and it gives

$$\int_{-\infty}^{+\infty} \frac{dp}{|p|^\alpha - \lambda^\alpha} = 2\lambda^{1-\alpha} \int_0^{+\infty} \frac{dq}{q^\alpha - 1} = -2\lambda^{1-\alpha} \frac{\pi}{\alpha} \cot \frac{\pi}{\alpha}, \quad (15)$$

where we have used formula 3.241.3 (pg. 322) of [15] - see Eq.(88). The constant  $K$  is therefore

$$K = \frac{2\pi\hbar(C_1 + C_2)\alpha\lambda^{\alpha-1}}{\alpha\lambda^{\alpha-1} - 2\pi\gamma \cot \pi/\alpha}, \quad (16)$$

and we have

$$\phi(p) = 2\pi\hbar C_1 \delta(p - \lambda) + 2\pi\hbar C_2 \delta(p + \lambda) - \frac{2\pi\hbar\gamma(C_1 + C_2)\alpha\lambda^{\alpha-1}}{(\alpha\lambda^{\alpha-1} - 2\pi\gamma \cot \pi/\alpha)} \frac{1}{(|p|^\alpha - \lambda^\alpha)}. \quad (17)$$

Next we need to calculate the inverse Fourier transform of  $\phi(p)$  to obtain  $\psi(x)$ , that is,

$$\psi(x) = C_1 e^{i\lambda x/\hbar} + C_2 e^{-i\lambda x/\hbar} - \frac{\gamma(C_1 + C_2)\alpha\lambda^{\alpha-1}}{(\alpha\lambda^{\alpha-1} - 2\pi\gamma \cot \pi/\alpha)} \int_{-\infty}^{+\infty} \frac{e^{ipx/\hbar}}{|p|^\alpha - \lambda^\alpha} dp. \quad (18)$$

The above integral is calculated in the Appendix B, and is given by Eq.(91). Using this result, and the definition of  $\gamma$  in Eq.(11) and  $\lambda$  in Eq.(12), we can write that

$$\psi(x) = C_1 e^{i\lambda x/\hbar} + C_2 e^{-i\lambda x/\hbar} + \Omega_\alpha \frac{(C_1 + C_2)}{2} \Phi_\alpha \left( \frac{\lambda|x|}{\hbar} \right), \quad (19)$$

where

$$\begin{aligned} \Phi_\alpha \left( \frac{\lambda|x|}{\hbar} \right) = & \frac{\alpha\hbar}{\lambda|x|} \left( H_{2,3}^{2,1} \left[ \left( \frac{\lambda|x|}{\hbar} \right)^\alpha \middle| \begin{array}{l} (1, 1), (1, (2 + \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 + \alpha)/2) \end{array} \right] \right. \\ & \left. - H_{2,3}^{2,1} \left[ \left( \frac{\lambda|x|}{\hbar} \right)^\alpha \middle| \begin{array}{l} (1, 1), (1, (2 - \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 - \alpha)/2) \end{array} \right] \right), \end{aligned} \quad (20)$$

and

$$\Omega_\alpha = \left[ \left( \frac{E}{U} \right)^{\frac{\alpha-1}{\alpha}} - \cot \frac{\pi}{\alpha} \right]^{-1}, \quad (21)$$

and

$$U = \left( \frac{V_0}{\alpha\hbar D_\alpha^{1/\alpha}} \right)^{\alpha/(\alpha-1)}. \quad (22)$$

## 2.2. FSE for Double Delta Potential

Now let the potential be given by

$$V(x) = V_0[\delta(x + R/2) + \mu\delta(x - R/2)], \quad (23)$$

with  $\mu$ ,  $V_0$  and  $R$  real constants. When  $V_0 < 0$  this potential can be seen as a model for the one-dimensional limit of the molecular ion  $H_2^+$  [16]. The parameter  $R$  is interpreted as the internuclear distance and the coupling parameters are  $V_0$  and  $\mu V_0$ . Its Fourier transform is

$$W(p) = V_0 e^{ipR/2\hbar} + V_0 \mu e^{-ipR/2\hbar} \quad (24)$$

and for the convolution

$$(W * \phi)(p) = V_0 e^{ipR/2\hbar} K_1(R) + V_0 \mu e^{-ipR/2\hbar} K_2(R), \quad (25)$$

where  $K_1(R)$  and  $K_2(R)$  are constants given by

$$K_1(R) = K_2(-R) = \int_{-\infty}^{+\infty} e^{-iRq/2\hbar} \phi(q) dq. \quad (26)$$

The FSE in momentum space is

$$\left( |p|^\alpha - \frac{E}{D_\alpha} \right) \phi(p) = -\gamma e^{iRp/2\hbar} K_1(R) - \gamma \mu e^{-iRp/2\hbar} K_2(R), \quad (27)$$

where we used the notation introduced in Eq.(11).

Since we are interested in scattering states, we have  $E > 0$  and we write  $\lambda$  as in Eq.(12) and for the solution of Eq.(27) we have

$$\phi(p) = 2\pi\hbar C_1 \delta(p - \lambda) + 2\pi\hbar C_2 \delta(p + \lambda) - \frac{\gamma e^{iRp/2\hbar} K_1(R)}{|p|^\alpha - \lambda^\alpha} - \frac{\mu\gamma e^{-iRp/2\hbar} K_2(R)}{|p|^\alpha - \lambda^\alpha}. \quad (28)$$

Using this expression for  $\phi(p)$  in Eq.(26) of definition of  $K_1(R)$  and  $K_2(R)$  we have

$$(1 + 2\pi\gamma\lambda^{1-\alpha} J_\alpha(0)) K_1(R) + \mu 2\pi\gamma\lambda^{1-\alpha} J_\alpha(\lambda R/\hbar) K_2(R) = 2\pi\hbar C'_1, \quad (29)$$

$$2\pi\gamma\lambda^{1-\alpha} J_\alpha(\lambda R/\hbar) K_1(R) + (1 + \mu 2\pi\gamma\lambda^{1-\alpha} J_\alpha(0)) K_2(R) = 2\pi\hbar C'_2, \quad (30)$$

where

$$C'_1 = C_1 e^{-iR\lambda/2\hbar} + C_2 e^{iR\lambda/2\hbar}, \quad C'_2 = C_1 e^{iR\lambda/2\hbar} + C_2 e^{-iR\lambda/2\hbar}, \quad (31)$$

and  $J_\alpha(w)$  is the Cauchy principal value of the integral

$$J_\alpha(w) = \frac{1}{\pi} \int_0^{+\infty} \frac{\cos wq}{q^\alpha - 1} dq. \quad (32)$$

In order to write the solution of these equations it is convenient to define

$$\varepsilon = \frac{\lambda^{\alpha-1}}{2\pi\gamma} = \frac{1}{\alpha} \left( \frac{E}{U} \right)^{\frac{\alpha-1}{\alpha}}, \quad (33)$$

where  $U$  was defined in Eq.(22), in such a way that we have

$$K_1(R) = \frac{2\pi\hbar\varepsilon}{W} [(\varepsilon\mu^{-1} + J_\alpha(0))C'_1 - J_\alpha(\lambda R/\hbar)C'_2], \quad (34)$$

$$K_2(R) = \frac{2\pi\hbar\varepsilon}{\mu W} [(\varepsilon + J_\alpha(0))C'_2 - J_\alpha(\lambda R/\hbar)C'_1], \quad (35)$$

where

$$W = (\varepsilon + J_\alpha(0))(\varepsilon\mu^{-1} + J_\alpha(0)) - (J_\alpha(\lambda R/\hbar))^2. \quad (36)$$

Using  $K_1(R)$  and  $K_2(R)$  in Eq.(28) gives  $\phi(p)$ . Then, for  $\psi(x)$ , we have

$$\begin{aligned} \psi(x) &= C_1 e^{i\lambda x/\hbar} + C_2 e^{-i\lambda x/\hbar} \\ &+ \frac{1}{2\alpha W} [(\varepsilon\mu^{-1} + J_\alpha(0))C'_1 - J_\alpha(\lambda R/\hbar)C'_2] \Phi_\alpha \left( \frac{\lambda|x + R/2|}{\hbar} \right) \\ &+ \frac{1}{2\alpha W} [(\varepsilon + J_\alpha(0))C'_2 - J_\alpha(\lambda R/\hbar)C'_1] \Phi_\alpha \left( \frac{\lambda|x - R/2|}{\hbar} \right), \end{aligned} \quad (37)$$

where we have expressed the result in terms of the function  $\Phi_\alpha$  defined in Eq.(20).

### 3. Calculation of the Transmission Coefficients

In order to calculate the transmission coefficients, we need to know the asymptotic behaviour of the solutions. The asymptotic behaviour of Fox's  $H$ -function is given, if  $\Delta > 0$ , by Eq.(80) or Eq.(82) according to  $\Delta^* > 0$  or  $\Delta^* = 0$ , respectively – see Eq.(79). In  $\Phi_\alpha(\lambda|x|/\hbar)$  we have the difference between two Fox's  $H$ -functions of the form

$$H_{2,3}^{2,1} \left[ w^\alpha \left| \begin{array}{l} (1, 1), (1, \mu) \\ (1, \alpha), (1, 1), (1, \mu) \end{array} \right. \right],$$

for  $\mu = (2 + \alpha)/2$  and  $\mu = (2 - \alpha)/2$ . In both cases we have  $\Delta = \alpha > 0$ , but  $\Delta^* = 0$  for  $\mu = (2 + \alpha)/2$  and  $\Delta^* > 0$  for  $\mu = (2 - \alpha)/2$ . Therefore, using Eq.(80) when  $\mu = (2 - \alpha)/2$  and Eq.(82) when  $\mu = (2 + \alpha)/2$  we have, respectively, that

$$H_{2,3}^{2,1} \left[ w^\alpha \left| \begin{array}{l} (1, 1), (1, (2 + \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 + \alpha)/2) \end{array} \right. \right] = \frac{2w}{\alpha} \sin w + o(1), \quad |w| \rightarrow \infty, \quad (38)$$

$$H_{2,3}^{2,1} \left[ z^\alpha \left| \begin{array}{l} (1, 1), (1, (2 - \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 - \alpha)/2) \end{array} \right. \right] = o(1), \quad |w| \rightarrow \infty, \quad (39)$$

and then

$$\Phi_\alpha \left( \frac{\lambda|x|}{\hbar} \right) = 2 \sin \frac{\lambda|x|}{\hbar} + o(|x|^{-1}), \quad |x| \rightarrow \infty. \quad (40)$$

#### 3.1. Transmission Coefficient for the Delta Potential

The behaviour of the wave function  $\psi(x)$  given by Eq.(19) for  $x \rightarrow \pm\infty$  is therefore

$$\psi(x) = C_1 e^{i\lambda x/\hbar} + C_2 e^{-i\lambda x/\hbar} \pm \Omega_\alpha (C_1 + C_2) \sin \frac{\lambda x}{\hbar} + o(x^{-1}), \quad x \rightarrow \pm\infty. \quad (41)$$

or

$$\psi(x) = A e^{i\lambda x/\hbar} + B e^{-i\lambda x/\hbar} + o(x^{-1}), \quad x \rightarrow -\infty, \quad (42)$$

$$\psi(x) = C e^{i\lambda x/\hbar} + D e^{-i\lambda x/\hbar} + o(x^{-1}), \quad x \rightarrow +\infty, \quad (43)$$

where we defined

$$A = C_1 + i(C_1 + C_2)\Omega_\alpha/2, \quad B = C_2 - i(C_1 + C_2)\Omega_\alpha/2, \quad (44)$$

$$C = C_1 - i(C_1 + C_2)\Omega_\alpha/2, \quad D = C_2 + i(C_1 + C_2)\Omega_\alpha/2. \quad (45)$$

Now let us consider the situation of particles coming from the left and scattered by the delta potential. In this case  $D = 0$  (no particles coming from the right) and  $B = rA$  and  $C = tA$ , where the reflexion  $\mathcal{R}$  and transmission  $\mathcal{T}$  coefficients are given by  $\mathcal{R} = |r|^2$  and  $\mathcal{T} = |t|^2$  (see, for example, [17]). The result is

$$r = \frac{-i\Omega_\alpha}{1 + i\Omega_\alpha}, \quad t = \frac{1}{1 + i\Omega_\alpha}, \quad (46)$$

and then

$$\mathcal{R} = \frac{\Omega_\alpha^2}{1 + \Omega_\alpha^2}, \quad \mathcal{T} = \frac{1}{1 + \Omega_\alpha^2}. \quad (47)$$

In Fig.(1) we show the behaviour of these coefficients for different values of  $\alpha$ .

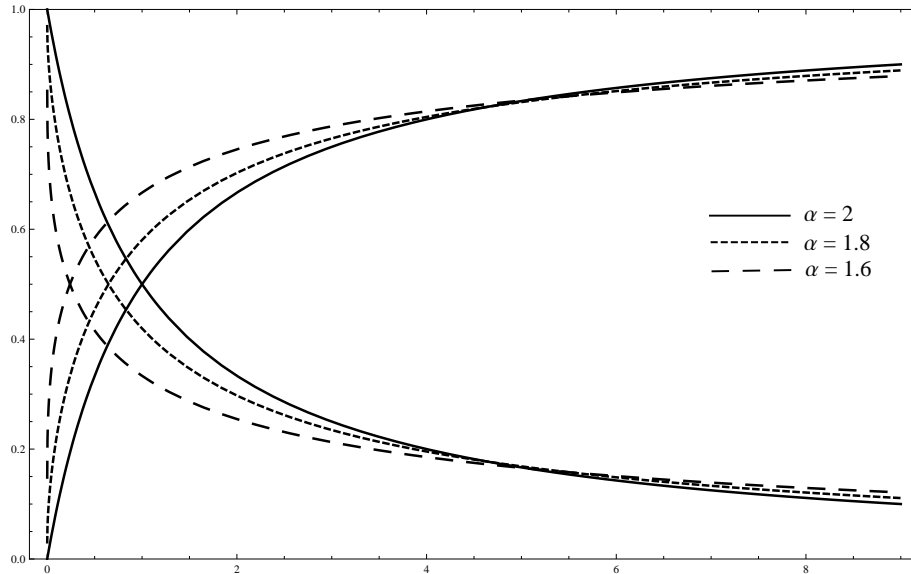


Figure 1: Reflection and transmission coefficients as function of  $E/U$ , as given by Eq.(47), for different values of  $\alpha$ .

We must note that transmission coefficient has a very interesting behaviour at zero energy. If we take the limit  $E \rightarrow 0$  in the expression for  $\Omega_\alpha$  in Eq.(21) we see that

$$\lim_{E \rightarrow 0} \Omega_\alpha = -\cot \frac{\pi}{\alpha}, \quad (48)$$

and then for the transmission coefficient  $\mathcal{T}$  we have

$$\lim_{E \rightarrow 0} \mathcal{T} = \cos^2 \frac{\pi}{\alpha}. \quad (49)$$

This is an unexpected and very interesting effect, which demands further interpretation.

### 3.2. Transmission Coefficient for the Double Delta Potential

Let us introduce the following notations:

$$\mathcal{U} = \frac{\varepsilon\mu^{-1} + J_\alpha(0)}{\alpha W}, \quad \mathcal{V} = \frac{\varepsilon + J_\alpha(0)}{\alpha W}, \quad \mathcal{X} = \frac{J_\alpha(\lambda R/\hbar)}{\alpha W}. \quad (50)$$

The asymptotic behaviour of the wave function  $\psi(x)$  given by Eq.(37) for  $x \rightarrow \pm\infty$  is therefore

$$\begin{aligned} \psi(x) = & C_1 e^{i\lambda x/\hbar} + C_2 e^{-i\lambda x/\hbar} + (UC'_1 - XC'_2) \sin \frac{\lambda|x + R/2|}{\hbar} \\ & + (VC'_2 - XC'_1) \sin \frac{\lambda|x - R/2|}{\hbar} + o(x^{-1}), \quad x \rightarrow \pm\infty, \end{aligned} \quad (51)$$

or

$$\psi(x) = A' e^{i\lambda x/\hbar} + B' e^{-i\lambda x/\hbar} + o(x^{-1}), \quad x \rightarrow -\infty, \quad (52)$$

$$\psi(x) = C' e^{i\lambda x/\hbar} + D' e^{-i\lambda x/\hbar} + o(x^{-1}), \quad x \rightarrow +\infty, \quad (53)$$

where we defined

$$A' = C_1 + M_1, \quad B' = C_2 + M_2, \quad (54)$$

$$C' = C_1 - M_1, \quad D' = C_2 - M_2, \quad (55)$$

and

$$M_1 = i(\rho C_1 + \sigma C_2 + i\tau C_2), \quad (56)$$

$$M_2 = -i(\sigma C_1 + \rho C_2 - i\tau C_1), \quad (57)$$

with

$$\begin{aligned} \rho = & \left( \frac{\mathcal{U} + \mathcal{V}}{2} \right) - \mathcal{X} \cos \frac{\lambda R}{\hbar}, \quad \sigma = \left( \frac{\mathcal{U} + \mathcal{V}}{2} \right) \cos \frac{\lambda R}{\hbar} - \mathcal{X}, \\ \tau = & \left( \frac{\mathcal{U} - \mathcal{V}}{2} \right) \sin \frac{\lambda R}{\hbar}. \end{aligned} \quad (58)$$

As in the case of the delta potential, let us consider the situation of particles coming from the left and scattered by the double delta potential. In complete analogy we have  $D' = 0$  (no particles coming from the right) and  $B' = rA'$  and  $C' = tA'$ , where the reflexion  $\mathcal{R}$  and transmission  $\mathcal{T}$  coefficients are given by  $\mathcal{R} = |r|^2$  and  $\mathcal{T} = |t|^2$ . The result is

$$r = \frac{2(\tau + i\sigma)}{(\rho^2 - \sigma^2 - \tau^2 - 1) - 2i\rho}, \quad t = -\frac{(\rho^2 - \sigma^2 - \tau^2 + 1)}{(\rho^2 - \sigma^2 - \tau^2 - 1) - 2i\rho}. \quad (59)$$

and  $\mathcal{R}$  and  $\mathcal{T}$  can be written as

$$\mathcal{R} = \frac{4(\sigma^2 + \tau^2)}{(\rho^2 - \sigma^2 - \tau^2 + 1)^2 + 4(\sigma^2 + \tau^2)}, \quad \mathcal{T} = \frac{(\rho^2 - \sigma^2 - \tau^2 + 1)^2}{(\rho^2 - \sigma^2 - \tau^2 + 1)^2 + 4(\sigma^2 + \tau^2)}. \quad (60)$$



We can simplify these expressions a little bit once we note that

$$\rho^2 - \sigma^2 - \tau^2 = \frac{\sin^2 \lambda R/\hbar}{\alpha^2 W}, \quad (61)$$

and then

$$\mathcal{R} = \frac{\Delta_\alpha^2}{1 + \Delta_\alpha^2}, \quad \mathcal{T} = \frac{1}{1 + \Delta_\alpha^2}, \quad (62)$$

with

$$\Delta_\alpha^2 = \frac{4\alpha^4 W^2 (\sigma^2 + \tau^2)}{(\alpha^2 W + \sin^2 \lambda R/\hbar)^2}. \quad (63)$$

In Fig.(2) we show the behaviour of the transmission coefficient for different values of  $\alpha$  and in Fig.(3) for different values of  $\mu$ .

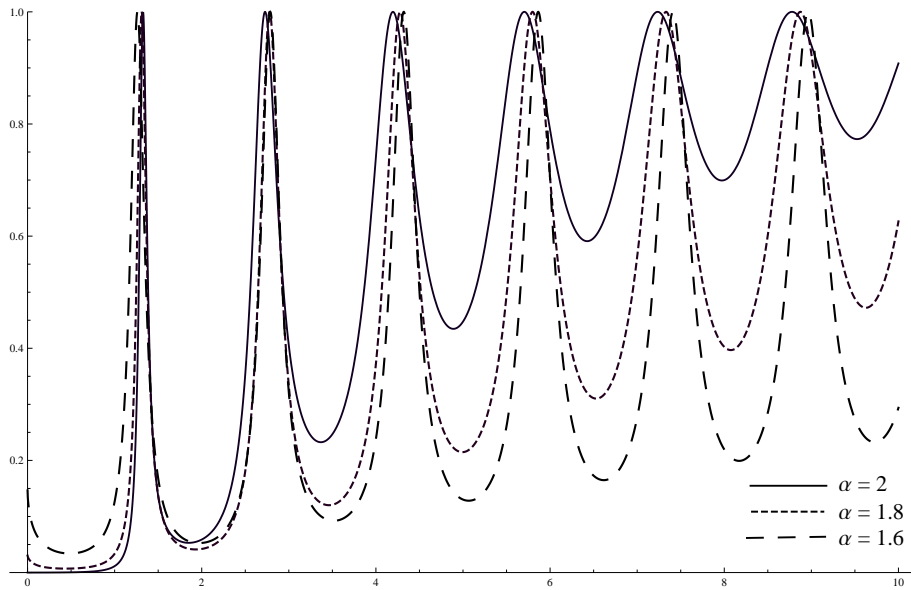


Figure 2: Transmission coefficients as function of  $\lambda R/2\hbar$ , as given by Eq.(62), for different values of  $\alpha$ , when  $2\pi\gamma = 10$  and  $\mu = 1$ .

As in the case of the delta potential, we have a very interesting behaviour for these coefficients when  $E \rightarrow 0$ . Firstly, using Eq.(98) in the Appendix B, we have

$$J_\alpha(\lambda R/\hbar) = J_\alpha(0) + A_1 \lambda^{\alpha-1} + A_2 \lambda^2 + \mathcal{O}(\lambda^{3\alpha-1}), \quad (64)$$

where

$$A_1 = \frac{(R/\hbar)^{\alpha-1}}{2\Gamma(\alpha) \cos \pi\alpha/2}, \quad A_2 = \frac{(R/\hbar)^2 \cot 3\pi/\alpha}{2\alpha}. \quad (65)$$

Using this, the expression Eq.(36) for  $W$  gives

$$W = J_\alpha(0)(H(1+\mu^{-1}) - 2A_1)\lambda^{\alpha-1} + (H^2\mu^{-1} - A_1^2)\lambda^{2(\alpha-1)} - 2J_\alpha(0)A_2\lambda^2 + \mathcal{O}(\lambda^{\alpha+1}), \quad (66)$$

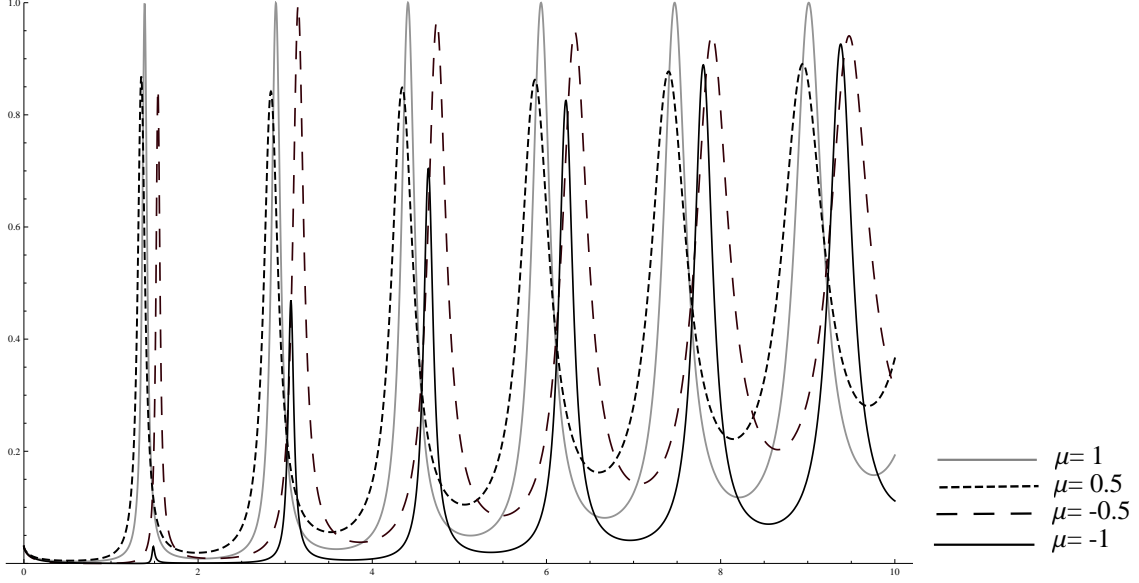


Figure 3: Transmission coefficients as function of  $\lambda R/2\hbar$ , as given by Eq.(62), for different values of  $\mu$ , when  $2\pi\gamma = 20$  and  $\alpha = 1.8$ .

where  $H = 1/2\pi\gamma$ . Then, with some calculations, we obtain that

$$4\alpha^2 W^2(\sigma^2 + \tau^2) = B_1 \lambda^{2(\alpha-1)} + B_2 \lambda^{\alpha+1} + \mathcal{O}(\lambda^{2\alpha}), \quad (67)$$

with

$$B_1 = (H(1 + \mu^{-1}) - 2A_1)^2, \quad B_2 = -2(H(1 + \mu^{-1}) - 2A_1)(2A_2 + J_\alpha(0)(R/\hbar)^2), \quad (68)$$

and

$$\alpha^2 W^2(\rho^2 - \sigma^2 - \tau^2 + 1) = B'_1 \lambda^{2(\alpha-1)} + B'_2 \lambda^{4(\alpha-1)} + B'_3 \lambda^{\alpha+1} + \mathcal{O}(\lambda^{2\alpha}), \quad (69)$$

with

$$\begin{aligned} B'_1 &= \alpha^2 J_\alpha^2(0)(H(1 + \mu^{-1}) - 2A_1)^2, & B'_2 &= \alpha^2 (H^2 \mu^{-1} - A_1)^2, \\ B'_3 &= -2\alpha J_\alpha(0)(H(1 + \mu^{-1}) - 2A_1)(2\alpha^2 J_\alpha(0)A_2 - (R/\hbar)^2). \end{aligned} \quad (70)$$

Using these results we can easily see that, for  $E/D_\alpha = \lambda^\alpha \rightarrow 0$ ,

$$\lim_{E \rightarrow 0} \Delta_\alpha^2 = \frac{1}{\alpha^2 J_\alpha^2(0)}, \quad (71)$$

and, since  $J_\alpha(0) = -(1/\alpha) \cot \pi/\alpha$ , that

$$\lim_{E \rightarrow 0} T = \cos^2 \frac{\pi}{\alpha}. \quad (72)$$

This the same result we obtained for the zero energy limit of the transmission coefficient for a single delta potential. Moreover, this limit does not depends on  $\mu$ , which is the parameter that relates the two delta functions in the potential in Eq.(23). Again, this is a very interesting and unexpected result.

## 4. Conclusions

The tunneling effect in fractional quantum mechanics has some very interesting properties which are not observed in the usual  $\alpha = 2$  quantum mechanics. The most interesting is the presence of tunneling through delta and double delta potentials even at zero energy. Moreover, in the case of the double delta potential, this zero energy tunneling is independent of the relation of the two delta functions. These results demands for physical explanations which are beyond the scope of the present paper.

### A. Fox's $H$ -Function

The Fox's  $H$ -function, also known as  $H$ -function or Fox's function, was introduced in the literature as an integral of Mellin-Barnes type [18].

Let  $m, n, p$  and  $q$  be integer numbers. Consider the function

$$\Lambda(s) = \frac{\prod_{i=1}^m \Gamma(b_i + B_i s) \prod_{i=1}^n \Gamma(1 - a_i - A_i s)}{\prod_{i=m+1}^q \Gamma(1 - b_i - B_i s) \prod_{i=n+1}^p \Gamma(a_i + A_i s)} \quad (73)$$

with  $1 \leq m \leq q$  and  $0 \leq n \leq p$ . The coefficients  $A_i$  and  $B_i$  are positive real numbers;  $a_i$  and  $b_i$  are complex parameters.

The Fox's  $H$ -function, denoted by,

$$H_{p,q}^{m,n}(x) = H_{p,q}^{m,n} \left( x \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right) = H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right. \right] \quad (74)$$

is defined as the inverse Mellin transform, i.e.,

$$H_{p,q}^{m,n}(x) = \frac{1}{2\pi i} \int_L \Lambda(s) x^{-s} ds \quad (75)$$

where  $\Lambda(s)$  is given by Eq.(73), and the contour  $L$  runs from  $L - i\infty$  to  $L + i\infty$  separating the poles of  $\Gamma(1 - a_i - A_i s)$ , ( $i = 1, \dots, n$ ) from those of  $\Gamma(b_i + B_i s)$ , ( $i = 1, \dots, m$ ). The complex parameters  $a_i$  and  $b_i$  are taken with the imposition that no poles in the integrand coincide.

There are some interesting properties associated with the Fox's  $H$ -function. We consider here the following ones:

#### P.1. Change the independent variable

Let  $c$  be a positive constant. We have

$$H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] = c H_{p,q}^{m,n} \left[ x^c \left| \begin{matrix} (a_p, c A_p) \\ (b_q, c B_q) \end{matrix} \right. \right]. \quad (76)$$

To show this expression one introduce a change of variable  $s \rightarrow cs$  in the integral of inverse Mellin transform.

## P.2. Change the first argument

Set  $\alpha \in \mathbb{R}$ . Then we can write

$$x^\alpha H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] = H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p + \alpha A_p, A_p) \\ (b_q + \alpha B_q, B_q) \end{matrix} \right. \right]. \quad (77)$$

To show this expression first we introduce the change  $a_p \rightarrow a_p + \alpha A_p$  and take  $s \rightarrow s - \alpha$  in the integral of inverse Mellin transform.

## P.3. Lowering of Order

If the first factor  $(a_1, A_1)$  is equal to the last one,  $(b_q, B_q)$ , we have

$$H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), (a_1, A_1) \end{matrix} \right. \right] = H_{p-1,q-1}^{m,n-1} \left[ x \left| \begin{matrix} (a_2, A_2), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}) \end{matrix} \right. \right]. \quad (78)$$

To show this identity is sufficient to simplify the common arguments in the Mellin-Barnes integral.

## P.4. Asymptotic Expansions

The asymptotic expansions for Fox's  $H$ -functions have been studied in [19]. Let  $\Delta$  and  $\Delta^*$  be defined as

$$\Delta = \sum_{i=1}^q B_i - \sum_{i=1}^p A_i, \quad \Delta^* = \sum_{i=1}^n A_i - \sum_{i=n+1}^p A_i + \sum_{i=1}^m B_i - \sum_{i=m+1}^q B_i. \quad (79)$$

If  $\Delta > 0$  and  $\Delta^* > 0$  we have [20]

$$H_{p,q}^{m,n}(x) = \sum_{r=1}^n [h_r x^{(a_r-1)/A_r} + o(x^{(a_r-1)/A_r})], \quad |x| \rightarrow \infty. \quad (80)$$

where

$$h_r = \frac{1}{A_r} \frac{\prod_{j=1}^m \Gamma(b_j + (1-a_r)B_j/A_r) \prod_{j=1, j \neq r}^n \Gamma(1-a_j - (1-a_r)A_j/A_r)}{\prod_{j=n+1}^p \Gamma(a_j - (1-a_r)A_j/A_r) \prod_{j=m+1}^q \Gamma(1-b_j - (1-a_r)B_j/A_r)}, \quad (81)$$

and if  $\Delta > 0$  and  $\Delta^* = 0$  we have [20]

$$\begin{aligned} H_{p,q}^{m,n}(x) &= \sum_{r=1}^n [h_r x^{(a_r-1)/A_r} + o(x^{(a_r-1)/A_r})] \\ &+ Ax^{(\nu+1/2)/\Delta} (c_0 \exp[i(B+Cx^{1/\Delta})] - d_0 \exp[-i(B+Cx^{1/\Delta})]) \\ &+ o(x^{(\nu+1/2)/|\Delta|}), \quad |x| \rightarrow \infty, \end{aligned} \quad (82)$$

where

$$\begin{aligned}
c_0 &= (2\pi i)^{m+n-p} \exp \left[ \pi i \left( \sum_{r=n+1}^p a_r - \sum_{j=1}^m b_j \right) \right], \\
d_0 &= (-2\pi i)^{m+n-p} \exp \left[ -\pi i \left( \sum_{r=n+1}^p a_r - \sum_{j=1}^m b_j \right) \pi i \right], \\
A &= \frac{1}{2\pi i \Delta} (2\pi)^{(p-q+1)/2} \Delta^{-\nu} \prod_{r=1}^p A_r^{-a_r+1/2} \prod_{j=1}^q B_j^{b_j-1/2} \left( \frac{\Delta \Delta}{\delta} \right)^{(\nu+1/2)/\Delta}, \\
B &= \frac{(2\nu+1)\pi}{4}, \quad C = \left( \frac{\Delta \Delta}{\delta} \right)^{1/\Delta}, \\
\delta &= \prod_{l=1}^p |A_l|^{-A_l} \prod_{j=1}^q |B_j|^{B_j}, \quad \nu = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j + \frac{p-q}{2}.
\end{aligned}$$

### P.5. Series Expansion

In [18] we can see that in some cases there is a series expansion for Fox's  $H$ -function. For example, when the poles of  $\prod_{j=1}^m \Gamma(b_j + B_j s)$  are simple, we can write

$$H_{p,q}^{m,n}(x) = \sum_{j=1}^m \sum_{\nu=0}^{\infty} h_{j\nu} x^{(b_j+\nu)/B_j}, \quad (83)$$

where

$$h_{j\nu} = \frac{(-1)^\nu \prod_{i=1, i \neq j}^m \Gamma(b_i - B_i(b_j + \nu)/B_j) \prod_{i=1}^n \Gamma(1 - a_i + A_i(b_j + \nu)/B_j)}{\nu! B_j \prod_{i=m+1}^q \Gamma(1 - b_i + B_i(b_j + \nu)/b_j) \prod_{i=n+1}^p \Gamma(a_i - A_i(b_j + \nu)/B_j)}. \quad (84)$$

## B. Calculation of the Integral in Eq.(18)

Let  $J_\alpha(w)$  be given by

$$J_\alpha(w) = \frac{1}{\pi} \int_0^{+\infty} \frac{\cos wy}{y^\alpha - 1} dy. \quad (85)$$

Then the integral in Eq.(18) can be written as

$$\int_{-\infty}^{+\infty} \frac{e^{ipx/\hbar}}{|p|^\alpha - \lambda^\alpha} dp = 2\pi \lambda^{1-\alpha} J_\alpha(\lambda x/\hbar), \quad (86)$$

where we remember that  $\lambda > 0$ . Taking the Mellin transform we have that

$$\mathcal{M}_w[J_\alpha(w)](z) = \frac{1}{\pi} \Gamma(z) \cos \frac{\pi z}{2} \int_0^{+\infty} \frac{y^{-z}}{y^\alpha - 1} dy. \quad (87)$$

This last integral is given by formula 3.241.3 (pg.322) of [15], that is,

$$\int_0^{+\infty} \frac{x^{\mu-1}}{1-x^\nu} dx = \frac{\pi}{\nu} \cot \frac{\mu\pi}{\nu}, \quad (88)$$

where the integration is understood as the Cauchy principal value<sup>3</sup>. Therefore we have

$$\mathcal{M}_w[J_\alpha(w)](z) = -\frac{1}{\alpha} \Gamma(z) \sin \frac{\pi(1-z)}{2} \cot \frac{\pi(1-z)}{\alpha}. \quad (89)$$

Using the relation  $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$  and writing the sine function in terms of the product of gamma functions we can write that

$$\begin{aligned} \mathcal{M}_w[J_\alpha(w)](z) &= -\frac{1}{2\alpha} \frac{\Gamma(z)\Gamma(\frac{1-z}{\alpha})\Gamma(1-\frac{1-z}{\alpha})}{\Gamma((1-z)\frac{(2+\alpha)}{2\alpha})\Gamma(1-(1-z)\frac{(2+\alpha)}{2\alpha})} \\ &\quad + \frac{1}{2\alpha} \frac{\Gamma(z)\Gamma(\frac{1-z}{\alpha})\Gamma(1-\frac{1-z}{\alpha})}{\Gamma((1-z)\frac{(2-\alpha)}{2\alpha})\Gamma(1-(1-z)\frac{(2-\alpha)}{2\alpha})} = F_2(z). \end{aligned}$$

Taking the inverse Mellin transform and using the definition of the Fox's  $H$ -function we have that

$$\begin{aligned} J_\alpha(w) &= -\frac{1}{2\alpha} H_{2,3}^{2,1} \left[ w \left| \begin{array}{l} (1-1/\alpha, 1/\alpha), (1-(2+\alpha)/2\alpha, (2+\alpha)/2\alpha) \\ (0, 1), (1-1/\alpha, 1/\alpha), (1-(2+\alpha)/2\alpha, (2+\alpha)/2\alpha) \end{array} \right. \right] + \\ &\quad + \frac{1}{2\alpha} H_{2,3}^{2,1} \left[ w \left| \begin{array}{l} (1-1/\alpha, 1/\alpha), (1-(2-\alpha)/2\alpha, (2-\alpha)/2\alpha) \\ (0, 1), (1-1/\alpha, 1/\alpha), (1-(2-\alpha)/2\alpha, (2-\alpha)/2\alpha) \end{array} \right. \right]. \end{aligned}$$

Using the properties given by Eqs.(76,77) and replacing  $w$  by  $|w|$  since  $J_\alpha(-w) = J_\alpha(w)$  we obtain

$$\begin{aligned} J_\alpha(w) &= -\frac{1}{2|w|} H_{2,3}^{2,1} \left[ |w|^\alpha \left| \begin{array}{l} (1, 1), (1, (2+\alpha)/2) \\ (1, \alpha), (1, 1), (1, (2+\alpha)/\alpha) \end{array} \right. \right] + \\ &\quad + \frac{1}{2|w|} H_{2,3}^{2,1} \left[ |w|^\alpha \left| \begin{array}{l} (1, 1), (1, (2-\alpha)/2) \\ (1, \alpha), (1, 1), (1, (2-\alpha)/2) \end{array} \right. \right]. \end{aligned} \quad (90)$$

Finally, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{ipx/h}}{|p|^\alpha - \lambda^\alpha} dp &= -\frac{\hbar\pi}{\lambda^\alpha|x|} \left( H_{2,3}^{2,1} \left[ (\hbar^{-1}\lambda)^\alpha |x|^\alpha \left| \begin{array}{l} (1, 1), (1, (2+\alpha)/2) \\ (1, \alpha), (1, 1), (1, (2+\alpha)/\alpha) \end{array} \right. \right] + \right. \\ &\quad \left. - H_{2,3}^{2,1} \left[ (\hbar^{-1}\lambda)^\alpha |x|^\alpha \left| \begin{array}{l} (1, 1), (1, (2-\alpha)/2) \\ (1, \alpha), (1, 1), (1, (2-\alpha)/2) \end{array} \right. \right] \right). \end{aligned} \quad (91)$$

Let us see what happens in the particular case  $\alpha = 2$ . From the definition of Fox's  $H$ -function we can see that

$$H_{2,3}^{2,1} \left[ |w|^2 \left| \begin{array}{l} (1, 1), (1, 0) \\ (1, 2), (1, 1), (1, 0) \end{array} \right. \right] = 0 \quad (92)$$

<sup>3</sup>We remember that in the inversion of the Fourier transform the integration is to be done in the sense of the Cauchy principal value [21].

and that

$$H_{2,3}^{2,1} \left[ w^2 \left| \begin{array}{l} (1, 1), (1, 2) \\ (1, 2), (1, 1), (1, 2) \end{array} \right. \right] = H_{1,2}^{1,1} \left[ w^2 \left| \begin{array}{l} (1, 1) \\ (1, 1), (1, 2) \end{array} \right. \right] = w^2 H_{1,2}^{1,1} \left[ w^2 \left| \begin{array}{l} (0, 1) \\ (0, 1), (-1, 2) \end{array} \right. \right].$$

But [18]

$$H_{1,2}^{1,1} \left[ -z \left| \begin{array}{l} (0, 1) \\ (0, 1), (1 - b, a) \end{array} \right. \right] = E_{a,b}(z), \quad (93)$$

where  $E_{a,b}(z)$  is the two-parameter Mittag-Leffler function. However, it is known [22] that

$$E_{2,2}(z) = \frac{\sinh \sqrt{z}}{\sqrt{z}}. \quad (94)$$

Consequently, we have

$$H_{2,3}^{2,1} \left[ w^2 \left| \begin{array}{l} (1, 1), (1, 2) \\ (1, 2), (1, 1), (1, 2) \end{array} \right. \right] = |w|^2 E_{2,2}(-|w|^2) = |w| \sin |w|. \quad (95)$$

Then for  $\alpha = 2$  we have

$$\int_{-\infty}^{+\infty} \frac{e^{ipx/\hbar}}{|p|^2 - \lambda^2} dp = -\frac{\pi}{\lambda} \sin \frac{\lambda|x|}{\hbar}. \quad (96)$$

We are also interested in the expression of  $J_\alpha(w)$  for small  $w$ . From Eq.(90) we see that we need to know the behaviour of

$$H_{2,3}^{2,1} \left[ |w|^\alpha \left| \begin{array}{l} (1, 1), (1, \mu) \\ (1, \alpha), (1, 1), (1, \mu) \end{array} \right. \right]$$

for small  $w$ . This is given by the series expansion from Eq.(83), which gives

$$\begin{aligned} & H_{2,3}^{2,1} \left[ z \left| \begin{array}{l} (1, 1), (1, \mu) \\ (1, \alpha), (1, 1), (1, \mu) \end{array} \right. \right] \\ &= \frac{\Gamma(1 - 1/\alpha)\Gamma(1/\alpha)}{\Gamma(1 - \mu/\alpha)\Gamma(\mu/\alpha)} \frac{z^{1/\alpha}}{\alpha} - \frac{\Gamma(1 - 2/\alpha)\Gamma(2/\alpha)}{\Gamma(1 - 2\mu/\alpha)\Gamma(2\mu/\alpha)} \frac{z^{2/\alpha}}{\alpha} \\ &+ \frac{\Gamma(1 - 3/\alpha)\Gamma(3/\alpha)}{\Gamma(1 - 3\mu/\alpha)\Gamma(3\mu/\alpha)} \frac{z^{3/\alpha}}{2\alpha} + \mathcal{O}(z^{4/\alpha}) \\ &+ \frac{\Gamma(1 - \alpha)\Gamma(1)}{\Gamma(1 - \mu)\Gamma(\mu)} z - \frac{\Gamma(1 - 2\alpha)\Gamma(2)}{\Gamma(1 - 2\mu)\Gamma(2\mu)} z^2 + \frac{\Gamma(1 - 3\alpha)\Gamma(3)}{\Gamma(1 - 3\mu)\Gamma(3\mu)} \frac{z^3}{2} + \mathcal{O}(z^4). \end{aligned} \quad (97)$$

Using this in Eq.(90) we arrive, after some manipulations, to

$$J_\alpha(w) = J_\alpha(0) + \frac{1}{2\Gamma(\alpha) \cos \pi\alpha/2} w^{\alpha-1} + \frac{\cot 3\pi/\alpha}{2\alpha} w^2 + \mathcal{O}(w^{3\alpha-1}), \quad (98)$$

where

$$J_\alpha(0) = -\frac{1}{\alpha} \cot \frac{\pi}{\alpha}. \quad (99)$$

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