LORENTZ INVARIANCE, NONZERO MINIMAL UNCERTAINTY IN POSITION, AND INHOMOGENEITY OF SPACE AT THE PLANCK SCALE

ARKO BOSE

Abstract. The suspicion that the existence of a minimal uncertainty in position measurements violates Lorentz invariance seems unfounded. It is shown that the existence of such a nonzero minimal uncertainty in position is not only consistent with Lorentz invariance, but that the latter also fixes the algebra between position and momentum which gives rise to this minimal uncertainty. We also investigate how this algebra affects the underlying quantum mechanical structure, and why, at the Planck scale, space can no longer be considered homogeneous.

PACS number(s): 11.30.Cp, 03.65.Ca, 03.65.Ta, 02.40.Gh

1. Introduction

The possibility of the existence of a nonzero minimal uncertainty in position measurements - whose effects are believed to be pronounced near the Planck scale - has been widely studied. Such a minimal uncertainty in position measurements is incorporated into theory by generalizing the algebra between position and momentum. Several formulations of this algebra exist, some which break Lorentz invariance (see *e.g.*, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]), while others which respect it (see *e.g.*, [11, 12, 13]). The common feature of such generalized algebras is that they render spacetime noncommutative. One pervasive notion is that the existence of this nonzero minimal uncertainty in position measurements - an ultraviolet (UV) cutoff in Nature - violates Lorentz invariance.

Rejection of Lorentz invariance seems premature, however, in light of recent experiments ([14]) which show no evidence for Lorentz violation. As we shall see in Section 2, a nonzero minimal uncertainty in position measurements is consistent with Lorentz invariance. This is achieved by first deriving the most general Lorentz contravariant representation of position in the momentum-space. This procedure generates a nondenumerably infinite set of geometries. It will then be shown, in Section 3, that among these nondenumerably infinite geometries, Snyder-geometry is one of only two Lorentz invariant geometries which retain the symmetry of the position operators. Finally, with the algebra between position and momentum thus fixed, it will be seen that a nonzero minimal uncertainty in position measurements emerges naturally.

It was Snyder ([15]) who first constructed a model of discrete spacetime which respected Lorentz invariance. However, the model was flawed. Though it successfully predicted - a prediction to which Snyder did not pay attention, at least in the paper - a minimal uncertainty in position measurements, it also assumed a lattice structure of space - two mutually incompatible concepts.

One relevant issue here is that of translational invariance. As remarked by Snyder himself (and later elaborated by Yang [16]), his theory lacks translational invariance. It shall be argued in Section 2 that the existence of a nonzero minimal uncertainty in position measurements necessarily violates translational invariance, thus rendering space inhomogeneous near the Planck scale.

There have been investigations into the possibility of minimal uncertainties in both position and momentum measurements. However, in this paper, we restrict ourselves to the case of nonzero minimal uncertainty in position measurements. We discuss the possibility of a nonzero minimal uncertainty in momentum measurements in Section 8.

Lastly, certain fundamental misconceptions in existing literature will be discussed (in sections 2, 6, and 7), and rectified.

2. Lorentz invariance and the generalized representation of position in momentum space

As has been remarked in [5], the existence of a nonzero minimal uncertainty in position measurements implies that position is no longer an observable, since position can no longer have eigenstates.

It is a simple argument to show that the existence of a nonzero minimal uncertainty in position measurements necessarily violates translational invariance. For a theory to have translational invariance, the underlying algebra must permit a representation of momentum in the position-space, allowing momentum to act as the generator of infinitesimal spatial translations. However, the existence of such a position-space representation of momentum presupposes the existence of position eigenstates, which we have just seen do not exist, due to the nonzero minimal uncertainty in position measurements. Thus, unless the theory can neglect effects of this nonzero minimal uncertainty in position measurements, which is essentially the low-energy limit, space can not be assumed to be homogeneous. (See *e.g.* [17], wherein the generalized position-momentum commutation relation is claimed to respect translational invariance. It is easy to see that this is not a correct claim: the algebra does not permit a position representation of momentum.) Geometrically, this small-scale inhomogeneity of space is a manifestation of the fact that - owing to the nonzero minimal uncertainty - points lose their meaning as geometrically localizable entities (see [12]). In Section 8, it will be shown that a nonzero minimal uncertainty in momentum measurements is consistent with both Lorentz invariance and translational invariance, owing to the fact that the algebra permits a representation of momentum in the position-space.

In this connection see [16], wherein the theory maintains translational invariance by permitting a representation of momentum in the position-space. However, in Yang's theory, position is an observable with a discrete spectrum. See also [18], wherein a global counterpart of twisted Poincaré algebra is constructed, however, on a spacetime controlled by a constant, antisymmetric matrix $\theta^{\mu\nu}$: $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. We note here that as long as $\theta^{\mu\nu}$ is constant, position acts as an observable.

At this point, we make two assumptions:

(1) Momentum is an observable. This assumption will be justified later, in Section 8.
(2) All components of the momentum 4-vector commute:

$$[p^\mu, p^\nu] = 0 \tag{2.1}$$

It follows from the above discussion that to obtain information about position, one must work in the momentum representation, owing to the fact that position no longer has eigenstates, while momentum is assumed to have eigenstates. Thus, what we must seek is a generalized, Lorentz contravariant representation of position in the momentum-space, which reduces to the ordinary quantum mechanical representation in the low-energy (large-scale) limit. This would be (we use the Minkowski metric signature $\{+,-,-,-\}$):

$$\hat{x}^{\mu} \equiv i\hbar[-\partial_p^{\mu} + bA_0^{\mu\nu}\partial_{p\,\nu} + b^2 A_1^{\mu\rho\sigma}\partial_{p\,\rho}\partial_{p\,\sigma} + b^3 A_2^{\mu\lambda\gamma\alpha}\partial_{p\,\lambda}\partial_{p\,\gamma}\partial_{p\,\alpha} + \ldots] \tag{2.2}$$

where $\partial_{p\,\nu} \equiv \frac{\partial}{\partial p^{\nu}}$, and $A_0^{\mu\nu}$, $A_1^{\mu\nu\sigma}$, $A_2^{\mu\nu\sigma\rho}, \ldots$ are, as yet, arbitrary functions, and b is a dimensionless parameter.

Lorentz contravariance of $\hat{x}^{\mu}$ gives:

$$\begin{aligned}
\hat{x}'^{\,\mu} =& i\hbar[-\partial_p'^{\mu} + bA_0'^{\mu\nu}\partial'_{p\,\nu} + b^2 A_1'^{\mu\rho\sigma}\partial'_{p\,\rho}\partial'_{p\,\sigma} + b^3 A_2'^{\mu\lambda\gamma\alpha}\partial'_{p\,\lambda}\partial'_{p\,\gamma}\partial'_{p\,\alpha} + \ldots] \\
=& \Lambda^{\mu}{}_{\tau}\hat{x}^{\tau} + a^{\mu} \\
=& a^{\mu} + i\hbar\Lambda^{\mu}{}_{\tau}[-\partial_p^{\tau} + bA_0^{\tau\tilde{\nu}}\partial_{p\,\tilde{\nu}} b^2 A_1^{\tau\tilde{\rho}\tilde{\sigma}}\partial_{p\,\tilde{\rho}}\partial_{p\,\tilde{\sigma}} + b^3 A_2^{\tau\tilde{\lambda}\tilde{\gamma}\tilde{\alpha}}\partial_{p\,\tilde{\lambda}}\partial_{p\,\tilde{\gamma}}\partial_{p\,\alpha} + \ldots]
\end{aligned} \tag{2.3}$$

We get $a^{\mu} = 0$. Also, comparing the coefficients of differential operators of same order, we get

$$\begin{aligned}
& A_0'^{\mu\nu}\Lambda_{\nu}{}^{\nu_1}\partial_{p\,\nu_1} = \Lambda^{\mu}{}_{\tau}A_0^{\tau\tilde{\nu}}\partial_{p\,\tilde{\nu}} \\
\Rightarrow & A_0'^{\mu\nu}\Lambda_{\nu}{}^{\nu_1}\Lambda^{\nu}{}_{\tilde{\nu}}\partial_{p\,\nu_1} = \Lambda^{\mu}{}_{\tau}\Lambda^{\nu}{}_{\tilde{\nu}}A_0^{\tau\tilde{\nu}}\partial_{p\,\tilde{\nu}} \\
\Rightarrow & A_0'^{\mu\nu}\delta^{\nu_1}{}_{\tilde{\nu}}\partial_{p\,\nu_1} = \Lambda^{\mu}{}_{\tau}\Lambda^{\nu}{}_{\tilde{\nu}}A_0^{\tau\tilde{\nu}}\partial_{p\,\tilde{\nu}} \\
\Rightarrow & A_0'^{\mu\nu}\partial_{p\,\tilde{\nu}} = \Lambda^{\mu}{}_{\tau}\Lambda^{\nu}{}_{\tilde{\nu}}A_0^{\tau\tilde{\nu}}\partial_{p\,\tilde{\nu}} \\
\Rightarrow & A_0'^{\mu\nu} = \Lambda^{\mu}{}_{\tau}\Lambda^{\nu}{}_{\tilde{\nu}}A_0^{\tau\tilde{\nu}}
\end{aligned} \tag{2.4}$$

Therefore, $A_0^{\mu\nu}$ transforms as a second rank tensor. Similarly, it can be shown that $A_1^{\mu\rho\sigma}$ transforms as a third rank tensor, $A_2^{\mu\lambda\gamma\alpha}$ transforms as a fourth rank tensor and so on.

Further, simple dimensional analysis shows that $A_0^{\mu\nu}$ must be dimensionless, while $A_1^{\mu\nu\sigma}$ must have the dimensions of momentum, $A_2^{\mu\nu\sigma\rho}$ the dimensions of momentum squared, and so on. Thus, Lorentz invariance and dimensional analysis give us:

$$\begin{aligned}
A_0^{\mu\nu} = \frac{G}{\hbar c^3}\Omega_0^{\mu\nu\sigma\rho}p_{\sigma}p_{\rho} + P_0^{\mu\nu}; \quad A_1^{\mu\nu\sigma} = \frac{G}{\hbar c^3}\Omega_1^{\mu\nu\sigma\rho\delta\lambda}p_{\rho}p_{\delta}p_{\lambda} + P_1^{\mu\nu\sigma}; \\
A_2^{\mu\nu\sigma\rho} = \frac{G}{\hbar c^3}\Omega_2^{\mu\nu\sigma\rho\delta\lambda\gamma\kappa}p_{\delta}p_{\lambda}p_{\gamma}p_{\kappa} + P_2^{\mu\nu\sigma\rho}; \ldots
\end{aligned} \tag{2.5}$$

where $\Omega_0^{\mu\nu\sigma\rho}$, $\Omega_1^{\mu\nu\sigma\rho\delta\lambda}$, $\Omega_2^{\mu\nu\sigma\rho\delta\lambda\gamma\kappa}$, etc. are all dimensionless scalars of ranks 4, 6, 8, etc., and $P_0^{\mu\nu}$, $P_1^{\mu\nu\sigma}$, $P_2^{\mu\nu\sigma\rho}$, etc. are all scalars, of dimensions, respectively, p^0, p^1, p^2, etc.

Following above arguments, we thus find that Lorentz invariance fixes the generalized momentum representation of position to be:

$$\begin{aligned}
\hat{x}^{\mu} \equiv \quad & i\hbar\Bigg[-\partial_p^{\mu} + b\left(\frac{G}{\hbar c^3}\Omega_0^{\mu\nu\sigma\rho}p_{\sigma}p_{\rho} + P_0^{\mu\nu}\right)\partial_{p\,\nu} + \\
& b^2\left(\frac{G}{\hbar c^3}\Omega_1^{\mu\nu\sigma\rho\delta\lambda}p_{\rho}p_{\delta}p_{\lambda} + P_1^{\mu\nu\sigma}\right)\partial_{p\,\nu}\partial_{p\,\sigma} \\
& + b^3\left(\frac{G}{\hbar c^3}\Omega_2^{\mu\nu\sigma\rho\delta\lambda\gamma\kappa}p_{\delta}p_{\lambda}p_{\gamma}p_{\kappa} + P_2^{\mu\nu\sigma\rho}\right)\partial_{p\,\nu}\partial_{p\,\sigma}\partial_{p\,\rho} + \ldots\Bigg]
\end{aligned} \tag{2.6}$$

Now from definition (2.6), up to all orders in b, the following algebra between position and momentum immediately follows:

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar\left[-\eta^{\mu\nu} + b\left(\frac{G}{\hbar c^3}\Omega_0^{\mu\nu\sigma\rho}\hat{p}_\sigma\hat{p}_\rho + P_0^{\mu\nu}\right)\right] = i\hbar[-\eta^{\mu\nu} + bA_0^{\mu\nu}] \tag{2.7}$$

Since b and $P_0^{\mu\nu}$ are both constants, we can liberally redefine $bP_0^{\mu\nu} = \tilde{P}_0^{\mu\nu}$. Snyder-geometry is, thus, a special case wherein $\Omega_0^{\mu\nu\sigma\rho}$ is identically $\eta^{\mu\sigma}\eta^{\nu\rho}$, whereas $\tilde{P}_0^{\mu\nu}$ is identically zero.

Since we can only work in the momentum representation, it is clear that the entire phenomenology will be governed only by the algebra defined by equations (2.1) and (2.7). Also, since terms of order higher than one in b identically vanish in equation (2.7), we conclude that the exact, generalized momentum representation of position is defined by equation (2.6), by terms up to first order in b. Henceforth, we define $\beta = bG/\hbar c^3$. Therefore, the exact, Lorentz contravariant, generalized momentum representation of position is given by:

$$\hat{x}^\mu = i\hbar[-\eta^{\mu\nu} + \beta\Omega_0^{\mu\nu\sigma\rho}p_\sigma p_\rho + \tilde{P}_0^{\mu\nu}]\frac{\partial}{\partial p^\nu} \tag{2.8}$$

We can also compute the algebra between the position operators now:

$$\begin{aligned}[\hat{x}^\mu, \hat{x}^\nu] = -\hbar^2\beta\Bigg[&[-\eta^{\mu\alpha} + \beta\Omega_0^{\mu\alpha\sigma\rho}p_\sigma p_\rho + \tilde{P}_0^{\mu\alpha}]\Omega_0^{\mu\lambda\kappa\gamma}(\eta_{\kappa\alpha}p_\gamma + \eta_{\gamma\alpha}p_\kappa)\frac{\partial}{\partial p^\lambda} \\ &-[-\eta^{\nu\lambda} + \beta\Omega_0^{\mu\lambda\kappa\gamma}p_\kappa p_\gamma + \tilde{P}_0^{\mu\lambda}]\Omega_0^{\mu\alpha\sigma\rho}(\eta_{\sigma\lambda}p_\rho + \eta_{\rho\lambda}p_\sigma)\frac{\partial}{\partial p^\alpha}\Bigg]\end{aligned} \tag{2.9}$$

In order to derive an uncertainty relation, we require the position and momentum operators to have real expectation values. The sufficient condition for this is that both the operators be symmetric (see [5, 19]). Since momentum is self-adjoint, it is obviously symmetric. In the next section, it is shown that the requirement of symmetry of the position operator forces $\Omega_0^{\mu\nu\sigma\rho}$ and $\tilde{P}_0^{\mu\nu}$ to have specific values.

3. Symmetry of the position operator

The symmetry of the position operator is expressed as:

$$(\langle\phi|x^\mu)|\psi\rangle = \langle\phi|(x^\mu|\psi\rangle) \tag{3.1}$$

where both ϕ and ψ are L^2 functions.

Let $\mathcal{F} \equiv \mathcal{F}(\beta\eta_{\sigma\rho}p^\sigma p^\rho)$ be a dimensionless, scalar function which, from equation (3.1), gives us:

$$\int d^4p\mathcal{F}(x^\mu\phi)^*\psi = \int d^4p\mathcal{F}\phi^*(x^\mu\psi) \tag{3.2}$$

It follows that the above equation is satisfied if and only if

$$-2\frac{\mathcal{F}'}{\mathcal{F}} = 2\frac{[\beta\Omega_0^{'\mu\nu\rho\sigma}p_\sigma p_\rho + (\tilde{P}_0^{\mu\nu} - \eta^{\mu\nu})']p_\nu}{[\beta\Omega_0^{\mu\nu\rho\sigma}p_\sigma p_\rho + (\tilde{P}_0^{\mu\nu} - \eta^{\mu\nu})]p_\nu} + \frac{b\Omega^{\mu\nu\sigma\rho}(\eta_{\sigma\nu}p_\rho + \eta_{\rho\nu}p_\sigma)}{[\beta\Omega_0^{\mu\nu\rho\sigma}p_\sigma p_\rho + (\tilde{P}_0^{\mu\nu} - \eta^{\mu\nu})]p_\nu} \tag{3.3}$$

where the prime denotes differentiation with respect to $\frac{G}{\hbar c^3}p_\alpha p^\alpha$. Since $\mathcal{F} \equiv \mathcal{F}(\frac{G}{\hbar c^3}p_\sigma p^\sigma)$, it follows that the two numerators on the R.H.S of the above equation must satisfy:

$$\begin{aligned}\beta\Omega_0^{'\mu\nu\rho\sigma}p_\sigma p_\rho p_\nu + (\tilde{P}_0^{\mu\nu} - \eta^{\mu\nu})' p_\nu &= \{[\beta\Omega_0^{\mu\nu\rho\sigma}p_\sigma p_\rho + (\tilde{P}_0^{\mu\nu} - \eta^{\mu\nu})]p_\nu\}';\\ [\beta\Omega_0^{\mu\nu\rho\sigma}p_\sigma p_\rho + (\tilde{P}_0^{\mu\nu} - \eta^{\mu\nu})]p_\nu &= b\Omega^{\mu\nu\sigma\rho}(\eta_{\sigma\nu}p_\rho + \eta_{\rho\nu}p_\sigma)[M\frac{G}{\hbar c^3}p_\alpha p^\alpha + L];\\ (\tilde{P}_0^{\mu\nu} - \eta^{\mu\nu})p_\nu &= bL\Omega_0^{\mu\nu\sigma\rho}(\eta_{\sigma\nu}p_\rho + \eta_{\rho\nu}p_\sigma)p_\alpha p^\alpha\end{aligned} \tag{3.4}$$

where M and L are constants. The solutions are

$$\begin{aligned}\Omega_0^{\mu\nu\sigma\rho} &= \eta^{\mu\sigma}\eta^{\nu\rho};\\ \tilde{P}_0^{\mu\nu} &= b\eta^{\mu\nu}, 0\end{aligned} \tag{3.5}$$

Thus, Snyder-geometry ($\Omega_0^{\mu\nu\sigma\rho} = \eta^{\mu\sigma}\eta^{\nu\rho}$, $\tilde{P}_0^{\mu\nu} = 0$) is one of only two Lorentz-invariant geometries which retain the symmetry of the position operators, under the assumptions that momentum is an observable and that the momenta commute among themselves. From hereon, we work with the case of Snyder-geometry.

We obtain

$$\mathcal{F} = \Upsilon[1 - \beta\eta_{\mu\nu}p^\mu p^\nu]^{-5/2} \tag{3.6}$$

where Υ is a dimensionless constant, which, for convenience, can be taken to be 1.

4. The generalized uncertainty relation

We have seen that the symmetry of the position operator implies that $\Omega_0^{\mu\nu\sigma\rho} = \eta^{\mu\sigma}\eta^{\nu\rho}$ and $\tilde{P}_0^{\mu\nu} = 0$. With these values, equation (2.8) gives

$$\hat{x}^\mu \equiv i\hbar[-\eta^{\mu\nu} + \beta p^\mu p^\nu]\frac{\partial}{\partial p^\nu} \tag{4.1}$$

from where we obtain

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar[-\eta^{\mu\nu} + \beta p^\mu p^\nu] \tag{4.2}$$

Further, the algebra between the position operators is, as shown in [15],

$$[\hat{x}^\mu, \hat{x}^\nu] = i\hbar\beta(\hat{p}^\mu\hat{x}^\nu - \hat{p}^\nu\hat{x}^\mu) \tag{4.3}$$

Snyder's motivation was the construction of a lattice space based upon the assumption of a minimal length in Nature, which is, as the existence of the minimal uncertainty in position measurements shows, a flawed premise to work upon. While the theory is sound both in being Lorentz invariant and in lacking translational invariance, it is an incorrect assumption that we are working in a lattice space.

The algebra given by equation (4.2) immediately gives us the following generalized uncertainty relation:

$$\sqrt{\langle(\Delta x^\mu)^2\rangle}\sqrt{\langle(\Delta p^\nu)^2\rangle} \geq \frac{\hbar}{2}| - \eta^{\mu\nu} + \beta\sqrt{\langle(\Delta p^\mu)^2\rangle}\sqrt{\langle(\Delta p^\nu)^2\rangle} + \beta\langle p^\mu\rangle\langle p^\nu\rangle| \tag{4.4}$$

Considering the space operators, from inequality (4.4), we can write (hereon, we simply write Δx for $\sqrt{\langle(\Delta x)^2\rangle}$, keeping in mind the distinction from the operator $\Delta x = x - \langle x\rangle$):

$$(\Delta p_x)^2 - \frac{2}{\hbar\beta}\Delta x\Delta p_x + \frac{1 + \beta\langle p_x\rangle^2}{\beta} \leq 0 \tag{4.5}$$

$$\Rightarrow \Delta p_x \leq \frac{\Delta x}{\hbar\beta} \pm \sqrt{\left(\frac{\Delta x}{\hbar\beta}\right)^2 - \frac{1}{\beta} - \langle p_x\rangle^2} \tag{4.6}$$

See [5]. Thus, we must have

$$\Delta x = \sqrt{\langle(\Delta x)^2\rangle} \geq \hbar\sqrt{\beta}\sqrt{1 + \beta\langle p_x\rangle^2} \tag{4.7}$$

Thus, the absolute minimal uncertainty in position measurements is $\hbar\sqrt{\beta}$, for $\langle p_x\rangle = 0$. Since the same arguments hold for the y- and z-coordinates, we see that the absolute minimal uncertainty in measuring x^j is given by $\Delta x_0^j = \hbar\sqrt{\beta}$, for $\langle p^j \rangle = 0$. Since $\hbar\sqrt{\beta} = l_P\sqrt{b}$, where l_P is the Planck length, we see that there is a *cube of absolute minimal uncertainty*, of scale-dependent side-length, which can not be breached.

We now discuss in detail the implications of the above conclusions.

Since the equality in (4.6) only holds when $\Delta p^j = 1/\sqrt{\beta}$ and $\Delta x^j = \hbar\sqrt{\beta}$, it follows that at any other value, only the strict inequality will hold. Now, for $\langle p^j \rangle^2 = 0$ one gets $\langle (p^j)^2 \rangle = \langle (\Delta p^j)^2 \rangle$. We therefore have, in an inertial frame:

$$\langle E^2 \rangle = \langle (\Delta \mathbf{p})^2 \rangle c^2 + m^2 c^4 \tag{4.8}$$

Now, let us assume that the uncertainty in position measurements could be lowered below the absolute minimal uncertainty $\hbar\sqrt{\beta}$. That is, let us assume that the side-length of the cube of absolute minimal uncertainty falls below $\hbar\sqrt{\beta}$: $\langle (\Delta x^j)^2 \rangle = \hbar^2\beta - \varepsilon$, where $\varepsilon > 0$. We immediately see that $\langle (\Delta p^j)^2 \rangle$ becomes imaginary. This implies, from equation (4.8) above, that $\langle E^2 \rangle$ ceases to exist. We thus conclude that in any inertial frame, a region defined by a cube of side-length less than $\hbar\sqrt{\beta}$ is forbidden from access. Thus, if we try to measure the position of a particle with a photon of wavelength less than $\hbar\sqrt{\beta}$, this photon will be scattered by the cube of absolute minimal uncertainty. As explained in [5], this does not mean that we are working in a lattice space, since the position of the particle is not known within the cube of absolute minimal uncertainty.

At this point, the reason why momentum can no longer be considered the generator of momentum, becomes readily amenable to comprehension. Once a particle is maximally localized - *i.e.* localized with an uncertainty of $\hbar\sqrt{\beta}$ in its position - it can freely move about within the cube of absolute minimal uncertainty: its momentum is no longer a generator of its translation because it can be localized no further.

We recognize that the minimal uncertainty in position is not a physically measurable "length": it is minimum when $\langle p_x \rangle$ is zero, and increases with $\langle p_x \rangle$ (however, we can not say that *translation* leads to an increase in the uncertainty, since momentum is no longer the generator of translation). Thus, the existence of a nonzero minimal uncertainty does not violate Lorentz invariance: a result consistent with recent experiments ([14]) which have found no evidence for Lorentz violation down to the Planck scale.

We also observe that multiplying the inequality (4.5) throughout by $\hbar\beta$ and letting $\beta \to 0$, we regain the large scale (low energy) uncertainty relation.

A pertinent question at this point would be: how do we know *a priori* that a momentum representation of position operators, of the form given by equation (4.1), exists? Indeed, we have been able to derive a representation of position in momentum-space only because of the assumption that momentum is an observable. To elaborate, let us, hypothetically speaking, write a generalized representation of momentum in the position space, which, following the exact same arguments as above, will be (we only study one of the two possible representations, both of which yield the same qualitative result):

$$\hat{p}^\mu \equiv i\hbar[\eta^{\mu\nu} + \alpha x^\mu x^\nu]\partial_\nu \tag{4.9}$$

where $\alpha = ac^3/\hbar G$, and a is a dimensionless parameter. It is trivial to see that the above representation is not consistent with the representation (4.1); it leads to nonzero minimal uncertainty in momentum measurements, and last but not the least, allows a commutative spacetime, while making the momentum-space noncommutative. In fact, such a representation is rendered possible only by the assumption that position is an observable, and therefore has eigenstates. As will be shown in Section 8, this assumption, together with the representation in equation

(4.9), makes the theory Lorentz invariant while maintaining homogeneity of space. It is also clear that both representations maintain unitarity.

The question is: which do we choose? It is clear that the choice must be dictated by physical considerations. As has been explained in [4], it is probable that Nature exhibits minimal uncertainties in both position and momentum measurements. Thus in general, we may require mutually consistent Lorentz contravariant representations of both position and momentum. For the purpose of this paper, we maintain the assumption that momentum does not exhibit nonzero minimal uncertainty in measurements. This will allow us to retain the hermiticity of momentum operators. In Section 6, we briefly discuss the implications of nonzero minimal uncertainty in momentum measurements.

In the next section, we shall construct the Hilbert of the algebra defined by equations (2.1) and (4.2).

5. The Hilbert Space

The Hilbert space has to satisfy the following familiar requirements:

(1) Physical states are normalizable. In particular, the norm has to be positive definite.
(2) Expectation values of position ($\langle x^\mu \rangle$) and momentum ($\langle p^\mu \rangle$) are well defined.
(3) Uncertainties in position ($\Delta x^\mu = \sqrt{\langle (x^\mu)^2 \rangle - \langle x^\mu \rangle^2}$) and momentum ($\Delta p^\mu = \sqrt{\langle (p^\mu)^2 \rangle - \langle p^\mu \rangle^2}$) are well defined.

Thus, physical states must lie in the common domain $D_{\mathbf{x},\mathbf{x}^2,\mathbf{p},\mathbf{p}^2} \bigcap D_{t,t^2,p_t,p_t^2}$, where $\forall \mu\ x^\mu$, x_μ^2, p^μ, p_μ^2 are symmetric.

The representation theoretic consequences of the uncertainty relations similar to the one given by inequality (4.4) have been analyzed before ([5]), and shall not be reviewed in complete detail here. However, we note the most important consequence of all: a nonzero minimal uncertainty in position measurements implies that the position operators have no eigenstates. This means that the position operators, though still symmetric, are no longer self-adjoint. Thus all information about position must be obtained in the momentum space.

5.1. **Representation on momentum space.** Momentum operators are evidently self-adjoint. However, as has been hinted by Snyder ([15]), it can be shown (see Section 3) that the position operators are symmetric (but not self-adjoint) only with respect to the "scaled" scalar product:

$$\langle \psi | \varphi \rangle = \int \frac{d^4 p}{[1 + \beta(\mathbf{p}^2 - p_t^2)]^{5/2}} \psi^*(p)\varphi(p) \tag{5.1}$$

Thus, owing to the requirement that all physical states must lie in the common domain $D_{\mathbf{x},\mathbf{x}^2,\mathbf{p},\mathbf{p}^2} \bigcap D_{t,t^2,p_t,p_t^2}$, we take equation (5.1) as the definition of scalar product, since both momentum and position are symmetric with respect to it.

The momentum eigenstates are mutually orthogonal:

$$\langle p | p' \rangle = \delta(p - p') \tag{5.2}$$

From the above orthogonality, we get the usual completeness relation (we do not require the factor of $[1 + \beta(\mathbf{p}^2 - p_t^2)]^{-5/2}$, since the momentum eigenstates, not being normalizable, are not physical states):

$$\int dp |p\rangle\langle p| = 1 \tag{5.3}$$

Using equation (4.1), we can write:

$$\hat{p}^\mu \psi(p) = p^\mu \psi(p) \tag{5.4}$$

$$\hat{x}^\sigma \psi(p) = i\hbar(-\eta^{\sigma\nu} + \beta p^\sigma p^\nu)\frac{\partial}{\partial p^\nu}\psi(p) = \lambda^\sigma \psi(p) \tag{5.5}$$

Though equation (5.5) can be solved for formal position eigenstates, we see immediately why they are of no consequence. Such formal eigenstates will yield a vanishing uncertainty in position measurements, which, as we have seen in inequality (4.7) (and the discussion that follows it), is forbidden.

In the next section, we calculate the state which represents the state of maximal localization, *i.e.*, the state with the absolute minimal uncertainty in position. As we will see, these are the only states which yield useful information about position. Though maximal localization states have been calculated before (see [5]), most existing calculations attempt to calculate the maximal localization state by solving the eigenvalue equation $\Delta x\psi = \hbar\sqrt{\beta}\psi$. As will be argued in the next section, this practice is fundamentally wrong.

6. The maximal localization states

As we saw in the last section, formal eigenstates of the position operator (in momentum representation) are forbidden by the relation (4.7). This means that the only information which we can obtain about position is, so to say, the region of maximal proximity around any given point λ. This region is defined by a circle centered at λ:

$$\langle (x-\lambda)^2\rangle \geq \hbar^2\beta \tag{6.1}$$

Since $\langle \Delta x\rangle = 0$ identically, and in general is not equal to $\sqrt{\langle(\Delta x)^2\rangle}$, whose minimum, as we have seen, is nonzero, it is clear that we must use $(\Delta x)^2$ to find the minimal localization states. Another reason for this is that $[\Delta x, x] = 0$, implying that eigenstates of Δx are the same as the formal eigenstates of the position operator, which we know are of no consequence since they always yield a vanishing uncertainty (see *e.g.*, [5] wherein Δx, instead of $(\Delta x)^2$, is incorrectly used to compute the maximal localization states).

Now, since the lowest eigenvalue of an observable is equal to the minimum of the expectation value of that observable, we conclude that the lowest eigenvalue of $(\Delta x)^2$ is $\hbar^2\beta$. Thus, the maximal localization state will be the state with this eigenvalue for $(\Delta x)^2$. Further, if $\langle(\Delta x)^2\rangle = \hbar^2\beta$ around the point λ, we can denote the corresponding maximal localization state by $|\hbar^2\beta\rangle_\lambda$. We also have ${}_\lambda\langle\hbar^2\beta|x|\hbar^2\beta\rangle_\lambda = \lambda$.

Thus, the maximal localization state is the solution of the following equation:

$$(\Delta x)^2|\hbar^2\beta\rangle_\lambda = (x^2 - 2x\langle x\rangle + \langle x\rangle^2)|\hbar^2\beta\rangle_\lambda = (x^2 - 2x\lambda + \lambda^2)|\hbar^2\beta\rangle_\lambda = \hbar^2\beta|\hbar^2\beta\rangle_\lambda \tag{6.2}$$

where, following equation (4.1),

$$x\psi(p) = i\hbar\left[(1+\beta p_x^2)\frac{\partial}{\partial p_x} + \beta p_x p_y\frac{\partial}{\partial p_y} + \beta p_x p_z\frac{\partial}{\partial p_z} + \beta p_x p_t\frac{\partial}{\partial p_t}\right]\psi(p) \tag{6.3}$$

The solution, derived in the Appendix, is:

$$|\hbar^2\beta\rangle_\lambda = \Pi p_t^{\zeta_0} p_y^{\zeta_2} p_z^{\zeta_3} 2^{-(\alpha_1+\gamma_1)}(1+\beta p_x^2)^{\frac{\alpha_1+\gamma_1}{2}} e^{i(\alpha_1-\gamma_1)ArcTan(\sqrt{\beta}p_x)}$$
$${}_2F_1(1-n, \beta_2-\alpha_1-\gamma_1; 1-\alpha_1-\alpha_2-2\gamma_1; \sqrt{1+\beta p_x^2}e^{iArcTan(\sqrt{\beta}p_x)}/2) \tag{6.4}$$

where, ζ_0, ζ_2, and ζ_3 are constants (see equation (11.15)), Π is a normalization constant, α_i, β_i and γ_i are exponents defined in the Appendix (equations (11.11)), and n is a natural number.

As is shown in the Appendix, the maximal localization states are not orthogonal, except in the limit $\beta \to 0$. The physical interpretation of this is rather simple. Since the maximal localization state denotes the state in which the square of the uncertainty in position of a particle, around a point $x = \lambda$, is $\hbar^2\beta$, we immediately see that this particle is also maximally localized around any of the infinitely many points $x = \lambda'$ for which $|\lambda - \lambda'| < \hbar\sqrt{\beta}$. The question of orthogonality of two states when $|\lambda - \lambda'| \geq \hbar\sqrt{\beta}$ will be discussed in a future publication.

In the next section, we investigate the Schrödinger equation for a simple harmonic oscillator. It will be shown that, except in the limit $\beta \to 0$, one can not find creation and annihilation operators which can provide a representation-free basis of states.

7. The harmonic oscillator

Before we proceed, we prove an important lemma:

Lemma 1. *No physical state may depend on the free-particle/-field energy $cp^0 = cp_t$. By a physical state, we mean here a solution of the equation $H\psi = E\psi$*

Proof: Consider the formal energy eigenvalue equation:

$$H\psi = E\psi = [T + U(\mathbf{r})]\psi \tag{7.1}$$

where $U(\mathbf{r})$ is the interaction energy. Let us assume that $\psi = \psi(\mathbf{p}, p_t)$. Since each component of $\mathbf{r}$ has a p_t-dependence through the operator $p_t\frac{\partial}{\partial p_t}$, we can write in the momentum representation

$$H\psi(\mathbf{p}, p_t) = E\psi(\mathbf{p}, p_t) = [T + U(\mathbf{r}(\mathbf{p}, p_t))]\psi(\mathbf{p}, p_t) \tag{7.2}$$

However, the free-particle energy $cp_t = H - U$. Therefore, we get:

$$H\psi(\mathbf{p}, U) = E\psi(\mathbf{p}, U) = [T + U(\mathbf{r}(\mathbf{p}, U))]\psi(\mathbf{p}, U) \tag{7.3}$$

It is clear that the self-recursion in equation (7.3) can not be removed, and hence no $\psi = \psi(\mathbf{p}, p_t)$ can be found which is a solution of equation (7.1). However, if $\psi = \psi(\mathbf{p})$, then equation (7.1) gets rid of its dependence on p_t (since each derivative of ψ with respect to p_t vanishes), and hence has solutions.

The one-dimensional harmonic oscillator Hamiltonian, with mass m and frequency ω reads:

$$H = \frac{m\omega^2}{2}x^2 + \frac{p_x^2}{2m} \tag{7.4}$$

From equation (4.1), the representation of x is:

$$x \equiv i\hbar\left[(1 + \beta p_x^2)\frac{\partial}{\partial p_x} + \beta p_x p_t \frac{\partial}{\partial p_t}\right] \tag{7.5}$$

However, due to Lemma 1 above, the Schrödinger equation in momentum representation, for the Hamiltonian (7.4), becomes:

$$\frac{d^2\psi}{dp_x^2} + \frac{2\beta p_x}{1 + \beta p_x^2}\frac{d\psi}{dp_x} + \frac{2}{m\hbar^2\omega^2(1 + \beta p_x^2)^2}\left[E - \frac{p_x^2}{2m}\right]\psi = 0 \tag{7.6}$$

For details of the solution of the above equation, see [5]. It might be hoped that a representation-free solution for the harmonic oscillator could be found in the Bargmann-Fock space of generalized creation and annihilation operators.

To find generalized creation and annihilation operators consistent with the algebra given by equation (4.2), we first note that two arbitrary operators $A \equiv$

$a_1x + ia_2f(p_x)$ and $B \equiv b_1x + ib_2f(p_x)$, with a_1, a_2, b_1, and b_2 all real constants, can be called creation and annihilation operators, respectively, if and only if they satisfy both of the following requirements:

(1) $[A, B] = 1$.
(2) $[H, AB] = [H, BA] = 0$. Thus B creates, while A annihilates, the eigenstates of the Hamiltonian. Further, this also means that the operator BA functions as the number operator.

If the first requirement is violated, then the vacuum state can not be uniquely identified, as a result of which states above the vacuum can not be defined. If the second requirement is violated, then the eigenstates of BA are no longer the eigenstates of the Hamiltonian. See the result derived in [8], in which the operator $a^\dagger a$ violates this second requirement, and is thus not the number operator of the oscillator, as is incorrectly reported.

Any two operators A and B satisfying both of the above requirements then allow the Hamiltonian (7.4) to be written in the form:

$$H = \hbar\omega\left[BA + \frac{g}{2}\right] \tag{7.7}$$

where g is a dimensionless constant, such that $g = 1$ if and only if $\beta = 0$. We also have the following constraints:

$$\lim_{\beta\to 0} B = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p_x}{m\omega}\right) \tag{7.8}$$

$$\lim_{\beta\to 0} A = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p_x}{m\omega}\right) \tag{7.9}$$

We now prove that unless $\beta = 0$, such operators A and B, satisfying both of the above two requirements, do not exist.

To see this, we note that the Hamiltonian defined by equation (7.7) is:

$$H = \hbar\omega\left[a_1b_1x^2 - a_2b_2f^2 + i(a_2b_1xf + a_1b_2fx) + \frac{g}{2}\right] \tag{7.10}$$

Therefore, we get, from (7.4):

$$a_1b_1 = \frac{m\omega}{2\hbar} \tag{7.11}$$

$$-a_2b_2f^2 + i(a_2b_1xf + a_1b_2fx) + \frac{g}{2} = \frac{p_x^2}{2m} \tag{7.12}$$

It is clear from equation (7.12) that $a_1b_2 = -a_2b_1$. Further, it is clear that the above two equations complete the second requirement: $[H, BA] = 0$.

Now, the first requirement gives

$$[A, B] = i(a_1b_2 - a_2b_1)[x, f] = i2a_1b_2[x, f] = 1 \tag{7.13}$$

which is nothing but the first order partial differential equation:

$$(1 + \beta p_x^2)\frac{\partial f}{\partial p_x} = -\frac{1}{2\hbar a_1b_2} \tag{7.14}$$

The solution is:

$$f = -\frac{1}{2\hbar\sqrt{\beta}a_1b_2}ArcTan(\sqrt{\beta}p_x) + \mathcal{C} \tag{7.15}$$

where $\mathcal{C}$ is an arbitrary constant. This solution is the same as reported in [8], with $\mathcal{C} = 0$. However, substituting f from equation (7.15) into equation (7.12) gives:

$$\begin{aligned}
&\frac{1}{a_2 b_2}\left[\frac{p_x^2}{2m\hbar\omega} + \frac{1-g}{2}\right] + \frac{1}{4\hbar^2 \beta a_1^2 b_2^2} ArcTan^2(\sqrt{\beta} p_x) + \mathcal{C}^2 \\
&\quad = \frac{\mathcal{C}}{2\hbar\sqrt{\beta} a_1 b_2} ArcTan(\sqrt{\beta} p_x) \qquad (7.16) \\
\Rightarrow \quad &\mathcal{C} = \frac{1}{2}\left[\frac{1}{\hbar\sqrt{\beta} a_1 b_2} ArcTan(\sqrt{\beta} p_x) \pm \sqrt{\frac{2}{a_2 b_2}}\sqrt{g - 1 - \frac{p_x^2}{m\hbar\omega}}\right]
\end{aligned}$$

If $\mathcal{C}$ is a constant, then $d\mathcal{C}/dp_x = 0$ identically. Thus, for all p_x, we must have

$$2\beta^2 p_x^6 + 4\beta p_x^4 + \left[2 + \frac{a_2 m\omega}{a_1^2 b_2 \hbar}\right] p_x^2 + \frac{a_2 m^2 \omega^2}{a_1^2 b_2}(1-g) = 0 \qquad (7.17)$$

Since coefficients of all powers of p_x must individually vanish, we get $\beta = 0$, and $g = 1$, which is just the result of ordinary, low-energy quantum mechanics.

8. Nonzero minimal uncertainty in momenta and homogeneity of space

A note about the possibility of nonzero minimal uncertainty in momentum measurements is in order. We saw in Section 2 that the existence of a nonzero minimal uncertainty in position measurements necessarily renders space inhomogeneous, because points lose their meaning as geometrically localizable entities. However, if we assume that position is an observable, then what can we say about the possibility of a nonzero minimal uncertainty in momentum measurements?

Translational invariance in quantum mechanics is generally expressed by

$$\psi'(x - \delta x) \approxeq \psi(x) - \eta^{\mu\nu} \delta x_\mu \partial_\nu \psi(x) \qquad (8.1)$$

where δx^μ is an infinitesimal translation. We observe that equation (8.1) is manifestly Lorentz invariant. However, on using equation (4.9) (keeping in mind that it results in a nonzero minimal uncertainty in momentum measurements), we find that it is just an approximation to the more general statement:

$$\psi'(x - \delta x) \approxeq \psi(x) - \delta x_\mu [\eta^{\mu\nu} + \alpha x^\mu x^\nu] \partial_\nu \psi(x) \qquad (8.2)$$

Therefore, we find that homogeneity of space simply hinges on whether or not position is an observable. Conversely, if space is exactly homogeneous, then position is an observable. In addition, a nonzero minimal uncertainty in momentum measurements too is consistent with Lorentz invariance and translational invariance. Algebraically, we see that this is because momentum now has a representation in the position-space, which makes momentum act as the generator of infinitesimal translations.

At this point, we see that one can generalize the notion of homogeneity to momentum-space as well. Whereas the existence of a nonzero minimal uncertainty in position measurements renders position-space inhomogeneous, the existence of momentum eigenstates maintains the homogeneity of the momentum space. To see this, we observe that the invariance of wavefunctions in momentum-space, under infinitesimal variations of momenta, can be expressed as:

$$\Psi'(p - \delta p) \approxeq \Psi(p) - \eta^{\mu\nu} \delta p_\mu \partial_{p\,\nu} \Psi(p) \qquad (8.3)$$

which, using equation (4.1), may be generalized to:

$$\Psi'(p - \delta p) \approxeq \Psi(p) + \delta p_\mu [-\eta^{\mu\nu} + \beta p^\mu p^\nu] \partial_{p\,\nu} \Psi(p) \qquad (8.4)$$

which is manifestly Lorentz-invariant. We also observe that both equations (8.2) and (8.4) preserve the norm for infinitesimal δx and δp.

Now we address the larger question: how likely is Nature to exhibit a nonzero minimal uncertainty in momentum measurements? For the purpose of generality of argument, let us include the possibility that Nature may exhibit nonzero minimal uncertainties in measurements of both position and momentum. The generalized uncertainty relation, with nonzero minimal uncertainties in both position and momentum measurements, will take the following form:

$$\begin{aligned}\Delta x^\mu \Delta p^\nu \geq \frac{\hbar}{2}\bigg| -\eta^{\mu\nu} + a\frac{c^3}{\hbar G}\Delta x^\mu \Delta x^\nu + a\frac{c^3}{\hbar G}\langle x^\mu\rangle\langle x^\nu\rangle + \mathcal{O}(a^2) \\ +b\frac{G}{\hbar c^3}\Delta p^\mu \Delta p^\nu + b\frac{G}{\hbar c^3}\langle p^\mu\rangle\langle p^\nu\rangle + \mathcal{O}(b^2)\bigg|\end{aligned} \tag{8.5}$$

Though one may write an uncertainty relation which breaks Lorentz symmetry, what is important to realize is that the factors $G/\hbar c^3$ and $c^3/\hbar G$ will always be present.

We observe that the constant $G/\hbar c^3$ is of the order of 10^{-1}kg^{-2}m^{-2}s^{2}. Since the nonzero minimal uncertainty in position measurements is expected to be observable only at extremely small scales, it is plausible that the dimensionless constant b is an decreasing function of l_P/L, where L represents the size of the object whose position is being measured, and l_P is the Planck length. Thus we see that unless L becomes comparable to l_P, b remains negligible.

On the other hand, the constant $c^3/\hbar G$, which is $1/l_P^2$, is of the order of 10^{69}m^{-2}. Since the nonzero minimal uncertainty in momentum measurements is expected to be observable only at large scales, it becomes extremely difficult to explain how, at laboratory scales, this nonzero minimal uncertainty in momentum eludes observation. This may be an indication, as suggested by the incompatibility between representations (4.1) and (4.9) (unless $a = b = 0$), that Nature may exhibit a nonzero minimal uncertainty only in position measurements.

9. Concluding remarks

By simply deriving the most general Lorentz contravariant representation of position in momentum space, and then imposing the condition of symmetry on it, we have shown that the notion of a nonzero minimal uncertainty in position measurements is consistent with Lorentz invariance. Lorentz invariance alone does not predict the existence of the nonzero minimal uncertainty in position: it merely presents us with two alternate - and mutually incompatible - cutoffs, one in position measurements and the other in momentum measurements. The question of the existence of one cutoff, and the absence of the other, seems to be arbitrated by the magnitudes of the constants $G/\hbar c^3$ and $c^3/\hbar G$.

Though Lorentz invariance itself allows nondenumerably infinite geometries, as is evident from the arbitrary scalars $\Omega_0^{\mu\nu\sigma\rho}$ and $\tilde{P}_0^{\mu\nu}$ in equation (2.8), we have seen how the symmetry of position operators uniquely picks up two geometries, one among which is the Snyder-geometry (coefficients of correction terms, in any generalized algebra, must be similarly determined by imposing the nontrivial condition of symmetry on the operator, instead of using the Jacobi identity, which, being an identity, will always be satisfied. E.g., see [20, 17], wherein the position operator is not symmetric.). Further, we have seen that the uncertainty in position is an increasing function of momentum. Thus, minimal uncertainty is not physically analogous to a measurable length. Lastly, whether or not Lorentz invariance breaks down at, or below, the Planck scale, we have seen why the existence of a nonzero minimal uncertainty necessarily renders space inhomogeneous.

Owing to the fact that position no longer is an observable, we have been forced to content ourselves with measuring position with the absolute minimal uncertainty, instead of arbitrary accuracy. We have seen, in Section 4, how this leads to the maximally localized wavefunctions, which are in general not orthogonal, but are normalizable. This result has been reported before too ([5]), however, as explained in Section 6, the calculations used in that paper are fundamentally incorrect. The concept of maximal localization states is nothing new, and has been investigated before. However, as has been argued in Section 6, most existing calculations suffer from a fundamental error of computing the maximal localization state from the eigenvalue equation $\Delta x\psi = \hbar\sqrt{\beta}\psi$, instead of the correct equation $(\Delta x)^2\psi = \hbar^2\beta\psi$.

The usual method of doing quantum field theory in a noncommutative spacetime is by defining an associative *-product of fields. However, with a nonzero minimal uncertainty in position measurements, we can no longer integrate over a smooth manifold, but, as explained in [6], over the space of position expectation values of the maximal localization states. Once the *-product is defined this way, as shown in [6], the normalizability of the maximal localization state (shown in the Appendix), which is a consequence of the nonzero minimal uncertainty in position, regularizes the ultraviolet divergence of the Feynman tadpole diagram. Thus, the nonzero minimal uncertainty in position measurements effectively acts as a UV cutoff.

It is worthwhile here to note that the result shown in [21], that noncommutativity of spacetime does not necessarily regularize UV divergences, does in no way contradict the results shown in the current work. This is because in the geometry we have considered, $[\hat{x}^\mu, \hat{x}^\nu] \propto (\hat{x}^\mu\hat{p}^\nu - \hat{x}^\nu\hat{p}^\mu)$, which is not a constant. This makes the transformation to phase space operators, of the type considered in [21], which follow the algebra of ordinary quantum mechanics, impossible.

10. Acknowledgements

I am indebted to Prof. A. Kempf of the University of Waterloo, Prof. Oskar Vafek of Florida State University, Tallahassee, and Prof. V. Subrahmanyam of the Indian Institute of Technology, Kanpur, for invaluable discussions.

11. Appendix

We now derive the solution (6.4) of equation (6.2). Since the equation is linear and homogeneous, we can find a variable separable solution of the form

$$|\hbar^2\beta\rangle_\lambda = \psi_0(p_t)\psi_1(p_x)\psi_2(p_y)\psi_3(p_z) \tag{11.1}$$

On dividing equation (6.2) throughout by $\psi_0\psi_1\psi_2\psi_3$, with x represented as in equation (6.3), we obtain, after collecting all terms with dependence on p_x:

$$\begin{aligned}\frac{d^2\psi_1}{dp_x^2} + \frac{2}{\hbar(1+\beta p_x^2)}[\hbar\beta p_x(1+\zeta) + i\lambda]\frac{d\psi_1}{dp_x} + \\ \frac{\hbar^2\beta^2 p_x^2(2\zeta+\theta) + i2\hbar\beta\lambda\zeta p_x + \hbar^2\beta(1+\zeta) - \lambda^2}{\hbar^2(1+\beta p_x^2)^2}\psi_1 = 0\end{aligned} \tag{11.2}$$

where

$$\zeta \equiv \frac{p_t}{\psi_0}\frac{d\psi_0}{dp_t} + \frac{p_y}{\psi_2}\frac{d\psi_2}{dp_y} + \frac{p_z}{\psi_3}\frac{d\psi_3}{dp_z} \tag{11.3}$$

and

$$\begin{aligned}\theta =& \frac{p_t^2}{\psi_0}\frac{d^2\psi_0}{dp_t^2} + \frac{p_y^2}{\psi_2}\frac{d^2\psi_2}{dp_y^2} + \frac{p_z^2}{\psi_3}\frac{d^2\psi_3}{dp_z^2} + \\ & 2\frac{p_t}{\psi_0}\frac{p_y}{\psi_2}\frac{d\psi_0}{dp_t}\frac{d\psi_2}{dp_y} + 2\frac{p_y}{\psi_2}\frac{p_z}{\psi_3}\frac{d\psi_2}{dp_y}\frac{d\psi_3}{dp_z} + 2\frac{p_z}{\psi_3}\frac{p_t}{\psi_2}\frac{d\psi_3}{dp_z}\frac{d\psi_0}{dp_t}\end{aligned} \tag{11.4}$$

from where we get

$$\theta = \zeta^2 - \zeta + p_t \frac{\partial \zeta}{\partial p_t} + p_y \frac{\partial \zeta}{\partial p_y} + p_z \frac{\partial \zeta}{\partial p_z} \tag{11.5}$$

On making the substitution $\chi = (1 + i\sqrt{\beta} p_x)/2$, equation (11.2) becomes the hypergeometric equation ([22]):

$$\begin{aligned} \frac{d^2\psi_1}{d\chi^2} + &\frac{[2(1+\zeta)\chi - (1+\zeta+\frac{\lambda}{\hbar\sqrt{\beta}})]}{\chi(\chi-1)} \frac{d\psi_1}{d\chi} + \Bigg[(2\zeta+\theta)\chi^2 - (2\zeta+\theta+\zeta\frac{\lambda}{\hbar\sqrt{\beta}})\chi \\ &+ \frac{1}{4}\left(\zeta+\theta+2\zeta\frac{\lambda}{\hbar\sqrt{\beta}} + \frac{\lambda^2}{\hbar^2\beta} - 1\right)\Bigg] \frac{\psi_1}{\chi^2(\chi-1)^2} = 0 \end{aligned} \tag{11.6}$$

The above equation has three regular singularities at 0, 1 and ∞, at which the indicial equations are, respectively:

$$r^2 + \left[\zeta + \frac{\lambda}{\hbar\sqrt{\beta}}\right] r + \frac{1}{4}\left[\zeta + \theta + \zeta\frac{\lambda}{\hbar\sqrt{\beta}} + \frac{\lambda^2}{\hbar^2\beta} - 1\right] = 0; \tag{11.7}$$

$$r^2 + \left[\zeta - \frac{\lambda}{\hbar\sqrt{\beta}}\right] r + \frac{1}{4}\left[\zeta + \theta + \frac{\lambda^2}{\hbar^2\beta} - 1\right] = 0; \tag{11.8}$$

$$r^2 - (2\zeta+\theta) r + 2\zeta + \theta = 0 \tag{11.9}$$

with respective roots

$$\begin{aligned} \alpha_1,\ \alpha_2 &= \frac{1}{2}\left[-\left(\zeta + \frac{\lambda}{\hbar\sqrt{\beta}}\right) \pm \sqrt{\zeta^2 - \zeta - \theta + 1 + \zeta\frac{\lambda}{\hbar\sqrt{\beta}}}\right]; \\ \gamma_1,\ \gamma_2 &= \frac{1}{2}\left[-\left(\zeta - \frac{\lambda}{\hbar\sqrt{\beta}}\right) \pm \sqrt{\zeta^2 - \zeta - \theta + 1 - 2\zeta\frac{\lambda}{\hbar\sqrt{\beta}}}\right]; \\ \beta_1,\ \beta_2 &= \frac{1}{2}[2\zeta + 1 \pm \sqrt{4(\zeta^2 - \zeta - \theta) + 1}] \end{aligned} \tag{11.10}$$

We note in passing that the sum of the pairs of exponents with respect to the singular points 0, 1 and ∞ is $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 1$, just as required. Now, ψ_1 is independent of p_t, p_y, and p_z. It is clear from equation (11.5) that unless ζ is a constant, this condition can not be satisfied, since all the exponents will otherwise be functions of p_t, p_y, and p_z. Taking ζ to be a constant gives, from equation (11.5), $\zeta^2 - \zeta - \theta = 0$. This reduces the exponents to:

$$\begin{aligned} \alpha_1,\ \alpha_2 &= \frac{1}{2}\left[-\left(\zeta + \frac{\lambda}{\hbar\sqrt{\beta}}\right) \pm \sqrt{1 + \zeta\frac{\lambda}{\hbar\sqrt{\beta}}}\right]; \\ \gamma_1,\ \gamma_2 &= \frac{1}{2}\left[-\left(\zeta - \frac{\lambda}{\hbar\sqrt{\beta}}\right) \pm \sqrt{1 - 2\zeta\frac{\lambda}{\hbar\sqrt{\beta}}}\right]; \\ \beta_1,\ \beta_2 &= \frac{1}{2}[2\zeta + 1 \pm \sqrt{1}] = \zeta + 1,\ \zeta \end{aligned} \tag{11.11}$$

With the above values of the exponents, we can immediately write the solution (up to a multiplicative constant) of equation (11.6) as the generalized hypergeometric series ([23]):

$$\begin{aligned}\psi_1(\chi) &= P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha_1 & \beta_1 & \gamma_1 & ;\chi \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} \\ &= \chi^{\alpha_1}(1-\chi)^{\gamma_1}{}_2F_1(\alpha_1+\beta_1+\gamma_1, \beta_2-\alpha_1-\gamma_1; 1-\alpha_1-\alpha_2-2\gamma_1; \chi)\end{aligned} \tag{11.12}$$

where ${}_2F_1(a, b; c; \chi)$ is the Gauss hypergeometric function, which has the following properties:

(1) For $a - 1 = -n$, or $b - 1 = -n$, with $n \in \mathbb{N}$, the function reduces to a polynomial, and for c a non-positive integer, the function is not defined.
(2) In all other cases, the function is convergent if $|\chi|^2 < 1$.

Since $-\infty < p_x < \infty$, it follows that for $\psi_1(\chi)$ to converge, ${}_2F_1(\alpha_1+\beta_1+\gamma_1, \beta_2-\alpha_1-\gamma_1; 1-\alpha_1-\alpha_2-2\gamma_1; \chi)$ must reduce to a polynomial.

Thus, we impose the following conditions:

$$\begin{aligned}\alpha_1+\beta_1+\gamma_1-1 &= -n, \ \textit{or} \\ \beta_2-\alpha_1-\gamma_1-1 &= -n\end{aligned} \tag{11.13}$$

for $n \ \in \mathbb{N}$, and

$$\begin{gathered}1-\alpha_1-\alpha_2-2\gamma_1 \neq -n, \quad n \in \mathbb{N}\cup\{0\} \\ \Rightarrow \zeta \neq \frac{1}{4}\left[-2(n+1) - \frac{\lambda}{\hbar\sqrt{\beta}} \pm \sqrt{\left(\frac{\lambda}{\hbar\sqrt{\beta}}+2\right)^2 + 4n\frac{\lambda}{\hbar\sqrt{\beta}}}\right]\end{gathered} \tag{11.14}$$

This also makes physical sense, since we require ζ to vanish in the limit $\beta \to 0$. We shall return to these considerations shortly. Meanwhile, from equation (11.3), with ζ a constant, we can write:

$$\begin{aligned}&\frac{p_t}{\psi_0}\frac{d\psi_0}{dp_t} + \frac{p_y}{\psi_2}\frac{d\psi_2}{dp_y} + \frac{p_z}{\psi_3}\frac{d\psi_3}{dp_z} - \zeta = \\ &\qquad \left(\frac{p_t}{\psi_0}\frac{d\psi_0}{dp_t} - \zeta_0\right) + \left(\frac{p_y}{\psi_2}\frac{d\psi_2}{dp_y} - \zeta_2\right) + \left(\frac{p_z}{\psi_3}\frac{d\psi_3}{dp_z} - \zeta_3\right) = 0\end{aligned} \tag{11.15}$$

where $\zeta = \zeta_0 + \zeta_2 + \zeta_3$. Since the quantities within the three pairs of parentheses are functionally independent, the solutions are

$$\psi_0(p_t) = \Pi_0 p_t^{\zeta_0}, \quad \psi_2(p_y) = \Pi_2 p_y^{\zeta_2}, \quad \psi_3(p_z) = \Pi_3 p_z^{\zeta_3} \tag{11.16}$$

where Π_0, Π_2, and Π_3 are all constants. Since in the limit $\beta \to 0$, ψ_0, ψ_2, and ψ_3 must all reduce to constants, we conclude

$$\lim_{\beta\to 0}\zeta_\nu = 0 \quad \forall\nu \ \in \{0,2,3\} \tag{11.17}$$

Thus, the solution to equation (6.2) is

$$|\hbar^2\beta\rangle_\lambda = \Pi p_t^{\zeta_0} p_y^{\zeta_2} p_z^{\zeta_3}\chi^{\alpha_1}(1-\chi)^{\gamma_1}{}_2F_1(\alpha_1+\beta_1+\gamma_1, \beta_2-\alpha_1-\gamma_1; 1-\alpha_1-\alpha_2-2\gamma_1; \chi) \tag{11.18}$$

where Π is a normalization constant. Since $|\hbar^2\beta\rangle_\lambda$ must not diverge for large momenta and energies, we require $Re(\zeta_\nu) \leq 0, \ \forall\nu \ \in \{0,2,3\}$. This in turn means that $Re(\zeta) \leq 0$. The first of conditions (11.13) yields a quadratic equation in ζ:

$$\zeta^2 + \frac{8\hbar\sqrt{\beta}(n+\delta)^2}{9\lambda}\zeta + \frac{16(n+\delta)^2[(n+\delta)^2-1]\hbar^2\beta}{9\lambda^2} = 0, \quad \delta \in \{0,1\} \tag{11.19}$$

$$\Rightarrow \zeta = \frac{4\hbar\sqrt{\beta}}{9\lambda}(n+\delta)^2\left[-1 \pm \sqrt{\frac{9}{(n+\delta)^2} - 8}\right] \tag{11.20}$$

whose solution, as we observe, has a non-positive real part, as required. The second of conditions (11.13) yields a quartic equation in ζ whose solutions diverge in the limit $\beta \to 0$, and are thus, unacceptable. Thus we see that ζ is a function of a natural number, which in turn implies that all the exponents in equations (11.11) are functions of a natural number. With ζ given by equation (11.20), we get an equivalent form of equation (11.18) (the subscript λ is appended to emphasize the fact that ζ and all exponents depend explicitly on λ):

$$\begin{aligned}|\hbar^2\beta\rangle_\lambda = \Pi p_t^{\zeta_{\lambda 0}} p_y^{\zeta_{\lambda 2}} p_z^{\zeta_{\lambda 3}} 2^{-(\alpha_{\lambda 1}+\gamma_{\lambda 1})}(1+\beta p_x^2)^{\frac{\alpha_{\lambda 1}+\gamma_{\lambda 1}}{2}} e^{i(\alpha_{\lambda 1}-\gamma_{\lambda 1})ArcTan(\sqrt{\beta}p_x)} \\ {}_2F_1(1-n, B_\lambda^{(1)}; B_\lambda^{(2)}; \sqrt{1+\beta p_x^2}e^{iArcTan(\sqrt{\beta}p_x)}/2)\end{aligned} \tag{11.21}$$

where

$$B_\lambda^{(1)} = \beta_{\lambda 2} - \alpha_{\lambda 1} - \gamma_{\lambda 1}; \quad B_\lambda^{(2)} = 1 - \alpha_{\lambda 1} - \alpha_{\lambda 2} - 2\gamma_{\lambda 1} \tag{11.22}$$

We observe that in the limit $\beta \to 0$, the above state reduces to a plane wave.

The scalar product ${}_{\lambda'}\langle\hbar^2\beta|\hbar^2\beta\rangle_\lambda$ is given by

$$\begin{aligned}{}_{\lambda'}\langle\hbar^2\beta|\hbar^2\beta\rangle_\lambda =& |\Pi|^2 \int \frac{d^4p}{[1+\beta(\mathbf{p}^2-p_t^2)]^{5/2}}\Bigg[p_t^{\zeta_{\lambda 0}+\zeta^*_{\lambda' 0}} p_y^{\zeta_{\lambda 2}+\zeta^*_{\lambda' 2}} p_z^{\zeta_{\lambda 3}+\zeta^*_{\lambda' 3}} \\ &\left(\frac{\sqrt{1+\beta p_x^2}}{2}\right)^{\alpha_{\lambda 1}+\gamma_{\lambda 1}+\alpha^*_{\lambda' 1}+\gamma^*_{\lambda' 1}} e^{i[(\alpha_{\lambda 1}-\gamma_{\lambda 1})-(\alpha^*_{\lambda' 1}-\gamma^*_{\lambda' 1})]ArcTan(\sqrt{\beta}p_x)} \\ &{}_2F_1(1-n, B_\lambda^{(1)}; B_\lambda^{(2)}; \sqrt{1+\beta p_x^2}e^{iArcTan(\sqrt{\beta}p_x)}/2) \\ &{}_2F_1(1-n, B_{\lambda'}^{*(1)}; B_{\lambda'}^{*(2)}; \sqrt{1+\beta p_x^2}e^{-iArcTan(\sqrt{\beta}p_x)}/2)\Bigg]\end{aligned} \tag{11.23}$$

This integral converges only for $Re(\zeta_{\lambda 0} + \zeta^*_{\lambda' 0}) > -1$. We assume this to be true. Then the integral becomes

$$\begin{aligned}{}_{\lambda'}\langle\hbar^2\beta|\hbar^2\beta\rangle_\lambda =& 2^{\zeta_\lambda+\zeta^*_{\lambda'}-2}|\Pi|^2 C\int_{-\infty}^{\infty} dp_x \Bigg[\left(\frac{\sqrt{1+\beta p_x^2}}{2}\right)^{Re(N_\gamma+N_\alpha)-2} \\ &e^{i[\frac{\lambda-\lambda'}{\hbar\sqrt{\beta}}+iIm(N_\gamma-N_\alpha)]ArcTan(\sqrt{\beta}p_x)} \\ &{}_2F_1(1-n, B_\lambda^{(1)}; B_\lambda^{(2)}; \sqrt{1+\beta p_x^2}e^{iArcTan(\sqrt{\beta}p_x)}/2) \\ &{}_2F_1(1-n, B_{\lambda'}^{*(1)}; B_{\lambda'}^{*(2)}; \sqrt{1+\beta p_x^2}e^{-iArcTan(\sqrt{\beta}p_x)}/2)\Bigg]\end{aligned} \tag{11.24}$$

where

$$
\begin{aligned}
N_\alpha &= (-1)^{\delta_\alpha}\sqrt{1+\frac{4}{9}(n+\delta)^2\left[-1+(-1)^{\delta_0}\sqrt{\frac{9}{(n+\delta)^2}-8}\right]} \\
N_\gamma &= (-1)^{\delta_\gamma}\sqrt{1-\frac{8}{9}(n+\delta)^2\left[-1+(-1)^{\delta_0}\sqrt{\frac{9}{(n+\delta)^2}-8}\right]} \\
&\delta_0, \delta_\alpha, \delta_\gamma \in \{0,1\}
\end{aligned}
\tag{11.25}
$$

and

$$
\begin{aligned}
C = -&\frac{i^{1-\zeta_{\lambda 3}-\zeta^*_{\lambda' 3}}}{3\beta^{(3+\zeta_\lambda+\zeta^*_{\lambda'})/2}\sqrt{\pi}}[1+(-1)^{\zeta_{\lambda 0}+\zeta^*_{\lambda' 0}}][1+(-1)^{\zeta_{\lambda 2}+\zeta^*_{\lambda' 2}}][1+(-1)^{\zeta_{\lambda 3}+\zeta^*_{\lambda' 3}}] \\
&\Gamma\left(\frac{1+\zeta_{\lambda 0}+\zeta^*_{\lambda' 0}}{2}\right)\Gamma\left(\frac{1+\zeta_{\lambda 2}+\zeta^*_{\lambda' 2}}{2}\right)\Gamma\left(\frac{1+\zeta_{\lambda 3}+\zeta^*_{\lambda' 3}}{2}\right)\Gamma\left(\frac{2-\zeta_\lambda-\zeta^*_{\lambda'}}{2}\right)
\end{aligned}
\tag{11.26}
$$

Equation (11.24) reduces to

$$
\begin{aligned}
{}_{\lambda'}\langle \hbar^2\beta|\hbar^2\beta\rangle_\lambda =& 2^{\zeta_\lambda+\zeta^*_{\lambda'}-2}|\Pi|^2 C\sum_{r=s=0}^{n-1}\mathcal{C}_{rs}\int_{-\infty}^{\infty}dp_x(1+\beta p_x^2)^{\frac{Re(N_\gamma+N_\alpha)+r+s-2}{2}} \\
&e^{i(\frac{\lambda-\lambda'}{\hbar\sqrt{\beta}}+r-s)-Im(N_\gamma-N_\alpha)}
\end{aligned}
\tag{11.27}
$$

where

$$
\mathcal{C}_{rs} = \frac{\Gamma(1-n+r)\Gamma(1-n+s)\Gamma(B^{(1)}_\lambda+r)\Gamma(B^{*(1)}_{\lambda'}+s)}{r!s!\Gamma(B^{(2)}_\lambda+r)\Gamma(B^{*(2)}_{\lambda'}+s)}
\tag{11.28}
$$

The integral in equation (11.27) converges if $Re(N_\gamma+N_\alpha)+r+s<1$, which can be shown to hold. Equation (11.27) then evaluates to

$$
\begin{aligned}
{}_{\lambda'}\langle \hbar^2\beta|\hbar^2\beta\rangle_\lambda =& 2^{\zeta_\lambda+\zeta^*_{\lambda'}-2}|\Pi|^2\sqrt{\frac{\pi}{\beta}}C\sum_{r=s=0}^{n-1}\frac{\mathcal{C}_{rs}}{\Gamma(A_1)\Gamma(A_2+1/2)}\Bigg[\Gamma(A_1)\Gamma(A_2) \\
&{}_2F_1\left(\frac{A_1}{2},\frac{1+A_1}{2};1-A_2;1\right)+2^{2s-1-\frac{\lambda-\lambda'}{\hbar\sqrt{\beta}}+N^*_\gamma+N_\alpha}\Gamma(A_1+2A_2) \\
&\Gamma(-A_2){}_2F_1\left(\frac{A_1+2A_2}{2},\frac{1+A_1+2A_2}{2};A_2+1;1\right)\Bigg]
\end{aligned}
\tag{11.29}
$$

where

$$
\begin{aligned}
A_1 &= -\frac{\lambda-\lambda'}{\hbar\sqrt{\beta}}-r+s-iIm(N_\gamma-N_\alpha); \\
A_2 &= \frac{1}{2}\left[1+\frac{\lambda-\lambda'}{\hbar\sqrt{\beta}}-2s-(N_\gamma+N^*_\alpha)\right]
\end{aligned}
\tag{11.30}
$$

Therefore, ${}_{\lambda'}\langle\hbar^2\beta|\hbar^2\beta\rangle_\lambda$ is not necessarily zero. However, we find that the maximal localization state is normalizable:

$$\begin{aligned}{}_{\lambda}\langle\hbar^2\beta|\hbar^2\beta\rangle_\lambda =&2^{2[Re(\zeta_\lambda)-1]}|\Pi|^2\sqrt{\frac{\pi}{\beta}}\tilde{C}\sum_{r=s=0}^{n-1}\frac{\tilde{\mathcal{C}}_{rs}}{\Gamma(\tilde{A}_1)\Gamma(\tilde{A}_2+1/2)}\Bigg[\Gamma(\tilde{A}_1)\Gamma(\tilde{A}_2)\\ &{}_2F_1\left(\frac{\tilde{A}_1}{2},\frac{1+\tilde{A}_1}{2};1-\tilde{A}_2;1\right)+2^{2s-1+N_\gamma^*+N_\alpha}\Gamma(\tilde{A}_1+2\tilde{A}_2)\\ &\Gamma(-\tilde{A}_2)\,{}_2F_1\left(\frac{\tilde{A}_1+2\tilde{A}_2}{2},\frac{1+\tilde{A}_1+2\tilde{A}_2}{2};\tilde{A}_2+1;1\right)\Bigg]=1\end{aligned} \tag{11.31}$$

where

$$\begin{aligned}&\tilde{C}=\frac{-8i^{1-2Re(\zeta_{\lambda 3})}}{3\beta^{3/2+Re(\zeta_\lambda)}}\Gamma\Big(\frac{1}{2}+Re(\zeta_{\lambda 0})\Big)\Gamma\Big(\frac{1}{2}+Re(\zeta_{\lambda 2})\Big)\Gamma\Big(\frac{1}{2}+Re(\zeta_{\lambda 3})\Big)\Gamma(1-Re(\zeta_\lambda));\\ &\tilde{\mathcal{C}}_{rs}=\frac{\Gamma(1-n+r)\Gamma(1-n+s)\Gamma(B_\lambda^{(1)}+r)\Gamma(B_\lambda^{*(1)}+s)}{r!s!\Gamma(B_\lambda^{(2)}+r)\Gamma(B_\lambda^{*(2)}+s)};\\ &\tilde{A}_1=s-r-iIm(N_\gamma-N_\alpha);\\ &\tilde{A}_2=\frac{1}{2}[1-2s-(N_\gamma+N_\alpha^*)]\end{aligned} \tag{11.32}$$

References

[1] Michele Maggiore. A Generalized Uncertainty Principle in Quantum Gravity. *hep-th/9301067v1.*

[2] Michele Maggiore. Quantum Groups, Gravity and the Generalized Uncertainty Principle. *hep-th/9305163v3.*

[3] Michele Maggiore. The algebraic structure of the generalized uncertainty principle. *hep-th/9309034v1.*

[4] A. Kempf. Quantum field theory with nonzero minimal uncertainties in positions and momemta. *hep-th/9405067.*

[5] Achim Kempf, Gianpiero Mangano, and Robert B. Mann. Hilbert Space Representation of the Minimal Length Uncertainty Relation. *Phys. Rev. D*, 52(2):1108, December 1995.

[6] A. Kempf and Gianpiero Mangano. Minimal length uncertainty relation and ultraviolet regularisation. *hep-th/9612084.*

[7] F. Brau. Minimal length uncertainty relation and hydrogen atom. *quant-ph/9905033v1.*

[8] A. Camacho. Generalized Uncertainty Principle and Quantum Electrodynamics. *General Relativity and Gravitation*, 35(7):1153, 2003.

[9] Ahmed Farag Ali, Saurya Das, and Elias C. Vagenas. Discreteness of Space from the Generalized Uncertainty Principle. *hep-th/0906.5396v1.*

[10] Saurya Das and Elias C. Vagenas. Phenomenological implications of the generalized uncertainty principle. *hep-th/0901.1768v1.*

[11] N. Sasakura. Space-time uncertainty relation and lorentz invariance. *Journal of High Energy Physics*, 05:015, 2000.

[12] Shin'ichi Imai and N. Sasakura. Scalar field theories in a lorentz-invariant three-dimensional non-commutative space-time. *Journal of High Energy Physics*, 09:032, 2000.

[13] Marco Valerio Battisti and Stjepan Meljanac. Modification of heisenberg uncertainty relations in non-commutative snyder space-time geometry. *Phys. Rev. D*, 79:067505, 2009.
[14] V. Alan Kostelecký and Neil Russell. Data tables for Lorentz and CPT violation. *hep-th/0801.0287v3*.
[15] Hartland S. Snyder. Quantized Space-Time. *Physical Review*, 71(1):38–41, January 1947.
[16] C. N. Yang. On quantized space-time. *Phys. Rev.*, 72:874, 1947.
[17] Achim Kempf. Mode generating mechanism in inflation with cutoff. *astro-ph/0009209*, 2000.
[18] C. Gonera, P. Kosiński, P. Maślanka, and S.Giller. Global symmetries of noncommutative field theory. *Journal of Physics: Conference Series*, 53:865871, 2006.
[19] Haye Hinrichsen and Achim Kempf. Maximal localisation in the presence of minimal uncertainties in positions and in momenta. *hep-th/9510144v1*, 1995.
[20] A. Kempf. *J. Phys. A*, 30:2093, 1997. hep-th/9604045.
[21] M. Chaichian, A. Demichev, and P. Prešnajder. Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences. *hep-th/9812180v3*, 1999.
[22] Jeremy J. Gray. *Linear Differential Equations and Group Theory from Riemann to Poincaré.* Birkhäuser, 1999.
[23] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* New York, Dover, 1965.

E-mail address, Arko Bose: arkobose@gmail.com