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# Phase structure of black branes in grand canonical ensemble

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**Abstract**

This is a companion paper of our previous work [1] where we studied the thermodynamics and phase structure of asymptotically flat black $p$-branes in a cavity in arbitrary dimensions $D$ in a canonical ensemble. In this work we study the thermodynamics and phase structure of the same in a grand canonical ensemble. Since the boundary data in two cases are different (for the grand canonical ensemble boundary potential is fixed instead of the charge as in canonical ensemble) the stability analysis and the phase structure in the two cases are quite different. For the case of grand canonical ensemble, in some situation, the phase structure of charged black $p$-branes have some similarities as well as dissimilarities with that of the zero charge black $p$-branes in canonical ensemble. So, in the grand canonical ensemble there is an analog of Hawking-Page transition, even for the charged black $p$-brane, as opposed to the canonical ensemble. Our study applies to non-dilatonic as well as dilatonic black $p$-branes in $D$ space-time dimensions.

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## 1 Introduction

Black holes in asymptotically AdS space have attracted a lot of attention in recent years. The reason is partly due to the AdS/CFT correspondence proposed by Maldacena [2] and its consequences [3, 4, 5]. Black holes in AdS space are thermodynamically stable [6] and so, the equilibrium states and the phase structure of such space-times can be easily studied. By AdS/CFT correspondence this in turn will provide us with the information about the similar states and phase structure on the CFT or gauge theory side [7, 8]. Black hole space-time is naturally associated with a Hawking temperature [9], so the gauge theory will also be at a finite temperature [10]. In particular, AdS black holes are well-known to undergo a Hawking-Page transition [6] and by AdS/CFT, this corresponds to the confinement-deconfinement phase transition [10] in SU($N$) gauge theory at large $N$.

However, it is well-known that the phase structure just mentioned is not unique to the AdS black holes, but similar structure also arises in suitably stabilized flat as well as dS black holes [11, 12]. Higher dimensional theories like string or M-theory are known to admit higher dimensional black objects in the form of black $p$-branes [13, 14, 15] which are asymptotically flat and so, it would be interesting to see what kind of equilibria and phase structure they give rise to. With this motivation we obtained the equilibrium states and the phase structure of the black $p$-branes in the canonical ensemble in the previous work [1]. In this paper we study the same for the black $p$-branes in grand canonical ensemble. Since the boundary data in these two ensembles are quite different, their phase structures are also expected to be different and it is of interest to see how they differ with the change of the boundary data.

We will consider the dilatonic black $p$-branes (in $D$ dimensions) which are the solutions of $D$-dimensional gravity coupled to a dilaton (a scalar field) and a $(p+1)$-form gauge field [14]. These are asymptotically flat and so are thermodynamically unstable as an isolated black $p$-brane would radiate energy in the form of Hawking radiation [9]. In order to restore thermodynamic stability so that equilibrium thermodynamics and phase structure can be studied we must consider ensembles that include not only the branes but also their environment. As self-gravitating systems are spatially inhomogeneous, any specification of such ensembles requires not just thermodynamic quantities of interest but the place at which they take the specified values. In other words, we place the black brane in a cavity *a la* York [16] and its extension in the charged case. The wall of the cavity is fixed at a radial distance $\rho_B$, and we will keep the temperature and the gauge potential at the wall of the cavity (at $\rho_B$) fixed. This will define a grand canonical ensemble (Note that for the canonical ensemble, the charge enclosed in the cavity is fixed instead.) So, for the grand canonical ensemble charge can vary. We will study the phase structure of black $p$-branes in this ensemble. Charged black holes in the grand canonical ensemble have been studied in [17, 18]

We will employ the Euclidean action formalism [19, 16, 20] to the dilatonic black $p$-brane geometry in $D$ space-time dimensions. We first write the Euclidean action containing the gravitational part including a Gibbons-Hawking boundary term [19], the dilatonic part and the form-field part. By using the equation of motion we simplify the action and then evaluate it for the given black $p$-brane geometry. To the leading order this action is related to the grand potential or Gibbs free energy of the system and is an essential entity for the stability analysis. The stationary point of this action will determine the relevant configuration with the temperature and potential fixed at the wall of the cavity. The second derivative or derivatives of the action at the stationary point can be used to give us the information about the stability of the black brane at that point. We find the condition which determines at least the local stability of the black brane phase. In the case when there is a stable black brane phase, we find that there exists a minimum temperature below which there is only 'hot flat space' phase and no black brane phase. But above this temperature there exist two black brane phases with two different radii. The smaller one is unstable which corresponds to the local maximum of the free energy and the larger one is locally stable and corresponds to the local minimum of the free energy. This local minimum becomes a global one only after the temperature rises above a certain value. Below this value and above minimum temperature, the locally stable black brane eventually makes a transition to the 'hot flat space' by a topological phase transition. But above this value, the large black brane is globally stable. Upto this point the picture is very similar to the chargeless black brane case in the canonical ensemble [1]. But now for the grand canonical ensemble as the temperature rises more, the globally stable black brane phase disappears after a certain value and at this point there is only one unstable black brane phase and the stable phase here would be the 'hot flat space'. Also note that unlike in the case of canonical ensemble, here we do not have a situation where we have two locally stable black brane phases and so there is no such phase transition similar to the van der Waals-Maxwell liquid-gas phase transition or even a transition from one black brane phase to the other.

This paper is organized as follows. In section 2, we discuss the dilatonic black $p$-brane solution and evaluate the action in Euclidean signature. Then we discuss the stability of various equilibrium states from this action in section 3. The phase structure of the black $p$-brane is discussed in section 4. Then we conclude in section 5. A more general two-variable stability analysis (as opposed to one-variable stability analysis performed in section 3) is presented in the Appendix.

## 2 Black $p$-brane solution and the action

The black $p$-brane solution was originally constructed [13] as a solution to the ten dimensional supergravity containing a metric, a dilaton and a $(p+1)$-form gauge field and was generalized to arbitrary dimensions in [14]. These solutions are given in Lorentzian signature, but for the purpose of studying thermodynamics, we write the black $p$-brane solution in Euclidean signature as (see for example [21]),

$$\begin{aligned}
ds^2 &= \Delta_+ \Delta_-^{-\frac{d}{D-2}} dt^2 + \Delta_-^{\frac{\tilde{d}}{D-2}} \sum_{i=1}^{d-1} (dx^i)^2 + \Delta_+^{-1} \Delta_-^{\frac{a^2}{2\tilde{d}}-1} d\rho^2 + \rho^2 \Delta_-^{\frac{a^2}{2\tilde{d}}} \Omega_{\tilde{d}+1}^2, \\
A_{[p+1]} &= -i e^{a\phi_0/2} \left[ \left(\frac{r_-}{r_+}\right)^{\tilde{d}/2} - \left(\frac{r_- r_+}{\rho^2}\right)^{\tilde{d}/2} \right] dt \wedge dx^1 \wedge \ldots \wedge dx^p, \\
F_{[p+2]} &\equiv dA_{[p+1]} = -i e^{a\phi_0/2} \tilde{d} \, \frac{(r_- r_+)^{\tilde{d}/2}}{\rho^{\tilde{d}+1}} d\rho \wedge dt \wedge dx^1 \wedge \ldots \wedge dx^p, \\
e^{2(\phi-\phi_0)} &= \Delta_-^a,
\end{aligned} \tag{1}$$

Here we have defined

$$\Delta_\pm = 1 - \left(\frac{r_\pm}{\rho}\right)^{\tilde{d}} \tag{2}$$

where, $r_\pm$ are the two parameters characterizing the solution and are related to the mass and the charge of the black brane. In the metric (1) the Euclidean time is periodic and so, the metric has an isometry $S^1 \times SO(d-1) \times SO(\tilde{d}+2)$ indicating that it represents a $(d-1) \equiv p$-brane in Euclidean signature. The total space-time dimesion is $D = d + \tilde{d} + 2$, where the space transverse to the $p$-brane has the dimensionality $\tilde{d}+2$. $\phi$ is the dilaton and $\phi_0$ is its asymptotic value and related to the string coupling as $g_s = e^{\phi_0}$. $a$ is the dilaton coupling and is given for the supergravity theory with maximal supersymmetry by,

$$a^2 = 4 - \frac{2d\tilde{d}}{D-2}. \tag{3}$$

It is clear from the Lorentzian form of the above metric that when $r_- = 0$, and $a = 0$, it reduces to the $D$-dimensional Schwarzschild solution which has an event horizon at $\rho = r_+$, whereas, at $\rho = r_-$, there is a curvature singularity. So, the metric in (1) represents a black $p$-brane only for $r_+ > r_-$, with $r_+ = r_-$, being its extremal limit [13]. A $p$-brane naturally couples to the $(p+1)$-form gauge field whose form and its field strength are given in (1). Note that we have defined the gauge potential with a constant shift, following [18], in such a way that it vanishes on the horizon so that it is well-defined on the local inertial frame. The black $p$-brane will be placed in a cavity with its wall at $\rho = \rho_B$. It is clear from the metric in (1) that the physical radius of the cavity is

$$\bar{\rho}_B = \Delta_-^{\frac{a^2}{4\tilde{d}}} \rho_B \tag{4}$$

while $\rho_B$ is merely the coordinate radius. So, it is $\bar{\rho}_B$ which we should fix in the following discussion and not $\rho_B$. Note that when the black brane is non-dilatonic $a = 0$ and in that case $\rho_B = \bar{\rho}_B$. Also we fix the dilaton [4] at $\bar{\rho}_B$, which indicates that the asymptotic value of the dilaton is not fixed for the present consideration and this is crucial for our discussion in expressing relevant quantities in 'barred' parameters. By this argument we also have

$$\bar{r}_\pm = \Delta_-^{\frac{a^2}{4\tilde{d}}} r_\pm \tag{5}$$

and $\bar{r}_\pm$ are the proper parameters which we should use in the present context. In terms of the 'barred' parameters $\Delta_\pm$ remain the same as before,

$$\Delta_\pm = 1 - \frac{r_\pm^{\tilde{d}}}{\rho_B^{\tilde{d}}} = 1 - \frac{\bar{r}_\pm^{\tilde{d}}}{\bar{\rho}_B^{\tilde{d}}}. \tag{6}$$

Since the Euclidean time coordinate in (1) is periodic so, for the metric to be well-defined without a conical singularity at $\rho = r_+$, the Euclidean time must have a periodicity,

$$\beta^* = \frac{4\pi r_+}{\tilde{d}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}} - \frac{1}{2}}, \tag{7}$$

which is the inverse temperature at $\rho = \infty$. The local $\beta(\bar{\rho}_B)$ is given as

$$\beta = \beta(\bar{\rho}_B) = \Delta_+^{\frac{1}{2}} \Delta_-^{-\frac{d}{2(d+\tilde{d})}} \beta^* \tag{8}$$

which is the inverse of local temperature at $\bar{\rho}_B$ when in thermal equilibrium contact with the environment (the wall of cavity) at the inverse of temperature $\beta$. Note that in terms of the 'barred' parameters the inverse of temperature $\beta$ can be expressed from (8) and (4) as,

$$\beta = \frac{4\pi r_+}{\tilde{d}} \Delta_+^{\frac{1}{2}} \Delta_-^{-\frac{d}{2(d+\tilde{d})}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} = \frac{4\pi \bar{r}_+}{\tilde{d}} \Delta_+^{\frac{1}{2}} \Delta_-^{-\frac{1}{\tilde{d}}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}} - \frac{1}{2}}, \tag{9}$$

[4]This enables us to obtain its correct equation of motion from the corresponding action in the presence of a boundary.

where in the second equality $\Delta_\pm$ are also expressed in terms of 'barred' parameters. Also the charge is defined as,

$$\begin{aligned} Q_d &= \frac{i}{2\sqrt{\kappa}} \int e^{-a(d)\phi} * F_{[p+2]} = \frac{\Omega_{\tilde{d}+1}}{2\sqrt{\kappa}} e^{-a\phi_0/2} \tilde{d}(r_+ r_-)^{\tilde{d}/2} \\ &= \frac{\Omega_{\tilde{d}+1}\tilde{d}}{\sqrt{2}\kappa} e^{-a\bar{\phi}/2} (\bar{r}_+ \bar{r}_-)^{\tilde{d}/2}. \end{aligned} \tag{10}$$

In (10), $\kappa$ is a constant with $1/(2\kappa^2)$ appearing in front of the Hilbert-Einstein action in canonical frame but containing no string coupling $g_s$, $*F_{[p+2]}$ denotes the Hodge dual of the $(p+2)$-form field given in (1). Also, $\Omega_n$ denotes the volume of a unit $n$-sphere. In the last line of (10) we have expressed the asymptotic value of the dilaton by the fixed dilaton $\bar{\phi} \equiv \phi(\bar{\rho}_B)$ at the wall of the cavity from the relation (1) and then expressed $r_\pm$ by $\bar{r}_\pm$ from (5).

In the grand canonical ensemble, the fixed quantities are the physical radius of the wall of cavity $\bar{\rho}_B$, the temperature, the brane volume $V_p = \Delta_-^{\frac{\tilde{d}(d-1)}{2(D-2)}} V_p^*$ with $V_p^* = \int d^p x$ and the potential, all at the wall of cavity. The potential in the local inertial frame at the wall of cavity can be obtained from $A_{[p+1]}$ and the metric in (1) as,

$$\begin{aligned} A_{[p+1]} &= -ie^{a\bar{\phi}/2} \left(\Delta_- \Delta_+\right)^{-\frac{1}{2}} \left(\frac{r_-}{r_+}\right)^{\frac{\tilde{d}}{2}} \left(1 - \frac{r_+^{\tilde{d}}}{\rho_B^{\tilde{d}}}\right) d\bar{t} \wedge d\bar{x}^1 \ldots \wedge d\bar{x}^p \\ &\equiv -i\sqrt{2}\kappa \Phi d\bar{t} \wedge d\bar{x}^1 \ldots d\bar{x}^p \end{aligned} \tag{11}$$

where $(\bar{t},\ \bar{x}^1,\ \ldots, \bar{x}^p)$ are the coordinates in the local inertial frame and is related to the original coordinates as $\bar{t} = \Delta_+^{\frac{1}{2}} \Delta_-^{-\frac{d}{2(D-2)}} t$ and $\bar{x}^i = \Delta_-^{\frac{\tilde{d}}{2(D-2)}} x^i$ for $i = 1,\ 2, \ldots, p$ as can be seen from the metric in (1). So, $\Phi$ is the potential conjugate to the charge and is fixed. We will now evaluate the action with these boundary data.

The Euclidean action for the dilatonic black branes in the canonical ensemble has already been evaluated in the Appendix of our previous work [1]. We will use that result to evaluate the action for the grand canonical ensemble. The relevant action for the gravity coupled to the dilaton and a $(p+1)$-form gauge field in the canonical ensemble is,

$$I_E^C = I_E^C(g) + I_E^C(\phi) + I_E^C(F) \tag{12}$$

where, $I_E^C(g)$ is the gravitational part of the action which has a Hilbert-Einstein term and a Gibbons-Hawking boundary term [19], $I_E^C(\phi)$ is the dilatonic part and $I_E^C(F)$ is the form-field part. The first two terms remain exactly the same as given in the first two lines in eq.(101) of [1] in the grand canonical ensemble, however, the form-field part which in the canonical ensemble has the form,

$$
\begin{aligned}
I_E^C(F) &= \frac{1}{2\kappa^2}\frac{1}{2(d+1)!}\int_M d^D x\sqrt{g}\, e^{-a(d)\phi}\, F_{d+1}^2 \\
&\quad -\frac{1}{2\kappa^2}\frac{1}{d!}\int_{\partial M} d^{D-1}x\sqrt{\gamma}\, n_\mu\, e^{-a(d)\phi}\, F^{\mu\mu_1\mu_2\cdots\mu_d}A_{\mu_1\mu_2\cdots\mu_d},
\end{aligned} \tag{13}
$$

will differ in the grand canonical ensemble since the potential at the wall of the cavity is fixed and so, the last term in (13) will be absent in the grand canonical ensemble. Therefore, the action for the grand canonical ensemble can be obtained from that of the canonical ensemble by the relation,

$$
I_E^{GC} = I_E^C + \frac{1}{2\kappa^2}\frac{1}{d!}\int_{\partial M} d^{D-1}x\sqrt{\gamma}\, n_\mu\, e^{-a(d)\phi}\, F^{\mu\mu_1\mu_2\cdots\mu_d}A_{\mu_1\mu_2\cdots\mu_d}. \tag{14}
$$

Here $\partial M$ denotes the boundary of the whole space-time $M$ and $n_\mu$ is a space-like vector normal to the boundary. $\gamma$ is the determinant of the boundary metric. Evaluating the last term in (14) for the black $p$-brane configuration given in (1) we get,

$$
I_E^{GC} = I_E^C - \frac{\beta^* V_p^*}{2\kappa^2}\Omega_{\tilde{d}+1}\tilde{d}r_-^{\tilde{d}}\Delta_+ = I_E^C - \beta V_p Q_d \Phi, \tag{15}
$$

where we have used the charge expression and the potential from (10) and (11) as,

$$
\begin{aligned}
Q_d &= \frac{\Omega_{\tilde{d}+1}\tilde{d}}{\sqrt{2}\kappa}e^{-a\bar{\phi}/2}(\bar{r}_+\bar{r}_-)^{\frac{\tilde{d}}{2}}, \\
\Phi &= \frac{1}{\sqrt{2}\kappa}e^{a\bar{\phi}/2}\left(\frac{\bar{r}_-}{\bar{r}_+}\right)^{\frac{\tilde{d}}{2}}\left(\frac{\Delta_+}{\Delta_-}\right)^{\frac{1}{2}}.
\end{aligned} \tag{16}
$$

Now using the form of the Euclidean action for the canonical ensemble given in eqs.(110) and (114) in [1] we write (15) as

$$
\begin{aligned}
I_E(\beta,\Phi,\bar{\rho}_B;Q_d,\bar{r}_+) &= -\frac{\beta V_p\Omega_{\tilde{d}+1}}{2\kappa^2}\bar{\rho}_B^{\tilde{d}}\left[(\tilde{d}+2)\left(\frac{\Delta_+}{\Delta_-}\right)^{1/2} + \tilde{d}(\Delta_+\Delta_-)^{1/2} - 2(\tilde{d}+1)\right] \\
&\quad -\frac{4\pi V_p\Omega_{\tilde{d}+1}}{2\kappa^2}\bar{r}_+^{\tilde{d}+1}\Delta_-^{-\frac{1}{2}-\frac{1}{\tilde{d}}}\left(1-\frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} - \beta V_p Q_d\Phi.
\end{aligned} \tag{17}
$$

Since this is the Euclidean action for the black $p$-brane in the grand canonical ensemble we will be working with, for brevity, we have removed the superscript '$GC$' from $I_E$ in writing (17). Note that we have expressed everything on the r.h.s. in terms of the 'barred' parameters showing that the formalism works for both the non-dilatonic as well as dilatonic branes. In (17) $\beta$, $\bar{\rho}_B$, $\Phi$ are the inverse of temperature, the physical radius and the potential of the cavity, therefore are all fixed, and $Q_d$, $\bar{r}_+$ are variables. Note that $\bar{r}_-$ is not independent and can be expressed in terms of $Q_d$ and $\bar{r}_+$ from (16) as,

$$\bar{r}_-^{\tilde{d}} = \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1}\tilde{d}} e^{a\bar{\phi}/2}\right)^2 \frac{1}{\bar{r}_+^{\tilde{d}}}. \tag{18}$$

Substituting (18) in (17) we can express the action in terms of two variables $Q_d$ and $\bar{r}_+$. Then varying the action with respect to $Q_d$ and putting that to zero, i.e., at the stationary point we obtain (after some simplification),

$$\begin{aligned} &\frac{\partial I_E}{\partial Q_d} = 0 \\ &\Rightarrow \quad \Phi = \frac{Q_d e^{a\bar{\phi}}}{\tilde{d}\Omega_{\tilde{d}+1}\bar{r}_+^{\tilde{d}}} \left(\frac{\Delta_+}{\Delta_-}\right)^{\frac{1}{2}} \left[1 + \frac{\tilde{d}+2}{\tilde{d}} \Delta_-^{-1} \left(\frac{4\pi\bar{r}_+\Delta_+^{\frac{1}{2}}\Delta_-^{-\frac{1}{\tilde{d}}}}{\beta\tilde{d}\left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{2}-\frac{1}{\tilde{d}}}} - 1\right)\right]. \end{aligned} \tag{19}$$

Similarly, varying the action with respect to the other variable $\bar{r}_+$ and setting that to zero, i.e., at the stationary point we obtain,

$$\frac{\partial I_E}{\partial \bar{r}_+} = 0 \quad \Rightarrow \quad \beta = \frac{4\pi\bar{r}_+}{\tilde{d}} \Delta_+^{\frac{1}{2}} \Delta_-^{-\frac{1}{\tilde{d}}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}}-\frac{1}{2}}. \tag{20}$$

This is the correct form of $\beta$ we had given earlier in (9). Using this (20) in (19) and substituting the form of $Q_d$ given in the first equation of (16) we recover the correct form of the potential given in the second equation of (16). This is a verification that the action $I_E$ given in (17) is indeed correct. We will use this form of the action in the next section to study the stability of the equilibrium states of the black $p$-branes in the grand canonical ensemble.

# 3 Stability analysis of the black $p$-branes

For the purpose of the stability analysis we will define some new parameters following refs.[18, 12, 11] and rewrite the action, the potential and the inverse temperature in terms of those parameters. Let us first define the reduced charge as,

$$Q_d^* = \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1}\tilde{d}} e^{a\bar{\phi}/2}\right)^{\frac{1}{\tilde{d}}} \tag{21}$$

Then from (18) we have

$$\bar{r}_- = \frac{(Q_d^*)^2}{\bar{r}_+} \tag{22}$$

Next we define the following parameters,

$$x = \left(\frac{\bar{r}_+}{\bar{\rho}_B}\right)^{\tilde{d}}, \qquad \bar{b} = \frac{\beta}{4\pi\bar{\rho}_B}, \qquad q = \left(\frac{Q_d^*}{\bar{\rho}_B}\right)^{\tilde{d}} \tag{23}$$

where the dimensionless parameter $\bar{b}$ is fixed, but the other two parameters $x$ and $q$ vary. Note that the parameter $\bar{b}$ is related to the inverse of temperature of the environment, $q$ is related to the charge and $x$ is related to the horizon size. In terms of these parameters we have

$$\begin{aligned}
\Delta_+ &= 1 - \frac{\bar{r}_+^{\tilde{d}}}{\bar{\rho}_B^{\tilde{d}}} = 1 - x \\
\Delta_- &= 1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{\rho}_B^{\tilde{d}}} = 1 - \frac{(Q_d^*)^{2\tilde{d}}}{\bar{r}_+^{\tilde{d}}\bar{\rho}_B^{\tilde{d}}} = 1 - \frac{q^2}{x} \\
1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}} &= 1 - \frac{q^2}{x^2}
\end{aligned} \tag{24}$$

Using (24) the two equations at the equilibrium (at the stationary point) giving the constant temperature and the potential at the wall of the cavity given in (20) and (19) can be written as

$$\bar{b} = b(x, q), \qquad \bar{\varphi} = \varphi(x, q) \tag{25}$$

where we have defined $\bar{\varphi} = \sqrt{2}\kappa e^{-a\bar{\phi}/2}\Phi$ and

$$b(x, q) = \frac{1}{\tilde{d}} \frac{x^{\frac{1}{\tilde{d}}}(1-x)^{\frac{1}{2}}}{\left(1 - \frac{q^2}{x^2}\right)^{\frac{1}{2} - \frac{1}{\tilde{d}}}\left(1 - \frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}}, \qquad \varphi(x, q) = \frac{q}{x}\left(\frac{1-x}{1 - \frac{q^2}{x}}\right)^{\frac{1}{2}}. \tag{26}$$

The two equations in (25) can also be derived from the following reduced action in a similar fashion as (20) and (19) but now with respect to variables $x$ and $q$, respectively,

$$\begin{aligned}
\tilde{I}_E(x, q) &\equiv \frac{2\kappa^2 I_E}{4\pi\bar{\rho}_B^{\tilde{d}+1} V_p \Omega_{\tilde{d}+1}} \\
&= -\bar{b}\left[(\tilde{d}+2)\left(\frac{1-x}{1-\frac{q^2}{x}}\right)^{\frac{1}{2}} + \tilde{d}(1-x)^{\frac{1}{2}}\left(1 - \frac{q^2}{x}\right)^{\frac{1}{2}} - 2(\tilde{d}+1) + \tilde{d}q\bar{\varphi}\right] \\
&\qquad -x^{1+\frac{1}{\tilde{d}}}\left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}}\right)^{\frac{1}{2}+\frac{1}{\tilde{d}}},
\end{aligned} \tag{27}$$

where $I_E$ is as given in (17). These two equations, for given $\bar{b}$ and $\bar{\varphi}$, determine $x$ and $q$ completely. However, in the presence of two variables, the analysis of stability at the extremal points determined by these two equations is a bit more complicated than that given in [1], in the canonical ensemble. There the relation at the charge equilibrium for fixed charge has been employed to reduce the two variables to only one[5]. While the analysis of stability with two variables, which we will perform in the Appendix, is more complete, there actually exists an analogous simpler analysis as in canonical case. But for this we need to employ the potential relation at the equilibrium for fixed potential, as given in the second equation of (25), to reduce the two variables to one variable. In what follows, we will first perform the simpler one-variable analysis and find the stability condition. We will perform the more complete yet more complicated two-variable analysis in the Appendix and show that these two approaches give the same stability condition.

Note that since $x/q = (\bar{r}_+/Q_d^*)^{\tilde{d}} = (\bar{r}_+/\bar{r}_-)^{\tilde{d}/2} > 1$, so, $x > q$ and since $x = (\bar{r}_+/\bar{\rho}_B)^{\tilde{d}} < 1$, so, $x$ has the range $q < x < 1$. We have excluded the two end points $x = 1$ and $x = q$ from our consideration. The former corresponds to taking $\bar{\rho}_B = \bar{r}_+$, i.e., the cavity is placed on the horizon where the ensemble temperature is infinity. While the latter corresponds to the BPS case for which the temperature is zero. In either case, strictly speaking, the current description using zero order approximation breaks down. So, we limit ourselves to the relevant region of $0 \leq q < x < 1$ in what follows. Also note from $\varphi$ expression in (26) that since $1 - q^2/x = 1 - (q^2/x^2)x > 1 - x$, so we have $0 \leq \varphi < 1$, with $\varphi = 0$ corresponding to the chargeless ($q = 0$) case.

Let us now consider the one-variable case first, i.e., when the second equation in (25) $0 \leq \bar{\varphi} = \varphi(x, q) < 1$ is satisfied. We can now solve this equation to get,

$$\frac{q^2}{x^2} = \frac{\bar{\varphi}^2}{1 - (1 - \bar{\varphi}^2)x}. \tag{28}$$

Now substituting (28) in $b(x, q)$ equation in (26) we have,

$$b_{\bar{\varphi}}(x) \equiv b(x, q) = \frac{x^{\frac{1}{\tilde{d}}} \left[1 - \left(1 - \bar{\varphi}^2\right) x\right]^{\frac{1}{2}}}{\tilde{d} \left(1 - \bar{\varphi}^2\right)^{\frac{1}{2} - \frac{1}{\tilde{d}}}}, \tag{29}$$

and in the reduced action in (27), we have

$$\tilde{I}_E^{\bar{\varphi}}(x) \equiv \tilde{I}_E(x, q) = -2\bar{b}\left(\tilde{d} + 1\right)\left[\left(1 - \left(1 - \bar{\varphi}^2\right)x\right)^{1/2} - 1\right] - x^{1+1/\tilde{d}}\left(1 - \bar{\varphi}^2\right)^{1/2+1/\tilde{d}}. \tag{30}$$

[5]While the one-variable analysis of stability given in [1] appears natural in canonical ensemble, there exists also a more general two-variable analysis which is more complete yet more complicated and gives the same stability condition. For the present case, the two-variable analysis seems more natural but as we will show in the text that when the second equation in (25) for the fixed potential is used to reduce the two variables to only one, the same stability condition can be reached.

One can easily check that

$$\frac{d\tilde{I}_E^{\bar\varphi}(x)}{dx} = \frac{(1+\tilde d)(1-\bar\varphi^2)}{[1-(1-\bar\varphi^2)x]^{1/2}}\left[\bar b - b_{\bar\varphi}(x)\right], \tag{31}$$

where $b_{\bar\varphi}(x)$ is defined in (29). This vanishes at the stationary point of the action, i.e.,

$$\bar b = b_{\bar\varphi}(\bar x), \tag{32}$$

which is the first equation in (25) under the present consideration. The local stability at the stationary point is determined by requiring it to be a local minimum of the action, i.e.,

$$\left.\frac{d^2 I_E^{\bar\varphi}(x)}{dx^2}\right|_{x=\bar x} = -\frac{(1+\tilde d)(1-\bar\varphi^2)}{[1-(1-\bar\varphi^2)\bar x]^{1/2}}\frac{db_{\bar\varphi}(\bar x)}{d\bar x} > 0. \tag{33}$$

In other words, we need to have (since the other factors are all positive)

$$\frac{db_{\bar\varphi}(\bar x)}{d\bar x} = \frac{\bar x^{1/\tilde d - 1}\left[2-(2+\tilde d)\bar x(1-\bar\varphi^2)\right]}{2\tilde d^2(1-\bar\varphi^2)^{1/2-1/\tilde d}\,[1-(1-\bar\varphi^2)\bar x]^{1/2}} < 0, \tag{34}$$

which gives the local stability condition as

$$2-(2+\tilde d)\bar x(1-\bar\varphi^2) < 0. \tag{35}$$

In the above, $\bar x$ is a solution of the present equation of state as given in (32).

As we will show in detail in the Appendix that the above local stability condition (35) continues to hold when we consider the more general yet more complicated two-variable situation and this justifies the simplified one-variable analysis of stability performed above.

We therefore conclude that for the local stability of the system we must have (35) to be satisfied. To understand the meaning of the stability condition (35) we rewrite it as

$$\bar x\left(1-\bar\varphi^2\right) > \frac{2}{\tilde d + 2} \tag{36}$$

which further implies,

$$\frac{2}{\tilde d+2} < \bar x < 1, \qquad \text{and} \qquad 0 < \bar\varphi < \sqrt{\frac{\tilde d}{\tilde d + 2}}. \tag{37}$$

In other words, for given $\bar\varphi$ satisfying the above constraint, only $\bar x$ which is in the range specified in (37), gives at least a locally stable system. Otherwise the system is not stable.

This concludes our discussion on the stability of asymptotically flat black $p$-branes in the grand canonical ensemble. Using this information we will construct the phase structure of the equilibrium states of the black $p$-branes in this ensemble and compare with that in the canonical ensemble.

# 4 Phase structure of black $p$-branes

We have seen in the previous section that the stability condition for the black $p$-branes is given by (35). When we use the potential condition at the equilibrium, $\bar{\varphi} = \varphi(x, q)$, to eliminate $q$ and obtain (29), i.e,

$$b_{\bar{\varphi}}(x) = \frac{x^{\frac{1}{\tilde{d}}} \left[1 - (1 - \bar{\varphi}^2)\, x\right]^{\frac{1}{2}}}{\tilde{d}\, (1 - \bar{\varphi}^2)^{\frac{1}{2} - \frac{1}{\tilde{d}}}}, \tag{38}$$

we have the range of $x$ as $0 < x < 1$. This is different from the fixed charge case of the canonical ensemble where the range is $q < x < 1$, since here we are considering the grand canonical ensemble where the potential $\bar{\varphi}$ is fixed and not the charge $q$. Even for the zero charge case of the canonical ensemble, where the range of $x$ in the two cases are the same the phase structure in these two cases, as we will see, will be quite different due to the different boundary data in the two ensembles. Note that as $x \to 0$, $b_{\bar{\varphi}}(x \to 0) \to b_{\bar{\varphi}}(0) = 0$, exactly as in the canonical case, but as $x \to 1$, $b_{\bar{\varphi}}(x \to 1) \to b_{\bar{\varphi}}(1)$, where it has the value,

$$b_{\bar{\varphi}}(1) = \frac{\bar{\varphi}}{\tilde{d}\, (1 - \bar{\varphi}^2)^{\frac{1}{2} - \frac{1}{\tilde{d}}}} > 0 \tag{39}$$

and this is different from the fixed charge canonical case including the zero charge case. The local temperature at this point has the value

$$T_{\bar{\varphi}}(1) = \frac{\tilde{d}\, (1 - \bar{\varphi}^2)^{\frac{1}{2} - \frac{1}{\tilde{d}}}}{4\pi \bar{\rho}_B \bar{\varphi}}. \tag{40}$$

Also note from (38) that in the range $0 < x < 1$, $b_{\bar{\varphi}}(x) > 0$ and so, if we plot $b_{\bar{\varphi}}(x)$ vs. $x$ curve it will start from zero when $x \to 0$ and then it rises and ends at $b_{\bar{\varphi}}(1) > 0$ given above when $x \to 1$. In between there can be extrema for $b_{\bar{\varphi}}(x)$ and in fact, it is not difficult to check from

$$\frac{d b_{\bar{\varphi}}(x)}{dx} = \frac{b_{\bar{\varphi}}(x) \left[2 - (\tilde{d} + 2) x (1 - \bar{\varphi}^2)\right]}{2\tilde{d}\, x \left[1 - x(1 - \bar{\varphi}^2)\right]}, \tag{41}$$

that there can be only one extremum and that is a maximum if it exists at all. This maximum is not always guaranteed to exist unlike the chargeless case of canonical ensemble. However, when it exists indeed, the characteristic behavior of $b_{\bar{\varphi}}(x)$ vs. $x$ is given in Figure 1. The plot of $T_{\bar{\varphi}}(x)$ vs. $x$ curve will have opposite behavior. It will start from infinity when $x \to 0$ then the curve will drop and end at $T_{\bar{\varphi}}(1)$ as $x \to 1$. In between there can be an extremum and in this case it is a minimum. Let us now determine the condition for the existence of the maximum in $b_{\bar{\varphi}}(x)$ or minimum of $T_{\bar{\varphi}}(x)$. Equating (41) to zero gives us the maximum of $x$ as,

$$x_{\max} = \frac{2}{(2+\tilde{d})(1-\bar{\varphi}^2)}. \tag{42}$$

Now substituting this value in (38) we determine the maximum value of $b_{\bar{\varphi}}(x)$ and from there the minimum value of $T_{\bar{\varphi}}(x)$ as,

$$\begin{aligned} (b_{\bar{\varphi}})_{\max} &= \left(\frac{2}{2+\tilde{d}}\right)^{\frac{1}{\tilde{d}}} \left[\tilde{d}(\tilde{d}+2)\left(1-\bar{\varphi}^2\right)\right]^{-\frac{1}{2}} \quad \Rightarrow \\ (T_{\bar{\varphi}})_{\min} &= \left(\frac{2+\tilde{d}}{2}\right)^{\frac{1}{\tilde{d}}} \frac{\left[\tilde{d}(\tilde{d}+2)\left(1-\bar{\varphi}^2\right)\right]^{\frac{1}{2}}}{4\pi\bar{\rho}_B}. \end{aligned} \tag{43}$$

The condition for the existence of maximum is $x_{\max} < 1$, which gives $\bar{\varphi} < (\tilde{d}/(\tilde{d}+2))^{1/2}$, i.e., the same constraint we obtained earlier in (37), as expected. From this we have the following constraint on $(b_{\bar{\varphi}})_{\max}$ or $(T_{\bar{\varphi}})_{\min}$,

$$(b_{\bar{\varphi}})_{\max} < \frac{1}{\sqrt{2\tilde{d}}}\left(\frac{2}{2+\tilde{d}}\right)^{\frac{1}{\tilde{d}}} \quad \Rightarrow \quad (T_{\bar{\varphi}})_{\min} > \frac{\sqrt{2\tilde{d}}}{4\pi\bar{\rho}_B}\left(\frac{2+\tilde{d}}{2}\right)^{\frac{1}{\tilde{d}}} \tag{44}$$

where we have used the constraint on $\bar{\varphi}$ given in (37). So, when $(b_{\bar{\varphi}})_{\max}$ ($(T_{\bar{\varphi}})_{\min}$) given in eq.(43) satisfy the constraint (44) we will have the maximum (minimum). From Figure 1, we can see that when $x_{\max} < x < 1$, $db_{\bar{\varphi}}/dx < 0$ and we have a locally stable system and when $0 < x < x_{\max}$, $db_{\bar{\varphi}}/dx > 0$ and we have an unstable system consistent with what we discussed earlier.

For any given $\bar{\varphi}$ and $\bar{b}$, satisfying $b_{\bar{\varphi}}(1) < \bar{b} < (b_{\bar{\varphi}})_{\max}$ with $b_{\bar{\varphi}}(1)$ as given in (39), one expects two black brane solutions with radii $x_1$ and $x_2$, where $x_1 < x_2$ from $\bar{b} = b_{\bar{\varphi}}(x)$ as shown in Figure 1 and only the large one ($x_2$) will give a locally stable phase. The small one ($x_1$) will be unstable. Since here for each given $\bar{\varphi}$ and $\bar{b}$, we have only one locally stable phase, we don't have a thermodynamical phase transition like in the canonical ensemble[6], even though $x_1 = x_2 = x_{\max}$ seems to appear as a critical point. When $\bar{\varphi} > 1$ and/or $\bar{b} > (b_{\bar{\varphi}})_{\max}$ ($\bar{T} < (T_{\bar{\varphi}})_{\min}$) no black brane phase is possible in the grand canonical ensemble and the only possible thermally stable phase here is the 'hot flat space'. Also for $0 = b_{\bar{\varphi}}(0) < \bar{b} < b_{\bar{\varphi}}(1)$, assuming $\bar{\varphi} < 1$ from now on, no stable black brane phase

[6]In the canonical ensemble there is a phase transition between two stable black brane phases, one of which is locally stable and the other is globally stable. Which one is locally stable and which one is globally stable depend on the temperature of the environment in contact with the system [1].

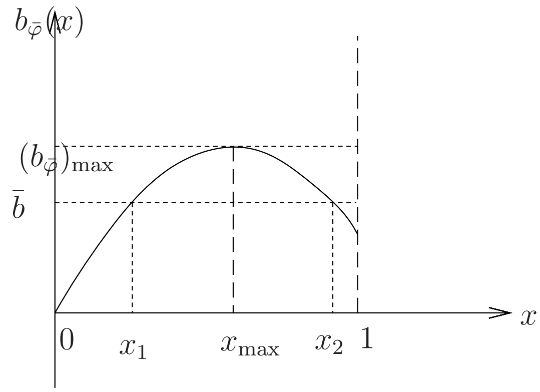


Figure 1: The typical behavior of $b_{\bar\varphi}(x)$ vs. $x$.

is possible and the 'hot flat space' is again the thermally stable phase. This clearly shows that the thermodynamics and the phase structure are quite different for the grand canonical ensemble than those in the canonical ensemble [1] and this is expected since the boundary data are different in the two cases.

To the leading order, the Euclidean action is directly related to the grand potential or Gibbs free energy as $\beta\Omega = I_E$. So, for a given temperature, smaller the value of $I_E$, the more stable the system is. We will consider the reduced action $\tilde I_E$. For a given $\bar b$ and $\bar\varphi$ satisfying their respective constraints such that the system is locally stable, if the reduced action is positive at the stationary point, it is metastable since the reduced action has smaller value, i.e., zero, for the 'hot flat space' with the same boundary data and will make a transition to this phase. In other words, if the system is initially at the locally stable black brane phase, after some time, a topological phase transition would occur so the black brane phase will become the 'hot flat space' phase with a different topology via a topological phase transition. So, for making sure that we have a stable black brane phase, we need to look for condition such that the minimum of the reduced action is actually a global minimum. This can be simply realized by conditions such that the stationary action at the minimum is negative.

The reduced action at the stationary point is described by $b_{\bar\varphi}(\bar x) = \bar b$ in the action (30) and is given as,

$$\tilde I_E^{\bar\varphi} = -\frac{\bar b}{y}(\tilde d + 2)(y-1)\left(y - \frac{\tilde d}{\tilde d + 2}\right) \tag{45}$$

where $y = \sqrt{1 - \bar x(1-\bar\varphi^2)}$. Given the fact that $y < 1$, so a negative reduced action requires,

$$y < \frac{\tilde{d}}{\tilde{d}+2} \tag{46}$$

which in turn gives,

$$\bar{x}\left(1-\bar{\varphi}^2\right) > \frac{4(\tilde{d}+1)}{(\tilde{d}+2)^2}, \tag{47}$$

where we have used the expression of $y$ given above. Note that this condition is consistent with the local stability condition given in (36) since $4(\tilde{d}+1)/(\tilde{d}+2)^2 > 2/(\tilde{d}+2)$. Also note that $4(\tilde{d}+1)/(\tilde{d}+2)^2 < 1$ for all $\tilde{d}$ and therefore, there is no contradiction. If we denote

$$\bar{x}_g = \frac{4(\tilde{d}+1)}{(\tilde{d}+2)^2\,(1-\bar{\varphi}^2)} > x_{\max} \tag{48}$$

then for $0 < \bar{\varphi} < \tilde{d}/(\tilde{d}+2)$ and $b_{\bar{\varphi}}(1) < \bar{b} < (b_{\bar{\varphi}})_g$ $((T_{\bar{\varphi}})_g < \bar{T} < T_{\bar{\varphi}}(1))$, where,

$$(b_{\bar{\varphi}})_g = \frac{\left(4(\tilde{d}+1)\right)^{\frac{1}{\tilde{d}}}}{(\tilde{d}+2)^{1+\frac{2}{\tilde{d}}}\sqrt{1-\bar{\varphi}^2}} < (b_{\bar{\varphi}})_{\max} \tag{49}$$

and

$$(T_{\bar{\varphi}})_g = \frac{(\tilde{d}+2)^{1+\frac{2}{\tilde{d}}}\sqrt{1-\bar{\varphi}^2}}{4\pi\bar{\rho}_B\left(4(\tilde{d}+1)\right)^{\frac{1}{\tilde{d}}}} \tag{50}$$

the reduced action has a global minimum at $\bar{x}_g < x_2 < 1$. In other words, this ensemble can give a sensible description of black brane thermodynamics only when the above conditions are satisfied. We would like to point out that when the maximum of $b_{\bar{\varphi}}(x)$ exists (or minimum of $T_{\bar{\varphi}}(x)$ exists), the grand canonical ensemble is in some sense similar to the chargeless case of canonical ensemble except around $x = 1$.

# 5 Conclusion

To conclude, in this paper we have studied the equilibria and the phase structure of the asymptotically flat dilatonic black $p$-branes in a fixed cavity in arbitrary dimensions $D$ in a grand canonical ensemble. For this, we have considered both the temperature and the potential fixed at the wall of the cavity and compared with our previous study of the same system in a canonical ensemble [1] for which the charge enclosed in the cavity, rather than the potential at the wall of the cavity, is fixed. We computed the Euclidean action corresponding to the dilatonic black $p$-brane solution of $D$-dimensional supergravity with maximal supersymmetry with proper consideration of the above fixed quantities at the wall of cavity in this ensemble. To the leading (or zeroth-loop) order this action is related to the grand potential or Gibbs free energy of the system and is an essential entity for the stability analysis. This action has two variables, namely, the horizon size $r_+$ (related to the variable $x$) and the charge $Q_d$ (related to the variable $q$). At the stationary point of this action, we derived the expected equations for which $x$ and $q$ have to satisfy with both the temperature and the potential fixed at the wall of cavity. The second derivatives of the action with respect to $x$ and $q$ at the stationary point can be used to analyze the local thermal stability of the system in this ensemble.

Unlike in the canonical case, the directly obtained action here is a function of two variables and this makes the stability analysis a bit involved. However, there exists also an analog of one-variable analysis as in the canonical case and we have shown in the text that the condition so obtained for the stability of the equilibrium states as given in (35) is the same as that of the rather complicated two-variable case. For the two-variable analysis given in the Appendix, we have made use of a trick to use the $b$, $\varphi$ variables as defined in (26) instead of origianl $x$, $q$ variables. This makes the analysis a bit simpler and straightforward. The stability condition can also be expressed as those given in (37). So, for a given potential satisfying the constraint given in (37), only the horizon size lying in the range also given in (37) gave locally stable system.

We then obtained the phase structure of the black $p$-branes in this ensemble. We found that locally stable black brane phase exists only when the minimum temperature of the system given in (43) satisfies the constraint (44). In this case, below the minimum temperature, there is no stable black brane phase and only stable phase is the 'hot flat space'. Above this temperature, and below the value $T_{\bar{\varphi}}(1)$ given in (40), there are two black brane phases, the larger one is locally stable and the smaller one is unstable. The locally stable black brane phase becomes globally stable only above the temperature $(T_{\bar{\varphi}})_g$ given in (50) and below $T_{\bar{\varphi}}(1)$. Below $(T_{\bar{\varphi}})_g$ and above $(T_{\bar{\varphi}})_{\min}$ (given in (43)), the stable black brane is only locally stable and will eventually make a transition to the 'hot flat space'. Finally, above $T_{\bar{\varphi}}(1)$ there is only one black brane phase, but this is unstable and will eventually decay to the stable 'hot flat space' phase. We also commented on the similarity of this phase structure with that of the zero charge canonical case except at one end of the $x$ variable, namely, near $x = 1$. This structure is reminiscent of the Hawking-Page phase transition of the AdS, dS and flat black holes discussed in [7, 11, 12]. However, unlike in the canonical case, here we found that at a given temperature when the stable phase exists it does not come in pairs – only a single stable phase appears. So, there is no thermodynamical phase transition between two stable black brane phases analogous to the van der Waals-Maxwell liquid-gas phase transition like that appeared in the canonical case.

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## Appendix

In this appendix we perform the more general yet more complicated two-variable analysis of the stability of black $p$-branes and show that the same local stability condition derived from the simple one-variable consideration as given in (35) continues to hold.

It is clear from (26) that both $b(x,q)$ and $\varphi(x,q)$ are smooth functions of $x$ and $q$ and so, in principle they can be inverted to give $x$ and $q$ as functions of $b$ and $\varphi$. Although we will not need their explicit form, however, we will need the form of $b$ as a function of $x$ and $\varphi$. For that we will eliminate $q$ from $b(x,q)$-equation given in (26) and express it as a function of $x$ and $\varphi$ instead. From the $\varphi$ equation in (26) we obtain $q^2/x^2$ as,

$$\frac{q^2}{x^2} = \frac{\varphi^2}{1-(1-\varphi^2)\,x} \tag{51}$$

Now substituting (51) in $b$ equation in (26) we obtain,

$$b = \frac{x^{\frac{1}{\tilde d}}\,[1-(1-\varphi^2)\,x]^{\frac{1}{2}}}{\tilde d\,(1-\varphi^2)^{\frac{1}{2}-\frac{1}{\tilde d}}}. \tag{52}$$

The reduced action (27) can now be re-expressed as,

$$\tilde I_E(x,\varphi) = -\bar b\left[(\tilde d+2)y + \tilde d\,(1-x(1-\varphi\bar\varphi))\,y^{-1} - 2(\tilde d+1)\right] + b\tilde d(y^2-1)y^{-1}, \tag{53}$$

where we have used (51) and (52). In the above, we have also defined

$$y = \sqrt{1-x\,(1-\varphi^2)}, \tag{54}$$

with $x(b,\varphi)$ determined by (52). We will use both of (52) and (53) later.

We now expand the reduced action (27) at the stationary point determined by the equation (25) with $x=\bar x$ and $q=\bar q$ to quadratic order as,

$$\tilde I_E(x,q) = \tilde I_E(\bar x,\bar q) + \tilde I_{ij}\Big|_{z_i=\bar z_i, z_j=\bar z_j}(z_i-\bar z_i)(z_j-\bar z_j) + \cdots \tag{55}$$

where $z_1 = x$, $z_2 = q$ with $i$, $j = 1, 2$ and we have used the stationary conditions for the first order terms. In the above

$$\tilde{I}_{ij} \equiv \frac{\partial^2 \tilde{I}_E(x,q)}{\partial z_i \partial z_j} \tag{56}$$

The local stability of the system is determined by whether the quadratic terms in the expansion are always positive definite. This can be easily understood if we diagonalize the matrix $\tilde{I}_{ij}$ and demand that each of the two eigenvalues $\lambda_1$ and $\lambda_2$ is positive definite. Now since,

$$\lambda_1 \lambda_2 = \det \tilde{I}_{ij}, \qquad \text{and} \qquad \lambda_1 + \lambda_2 = \tilde{I}_{xx} + \tilde{I}_{qq} \tag{57}$$

$\lambda_1$, $\lambda_2 > 0$ implies that both $\tilde{I}_{xx} + \tilde{I}_{qq} > 0$ and $\det \tilde{I}_{ij} > 0$. Now we will rewrite these two conditions such that our analysis becomes simpler as[7],

$$\tilde{I}_{qq} > 0, \tag{58}$$

$$\frac{\tilde{I}_{qq}}{\det \tilde{I}_{ij}} > 0 \tag{59}$$

In order to understand the meaning of the condition (58) we have to evaluate $\tilde{I}_{qq}$ and then set it to greater than zero. From the form of $\tilde{I}_E$ in (27) we obtain the first condition (58) as,

$$\begin{aligned} \tilde{I}_{qq} &\equiv \left.\frac{\partial^2 \tilde{I}_E}{\partial q^2}\right|_{x=\bar{x}, q=\bar{q}} \\ &= \frac{\bar{b}\tilde{d}\bar{\varphi}}{\bar{q}\left(1 - \frac{\bar{q}^2}{\bar{x}}\right)} \left[1 + \frac{(\tilde{d}+2)\bar{\varphi}^2 \left[1 - \frac{2}{\tilde{d}} + \frac{2}{\tilde{d}}\bar{x}\left(1-\bar{\varphi}^2\right)\right]}{\tilde{d}\left(1-\bar{\varphi}^2\right)\left(1-\bar{x}\right)}\right] > 0. \end{aligned} \tag{60}$$

Once this is satisfied the thermodynamic stability is completely determined by the condition (59). A direct evaluation of $\tilde{I}_{qq}/\det \tilde{I}_{ij}$ from the expression (in $(x,\, q)$ variables) of $\tilde{I}_E$ in (27) is complicated and tedious and we will make use of a trick in evaluating it. This will make the analysis of stability much more elegant and simpler. For this, we will instead evaluate the l.h.s. of (59) by first going to the $(b,\, \varphi)$ variable and evaluate $\tilde{I}_{ab}$, where

$$\tilde{I}_{ab} \equiv \frac{\partial^2 \tilde{I}_E(b,\varphi)}{\partial \xi_a \partial \xi_b} \tag{61}$$

[7]It is not difficult to see that the conditions given in (58) and (59) automatically implies $\det \tilde{I}_{ij} > 0$ and $\tilde{I}_{xx} + \tilde{I}_{qq} > 0$ . To see this note that we first have to have $\tilde{I}_{qq} > 0$ (if $I_{qq} < 0$, the above two conditions can never be satisfied), then (59) implies $\det \tilde{I}_{ij} > 0$ and so, $\det \tilde{I}_{ij} = \tilde{I}_{qq}\tilde{I}_{xx} - \tilde{I}_{qx}^2 = \tilde{I}_{qq}(\tilde{I}_{xx} - \tilde{I}_{qx}^2/\tilde{I}_{qq}) > 0$ $\Rightarrow \tilde{I}_{xx} - \tilde{I}_{qx}^2/\tilde{I}_{qq} > 0 \Rightarrow \tilde{I}_{xx} > \tilde{I}_{qx}^2/\tilde{I}_{qq} > 0$ and this in turn implies $\tilde{I}_{xx} + \tilde{I}_{qq} > 0$.

with $a$, $b = 1, 2$ and $\xi_1 = \varphi$, $\xi_2 = b$ and then relate this matrix to the original matrix $\tilde{I}_{ij}$ in $(x, q)$ variables by chain rules as follows,

$$\begin{aligned}
\tilde{I}_{ab} &= \begin{pmatrix} \tilde{I}_{\varphi\varphi} & \tilde{I}_{\varphi b} \\ \tilde{I}_{b\varphi} & \tilde{I}_{bb} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial q}{\partial \varphi} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial q}{\partial b} & \frac{\partial x}{\partial b} \end{pmatrix} \tilde{I}_{ij} \begin{pmatrix} \frac{\partial q}{\partial \varphi} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial q}{\partial b} & \frac{\partial x}{\partial b} \end{pmatrix}^T , \qquad (62)
\end{aligned}$$

where '$T$' denotes the transpose of a matrix. After that we invert this matrix relation to obtain,

$$\tilde{I}_{ij}^{-1} = \begin{pmatrix} \frac{\partial q}{\partial \varphi} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial q}{\partial b} & \frac{\partial x}{\partial b} \end{pmatrix}^T \tilde{I}_{ab}^{-1} \begin{pmatrix} \frac{\partial q}{\partial \varphi} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial q}{\partial b} & \frac{\partial x}{\partial b} \end{pmatrix}, \qquad (63)$$

Again inverting (63) we will evaluate $\tilde{I}_{ij}/\det \tilde{I}_{ij}$ in terms of $\tilde{I}_{ab}$ and other known functions given in (63). Now to evaluate $\tilde{I}_{ab}$, we first calculate

$$\begin{aligned}
\left(\frac{\partial \tilde{I}_E}{\partial \varphi}\right)_b &= \left(\frac{\partial \tilde{I}_E}{\partial x}\right)_q \left(\frac{\partial x}{\partial \varphi}\right)_b + \left(\frac{\partial \tilde{I}_E}{\partial q}\right)_x \left(\frac{\partial q}{\partial \varphi}\right)_b \\
&= \left[\bar{b} - b(x,q)\right] f(x,q) \left(\frac{\partial x}{\partial \varphi}\right)_b \\
&\quad - \left[\bar{b}\tilde{d}(\bar{\varphi} - \varphi(x,q)) + \frac{(\tilde{d}+2)\varphi(x,q)}{1 - \frac{q^2}{x}}(\bar{b} - b(x,q))\right] \left(\frac{\partial q}{\partial \varphi}\right)_b , \qquad (64)
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial \tilde{I}_E}{\partial b}\right)_\varphi &= \left(\frac{\partial \tilde{I}_E}{\partial x}\right)_q \left(\frac{\partial x}{\partial b}\right)_\varphi + \left(\frac{\partial \tilde{I}_E}{\partial q}\right)_x \left(\frac{\partial q}{\partial b}\right)_\varphi \\
&= \left[\bar{b} - b(x,q)\right] f(x,q) \left(\frac{\partial x}{\partial b}\right)_\varphi \\
&\quad - \left[\bar{b}\tilde{d}(\bar{\varphi} - \varphi(x,q)) + \frac{(\tilde{d}+2)\varphi(x,q)}{1 - \frac{q^2}{x}}(\bar{b} - b(x,q))\right] \left(\frac{\partial q}{\partial b}\right)_\varphi , \qquad (65)
\end{aligned}$$

where we have used the form of $\tilde{I}_E$ given in (27) and the function $f(x,q)$ is defined as,

$$f(x,q) = (1-x)^{-\frac{1}{2}} \left(1 - \frac{q^2}{x}\right)^{-\frac{1}{2}} \left[\tilde{d} + 2 - \frac{\tilde{d}+2}{2} \left(\frac{1 - \frac{q^2}{x^2}}{1 - \frac{q^2}{x}}\right) + \frac{\tilde{d}}{2}\left(1 - \frac{q^2}{x^2}\right)\right] > 0, \quad (66)$$

Thus we have from eqs.(64) and (65)

$$\frac{\partial^2 \tilde{I}_E}{\partial\varphi^2} = \bar{b}\tilde{d}\frac{\partial q}{\partial\varphi}, \qquad \frac{\partial^2 \tilde{I}_E}{\partial\varphi\partial b} = -f(\bar{x},\bar{q})\frac{\partial x}{\partial\varphi} + \frac{(\tilde{d}+2)\bar{\varphi}}{1-\frac{\bar{q}^2}{\bar{x}}}\frac{\partial q}{\partial\varphi},$$
$$\frac{\partial^2 \tilde{I}_E}{\partial b\partial\varphi} = \bar{b}\tilde{d}\frac{\partial q}{\partial b}, \qquad \frac{\partial^2 \tilde{I}_E}{\partial b^2} = -f(\bar{x},\bar{q})\frac{\partial x}{\partial b} + \frac{(\tilde{d}+2)\bar{\varphi}}{1-\frac{\bar{q}^2}{\bar{x}}}\frac{\partial q}{\partial b}, \tag{67}$$

where the second derivatives are evaluated at $b = \bar{b}$ and $\varphi = \bar{\varphi}$. So, from the above equations (67) we obtain $\tilde{I}_{ab}$ as,

$$\tilde{I}_{ab} = \begin{pmatrix} \bar{b}\tilde{d}\frac{\partial q}{\partial\varphi} & \bar{b}\tilde{d}\frac{\partial q}{\partial b} \\ \bar{b}\tilde{d}\frac{\partial q}{\partial b} & -f(\bar{x},\bar{q})\frac{\partial x}{\partial b} + \frac{(\tilde{d}+2)\bar{\varphi}}{1-\frac{\bar{q}^2}{\bar{x}}}\frac{\partial q}{\partial b} \end{pmatrix}, \tag{68}$$

where we have used

$$\bar{b}\tilde{d}\frac{\partial q}{\partial b} = -f(\bar{x},\bar{q})\frac{\partial x}{\partial\varphi} + \frac{(\tilde{d}+2)\bar{\varphi}}{1-\frac{\bar{q}^2}{\bar{x}}}\frac{\partial q}{\partial\varphi}. \tag{69}$$

From the form of $\tilde{I}_{ab}$ given in (68) we can calculate its inverse as,

$$\tilde{I}_{ab}^{-1} = \frac{1}{\det \tilde{I}_{ab}} \begin{pmatrix} -f(\bar{x},\bar{q})\frac{\partial x}{\partial b} + \frac{(\tilde{d}+2)\bar{\varphi}}{1-\frac{\bar{q}^2}{\bar{x}}}\frac{\partial q}{\partial b} & -\bar{b}\tilde{d}\frac{\partial q}{\partial b} \\ -\bar{b}\tilde{d}\frac{\partial q}{\partial b} & \bar{b}\tilde{d}\frac{\partial q}{\partial\varphi} \end{pmatrix}, \tag{70}$$

with det $\tilde{I}_{ab}$ having the from (as can be seen from (68))

$$\det \tilde{I}_{ab} = f\bar{b}\tilde{d}\left(\frac{\partial q}{\partial b}\frac{\partial x}{\partial\varphi} - \frac{\partial q}{\partial\varphi}\frac{\partial x}{\partial b}\right) \tag{71}$$

where we have made use of (69). Now substituting (70) in (63) we obtain,

$$\tilde{I}_{ij}^{-1} = \begin{pmatrix} \frac{1}{\bar{b}\tilde{d}}\frac{\partial q}{\partial\varphi} & \frac{1}{\bar{b}\tilde{d}}\frac{\partial x}{\partial\varphi} \\ \frac{1}{\bar{b}\tilde{d}}\frac{\partial x}{\partial\varphi} & \frac{(\tilde{d}+2)\bar{\varphi}}{f\bar{b}\tilde{d}\left(1-\frac{\bar{q}^2}{\bar{x}}\right)}\frac{\partial x}{\partial\varphi} - \frac{1}{f}\frac{\partial x}{\partial b} \end{pmatrix}, \tag{72}$$

where we also made use of (69). Inverting (72) we get

$$\tilde{I}_{ij} = \det \tilde{I}_{ij} \begin{pmatrix} \frac{(\tilde{d}+2)\bar{\varphi}}{f\bar{b}\tilde{d}\left(1-\frac{\bar{q}^2}{\bar{x}}\right)}\frac{\partial x}{\partial\varphi} - \frac{1}{f}\frac{\partial x}{\partial b} & -\frac{1}{\bar{b}\tilde{d}}\frac{\partial x}{\partial\varphi} \\ -\frac{1}{\bar{b}\tilde{d}}\frac{\partial x}{\partial\varphi} & \frac{1}{\bar{b}\tilde{d}}\frac{\partial q}{\partial\varphi} \end{pmatrix}. \tag{73}$$

From (72) we also obtain, $\det \tilde{I}_{ij}^{-1} = (\det \tilde{I}_{ij})^{-1} = \det \tilde{I}_{ab}/(f\bar{b}\tilde{d})^2$. Now from (73) we finally get,

$$\frac{\tilde{I}_{qq}}{\det \tilde{I}_{ij}} = \frac{(\tilde{d}+2)\bar{\varphi}}{f\bar{b}\tilde{d}\left(1-\frac{\bar{q}^2}{\bar{x}}\right)}\frac{\partial x}{\partial\varphi} - \frac{1}{f}\frac{\partial x}{\partial b}, \tag{74}$$

We can compute $\partial x/\partial b$ and $\partial x/\partial\varphi$ at the stationary point from (52) and obtain,

$$\frac{\partial x}{\partial b} = \frac{2\tilde{d}\bar{x}\left[1-\bar{x}(1-\bar{\varphi}^2)\right]}{\bar{b}\left[2-(\tilde{d}+2)\bar{x}(1-\bar{\varphi}^2)\right]}, \quad \frac{\partial x}{\partial\varphi} = -\frac{2\tilde{d}\bar{x}\bar{\varphi}\left[1-\frac{2}{\tilde{d}}+\frac{2}{\tilde{d}}\bar{x}(1-\bar{\varphi}^2)\right]}{(1-\bar{\varphi}^2)\left[2-(\tilde{d}+2)\bar{x}(1-\bar{\varphi}^2)\right]}, \tag{75}$$

Substituting (75) in (74) we have from the condition (59),

$$\frac{\tilde{I}_{qq}}{\det\tilde{I}_{ij}} = -\frac{2\tilde{d}\bar{x}\left[1-\bar{x}(1-\bar{\varphi}^2)\right]}{f\bar{b}\left[2-(\tilde{d}+2)\bar{x}(1-\bar{\varphi}^2)\right]}\left[1+\frac{(\tilde{d}+2)\bar{\varphi}^2\left[1-\frac{2}{\tilde{d}}+\frac{2}{\tilde{d}}\bar{x}(1-\bar{\varphi}^2)\right]}{\tilde{d}(1-\bar{\varphi}^2)(1-\bar{x})}\right] > 0. \tag{76}$$

Thus for the stability of the system the above condition has to be satisfied. Now since the second factor in the square bracket is positive definite by the first condition written in (60) it is clear that for (76) to hold the denominator of the first factor has to be negative definite. This gives, as promised, precisely the same condition as given in (35).

Now it can be checked that once (35) is satisfied the first condition written explicitly in (60) is automatically satisfied. To see this we look at the numerator of the second term in the square bracket in the expression of $\tilde{I}_{qq}$ given in (60),

$$\begin{aligned} 1-\frac{2}{\tilde{d}}+\frac{2}{\tilde{d}}\bar{x}\left(1-\bar{\varphi}^2\right) &= \frac{1}{\tilde{d}}\left[\tilde{d}-2\left(1-\bar{x}\left(1-\bar{\varphi}^2\right)\right)\right] \\ &= \left[1-\bar{x}\left(1-\bar{\varphi}^2\right)\right]+\frac{1}{\tilde{d}}\left[(\tilde{d}+2)\bar{x}\left(1-\bar{\varphi}^2\right)-2\right] \end{aligned} \tag{77}$$

and this is positive definite since the first term is positive definite and the second term is positive definite by (35). Therefore, $\tilde{I}_{qq}$ is positive definite. This shows that (35) is equivalent to both the stability conditions given in (58) and (59).

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