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Model building in AdS/CMT: DC Conductivity and Hall angle

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Abstract

Using the bottom-up approach in a holographic setting, we attempt to study both the transport and thermodynamic properties of a generic system in $3+1$ dimensional bulk spacetime. We show the exact $1/T$ and T^2 dependence of conductivity and Hall angle, as seen experimentally in most copper-oxide systems, which are believed to be close to quantum critical point. This particular temperature dependence of conductivities are possible for two different cases: (1) Background solutions with scale invariant and broken rotational symmetry, (2) solutions with pseudo-scaling and unbroken rotational symmetry but only at low density limit. Generically, the study of transport properties in a scale invariant background solution, using the probe brane approach, at high density and at low temperature limit suggests us to consider only metrics with two exponents. More precisely, the spatial part of the metric components should not be same i.e., $g_{xx} \neq g_{yy}$. In doing so, we have generated the above mentioned behavior to conductivity with a very special behavior to specific heat which at low temperature goes as: $C_V \sim T^3$. However, if we break the scaling symmetry of the background solution by including a nontrivial dilaton, axion or both and keep the rotational symmetry then also we can generate such a behavior to conductivity but only in the low density regime. As far as we are aware, this particular temperature dependence to both the conductivity and Hall angle is being shown for the first time using holography.

1 Introduction

There are interesting model building calculations that are being put forward using gauge/gravity duality, which suggests to have captured the experimental results close to quantum criticality and the associated quantum phase transitions. In particular, for the copper-oxide systems at low temperature, the resistivity, which is the inverse of conductivity, goes as $\sigma \sim T^{-1}$ [1], [2], [3], [4]. This interesting behavior has been reported in a controllable yet unrealistic setting for a very special kind of gravitational system that displays the Lifshitz like property and is possible only when the Lifshitz exponent takes a special value namely, $z = 2$, [1] [5]. However, it is also suggested in [1], [2], [3], [4] that for the copper-oxide systems the Hall angle, $cot\ \theta_H = \sigma^{xx}/\sigma^{xy}$, should have a quadratic dependence of temperature, $cot\ \theta_H \sim T^2$. But, unfortunately, use of the gravitational solutions showing the Lifshitz like scaling does not reproduce this behavior of Hall angle, rather it gives at low temperature a linear dependence of temperature and is not in complete agreement with the experimental results.

The experimental results for the transport properties of the copper-oxide systems near optimum doping at low temperature can be summarized as follows [1], [3], [4]

$$\sigma^{xx} \sim 1/T, \quad cot\ \theta_H = \sigma^{xx}/\sigma^{xy} \sim T^2 \quad \Longrightarrow \quad \sigma^{xy} \sim T^{-3}. \tag{1}$$

The basic reason of not getting the desired experimental behavior is due to the presence of a rotational symmetry in the $x,\ y$ plane of the metric while having the scaling symmetry of the background solution, where x and y are the only two spatial directions available in field theory. Even though this symmetry is broken explicitly by the presence of constant electric and magnetic field.

In this paper we shall show that eq(1) can only be reproduced in two different cases (1) background solutions respecting the scaling symmetry with broken rotational symmetry in the $x,\ y$ plane (2) pseudo-scaling background solutions with unbroken rotational symmetry at low density limit. Here the pseudo-scaling solutions means, the background geometry respects the scaling symmetry but not the dilaton and axion. Furthermore, the background solutions which shows the scaling symmetry, time translation, spatial translation, and the rotational symmetry are completely ruled out by eq(1), e.g., pure AdS and pure Lifshitz solutions. It is worth to emphasize that case (1) is the only choice that is permissible at high density, but at low density we can have either of the choices. We are discussing both the limits of densities because it is not *a priori* clear the scale of optimum doping in eq(1).

The basic philosophy of [5] is to introduce charge carriers via Dp branes, in the probe brane approximation. The charge carriers are in thermal contact with a heat bath, which is taken as the Lifshitz black hole. Translating it into the language of [7], it's the bi-fundamental degrees of freedom that are charged, interacting among themselves and with

[1] There is another paper [6], which does not require gravitational solution with Lifshitz scaling (rather with $z = 1$) in order to generate such a behavior to conductivity. More interestingly, it is shown that such a behavior follows at one loop.

the adjoint degrees of freedom gave us the desired feature of conductivity. In contrast to [6], where the authors have considered only the charged adjoint degrees of freedom to replicate the above mentioned experimental result at one loop[2]. In this paper, we have adopted the former approach (in the massless limit) and replace the heat bath of Lifshitz kind by another, more general, heat bath. The reason of such a replacement is that (1) Lifshitz type heat bath is a special type to this more general heat bath (2) It's the eq(1), which was not possible to reproduce fully with the Lifshitz type heat bath. Recall, the heat baths are essentially the source of studying physics around the quantum critical point at low temperature [8]. The consequences of replacing such a heat bath is also addressed, thermodynamically.

In the holographic setting [9], the authors of [10], [11] and [12] have proposed a beautiful algorithm to calculate the conductivities. Here we have modified it slightly and obtain an equivalent way to calculate the conductivities. The result of the calculation matches precisely as is done in [11] when the charge carriers move in a constant electric and magnetic field. Use of this equivalent prescription leads to the following dependence of conductivities on the metric components evaluated at some holographic energy scale, $r_\star$. At high densities compared to temperature

$$\sigma^{xx} \sim \frac{c_\phi e^{-2\Phi(r_\star)}}{g_{xx}(r_\star)}, \quad \sigma^{xy} \simeq \frac{B c_\phi e^{-4\Phi(r_\star)}}{g_{xx}(r_\star) g_{yy}(r_\star)}, \tag{2}$$

where c_ϕ is the charge density, B, the magnetic field, and Φ, the dilaton. In eq(2), the spatial parts of the metric components along x and y directions are denoted as g_{xx} and g_{yy} respectively. Note, this result follows when the probe brane action admits only the DBI type of action. If we do include the Chern-Simon part of the action to the probe brane as well then the results to conductivities gets slightly modified in the limit of high density compared to temperature

$$\sigma^{xx} \sim \frac{c_\phi e^{-2\Phi(r_\star)}}{g_{xx}(r_\star)}, \quad \sigma^{xy} \simeq \frac{B c_\phi e^{-4\Phi(r_\star)}}{g_{xx}(r_\star) g_{yy}(r_\star)} - \mu C_0(r_\star), \tag{3}$$

where μ is the coupling of the Chern-Simon action and C_0 is the axion field. Note that the scaling symmetry is broken for a non constant dilaton and axion field.

Now if we restrict ourselves to background solutions which possesses the scaling symmetry and consider the gravitational system that exhibits the rotational symmetry at the level of metric not the full system, then the off diagonal part of the conductivity in the high density limit goes as, $\sigma^{xy} \sim \left(\sigma^{xx}\right)^2$, which is not in accordance with the experimental result, see eq(1). This means to reproduce eq(1), in the high density limit we are forced to consider metric components for which $g_{xx} \neq g_{yy}$. This is one of the basic criteria that must be

[2]There arises a natural question: Is this behavior of charged bi-fundamental degrees of freedom in a heat bath=1-loop adjoint degrees of freedom in a different heat bath, generically ? Which we are not going to address.

imposed in choosing the background metric, i.e., the heat bath, in order to study the physics associated to transport properties around the quantum critical point.

In getting the results to conductivity and the Hall angle as in eq(1), we have assumed the background metric to respect the following scaling symmetry

$$t \to \lambda^z\, t, \quad x \to \lambda^w\, x, \quad y \to \lambda\, y, \quad r \to \frac{r}{\lambda}, \tag{4}$$

and also we have assumed that there is not any non trivial scalar field, dilaton or axion, in the entire set up. As the presence of such a non trivial background field would give rise to some-kind of pseudo-scaling theory. Of course, the charge density of the bi-fundamental degrees of freedom, i.e., two form field strength, F_2, that appear in the DBI action, breaks the scaling symmetry. More exactly, the gravitational solution without any non vanishing scalar field that gives us the desired result to conductivity and Hall angle has the exponents $z = 1,\ w = 1/2$. The zero temperature limit of the black hole solution, i.e., the solution without the thermal factor, has a boost symmetry along the $t,\ y$ plane with a form

$$ds^2 = L^2[-r^2dt^2 + rdx^2 + r^2dy^2 + \frac{dr^2}{r^2}], \tag{5}$$

where L is the size of the $3+1$ dimensional bulk system, which we shall set to unity in our calculations latter. The background geometry with two exponents z and w was proposed in [15] using a combination of Einstein-Hilbert action and several form field strengths. Since, the analytic non-extremal version of that solution is very difficult to obtain. So, here we have adopted a different path to generate such a solution by using only gravitons.

Let us do a little bit of dimensional analysis for various physical quantities. If the $d+1$ dimensional field theory spacetime coordinates (i.e the bulk is $d+2$ dimensional spacetime) behaves under scaling as

$$t \;\to\; \lambda^z\, t, \quad x \;\to\; \lambda^w\, x, \quad y_i \;\to\; \lambda\, y_i, \quad (i = 1, \cdots, d-1) \tag{6}$$

then the physical quantities possesses the following mass dimension

$$\begin{aligned}&[t] = -z,\ [x] = -w,\ [y_i] = -1\ [J^t] = w + d - 1,\ [J^x] = d + z - 1,\ [J^i] = d + w + z - 2,\\ &[A_t] = z,\ [A_x] = w,\ \ [A_i] = 1,\ [E_x] = w + z,\ [E_i] = 1 + z,\ [B_x] = 1 + w,\ [B_i] = 2,\\ &[T] = z = [\omega],\ [F] = z,\ [\sigma^{xx}] = d - w - 1,\ [\sigma^{xy}] = d - 2,\end{aligned} \tag{7}$$

where $J^t,\ J_i,\ A_t,\ A_x,\ A_i,\ E,\ B,\ T,\ \omega,\ F,\ \sigma$ are charge density, current density, time component of the gauge potential, x-component of the gauge potential, y_i-component of the gauge potential, electric field, magnetic field, temperature, frequency, free energy and conductivity respectively. The two form field strength has the following form i.e, $F_2 = -E_x dt\wedge dx - E_i dt \wedge dy_i + B_x dx \wedge dy_i + B_y dy_i \wedge dy_j + \cdots$.

In the small magnetic field and at low density limit with $c_\phi \gg B\mu C_0$, the conductivities are

$$\sigma^{xx} \sim \mathcal{N} e^{-\Phi(r_\star)} \sqrt{\frac{g_{yy}(r_\star)}{g_{xx}(r_\star)}}, \quad \sigma^{xy} \sim \frac{B c_\phi e^{-4\Phi(r_\star)}}{g_{xx}(r_\star) g_{yy}(r_\star)} - \mu C_0(r_\star). \tag{8}$$

Upon comparing with eq(1), we can generate the desired experimental behavior to transport quantities for background solutions showing the pseudo-scaling symmetry and unbroken rotational symmetry in the $x,\ y$ plane, see eq(78). So, only in the low density limit we need not have to consider two exponents solution as in eq(4). However, if we do then we can as well generate eq(1), even in this limit and the exponents for trivial dilaton and axion are $z = 4,\ w = 5$.

On summarizing the different possibilities with time translation and spatial translation symmetries are:

Symmetries	Density	Eq(1)	Symmetries	Density	Eq(1)
Pseudo-scaling and rotation	Low density	Possible	Scaling and rotation	Any density	Not possible
Pseudo-scaling and rotation	High density	Not possible	Scaling with broken rotation	Both low and high density	Possible

(9)

The holographic study of transport properties using the approach of [10], [11] and [12] gives us the non-linear behavior at the critical point and help us to understand the universal features, if any, in different limits of the parameter space, especially the quantity $dI/dV = 1/\mathcal{R} = \sigma$, where $\mathcal{R}$ is the resistance to the flow of current I with an applied voltage V. In this paper, we have generated successfully eq (1), and focused more on the model building than trying to find the universal features.

In the calculation of the conductivity, it is not *a priori* clear at what scale one should evaluate, i.e., how to choose the scale, $r_\star$, so as to capture the non-linear effect. Especially, for the system that is described by the Maxwell action. Of course, the gauge/gravity duality suggests us to do the calculations at the UV boundary. But, the result of this calculation produces only the linearized effect. However, for the system whose action is described by the DBI type there exists a very natural way to find the scale $r_\star$. This basically follows from the argument of [10],[11] and [12], which says that either the integrand of the action or the solution, which is in the form of $\sqrt{\frac{A}{B}}$ needed to be real. At a special value of the radial coordinate, $r = r_\star$, both A and B vanishes and there the action and the solution takes an indeterminate, $\frac{0}{0}$ form. Above or below this special scale $r_\star$, both A and B becomes positive or negative together. In this paper we give a physical argument to determine the scale $r_\star$ and show that it agrees precisely with the calculations done using the arguments of [10],[11] and [12]. We use the fact that the Legendre transformed action is same as the energy density, H_L, evaluated on the static solution, which comes as the square root of one term, importantly there is not any term in the denominator. On this energy we use the argument of [10],[11] or [12] to find the scale $r_\star$, instead of on the action or the solution. So, the scale $r_\star$ is the point on the holographic direction r for which the energy density vanishes and stay real

$$\Big(H_L\Big)_{r_\star} = 0. \tag{10}$$

The systems that are described by DBI kind of actions there exists another argument that precisely give the same result for $r_\star$ as suggested in the previous paragraph, even though the precise physical reason is not that clear. The argument is to find the on shell value of the norm of the field strength for which it takes a constant value, more precisely

$$\Big(F_{MN}F^{MN}\Big)_{r_\star} = -2. \tag{11}$$

There exists yet another way to determine the scale $r_\star$ that is to find a scale where the determinant of $det(g+F)_{ab}$ vanishes [11]. Here the indices a and b run only over the field theory directions. The equation for the condition is

$$\Big(det(g+F)_{ab}\Big)_{r_\star} = 0. \tag{12}$$

This can very easily be seen following the argument of vanishing H_L at the scale $r_\star$. Generically the energy density in the Legendre transformed frame can be written as

$$H_L = \int \sqrt{\mathcal{A}(r)[\mathcal{A}_\in(r)\mathcal{A}_\ni(r) - (\mathcal{A}_\triangle(r))^2]}. \tag{13}$$

Generically, the term $(\mathcal{A}_\triangle(r))^2$ is non-zero when we have got more than one spatial currents, more importantly this term is always positive. Whereas the term $\mathcal{A}_\in(r)$ and $\mathcal{A}_\ni(r)$ can change sign close to the horizon. So we can use the arguments of [10],[11] and [12] so as to have a real energy. Moreover, one of the term is nothing but $(-det(g+F)_{ab})$. Hence, the condition, eq(12), follows from H_L.

The prescription of holography [9] or that of [10] has been used to calculate the conductivity of several systems both in the top-down and bottom-up approaches. They include [16],–,[31] as a partial list.

This paper is organized as follows. In section 2, we shall review the calculation of the conductivity following [10] and compare it with that using eq(13) for systems that are described by DBI type of actions but in the absence of the charge density. In section 3, we study the systems in the presence of charge density both with and without Chern-Simon type of actions. Studies in section 2 and 3 are done for generic background solutions. Based on the calculations, in section 3, we give a toy example which is modeled in such a way that it gives us the desired behavior to conductivity and Hall angle in section 4. In section 5, we study the thermodynamics of the charge carriers in the presence of a constant magnetic field. Finally we conclude in section 6. Several details of the calculations are relegated to Appendices.

2 From Non-linear DBI action

In this section, we shall evaluate the expression to current, $J^\mu = \frac{\delta S}{\delta A_\mu}$, and then use the solution to equations of motion in it. In arbitrary spacetime dimensions, it is very difficult to solve the equations of motion that results from the DBI action, even in the massless and zero condensate limit i.e. for trivial embedding functions, Here, for simplicity, we shall restrict ourselves to $3+1$ dimensional bulk spacetime.

The DBI action is

$$S_{DBI} = -T\int e^{-\phi}\sqrt{-det([g]_{ab}+F_{ab})} \equiv -T\int e^{-\phi}\sqrt{-det(M_{ab})}, \tag{14}$$

where [ ] is used to denote the pull back of the bulk metric onto the world volume of the brane and T is the tension of brane. For simplicity, we have dropped the Chern-Simon part of the action. The equation of motion, its solution and the currents that follows are presented in Appendix B. This part of the subsection is also studied in [12], but here we shall be explicit.

In $3+1$ dimensions the form of the charge and current densities are

$$\begin{aligned}
J^\tau &= -T\ e^{-\phi}\Big[\frac{F_{xy}(F_{r\tau}F_{xy}+F_{x\tau}F_{yr}-F_{xr}F_{y\tau})+e^{4t}F_{r\tau}}{\sqrt{-det(M_{ab})}}\Big],\\
J^x &= -T\ e^{-\phi}\Big[\frac{F_{y\tau}(F_{r\tau}F_{xy}+F_{x\tau}F_{yr}-F_{xr}F_{y\tau})+e^{2t}(F_{x\tau}+hF_{xr})}{\sqrt{-det(M_{ab})}}\Big],\\
J^y &= -T\ e^{-\phi}\Big[\frac{F_{x\tau}(-F_{r\tau}F_{xy}+F_{y\tau}F_{xr}-F_{yr}F_{x\tau})+e^{2t}(F_{y\tau}+hF_{yr})}{\sqrt{-det(M_{ab})}}\Big],
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\sqrt{-det(M_{ab})} = \Big[&(1-F_{r\tau}^2)(e^{4t}+F_{xy}^2)-(F_{x\tau}F_{yr}-F_{y\tau}F_{xr})^2+\\
&2F_{r\tau}F_{xy}(F_{y\tau}F_{xr}-F_{x\tau}F_{yr})+e^{2t}(2F_{x\tau}F_{xr}+hF_{xr}^2+2F_{y\tau}F_{yr}+hF_{yr}^2)\Big]^{1/2}.
\end{aligned} \tag{16}$$

From which it follows the general form of the conductivities and are

$$\sigma^{xx} = \frac{J^x}{F_{x\tau}} = -T\ e^{-\phi}\Big[\frac{F_{y\tau}F_{yr}+e^{2t}}{\sqrt{-det(M_{ab})}}\Big],\quad \sigma^{yy} = \frac{J^y}{F_{y\tau}} = -T\ e^{-\phi}\Big[\frac{F_{x\tau}F_{xr}+e^{2t}}{\sqrt{-det(M_{ab})}}\Big],$$

$$\sigma^{xy} = \frac{J^x}{F_{y\tau}} = -T\ e^{-\phi}\Big[\frac{F_{r\tau}F_{xy} - F_{xr}F_{y\tau} + F_{x\tau}F_{yr}}{\sqrt{-det(M_{ab})}}\Big],$$
$$\sigma^{yx} = \frac{J^y}{F_{x\tau}} = -T\ e^{-\phi}\Big[\frac{-F_{r\tau}F_{xy} + F_{xr}F_{y\tau} - F_{x\tau}F_{yr}}{\sqrt{-det(M_{ab})}}\Big] \tag{17}$$

It follows trivially that the Hall conductivity is antisymmetric in the interchange of indices, i.e., $\sigma^{xy} = -\sigma^{yx}$. Using the explicit structure of the solution, from Appendix B, in the expression to currents, we ended up with

$$J^x = -T\ e^{-\phi}\ F_{x\tau}, \quad J^y = -T\ e^{-\phi}\ F_{y\tau}, \tag{18}$$

from which there follows the DC conductivities at the scale, $r = r_c$

$$\sigma^{xx}(r_c) = \sigma^{yy}(r_c) = -T\ e^{-\phi_0} \equiv \sigma. \tag{19}$$

This indeed reproduces the result of [12], i.e. unity conductivity.

2.1 Using the approach of [10]

In this subsection we shall try to derive the expression to conductivity from the DBI action in the absence of density. Let us work in a $d+2$ dimensional spacetime with dynamical exponent z. The exact form of the metric that we shall be considering is

$$ds^2_{d+2} = -r^{2z}dt^2 + r^2\sum_{i=1}^{d} dx_i^2 + \frac{dr^2}{r^2 f}, \tag{20}$$

we shall take $f = 1 - (r_0/r)^{d+z}$. This from of the metric gives us the Hawking temperature, $T_H = \left(\frac{d+z}{4\pi}\right)r_0^z$. In order to carry out the analysis for conductivity, we need to turn on a U(1) gauge potential which will give us the desired electric field in the field theory and for convenience we shall consider it as a constant field. Along with this, we shall turn on another component of the field strength, whose one leg is along the radial direction and the other along the spatial direction. For specificity, we shall turn on F_{xr}. So the complete form of the U(1) gauge field is $F_2 = -Edt \wedge dx - H'(r)dr \wedge dx$.

Let us consider a probe brane which is extended along time (t), radial direction (r) and $d_s - 1$ number of directions of the d number of spatial directions. Hence the probe brane is a d_s brane. For $d_s = d+1$, the probe brane will be a space filling brane. For simplicity we shall consider the massless limit scenario and the action in this case becomes

$$S = -N\int dtdrdxd^{d-1}y\sqrt{\prod_1^{d-1} g_{y_a y_a}}\sqrt{g_{tt}g_{rr}g_{xx} + H'^2 g_{tt} - E^2 g_{rr}}, \tag{21}$$

where we have considered the metric to be far more general than that of eq(20) but assumed to be in a diagonal form. Note, this form of the metric can very easily be re-written as is written in eq(134), by doing a coordinate transformations. The explicit form of the metric that we have considered has the following structure

$$ds^2_{d+2} = -g_{tt}(r)dr^2 + g_{rr}(r)dr^2 + g_{xx}(r)dx^2 + \sum_1^{d-1} g_{ab}(r)dy^a dy^b, \tag{22}$$

where $\sum_1^{d-1} g_{ab}(r)dy^a dy^b$ is assumed to be diagonal too, i.e., $\sum_1^{d-1} g_{ab}(r)dy^a dy^b = g_{11}(dy^1)^2 + \cdots + g_{d-1,d-1}(dy^{d-1})^2$. The normalization N includes the tension and the number of the probe branes. Since the action eq(21) do not depends on the function $H(r)$ means the 'momentum' associated to it must be a constant i.e $\frac{\delta S}{\delta H'} \equiv c$. From which it follows that the solution

$$H' = \pm c\sqrt{\frac{g_{rr}g_{tt}g_{xx} - E^2 g_{rr}}{N^2(\prod_a g_{y_a y_a})g^2_{tt} - c^2 g_{tt}}} \tag{23}$$

It is very easy to convince oneself that the constant c is nothing but the current density, J^x, in the field theory, which can be seen just using the equation of motion to the gauge field in the definition to c. Now using the arguments of [10] we obtain the necessary equations to fix c, which is J^x

$$E^2 = g_{tt}(r_\star)g_{xx}(r_\star), \quad J^2_x = N^2\Big(\prod_a g_{y_a y_a}(r_\star)\Big)g_{tt}(r_\star), \tag{24}$$

where $r_\star$ is the value of r, where both the numerator and denominator of H' changes sign. It is interesting to note that at $r_\star$, the solution takes $H' = \frac{0}{0}$ form which is an indeterminate structure. So, the better way to find $r_\star$ is to go over to an equivalent form of the action and demand that the energy density that follows is real as well as have a 'minimum' at some energy scale, which we denote it as $r_\star$.

The action eq(21) can equivalently be expressed by doing the Legendre transformation as

$$S_L = S - \int \frac{\delta S}{\delta H'}H' = -\int \sqrt{\Big[g_{rr}g_{tt}g_{xx} - E^2 g_{rr}\Big]\Big[N^2(\prod_a g_{y_a y_a}) - \frac{c^2}{g_{tt}}\Big]}. \tag{25}$$

Since we are working with a static configuration implies the energy is

$$H_L = \int \sqrt{\Big[g_{rr}(g_{tt}g_{xx} - E^2)\Big]\Big[N^2(\prod_a g_{y_a y_a}) - \frac{c^2}{g_{tt}}\Big]}, \tag{26}$$

where we have put an index L to the energy to denote it. For an illustration, let us take an example of the asymptotically AdS black hole, the first term in the square bracket under the square root changes sign some-where close to horizon and the same is true for the second term in the square bracket and their product is positive but diverges there. Since both the terms in the square bracket changes sign some-where close to the horizon we assume that this happens at the same energy scale, $r = r_\star$, so as to have a real energy. Asymptotically, the first term in the square bracket diverges so also the second term (for $d \geq 2$). Now the only place it can vanish (i.e minimum) is close to the horizon. For a discussion on the condition of minimization to energy, see Appendix C.

Demanding these two restrictions again gives the same two equations as written in eq(24). From which there follows the expression to current

$$J^x = \pm N \frac{\sqrt{\Big(\prod_a g_{y_a y_a}(r_\star)\Big)}}{\sqrt{g_{xx}(r_\star)}} E \tag{27}$$

The absence of singular behavior to observable J^x means the terms under the square-root should be regular. Upon choosing the positive sign, the conductivity is

$$\sigma = N \frac{\sqrt{\Big(\prod_a g_{y_a y_a}(r_\star)\Big)}}{\sqrt{g_{xx}(r_\star)}}. \tag{28}$$

The solution to the first equation of eq(24) gives the desired solution to $r_\star$ as a function of electric field E and Hawking temperature T_H, as g_{tt} is a function of T_H. If we assume that the metric components along the spatial directions are all same then the above formula to conductivity reduces to

$$\sigma = N \sqrt{\Big(\prod_1^{d-2} g_{y_a y_a}(r_\star)\Big)}. \tag{29}$$

This form of the conductivity is also found in the Maxwell system in [10],[13],[12], except the choice of $r_\star$ is not fixed. For the choice of our field strength, it is expected from eq(17) that there should not be any Hall conductivity. So, this gives a check of this procedure.

Let us find the complete form of the conductivity associated to the Lifshitz metric written in eq(20), as an example. In this case the relevant equation that gives $r_\star$ as a function of electric field E is

$$E^2 = r_\star^{2(1+z)} \left[1 - \left(\frac{r_0}{r_\star}\right)^{d+z}\right]. \tag{30}$$

This algebraic equation is very non-linear in nature and hence very difficult to find the exact solution, analytically. However, there exists exact solutions for few specific cases. In which case the number of spatial directions are tied to the exponent z as, $d = (n-1)z + n$

with $n = 0,\ 1,\ 2,\ 3$ and 4,

$$
\begin{aligned}
r_\star &= E^{\frac{1}{2(1+z)}}, \quad n=0,\\
r_\star &= \left[\frac{(\frac{4\pi}{1+z})^{\frac{1+z}{z}} T_H^{\frac{1+z}{z}} \pm \sqrt{4E^2 + (\frac{4\pi}{1+z})^{\frac{2(1+z)}{z}} T_H^{\frac{2(1+z)}{z}}}}{2}\right]^{\frac{1}{2(1+z)}}, \quad n=1,\\
r_\star &= \left[E^2 + \left(\frac{2\pi}{1+z}\right)^{\frac{2(1+z)}{z}} T_H^{\frac{2(1+z)}{z}}\right]^{\frac{1}{2(1+z)}} = \left[E^2 + \left(\frac{2\pi}{1+z}\right)^{\frac{2(1+z)}{z}} T_H^{\frac{2(1+z)}{z}}\right]^{\frac{1}{d+z}}, \quad n=2,\\
r_\star &= \left[\frac{2.3^{\frac{1}{3}}E^2 + 2^{\frac{1}{3}}\left[9(\frac{4\pi}{3(1+z)})^{\frac{3(1+z)}{z}} T_H^{\frac{3(1+z)}{z}} + \sqrt{81(\frac{4\pi}{3(1+z)})^{\frac{6(1+z)}{z}} T_H^{\frac{6(1+z)}{z}} - 12E^6}\right]^{\frac{2}{3}}}{6^{2/3}\left[9(\frac{4\pi}{3(1+z)})^{\frac{3(1+z)}{z}} T_H^{\frac{3(1+z)}{z}} + \sqrt{81(\frac{4\pi}{3(1+z)})^{\frac{6(1+z)}{z}} T_H^{\frac{6(1+z)}{z}} - 12E^6}\right]^{\frac{1}{3}}}\right]^{\frac{1}{2(1+z)}}, \quad n=3,\\
r_\star &= \left[\frac{E^2 \pm \sqrt{E^4 + 4(\frac{\pi}{1+z})^{\frac{4(1+z)}{z}} T_H^{\frac{4(1+z)}{z}}}}{2}\right]^{\frac{1}{2(1+z)}}, \quad n=4 \qquad (31)
\end{aligned}
$$

It is interesting to note that the choice $n = 0$ gives negative exponent $z = -d$, whereas $n = 1$ gives $d = 1$, which essentially says about a $1 + 1$ dimensional field theory for any exponent, the choice $n = 2,\ 3$ and 4 gives the exponent $z = d - 2,\ \frac{d-3}{2}$ and $z = \frac{d-4}{3}$ respectively.

Now using the spatial part of metric components from eq(20) in the expression to current, $J \equiv E^{\frac{d-1+z}{1+z}}\ Y_1$ with the function

$$
\begin{aligned}
Y_1 &= N\left[1 + \left(\frac{2\pi}{1+z}\right)^{\frac{2(1+z)}{z}} \left(\frac{T_H^{1+\frac{1}{z}}}{E}\right)^2\right]^{\frac{d-2}{2(1+z)}} \quad \text{for} \quad n=2,\\
Y_1 &= N\left[\frac{1 \pm \sqrt{1 + 4(\frac{\pi}{1+z})^{\frac{4(1+z)}{z}} \left(\frac{T_H^{1+\frac{1}{z}}}{E}\right)^4}}{2}\right]^{\frac{d-2}{2(1+z)}} \quad \text{for} \quad n=4, \qquad (32)
\end{aligned}
$$

for a couple of cases and the conductivity in these special cases are

$$
\begin{aligned}
\sigma &= N T_H^{\frac{d-2}{z}} \left[\left(\frac{2\pi}{1+z}\right)^{\frac{2(1+z)}{z}} + \left(\frac{E}{T_H^{1+\frac{1}{z}}}\right)^2\right]^{\frac{d-2}{2(1+z)}} \quad \text{for} \quad n=2,\\
\sigma &= \frac{N}{2^{\frac{d-2}{2(1+z)}}}\left[E^2 \pm \sqrt{E^4 + 4(\frac{\pi}{1+z})^{\frac{4(1+z)}{z}} T_H^{\frac{4(1+z)}{z}}}\right]^{\frac{1}{2(1+z)}} \quad \text{for} \quad n=4 \qquad (33)
\end{aligned}
$$

Hence for very small electric field and high temperature limit, $E << T^{1+\frac{1}{z}}$, the conductivity follows the power law behavior, in particular, $T_H^{\frac{d-2}{z}}$.

Let us go away from this special case of $d = (n-1)z + n$ and find the solution to $r_\star$ from eq(30). In the weak field limit $E << T^{1+\frac{1}{z}}$ the solution to $r_\star$ can be approximated as

$$r_\star \simeq r_0\left[1 + \left(\frac{E}{r_0^{1+z}}\right)^2\right]^{\frac{d-2}{d+z}} + \cdots = \left(\frac{4\pi T_H}{d+z}\right)^{\frac{1}{z}}\left[1 + \left(\frac{d+z}{4\pi}\right)^{\frac{2(1+z)}{z}}\left(\frac{E}{T_H^{1+\frac{1}{z}}}\right)^2\right]^{\frac{1}{d+z}} + \cdots \quad (34)$$

which gives the current to leading order

$$J_x \simeq NE\left(\frac{4\pi T_H}{d+z}\right)^{\frac{d-2}{z}}\left[1 + \left(\frac{d+z}{4\pi}\right)^{\frac{2(1+z)}{z}}\left(\frac{E}{T_H^{1+\frac{1}{z}}}\right)^2\right]^{\frac{d-2}{d+z}} + \cdots, \quad (35)$$

whereas in the strong field limit $E >> T^{1+\frac{1}{z}}$, the solution becomes

$$r_\star \simeq E^{\frac{1}{1+z}}\left[1 + \left(\frac{r_0}{E^{\frac{1}{1+z}}}\right)^{d+z}\right]^{\frac{1}{d+z}} + \cdots = E^{\frac{1}{1+z}}\left[1 + \left(\frac{4\pi}{d+z}\right)^{\frac{d+z}{z}}\left(\frac{T_H^{1+\frac{1}{z}}}{E}\right)^{\frac{d+z}{1+z}}\right]^{\frac{1}{d+z}} + \cdots \quad (36)$$

which gives the current to leading order

$$J_x \simeq NE^{\frac{d+z-1}{1+z}}\left[1 + \left(\frac{4\pi}{d+z}\right)^{\frac{d+z}{z}}\left(\frac{T_H^{1+\frac{1}{z}}}{E}\right)^{\frac{d+z}{1+z}}\right]^{\frac{d-2}{2(1+z)}} + \cdots, \quad (37)$$

this form of current essentially gives us the function

$$Y_1 = N\left[1 + \left(\frac{4\pi}{d+z}\right)^{\frac{d+z}{z}}\left(\frac{T_H^{1+\frac{1}{z}}}{E}\right)^{\frac{d+z}{1+z}}\right]^{\frac{d-2}{2(1+z)}} + \cdots. \quad (38)$$

On comparing with the expression to Y_1 for $n = 2$ case as in eq(32), it follows that the sub-leading terms to Y_1 in eq(38) vanishes exactly for $d = z + 2$.

2.2 Multiple electric fields

Let us consider another situation where we have turned on more than one constant electric field, for simplicity let us take the gauge potential as $A = -(E_1\, t + H(r))dx - E_2\, t\, dy$, which gives the field strength as

$$F_2 = -E_1 dt \wedge dx - E_2 dt \wedge dy - H'(r) dr \wedge dx \quad (39)$$

and considering the previous brane configuration again but with this new from of the gauge field strength in the background metric

$$ds^2_{d+2} = -g_{tt}(r)dr^2 + g_{rr}(r)dr^2 + g_{xx}(r)dx^2 + g_{yy}(r)dy^2 + \sum_1^{d-2} g_{ab}(r)dz^a dz^b. \quad (40)$$

On evaluating the DBI action

$$S = -N \int dtdrdxdyd^{d-2}z \sqrt{\prod_1^{d-2} g_{z_a z_a}} \sqrt{g_{tt}g_{rr}g_{xx}g_{yy} + H'^2(g_{tt}g_{yy} - E_2^2) - E_1^2 g_{rr}g_{yy} - E_2^2 g_{rr}g_{xx}} \tag{41}$$

Since the action do not depends on $H(r)$ implies the corresponding momentum is constant i.e. $\frac{\delta S}{\delta H'(r)} = c$, and this constant c is nothing but the current density along the x direction, J^x. Upon inverting to re-write

$$H' = \pm \frac{c}{[g_{tt}g_{yy} - E_2^2]} \sqrt{\frac{g_{rr}(g_{tt}g_{xx}g_{yy} - E_1^2 g_{yy} - E_2^2 g_{xx})}{N^2(\prod g_{z_a z_a}) - \frac{c^2}{g_{tt}g_{yy} - E_2^2}}} \tag{42}$$

Now demanding the reality of the solution as in [10] gives two different equations which needed to be satisfied at $r = r_\star$ and the equations are

$$g_{tt}(r_\star)g_{xx}(r_\star)g_{yy}(r_\star) = E_1^2 g_{yy}(r_\star) + E_2^2 g_{xx}(r_\star), \quad N^2(\prod g_{z_a z_a}(r_\star))(g_{tt}(r_\star)g_{yy}(r_\star) - E_2^2) = (J^x)^2 \tag{43}$$

These two equations again can be derived by putting the conditions on the energy that it vanishes and becomes real at the same energy scale $r = r_\star$. This follows by going over to the Legendre transformed action

$$S_L = S - \int \frac{\delta S}{\delta H'} H' = -\int \sqrt{g_{rr}(g_{tt}g_{xx}g_{yy} - E_1^2 g_{yy} - E_2^2 g_{xx})} \sqrt{N^2\Big(\prod g_{z_a z_a}\Big) - \frac{c^2}{(g_{tt}g_{yy} - E_2^2)}}, \tag{44}$$

which gives the energy as

$$H_L = \int \sqrt{g_{rr}(g_{tt}g_{xx}g_{yy} - E_1^2 g_{yy} - E_2^2 g_{xx})} \sqrt{N^2\Big(\prod g_{z_a z_a}\Big) - \frac{c^2}{(g_{tt}g_{yy} - E_2^2)}}, \tag{45}$$

Now demanding the condition as stated above yields the same two equations eq(43). The first equation gives the choice of $r_\star$ whereas the last equation gives the full non-linear expression to current density. Using these two equations, we ended up with

$$J^x = \pm N\ E_1 \sqrt{\prod_1^{d-2} g_{z_a z_a}(r_\star)} \sqrt{\frac{g_{yy}(r_\star)}{g_{xx}(r_\star)}}, \tag{46}$$

where $r_\star$ is to be determined by solving

$$g_{tt}(r_\star)g_{xx}(r_\star) = E_1^2 + E_2^2 \frac{g_{xx}(r_\star)}{g_{yy}(r_\star)}. \tag{47}$$

Now, note that the expression to current density remain same as is found in the DBI action with one electric field. Of course the condition of $r_\star$ is different. For $g_{xx}(r_\star) = g_{yy}(r_\star)$, the condition almost remains the same as for one electric field except with the substitution $E_1^2 \to E_1^2 + E_2^2$, but for unequal $g_{xx}(r_\star)$ and $g_{yy}(r_\star)$, one has to find the choice of cutoff $r_\star$ by solving eq(47).

2.3 With a constant electric and magnetic field

Let us re-run the argument of the previous subsection but with a constant electric and magnetic field, in which case the field strength comes as

$$F_2 = -Edt \wedge dx + Bdx \wedge dy - H'(r)dr \wedge dx \tag{48}$$

In this case the DBI action results

$$S = -N \int dtdrdxdyd^{d-2}z \sqrt{\prod_1^{d-2} g_{z_a z_a}} \sqrt{g_{tt}g_{rr}g_{xx}g_{yy} + H'^2 g_{tt}g_{yy} - E^2 g_{rr}g_{yy} + B^2 g_{rr}g_{tt}} \tag{49}$$

Once again the action does not depend on the field, $H(r)$, implies the corresponding momentum is constant, $\frac{\delta S}{\delta H'(r)} = c$. This constant c is nothing but the current density along the x direction, J^x. The gradient of the solution that results from it

$$H'(r) = c\sqrt{\frac{g_{rr}g_{tt}g_{xx}g_{yy} + B^2 g_{rr}g_{tt} - E^2 g_{rr}g_{yy}}{N^2(\prod g_{z_a z_a})g_{tt}^2 g_{yy}^2 - (J^x)^2 g_{tt}g_{yy}}} \tag{50}$$

Going through the logic of [10] gives two algebraic equations which are

$$g_{tt}(r_\star)g_{xx}(r_\star) = E^2 - B^2\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}, \quad \text{and} \quad J^x = \pm N\sqrt{\Big(\prod_{a=1}^{d-2} g_{z_a z_a}(r_\star)\Big)g_{tt}(r_\star)g_{yy}(r_\star)}, \tag{51}$$

These two equations again can be derived by putting the conditions on the energy that it vanishes and becomes real at the same energy scale, $r = r_\star$. Again, going over to the Legendre transformed action

$$S_L = S - \int \frac{\delta S}{\delta H'}H' = -\int \sqrt{g_{rr}(g_{tt}g_{xx}g_{yy} - E^2 g_{yy} + B^2 g_{tt})}\sqrt{N^2\Big(\prod g_{z_a z_a}\Big) - \frac{c^2}{g_{tt}g_{yy}}}, \tag{52}$$

gives the energy as

$$H_L = \int \sqrt{g_{rr}(g_{tt}g_{xx}g_{yy} - E^2 g_{yy} + B^2 g_{tt})}\sqrt{N^2\Big(\prod g_{z_a z_a}\Big) - \frac{c^2}{g_{tt}g_{yy}}}, \tag{53}$$

Now demanding the condition as stated above yields the same two equations as in eq(51), where the choice of $r_\star$ is determined by solving the first equation of eq(51). For a constant electric and magnetic field the formula to current density becomes

$$J^x = \pm N\ E\ \sqrt{\frac{\left(\prod_{a=1}^{d-2} g_{z_a z_a}(r_\star) g_{yy}(r_\star)\right)}{g_{xx}(r_\star)}}\sqrt{1-\frac{B^2}{E^2}\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}} \tag{54}$$

and has been modified with an additional multiplicative factor $\sqrt{1-\frac{B^2}{E^2}\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}}$ in comparison to the cases without any magnetic field. This is because the $r_\star$ is also modified by the same multiplicative factor on the right hand side to eq(24), but without the square-root.

At first sight it looks as if the result to currents in eq(18), after substituting the solution, are not compatible with eq(54), in $3+1$ dimensions. Actually to compare both the equations we need to go a frame where both the calculation are done in one frame not in different frames. To do that we can either do some change of coordinates or directly compute the current using the approach of [10] in the coordinate system that is used to do the calculations which is presented in Appendix B.

In either way note that the computation to eq(18) is done for which $F^{(\tau)}_{xr}$ vanishes in the frame of $(\tau,\ r,\ x_i)$. We have used a superscript (τ) in the expression to field strength to denote it, which in the frame $(t,\ r,\ x_i)$ e.g. as in eq(48) says that they are related as $F^{(\tau)}_{xr} = H'(r) - E/h(r)$. Vanishing of $F^{(\tau)}_{xr}$ means $H'^2(r) = E^2/h^2(r)$, which upon using in eq(50) gives

$$J^x = \pm N\ E, \tag{55}$$

for zero magnetic field. This precisely matches with eq(18) at the scale, $r = r_\star$ up to an over all identification of normalizations.

2.4 Subsummary

The summary to the study of this section is the result of current density in terms of one or more constant electric and magnetic field. Essentially, use of the prescription of [10] or equivalently that of eq(13), results a recipe to calculate the current density in $d+1$ dimensional field theory if the dual bulk geometry is of the form eq(40). With a constant electric field say along x, one of the spatial direction, the ratio of the current density to electric field is the square-root of the ratio of the product of metric component (up to an over all factor) of $d-1$ space, which is perpendicular to $t,\ x,\ r$ plane, to the metric component along x-axis i.e. eq(28). This quantity should be evaluated at an energy scale, $r_\star$, for which the product of metric components along t and x axis i.e. $g_{tt}(r_\star)g_{xx}(r_\star)$ becomes same as the square of the electric field i.e first equation of eq(24). This condition determining scale is generalized when there is more than one constant electric fields and a constant magnetic field in the theory.

It is worth to emphasize that $r_\star$ should be close to the horizon, r_h, rather than to boundary because in order for eq(54) to make sense. The factor $\sqrt{1-\frac{B^2}{E^2}\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}}$ should be a real quantity and can happen only when $r_\star$ is close to the horizon for any strength of the magnetic and electric field. This can be seen as follows, close to the horizon the ratio $\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}$ is very small whereas close to the boundary this ratio approaches unity. So for $B > E$ the second factor in the square root can become greater than unity.

3 With charge density

Let us discuss the case with non vanishing charge density first without the Chern-Simon term and then with it. The inclusion of Chern-Simon term makes an interesting change to the Hall conductivity that is it adds a piece and could potentially change the structure unless we take the axion to be constant. Moreover, the Chern-Simon term does not make any surprising changes to the Hall angle, $cot\ \theta_H = \sigma^{xy}/\sigma^{xx}$ at the leading order in the large density and small magnetic field limit.

3.1 Charge density without the Chern-Simon term

Let us work in the background metric, which we have taken to be diagonal for simplicity, as written in eq(40) and consider two electric fields that are turned on along two different directions along with two magnetic fields. So, the form of the gauge potential and the field strength are

$$\begin{aligned} A &= -\Big(E_1 t + \frac{B}{2}y + H(r)\Big)dx - \Big(E_2 t - \frac{B}{2}x + h(r)\Big)dy + \phi(r)dt \\ F_2 &= -E_1 dt\wedge dx - E_2 dt\wedge dy + B dx\wedge dy - H'(r)dr\wedge dx - h'(r)dr\wedge dy + \phi'(r)dr\wedge dt. \end{aligned} \tag{56}$$

The DBI action that describes the dynamics

$$\begin{aligned} S &= -\mathcal{N}\int dr\sqrt{\prod g_{z_a z_a}}\Big[g_{rr}(B^2 g_{tt} - E_1^2 g_{yy} - E_2^2 g_{xx} + g_{tt}g_{xx}g_{yy}) + (g_{tt}g_{xx} - E_1^2)h'^2 + \\ & \quad 2E_1E_2h'H' + (g_{tt}g_{yy} - E_2^2)H'^2 - 2B(E_1h' - E_2H')\phi' - (g_{xx}g_{yy} + B^2)\phi'^2\Big]^{1/2} \\ &\equiv -\mathcal{N}\int dr\sqrt{\prod g_{z_a z_a}}\quad \bar{\mathcal{L}}, \end{aligned} \tag{57}$$

where the normalization, $\mathcal{N}$, includes the tension of the probe brane and the volume of $R^{1,d}$. Since the action does not depend on the fields $H,\ h$ and ϕ implies the corresponding momentum are constants.

$$
\begin{aligned}
\frac{\delta S}{\delta \phi'} &\equiv c_\phi = -\frac{\mathcal{N}\sqrt{\prod g_{z_a z_a}}}{\bar{\mathcal{L}}}[-B(E_1 h' - E_2 H') - (g_{xx}g_{yy} + B^2)\phi'], \\
\frac{\delta S}{\delta h'} &\equiv c_h = -\frac{\mathcal{N}\sqrt{\prod g_{z_a z_a}}}{\bar{\mathcal{L}}}[(g_{tt}g_{xx} - E_1^2)h' + E_1 E_2 H' - BE_1\phi'], \\
\frac{\delta S}{\delta H'} &\equiv c_H = -\frac{\mathcal{N}\sqrt{\prod g_{z_a z_a}}}{\bar{\mathcal{L}}}[(g_{tt}g_{yy} - E_2^2)H' + BE_2\phi' + E_1 E_2 h']
\end{aligned} \tag{58}
$$

By taking the ratio of the momenta we determine h' and ϕ' in terms of H'

$$
\begin{aligned}
h' &= \frac{g_{yy}[-E_2(c_H E_1 + c_h E_2)g_{xx} + g_{tt}(B(Bc_h - c_\phi E_1) + c_h g_{xx}g_{yy})]H'}{g_{xx}[-E_1(c_H E_1 + c_h E_2)g_{yy} + g_{tt}(B(Bc_H + c_\phi E_2) + c_H g_{xx}g_{yy})]}, \\
\phi' &= \frac{g_{tt}[E_1(-Bc_h + c_\phi E_1)g_{yy} + g_{xx}(E_2(Bc_H + c_\phi E_2) - c_\phi g_{tt}g_{yy})]H'}{g_{xx}[-E_1(cHE_1 + c_h E_2)g_{yy} + g_{tt}(B(Bc_H + c_\phi E_2) + c_H g_{xx}g_{yy})]}
\end{aligned} \tag{59}
$$

The function H' can be determined by substituting eq(59) in the last equation of eq(58). We do not write down the exact form of H' because it is a very long expression and not that illuminating, even though one might think of using it to find the $r_\star$. We shall determine $r_\star$ using eq(13) and obtain the other relevant expressions so as to find the currents.

The Legendre transformed action

$$
\begin{aligned}
S_L &= S - \int \frac{\delta S}{\delta \phi'}\phi' - \int \frac{\delta S}{\delta h'}h' - \int \frac{\delta S}{\delta H'}H' \\
&= -\mathcal{N}\int dr \frac{\sqrt{\prod g_{z_a z_a}}}{\bar{\mathcal{L}}}\Big[g_{rr}(B^2 g_{tt} - E_1^2 g_{yy} - E_2^2 g_{xx} + g_{tt}g_{xx}g_{yy})\Big] \\
&= -\int dr\sqrt{\frac{g_{rr}}{g_{tt}g_{xx}g_{yy}}} \times \sqrt{[\mathcal{A}_\in(r)\mathcal{A}_\ni(r) - (\mathcal{A}_\triangle(r))^2]},
\end{aligned} \tag{60}
$$

where

$$
\begin{aligned}
\mathcal{A}_\in(r) &= B^2 g_{tt} - E_2^2 g_{xx} - E_1^2 g_{yy} + g_{tt}g_{xx}g_{yy}, \\
\mathcal{A}_\ni(r) &= \mathcal{N}^2(\prod g_{zz})g_{tt}g_{xx}g_{yy} - c_H^2 g_{xx} - c_h^2 g_{yy} + c_\phi^2 g_{tt}, \\
\mathcal{A}_\triangle(r) &= c_\phi B g_{tt} - c_h E_1 g_{yy} + c_H E_2 g_{xx}
\end{aligned} \tag{61}
$$

The prescription to find $r_\star$ and the expression to the currents are that each of the quantities $\mathcal{A}_\in(r)$, $\mathcal{A}_\ni(r)$ and $\mathcal{A}_\triangle(r)$ must vanish independently at the same value of r, which is $r_\star$. The solution to $\mathcal{A}_\ni(r_\star) = 0$ and $\mathcal{A}_\triangle(r_\star) = 0$ gives the expression to currents

$$
\begin{aligned}
c_H &= -\frac{Bc_\phi E_2 g_{tt}g_{xx} \pm \sqrt{E_1^2 g_{tt}g_{xx}g_{yy}[-B^2 c_\phi^2 g_{tt} + (E_2^2 g_{xx} + E_1^2 g_{yy})(c_\phi^2 + g_{xx}g_{yy}\mathcal{N}^2(\prod g_{zz}))]}}{g_{xx}(E_2^2 g_{xx} + E_1^2 g_{yy})} \\
c_h &= \frac{Bc_\phi E_1^2 g_{tt}g_{yy} \mp E_2\sqrt{E_1^2 g_{tt}g_{xx}g_{yy}[-B^2 c_\phi^2 g_{tt} + (E_2^2 g_{xx} + E_1^2 g_{yy})(c_\phi^2 + g_{xx}g_{yy}\mathcal{N}^2(\prod g_{zz}))]}}{E_1 g_{yy}(E_2^2 g_{xx} + E_1^2 g_{yy})}
\end{aligned} \tag{62}
$$

The scale $r_\star$ is determined by solving

$$\mathcal{A}_\in(r_\star) = [B^2 g_{tt} - E_2^2 g_{xx} - E_1^2 g_{yy} + g_{tt} g_{xx} g_{yy}]_{r_\star} = 0. \tag{63}$$

The constants c_H and c_h are nothing but the currents along the x and y direction respectively. Using the condition eq(63) in eq(62) gives the currents

$$\begin{aligned} J^x &= -\frac{B c_\phi E_2 \pm E_1 g_{yy}\sqrt{c_\phi^2 + \mathcal{N}^2(\prod g_{zz})(B^2 + g_{xx}g_{yy})}}{B^2 + g_{xx}g_{yy}}, \\ J^y &= \frac{B c_\phi E_1 \mp E_2 g_{xx}\sqrt{c_\phi^2 + \mathcal{N}^2(\prod g_{zz})(B^2 + g_{xx}g_{yy})}}{B^2 + g_{xx}g_{yy}}, \end{aligned} \tag{64}$$

The conductivities that follows from the expression to currents are

$$\begin{aligned} \sigma^{xx} &= \frac{dJ^x}{dE_1} = \mp\frac{g_{yy}}{B^2 + g_{xx}g_{yy}}\sqrt{c_\phi^2 + \mathcal{N}^2(B^2 + g_{xx}g_{yy})}, \quad \sigma^{xy} = \frac{dJ^x}{dE_2} = -\frac{B c_\phi}{B^2 + g_{xx}g_{yy}}, \\ \sigma^{yy} &= \frac{dJ^y}{dE_2} = \mp\frac{g_{xx}}{B^2 + g_{xx}g_{yy}}\sqrt{c_\phi^2 + \mathcal{N}^2(B^2 + g_{xx}g_{yy})}, \quad \sigma^{yx} = \frac{dJ^y}{dE_1} = \frac{B c_\phi}{B^2 + g_{xx}g_{yy}}. \end{aligned} \tag{65}$$

Now there follows trivially the Onsager relation, $\sigma^{xy}(B) = \sigma^{yx}(-B)$. For the choice of, $E_2 = 0$, the currents are

$$\begin{aligned} J^x &= \mp\frac{E_1 g_{yy}}{B^2 + g_{xx}g_{yy}}\sqrt{c_\phi^2 + \mathcal{N}^2(\prod g_{zz})(B^2 + g_{xx}g_{yy})}, \\ J^y &= \frac{B E_1 c_\phi}{B^2 + g_{xx}g_{yy}}, \end{aligned} \tag{66}$$

which precisely matches with the result of [11] for the special case when the spatial part of the metric components are same that is $g_{xx} = g_{yy} = g_{zz}$ and with the units $2\pi\alpha' = 1$. Upon restricting our selves to $3+1$ dimensional bulk systems we get the currents as

$$J^x = \mp\frac{E_1 g_{yy}}{B^2 + g_{xx}g_{yy}}\sqrt{c_\phi^2 + \mathcal{N}^2(B^2 + g_{xx}g_{yy})}, \quad J^y = \frac{B E_1 c_\phi}{B^2 + g_{xx}g_{yy}} \tag{67}$$

Now using the Ohm's law, follows the conductivities and are

$$\sigma^{xx} = \mp\frac{g_{yy}}{B^2 + g_{xx}g_{yy}}\sqrt{c_\phi^2 + \mathcal{N}^2(B^2 + g_{xx}g_{yy})}, \quad \sigma^{xy} = \frac{B c_\phi}{B^2 + g_{xx}g_{yy}}. \tag{68}$$

Let us look at a special corner of the parameter space of charge density c_ϕ and the magnetic field B for which $\mathcal{N}B$ is very small in comparison to density i.e, very large charge density. In this case the conductivity reduces to

$$\sigma^{xx} \simeq \mp\Big[\frac{c_\phi}{g_{xx}} + \mathcal{N}^2\frac{g_{yy}}{2c_\phi^2} + \cdots\Big], \quad \sigma^{xy} \simeq \frac{B c_\phi}{g_{xx}g_{yy}} + \cdots, \tag{69}$$

where the ellipses denote higher powers of magnetic field. Choosing the positive branch from σ^{xx} and dropping the second term in σ^{xx} gives us

$$\sigma^{xx} \sim \frac{c_\phi}{g_{xx}}, \quad \sigma^{xy} \simeq \frac{Bc_\phi}{g_{xx}g_{yy}}, \quad \frac{\sigma^{xx}}{\sigma^{xy}} \sim \frac{g_{yy}}{B}. \tag{70}$$

If we want different temperature dependence to the conductivity and the Hall angle, not as inverse, as reported by experiments restricts us to take the spatial part of the metric components to be different. Let us consider a situation where $g_{xx} \sim T_H^{2w/z}$ and $g_{yy} \sim T_H^{2/z}$ with T_H being the equilibrium temperature of the dual field theory. Then the conductivities are

$$\sigma^{xx} \sim T_H^{-2w/z}; \quad \frac{\sigma^{xx}}{\sigma^{xy}} \sim T_H^{2/z}. \tag{71}$$

On comparing with the experimental results gives us the condition on the exponents z and w as

$$z = 1, \quad w = 1/2. \tag{72}$$

So by considering different form of the metric components we can reproduce the precise form of the conductivities as seen in the NFL phase.

If one consider pseudo-scaling theories where the metric respects the scaling symmetry but not the scalar field, dilaton, Φ. Then the currents written in eq(66) gets changed to

$$\begin{aligned} J^x &= \mp \frac{E_1 e^{2\Phi} g_{yy}}{B^2 + e^{4\Phi} g_{xx} g_{yy}} \sqrt{c_\phi^2 + \mathcal{N}^2 e^{-2\Phi} (\prod_1^{d-2} e^{2\Phi} g_{zz})(B^2 + e^{4\Phi} g_{xx} g_{yy})}, \\ J^y &= \frac{BE_1 c_\phi}{B^2 + e^{4\Phi} g_{xx} g_{yy}}, \end{aligned} \tag{73}$$

as the action in eq(57) gets changed to $S = -\mathcal{N} \int dr e^{-\Phi} \sqrt{\prod(e^{2\Phi} g_{z_a z_a})} \;\; \bar{\mathcal{L}}$. Now the metric components that appear in $\bar{\mathcal{L}}$ can obtained from eq(57) by substituting $g^s_{ab} \to e^{2\Phi} g^E_{ab}$. This kind of changes to the action occurs because of doing the changes to the metric components i.e., from string frame to Einstein frame and performing the calculations for which the metric components are in Einstein frame.

Let us consider the conductivities that follows from eq(73), for $d = 2$, i.e., in $2+1$ dimensional dual field theory

$$\begin{aligned} \sigma^{xx} &= \frac{e^{2\Phi} g_{yy}}{B^2 + e^{4\Phi} g_{xx} g_{yy}} \sqrt{c_\phi^2 + \mathcal{N}^2 e^{-2\Phi} (B^2 + e^{4\Phi} g_{xx} g_{yy})} \longrightarrow \frac{e^{-2\Phi}}{g_{xx}} \sqrt{c_\phi^2 + \mathcal{N}^2 e^{2\Phi} g_{xx} g_{yy}}, \\ \sigma^{xy} &= \frac{Bc_\phi}{B^2 + e^{4\Phi} g_{xx} g_{yy}} \longrightarrow \frac{Bc_\phi e^{-4\Phi}}{g_{xx} g_{yy}}, \end{aligned} \tag{74}$$

where we have taken the very small magnetic field limit. Let us recall that $\mathcal{N}^2 \sim N_f^2/g_s^2$, where N_f is the number of flavor branes. It means in the very high density limit i.e., $c_\phi^2 \gg N_f^2 g_{xx} g_{yy}$, the conductivities reduces to

$$\sigma^{xx} \sim c_\phi \frac{e^{-2\Phi}}{g_{xx}}, \quad \sigma^{xy} \sim \frac{B c_\phi e^{-4\Phi}}{g_{xx} g_{yy}}, \quad \frac{\sigma^{xx}}{\sigma^{xy}} \sim \frac{e^{2\Phi} g_{yy}}{B}. \tag{75}$$

For a rotationally invariant geometry i.e., with $g_{xx} = g_{yy}$ and demanding that the conductivities in the very high density limit matches with eq(1), do not give us any solution to dilaton and metric components.

Instead of considering the very high density limit, let us consider the low density limit, $c_\phi^2 \ll N_f^2 g_{xx} g_{yy}$, in which case the conductivities eq(74) reduces to

$$\sigma^{xx} \sim \mathcal{N} e^{-\Phi} \sqrt{\frac{g_{yy}}{g_{xx}}}, \quad \sigma^{xy} \sim \frac{B c_\phi e^{-4\Phi}}{g_{xx} g_{yy}}, \quad \frac{\sigma^{xx}}{\sigma^{xy}} \sim \frac{\mathcal{N}}{B c_\phi} \frac{e^{3\Phi}}{g_{xx}} \frac{1}{\sqrt{g_{xx} g_{yy}}}. \tag{76}$$

Again considering the rotationally invariant geometry and demanding that the conductivities in this limit matches with eq(1) gives us the solution as

$$e^\Phi \sim \mathcal{N} T, \quad g_{xx}(r_\star) = g_{yy}(r_\star) \sim \frac{\sqrt{B c_\phi}}{\mathcal{N}^2} T^{-1/2}. \tag{77}$$

The prediction for the background solutions from eq(77) are very interesting, e.g., the smallness of the dilaton at low temperature. It would be interesting to generate such solutions and the predicted form of the metric in Einstein frame and dilaton evaluated at $r = r_\star$ are

$$ds_E^2(r_\star) \sim -g_{tt}(r_\star) dt^2 + g_{rr}(r_\star) dr^2 + \frac{dx^2 + dy^2}{r_\star^{1/2z}}, \quad \phi(r_\star) \sim r_\star^z \tag{78}$$

where we have used the choice that $T = T_H \sim r_h^z \sim r_\star^z$, with T_H being the Hawking temperature.

3.2 Charge density with the Chern-Simon term

It is not *a priori* clear whether the low energy effective action of the probe brane admits a Chern-Simon type term or not. We assume it does and takes the form similar to that in the string theory except that the target space here is $3+1$ dimensional. In this section we have made a small change to the form of the field strength, i.e., $h(r) \to -h(r)$

$$F_2 = -E_1 dt \wedge dx - E_2 dt \wedge dy + B dx \wedge dy - H'(r) dr \wedge dx + h'(r) dr \wedge dy + \phi'(r) dr \wedge dt. \tag{79}$$

The inclusion of the Chern-Simon term to the probe brane action adds the following term to the 3+1 dimensional action

$$S_{CS} = \mu \int \Big([C_0] F \wedge F + [C_2] \wedge F + [C_4] \Big), \tag{80}$$

in the absence of the B_2 field from the NS-NS sector. The bulk fields $[C_n]$ are to be understood as the pullback onto the world volume of the probe brane. Let us also assume for simplicity, C_4 vanishes, the C_2 has the following structure, $[C_2] = -2\tilde{C}_2(r) dt \wedge dy$ and $[C_0]$ depends only on the radial coordinate. Using the field strength as written in eq(79) results in

$$S_{CS} = 2\mu \int \Big[C_0(E_1 h' + E_2 H' - B\phi') + \tilde{C}_2 H' \Big] dt \wedge dx \wedge dy \wedge dr. \tag{81}$$

Let us redefine $\tilde{\mu} := 2\mu V_3$, where V_3 is the volume of $R^{1,2}$. Finally the Chern-Simon action becomes

$$S_{CS} = \tilde{\mu} \int \Big[C_0 E_1 h' + (C_0 E_2 + \tilde{C}_2) H' - C_0 B \phi' \Big] dr. \tag{82}$$

So, the full action of the probe brane is

$$\begin{aligned} S &= -\mathcal{N} \int dr \Big[g_{rr}(B^2 g_{tt} - E_1^2 g_{yy} - E_2^2 g_{xx} + g_{tt} g_{xx} g_{yy}) + (g_{tt} g_{xx} - E_1^2) h'^2 - \\ &\quad 2E_1 E_2 h' H' + (g_{tt} g_{yy} - E_2^2) H'^2 + 2B(E_1 h' + E_2 H')\phi' - (g_{xx} g_{yy} + B^2)\phi'^2 \Big]^{1/2} \\ &+ \tilde{\mu} \int dr \Big[C_0 E_1 h' + (C_0 E_2 + \tilde{C}_2) H' - C_0 B \phi' \Big] \\ &\equiv -\mathcal{N} \int dr \quad \bar{\mathcal{L}} + \tilde{\mu} \int dr \Big[C_0 E_1 h' + (C_0 E_2 + \tilde{C}_2) H' - C_0 B \phi' \Big] \end{aligned} \tag{83}$$

Once again the action does not depends on the field ϕ, H and h, so the corresponding momentum are constants. Let us denote the constant momentum for the field ϕ, h and H as c_ϕ, c_h and c_H, respectively

$$\begin{aligned} \frac{\delta S}{\delta \phi'} &\equiv c_\phi = -\frac{\mathcal{N}\sqrt{\prod g_{z_a z_a}}}{\bar{\mathcal{L}}} [B(E_1 h' + E_2 H') - (g_{xx} g_{yy} + B^2)\phi'] - \tilde{\mu} B C_0, \\ \frac{\delta S}{\delta h'} &\equiv c_h = -\frac{\mathcal{N}\sqrt{\prod g_{z_a z_a}}}{\bar{\mathcal{L}}} [(g_{tt} g_{xx} - E_1^2) h' - E_1 E_2 H' + B E_1 \phi'] + \tilde{\mu} E_1 C_0, \\ \frac{\delta S}{\delta H'} &\equiv c_H = -\frac{\mathcal{N}\sqrt{\prod g_{z_a z_a}}}{\bar{\mathcal{L}}} [(g_{tt} g_{yy} - E_2^2) H' + B E_2 \phi' - E_1 E_2 h'] + \tilde{\mu}(E_2 C_0 + \tilde{C}_2] \end{aligned} \tag{84}$$

From now on we shall drop the tildes from the field C_2 and the coupling μ so as to avoid cluttering of it. By taking the ratio of the momenta we determine h' and ϕ' in terms of H'

$$h' \equiv \frac{h_1}{h_2}, \quad \text{where}$$

$$
\begin{aligned}
h_1 &= g_{yy}\Big[E_2(c_H E_1 - c_h E_2 - E_1\mu C_2)g_{xx} + g_{tt}\Big(B(Bc_h + c_\phi E_1) + (c_h - E_1\mu C_0)g_{xx}g_{yy}\Big)\Big]H' \\
h_2 &= g_{xx}\Big[E_1(-c_H E_1 + c_h E_2 + E_1\mu C_2)g_{yy} + g_{tt}\Big(B(Bc_H + c_\phi E_2) + \\
&\quad (c_H - E_2\mu C_0)g_{xx}g_{yy} - \mu C_2(B^2 + g_{xx}g_{yy})\Big)\Big], \\
\phi' &\equiv \frac{\phi_1}{\phi_2}, \quad \text{where} \\
\phi_1 &= g_{tt}\Big[E_1(Bc_h + c_\phi E_1)g_{yy} + g_{xx}\Big(E_2(Bc_H + c_\phi E_2) - BE_2\mu C_2 - (c_\phi + B\mu C_0)g_{tt}g_{yy}\Big)\Big]H' \\
\phi_2 &= g_{xx}\Big[E_1(-cHE_1 + c_h E_2 + E_1\mu C_2)g_{yy} + g_{tt}\Big(B(Bc_H + c_\phi E_2) + \\
&\quad (c_H - E_2\mu C_0)g_{xx}g_{yy} - \mu C_2(B^2 + g_{xx}g_{yy})\Big)\Big] \qquad (85)
\end{aligned}
$$

The function H' can be evaluated by substituting eq(85) into the last equation of eq(84). Again we are not writing down the explicit form of H' as it is a very long expression. As is done previously the Legendre transformed action

$$
\begin{aligned}
S_L &= S - \int \frac{\delta S}{\delta \phi'}\phi' - \int \frac{\delta S}{\delta h'}h' - \int \frac{\delta S}{\delta H'}H' \\
&= -\mathcal{N}\int \frac{dr}{\bar{\mathcal{L}}}\Big[g_{rr}(B^2 g_{tt} - E_1^2 g_{yy} - E_2^2 g_{xx} + g_{tt}g_{xx}g_{yy})\Big] \\
&= -\int dr\sqrt{\frac{g_{rr}}{g_{tt}g_{xx}g_{yy}}} \times \sqrt{[\mathcal{A}_\in(r)\mathcal{A}_\ni(r) - (\mathcal{A}_\triangle(r))^2]}, \qquad (86)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}_\in(r) &= B^2 g_{tt} - E_2^2 g_{xx} - E_1^2 g_{yy} + g_{tt}g_{xx}g_{yy}, \\
\mathcal{A}_\ni(r) &= g_{tt}g_{xx}g_{yy} - \Big(c_H - \mu(E_2 C_0 + C_2)\Big)^2 g_{xx} - (c_h - E_1\mu C_0)^2 g_{yy} + (c_\phi + B\mu C_0)^2 g_{tt}, \\
\mathcal{A}_\triangle(r) &= (c_\phi + B\mu C_0)Bg_{tt} + (c_h - E_1\mu C_0)E_1 g_{yy} + \Big(c_H - \mu(E_2 C_0 + C_2)\Big)E_2 g_{xx} \qquad (87)
\end{aligned}
$$

From now on we shall set $E_2 = 0$ as it does not change much of the physics that we are going to do. The scale $r_\star$ is determined by solving

$$
\mathcal{A}_\in(r_\star) = [B^2 g_{tt} - E_1^2 g_{yy} + g_{tt}g_{xx}g_{yy}]_{r_\star} = 0, \qquad (88)
$$

whose functional form coincides with that of eq(63), which was considered without the Chern-Simon term. The form of the currents are

$$
J^x = \mu C_2 \mp \frac{E_1 g_{yy}}{B^2 + g_{xx}g_{yy}}\sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2(B^2 + g_{xx}g_{yy})},
$$

$$J^y = E_1\Big[\frac{B(c_\phi + B\mu C_0)}{B^2 + g_{xx}g_{yy}} - \mu C_0\Big] = \frac{E_1[Bc_\phi - \mu C_0 g_{xx}g_{yy}]}{B^2 + g_{xx}g_{yy}}, \tag{89}$$

where we have used eq(88). From which the conductivity follows upon using the Ohm's law

$$\sigma^{xx} = \mp\frac{g_{yy}}{B^2 + g_{xx}g_{yy}}\sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2(B^2 + g_{xx}g_{yy})}, \quad \sigma^{xy} = \frac{Bc_\phi - \mu C_0 g_{xx}g_{yy}}{B^2 + g_{xx}g_{yy}}. \tag{90}$$

Once again, let us look at a special corner of the parameter space of charge density c_ϕ and the magnetic field B for which $\mathcal{N}B$ is very small in comparison to density. In this case the conductivity reduces to

$$\sigma^{xx} \simeq \mp\Big[\frac{c_\phi}{g_{xx}} + \mathcal{N}^2\frac{g_{yy}}{2c_\phi^2} + \cdots\Big], \quad \sigma^{xy} \simeq \frac{Bc_\phi}{g_{xx}g_{yy}} - \mu C_0 + \cdots, \tag{91}$$

where the ellipses denote higher powers of magnetic field. Choosing the positive branch from σ^{xx} and dropping the second term in σ^{xx} gives us

$$\sigma^{xx} \sim \frac{c_\phi}{g_{xx}}, \quad \sigma^{xy} \simeq \frac{Bc_\phi}{g_{xx}g_{yy}} - \mu C_0, \tag{92}$$

and in the small μC_0 limit i.e. $\mu C_0 << Bc_\phi$ the Hall angle reduces to

$$\frac{\sigma^{xx}}{\sigma^{xy}} \sim \frac{g_{yy}}{B}. \tag{93}$$

So the presence of Chern-Simon term to the action parametrically does not change much to the conductivity in the small magnetic field and large density limit but adds a piece to the off diagonal part of the conductivity. The Hall angle in the limit, $\mu C_0 << \frac{Bc_\phi}{g_{xx}g_{yy}}$, remains same as in the case of without the Chern-Simon term.

In the presence of a non trivial dilaton, Φ, the form of the conductivities becomes

$$\begin{aligned}\sigma^{xx} &= \mp\frac{e^{2\Phi}g_{yy}}{B^2 + e^{4\Phi}g_{xx}g_{yy}}\sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2 e^{-2\Phi}(B^2 + e^{4\Phi}g_{xx}g_{yy})},\\ \sigma^{xy} &= \frac{Bc_\phi - \mu C_0 e^{4\Phi}g_{xx}g_{yy}}{B^2 + e^{4\Phi}g_{xx}g_{yy}},\end{aligned} \tag{94}$$

which in the small magnetic field and $c_\phi \gg B\mu C_0$ limit reduces to

$$\sigma^{xx} \sim \frac{e^{-2\Phi}}{g_{xx}}\sqrt{c_\phi^2 + \mathcal{N}^2 e^{2\Phi}g_{xx}g_{yy}}, \quad \sigma^{xy} \sim \frac{Bc_\phi e^{-4\Phi}}{g_{xx}g_{yy}} - \mu C_0. \tag{95}$$

If we take the axion as constant, $C_0 = \theta$, with a rotationally invariant geometry, very high density limit and assume the first term in the Hall conductivity dominates over the axionic term then there is no solution to dilaton and metric component that can give the result eq(1). However if we consider a small density limit (but $c_\phi \gg B\mu C_0$) in a rotationally invariant geometry with a non constant axion then the conductivities reduces to

$$\sigma^{xx} \sim \mathcal{N} e^{-\Phi}, \quad \sigma^{xy} \sim \frac{B c_\phi e^{-4\Phi}}{g_{xx}^2} - \mu C_0. \tag{96}$$

Upon comparing with eq(1), the dilaton goes $e^\Phi \sim \mathcal{N} T$ with the combination of metric component and axion as

$$\frac{B c_\phi e^{-4\Phi}}{g_{xx}^2} - \mu C_0 \sim T^{-3}. \tag{97}$$

It would be interesting to find such background solutions that shows the property as is being shown in eq(97).

4 Geometry with two exponents: An example

In this section we shall write down a gravitational solution for which the geometry exhibits the required two exponents explicitly. The extremal solution is already found in [15] in a specific setting that is with several form field strengths and metric. But to find the non-extremal solution in that setup is very cumbersome. Instead, here we shall find such solutions by adopting a different form of the gravitational action than that is considered in [15], but it comes up with a cost that is the entropy vanishes even though there is a finite size to the horizon. The on shell action vanishes identically as a result of the vanishing of the free energy and the energy. Similar kind of behavior was seen previously in the context of generating Lifshitz type solutions in [33] and [34].

The action that we shall consider is a Ricci squared corrected term to Einstein-Hilbert action with a cosmological constant

$$S = \frac{1}{2\kappa_4^2} \int \sqrt{-g}[R - 2\Lambda + \beta R^2] \equiv \int L. \tag{98}$$

The equation of motion that follows from it is

$$R_{MN} - \frac{1}{2} g_{MN} R + \Lambda g_{MN} + 2\beta g_{MN} \Box R - 2\beta \nabla_M \nabla_N R + 2\beta R R_{MN} - \frac{1}{2}\beta R^2 g_{MN} = 0. \tag{99}$$

The solution to the equation of motion comes as

$$ds^2 = L^2[-r^{2z} f(r) dt^2 + r^{2w} dx^2 + r^2 dy^2 + \frac{dr^2}{r^2 f(r)}], \tag{100}$$

which respect the scaling symmetry

$$t \to \lambda^z \; t, \quad x \to \lambda^w \; x, \quad y \to \lambda \; y, \quad r \to \frac{r}{\lambda}. \tag{101}$$

The function

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{\alpha_\pm}, \tag{102}$$

where

$$\alpha_\pm = 1 + w + \frac{3}{2}z \pm \frac{1}{2}\sqrt{-4(1+w^2) + 4z + 4wz + z^2}. \tag{103}$$

From which there follows a restriction on the exponents, $4z + 4wz + z^2 \;\; \geq \;\; 4(1+w^2)$ and the dimension full objects β and Λ are

$$\Lambda = -\frac{1}{2L^2}[1 + w + z + w^2 + z^2 + wz], \quad \beta = \frac{L^2}{4[1 + w + z + w^2 + z^2 + wz]} \tag{104}$$

with the Hawking temperature

$$T_H = \frac{\alpha_\pm}{4\pi} r_h^z. \tag{105}$$

It follows trivially that for a solution with exponents for which $z = 1$ and $w = 1/2$ satisfies the restrictions that $\alpha_\pm$ is a real quantity and hence the solution is real. For this choice to the exponents the cosmological constant and the coupling are

$$\Lambda = -\frac{17}{8L^2}, \quad \beta = \frac{L^2}{17}, \quad \alpha_\pm = \frac{3\sqrt{2} \pm 1}{\sqrt{2}}. \tag{106}$$

If we calculate the entropy of the system using Wald's formula [32]

$$S_{BH} = -2\pi \int_{r_h} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd}, \tag{107}$$

where the quantity ϵ_{ab} is binormal to the bifurcation surface, and is normalized in such a way that it obeys $\epsilon_{ab}\epsilon^{ab} = -2$. We use the convention of [35] to calculate it, which reads as

$$\epsilon_{ab} = \xi_a \eta_b - \xi_b \eta_a, \tag{108}$$

where ξ and η are null vectors normal to the bifurcate killing horizon, with $\xi.\eta = 1$. In our choice of $3+1$ dimensional metric, the non vanishing components of the null vectors are

$$\xi_t = -g_{tt} = -L^2 r^{2z} f(r), \quad \eta_t = 1, \quad \eta_r = -\sqrt{\frac{g_{rr}}{g_{tt}}} = -\frac{1}{f(r) r^{1+z}}. \tag{109}$$

In fact, for the action like eq(98) the entropy is

$$S_{BH} = \frac{2\pi^2}{\kappa_4^2}\Big(\sqrt{-g}[1+2\beta R]\Big)_{r_h}, \tag{110}$$

and using all this ingredients into this formula gives us vanishing entropy, which means the solution eq(100) has the constant curvature: $R = -1/2\beta$. From the trace of the equation of motion to metric eq(99), it follows that the scalar curvature obeys

$$R = 4\Lambda + 6\beta\Box R. \tag{111}$$

Now combining these two facts, we obtain the curvature

$$R = -1/2\beta = 4\Lambda, \tag{112}$$

which is precisely the behavior of the solution in eq(106).

4.1 Parameter Space

In this subsection we shall write down the exact form of both the conductivity and Hall angle that followed from section 3. Before the evaluation of the conductivity we need to know the scale, $r_\star$. From eq(88), it follows that for small electric field and magnetic field the scale

$$r_\star \sim r_h \sim T_H^{1/z}. \tag{113}$$

The corrections to this scale occurs in the dimensionless ratios of $E/T_H^{1+1/z}$ and $B/T_H^{(1+w)/z}$.

Now, substituting the explicit form of the metric components from eq(100) into eq(90) results in

$$\sigma^{xx} = \mp\frac{r_\star^2}{B^2 + r_\star^{2(1+w)}}\sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2\Big(B^2 + r_\star^{2(1+w)}\Big)}, \quad \sigma^{xy} = \frac{Bc_\phi - \mu C_0 r_\star^{2(1+w)}}{B^2 + r_\star^{2(1+w)}}. \tag{114}$$

In the small magnetic field, large density and at low temperature limit the expression to conductivities reduces to

$$\begin{aligned}&\sigma^{xx} \sim c_\phi r_\star^{-2w} + \cdots \sim c_\phi T_H^{-2w/z}, \quad \sigma^{xy} \sim Bc_\phi r_\star^{-2(1+w)} - \mu C_0 + \cdots \sim Bc_\phi T_H^{-2(1+w)/z} - \mu C_0,\\ &\Longrightarrow \sigma^{xx}/\sigma^{xy} \sim r_\star^2/c_\phi + \cdots \sim (T_H^{2/z})/c_\phi + \cdots,\end{aligned} \tag{115}$$

where in the last line we have assumed $Bc_\phi > \mu C_0 T_H^{2(1+w)/z}$. Demanding that this temperature dependence to conductivities should match the experimental results, eq(1), gives us the following values of exponent, $z = 1, \quad w = 1/2$. So, the above form of the exponents gives us the strange metal behavior of copper-oxide systems as seen in experiments

[1], [3], [4]. If we consider the other regime of parameter space with a constant axion at low temperature for which the magnetic field is small in comparison to charge density such that $Bc_\phi < \mu C_0 T_H^{2(1+w)/z}$ then the off diagonal conductivity does not depend on temperature, which is not of much interest as far as the experimental results are concerned. Hence, this regime of parameter space may not be that useful. However, if we consider the non constant axion field in the same limit, i.e., $Bc_\phi < \mu C_0 T_H^{2(1+w)/z}$, then by matching with eq(1), we get the exponents as $2w = z$ and the axion field should have the following behavior, $C_0 \sim T_H^{-3} \sim r_\star^{-3z}$. It would be interesting to find such background solutions.

4.2 Fermi Liquid

In this subsection we shall reproduce a well known transport properties of the Fermi liquid theory. It is known, see for example [38], that the conductivity at low temperature goes as $\sigma_{FL} \sim T^{-2}$. Now upon using eq(115), we see that in order to reproduce this particular behavior requires us to take the exponents as $w = z$. Here the exponents are not fixed to a particular value. In the next section, we shall demand that the specific heat heat should have a linear dependence to temperature, parametrically. The result of this, fixes the exponents to $z = w = -2$. Note, for this choice of exponents the quantity $\alpha_\pm$ defined in eq(103) becomes pure imaginary, which is an artifact of the action we used to construct it. However, in what follows we shall not be worried about the nature of $\alpha_\pm$ as we believe the above mentioned constraint on the exponents can be removed by looking at better solutions.

5 Probe brane thermodynamics

In this section we shall study some thermodynamic properties of the probe brane but without the Chern-Simon term and non trivial dilaton. Let us recall that the charge carriers are introduced via probe brane and the study of their thermodynamic behavior is very important so as to have a better understanding of the nature of quantum critical point. It is reported in [36] and [37], for a review see [38] that at low temperature the specific heat (for NFL) goes as $C_V \sim T\ Log\ T$. But unfortunately, with our choice of exponents as demanded by the transport properties: $z = 1,\ w = 1/2$, gives us the specific heat to go instead as $C_V \sim T_H^3$. This kind of behavior to specific heat resembles that of the Debye theory.

Let us see this particular behavior of specific heat in detail. We shall proceed to calculate the free energy of the probe brane system following [39]. The proper holographic treatment is also done in [23] and [40]. The Gibbs free energy, i.e., the thermodynamic potential, Ω, in the grand canonical ensemble is just the negative of the on shell value of the action times temperature. Here, we have chosen to work in the canonical ensemble. The easiest way to include the effect of charge density in a magnetic field is by using the field strength, $F_2 = \phi'(r) dr \wedge dt + B dx \wedge dy$ in the DBI action. In $3+1$ dimensions, using the metric as in eq(40) gives us the thermodynamic potential and chemical potential $\mu=\int_{r_h}^{\infty} dr F_{rt}=\int_{r_h}^{\infty} dr\phi'$. The chemical potential, μ, should not be confused with the Chern-Simon coupling that appeared in section 3.2.

$$\Omega=\mathcal{N}V_2\int_{r_h}^{\infty} dr\frac{(g_{xx}g_{yy}+B^2)\sqrt{g_{tt}g_{rr}}}{\sqrt{g_{xx}g_{yy}+B^2+\rho^2}},\quad \mu=\rho\int_{r_h}^{\infty} dr\frac{\sqrt{g_{tt}g_{rr}}}{\sqrt{g_{xx}g_{yy}+B^2+\rho^2}}, \tag{116}$$

where $\mathcal{N}\rho=c_\phi$, and c_ϕ is the charge density. V_2 is the flat space volume of $x,\ y$ plane. Using the metric structure as written in eq(100) gives

$$\Omega=\mathcal{N}V_2\int_{r_h}^{\infty} dr\frac{(r^{2+2w}+B^2)r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}},\quad \mu=\rho\int_{r_h}^{\infty} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}. \tag{117}$$

For generic choice of the exponents, the integral in the thermodynamic potential and in the chemical potential diverges at UV, so we need to regulate it. The way we shall do is to subtract an equivalent amount but without the charge density and magnetic field. It means

$$\begin{aligned}
\Omega &= \mathcal{N}V_2\int_0^{\infty} dr\Big(\frac{r^{1+2w+z}}{\sqrt{r^{2+2w}+B^2+\rho^2}}-r^{z+w}\Big)-\mathcal{N}V_2\int_0^{r_h} dr\frac{r^{1+2w+z}}{\sqrt{r^{2+2w}+B^2+\rho^2}}\\
&+ \mathcal{N}V_2B^2\int_0^{\infty} dr\Big(\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}-r^{z-w-2}\Big)-\mathcal{N}V_2B^2\int_0^{r_h} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}},\\
\mu &= \rho\int_0^{\infty} dr\Big(\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}-r^{z-w-2}\Big)-\rho\int_0^{r_h} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}
\end{aligned} \tag{118}$$

The second term in the square bracket of the first equation should not be there when $z+w=0$. Similarly, the second term in the square bracket of the first equation comes into picture only when $z>2+w$, so also for the second term in the second square bracket. Let us assume the case, where $z\not>2+w$ and $z\neq -w$. It means we want to regulate it in the following way

$$\begin{aligned}
\Omega &= \mathcal{N}V_2\int_0^{\infty} dr\Big(\frac{r^{1+2w+z}}{\sqrt{r^{2+2w}+B^2+\rho^2}}-r^{z+w}\Big)-\mathcal{N}V_2\int_0^{r_h} dr\frac{r^{1+2w+z}}{\sqrt{r^{2+2w}+B^2+\rho^2}}\\
&+ \mathcal{N}V_2B^2\int_0^{\infty} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}-\mathcal{N}V_2B^2\int_0^{r_h} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}},\\
\mu &= \rho\int_0^{\infty} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}-\rho\int_0^{r_h} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}
\end{aligned} \tag{119}$$

After doing these integrals

$$\frac{\Omega}{\mathcal{N}V_2}=\alpha(w,z)\ (B^2+\rho^2)^{\frac{z+w+1}{2+2w}}-\frac{1}{(2+2w+z)}\ \frac{r_h^{(2+2w+z)}}{\sqrt{B^2+\rho^2}}\times$$

$$
{}_2F_1[1+\frac{z}{2+2w},\frac{1}{2};2+\frac{z}{2+2w};-\frac{r_h^{2+2w}}{B^2+\rho^2}]+\frac{B^2}{z\sqrt{\pi}}\,(B^2+\rho^2)^{\frac{z-w-1}{2+2w}}\,\Gamma\Big(\frac{1+w-z}{2+2w}\Big)
$$
$$
\Gamma\Big(\frac{2+2w+z}{2+2w}\Big)-\frac{r_h^z}{z}\frac{B^2\rho}{\sqrt{B^2+\rho^2}}\,{}_2F_1\Big[\frac{z}{2+2w},\ \frac{1}{2};\ 1+\frac{z}{2+2w};\ \ -\frac{r_h^{2+2w}}{B^2+\rho^2}\Big],
$$
$$
\mu=\frac{1}{z\sqrt{\pi}}\,(B^2+\rho^2)^{\frac{z-w-1}{2+2w}}\,\Gamma\Big(\frac{1+w-z}{2+2w}\Big)\,\Gamma\Big(\frac{2+2w+z}{2+2w}\Big)-
$$
$$
\frac{r_h^z}{z}\frac{\rho}{\sqrt{B^2+\rho^2}}\,{}_2F_1\Big[\frac{z}{2+2w},\ \frac{1}{2};\ 1+\frac{z}{2+2w};\ \ -\frac{r_h^{2+2w}}{B^2+\rho^2}\Big],
$$
(120)

where $\alpha(w,z)$ is a function of the exponents, whose explicit structure is not that important for the understanding of thermodynamics. $\Gamma(x)$ and ${}_2F_1[a,b;c;x]$ are the gamma function and hypergeometric function, respectively. In the limit of high density, low magnetic field and low temperature, i.e.,$T^{\frac{1+w}{z}}/\sqrt{B^2+\rho^2}\ \ll\ 1$, eq(120) can be expanded in the series form. The free energy in the canonical ensemble, $F=\Omega+\mu J^t$, where $J^t=\mathcal{N}V_2\rho$ is the charge. From this the entropy density goes as

$$
s=-\frac{1}{V_2}\Big(\frac{\partial F}{\partial T_H}\Big)=s_0+\frac{\mathcal{N}}{2z\sqrt{B^2+\rho^2}}\,\Big(4\pi/\alpha_\pm\Big)^{\frac{2+2w+z}{z}}\,T_H^{\frac{2+2w}{z}}, \tag{121}
$$

where $s_0=\frac{4\pi\mathcal{N}}{z\alpha_\pm}\sqrt{B^2+\rho^2}$ is the entropy density at zero temperature. The specific heat defined as the heat capacity per unit volume, at low temperature, goes as

$$
C_V=T_H\Big(\frac{\partial s}{\partial T_H}\Big)=\frac{\mathcal{N}}{\sqrt{B^2+\rho^2}}\,\Big(\frac{1+w}{z^2}\Big)\,\Big(4\pi/\alpha_\pm\Big)^{\frac{2+2w+z}{z}}\,T_H^{\frac{2+2w}{z}}. \tag{122}
$$

The magnetic susceptibility, which we shall call as susceptibility, at low temperature

$$
\chi/V_2=-\Big(\frac{\partial^2 F}{\partial B^2}\Big)=-\chi_0(B,\rho)+\frac{\mathcal{N}4\pi}{z\alpha_\pm}\frac{\rho^2}{(B^2+\rho^2)^3/2}T_H, \tag{123}
$$

where χ_0 is some function of B and ρ, whose exact form is not that illuminating. The effect of the Chern-Simon term with the field strength, $F_2=\phi'(r)dr\wedge dt+Bdx\wedge dy$, is to replace ρ in all of the above formulas by $\rho+\mu\theta B$, where we have considered the axion field to be a constant and identified it with $C_0\equiv\theta$.

5.1 At high temperature, low magnetic field and low density

In this subsection, we shall write down the behavior to thermodynamic quantities in the high temperature but low magnetic field limit. One of the main reason of study of this regime of parameter space is to see the behavior of susceptibility. Probably, it is correct to say that when we are in the proximity of quantum critical point the magnetization should not obey the Curie-Weiss type behavior in the high temperature limit.

The temperature dependence to free energy in this regime can be obtained very easily by looking at the following integrals

$$\begin{aligned}
\frac{\Omega}{\mathcal{N}V_2} &\sim -\int_0^{r_h} dr \frac{r^{1+2w+z}}{\sqrt{r^{2+2w}+B^2+\rho^2}} - B^2 \int_0^{r_h} dr \frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}}, \\
&= -\frac{r_h^{z+w+1}}{z+w+1} - \frac{B^2 r_h^{z-w-1}}{z-w-1} + \frac{(B^2+\rho^2)}{2(z-w-1)} r_h^{z-w-1} + \frac{B^2(B^2+\rho^2)}{2(z-3w-3)} r_h^{z-3w-3} + \cdots, \\
\mu &\sim -\rho \int_0^{r_h} dr \frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}} = -\frac{\rho r_h^{z-w-1}}{z-w-1} + \frac{\rho(B^2+\rho^2)}{2(z-3w-3)} r_h^{z-3w-3} + \cdots. \quad (124)
\end{aligned}$$

So, the Free energy in the canonical ensemble has the following behavior in the high temperature limit

$$\begin{aligned}
\frac{F}{\mathcal{N}V_2} &= -\frac{1}{z+w+1}\Big(4\pi/\alpha_\pm\Big)^{\frac{z+w+1}{z}} T_H^{\frac{z+w+1}{z}} - \frac{(B^2+\rho^2)}{2(z-w-1)}\Big(4\pi/\alpha_\pm\Big)^{\frac{z-w-1}{z}} T_H^{\frac{z-w-1}{z}} + \\
&\quad \frac{(B^2+\rho^2)^2}{2(z-3w-3)}\Big(4\pi/\alpha_\pm\Big)^{\frac{z-3w-3}{z}} T_H^{\frac{z-3w-3}{z}} + \cdots. \quad (125)
\end{aligned}$$

From which the magnetization, $\frac{M}{\mathcal{N}V_2} = -\left(\frac{\partial F/(\mathcal{N}V_2)}{\partial B}\right)$ and the susceptibility, $\frac{\chi}{\mathcal{N}V_2} = -\left(\frac{\partial^2 F/(\mathcal{N}V_2)}{\partial B^2}\right)$ are

$$\begin{aligned}
\frac{M}{\mathcal{N}V_2} &= -\frac{B}{(z-w-1)}\Big(4\pi/\alpha_\pm\Big)^{\frac{z-w-1}{z}} T_H^{\frac{z-w-1}{z}} - \frac{2B(B^2+\rho^2)}{(z-3w-3)}\Big(4\pi/\alpha_\pm\Big)^{\frac{z-3w-3}{z}} T_H^{\frac{z-3w-3}{z}}, \\
\frac{\chi}{\mathcal{N}V_2} &= \frac{1}{(z-w-1)}\Big(4\pi/\alpha_\pm\Big)^{\frac{z-w-1}{z}} T_H^{\frac{z-w-1}{z}} - \frac{2(3B^2+\rho^2)}{(z-3w-3)}\Big(4\pi/\alpha_\pm\Big)^{\frac{z-3w-3}{z}} T_H^{\frac{z-3w-3}{z}} \\
&\equiv \widetilde{\chi}_0 T_H^{\frac{z-w-1}{z}} - \widetilde{\chi}_1 T_H^{\frac{z-3w-3}{z}} \quad (126)
\end{aligned}$$

Now, if we demand that the magnetization or more precisely, the susceptibility has the Curie-Weiss type behavior, the above result forces us to put the following constraints on the exponents

$$2z = 1 + w. \quad (127)$$

Recalling the results to the exponents that follows from the study of conductivity and Hall angle in section 4, suggests that near to quantum critical point the system does not show the Curie-Weiss type behavior. In fact the behavior of susceptibility using the exponents, $z = 1,\ w = 1/2$, gives

$$\chi = \widetilde{\chi}_0\ T_H^{-1/2}, \quad (128)$$

and for Fermi liquid, $z=-2=w$, at high temperature limit goes as

$$\chi = \widetilde{\chi}_0\ T_H^{1/2}. \tag{129}$$

The Curie-Weiss type behavior is possible only when eq(127) is obeyed. From which it follows trivially that the asymptotically AdS spacetime possesses such kind of behavior, as an example, for which $z=1=w$. Once again the effect of the Chern-Simon term is to replace ρ in all of the above formulas by $\rho+\mu\theta B$, for constant axion as stated in the previous section.

6 Conclusion

In this paper we have shown that there exists only two possible ways with different symmetries to find the precise temperature dependence to conductivity and Hall angle, $1/T$ and T^2, respectively as seen in the non-Fermi liquid. The calculation is done similar in spirit to the proposal of [10], where the charge density is introduced via flavor brane. It is done in a generic background demanding the symmetries, like scaling, time translation, spatial translations and rotations. The result of the calculation suggests us that for a theory having scaling, time and spatial translation symmetry at high density limit, we should not take the spatial part of the metric components to be same, i.e. $g_{xx}\neq g_{yy}$, in order to get the desired experimental result for transport quantities that is mentioned above. For this purpose, we have considered a metric with two exponents, z and w, as defined in eq(4). The end result of this requirement is that the exponents takes the values, $z=1,\ w=1/2$.

The study of the thermodynamic behavior of various physical quantities are also of equal importance in the study of quantum critical point or otherwise. For the above choice of exponents, the specific heat at low temperature goes as $C_V\sim T_H^3$, which resembles that of the Debye type. The susceptibility at zero magnetic field and at low temperature goes as $\chi=-\chi_0+(\text{constants})\times T_H/\rho$, where χ_0 is a function of charge density.

However, if we consider the theory to have the symmetries like pseudo-scaling, time translation, spatial translation and rotation in the low density limit, we can reproduce eq(1) without the need to introduce two exponents. We leave the detailed study of the thermodynamic behavior of this class of solution for future research.

From this study there follows one interesting outcome that is we are completely ruling out the possibility to see eq(1) for background solutions showing scaling symmetry, time translation, spatial translation and rotational symmetry. In other words these symmetries are not consistent with eq(1).

The transport and thermodynamic behavior of various physical quantities at high density and at low temperature can be summarized in this two exponent model as follows :

Type	Physical quantity	Expt. result	Ref	In this model	Experimental result forces the choice of Exponents
NFL	Conductivity	T^{-1}	[1],[3], [4]	$T_H^{-2w/z}$	$z=1,\ w=1/2$
NFL	Hall Angle	T^2	[1],[3], [4]	$T_H^{2/z}$	$z=1,\ w=1/2$
FL	Conductivity	T^{-2}	[38]	$T_H^{-2w/z}$	$z=-2,\ w=-2$
FL	Hall Angle	Not known to author		$T_H^{2/z}\sim T_H^{-1}$	

(130)

and

Type	Physical quantity	Expt. result	Ref	In this model	Experimental result forces the choice of Exponents
FL	Specific heat	T	[38]	$-T_H^{(2+2w)/z}$	$z=-2,\ w=-2$
FL	Susceptibility: $\chi(B=0)$	independent of T	[38]	$-\chi_0+\text{const}/\alpha_\pm \times T_H/\rho$	$z=-2,\ w=-2$
NFL	Specific heat	should not be as T	[38]	$T_H^{(2+2w)/z}\sim T_H^3$	
NFL	Susceptibility: $\chi(B=0)$	Not known to author		$-\chi_0+$ const $\times\, T_H/\rho$	

(131)

At high temperature, low density and low magnetic field limit

Type	Physical quantity	Exponents	In this model	Prediction
NFL	Specific heat	$z=1,\ w=1/2$	$T_H^{(1+w)/z}$	$T_H^{3/2}$
NFL	Susceptibility	$z=1,\ w=1/2$	$T_H^{(z-w-1)/z}$	$T_H^{-1/2}$
FL	Specific heat	$z=-2=w$	$T_H^{(1+w)/z}$	$T_H^{1/2}$
FL	Susceptibility	$z=-2=w$	$T_H^{(z-w-1)/z}$	$T_H^{1/2}$
AdS Spacetime	Specific heat	$z=1=w$	$T_H^{(1+w)/z}$	T_H^2
AdS Spacetime	Susceptibility	$z=1=w$	$T_H^{(z-w-1)/z}$	T_H^{-1}

(132)

From eq(132), it follows that it's only the asymptotically AdS spacetime that shows the Curie-Weiss type behavior. From which it is natural to think that the asymptotically AdS spacetime may be associated to metals, more specifically to the paramagnets, even though the specific heat shows a quadratic dependence to temperature.

We have constructed a background geometry with two exponents, for illustration. The future goal would be to construct other background solutions having non trivial spacetime thermodynamics, i.e., the thermodynamics of adjoint degrees of freedom, in the sense of having non zero entropy for finite horizon size and may be a non zero free energy, depending on the requirement of the model, which is *a priori* not clear at present. Moreover, the thermodynamic quantities in the Fermi liquid phase need to be real.

There are several other checks that needed to be done. In particular, the AC conductivity $\sigma(\omega)$, which in the interval $T_H < \ \omega < \ \tilde{\Omega}$, shows a very specific behavior [41], where ω and $\tilde{\Omega}$ are the frequency and some high energy cutoff scale. This result of [41] for copper oxide systems puts some serious restrictions on the form of the (bulk) geometry. In the study of superconductors [42] at low temperature (even at extremality), it was suggested that if the potential energy close to IR behaves as, $V = V_0/r^2$, then the real part of AC conductivity goes as, $\Re[\sigma(\omega)] \propto \omega^{\sqrt{4V_0+1}-1}$. Now upon matching with the results of [41], we get $V_0 = -2/9$, i.e., there should be an attractive potential energy close to IR. Which is an interesting prediction but we leave this aspect of holographic model building for future research.

There is one further comment that deserve to be mentioned. In [5], it is shown that both for scaling and presudo-scaling theories with unbroken rotational symmetry in the $x,\ y$ plane, the resistivity and the AC conductivity have the following temperature and frequency dependence at low temperature as $\rho \sim T^{\nu_1}$, and $\sigma(\omega) \sim \omega^{-\nu_1}$ for $\nu_1 \leq 1$. Now if we demand eq(1) on this result, then it fixes $\nu_1 = 1$, it means, $\sigma(\omega) \sim \omega^{-1}$, which is not what [41] suggests. So, it is natural to think of some more exotic models that are either proposed in [5] or that of discussed in this paper so as to get as close as possible to the experimental results.

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8 Appendix A: Solution to Maxwell system

In this section we shall write down the exact solution to Maxwell's equation of motion in the notation of [14]. Let us start with a system whose dynamics is described by Maxwell's action

$$S = -\frac{1}{4}\int d^{d+1}x \frac{\sqrt{-g}}{g_{YM}^2} F_{MN}F^{MN}, \tag{133}$$

with coordinate dependent coupling, g_{YM}, whose explicit dependence we do not specify. Also assume that the $d+1$ dimensional spacetime possessing the symmetries like rotation in $d-1$ dimensional space, time and spatial translations has a structure like

$$ds_{d+1}^2 = -h(r)d\tau^2 + 2d\tau dr + e^{2t(r)}\delta_{ij}dx^i dx^j, \tag{134}$$

where the radial coordinate can have a range, $r_h \leq r \leq r_c$, with r_h denote the horizon of a black hole and r_c is the upper cutoff, which is the UV. The non-vanishing components of the field strength's are $F_{\tau r},\ F_{\tau i},\ F_{ij},\ F_{ir}$.

In terms which the equations of motion are

$$\begin{aligned}
&e^{2t}\partial_\tau F_{\tau r} + \partial^i F_{\tau i} = h\partial^i F_{ir},\\
&\partial_\tau F_{ir} + [(2-p)t' - \phi']F_{\tau i} - \partial_r F_{\tau i} + h' F_{ir} = -h\partial_r F_{ir} + h[(2-p)t' - \phi']F_{ir} + e^{-2t}\partial^j F_{ji},\\
&\partial_r F_{\tau r} + (pt' + \phi')F_{\tau r} + e^{-2t}\partial^i F_{ir} = 0,
\end{aligned} \tag{135}$$

where we have used $1/g_{YM}^2 = e^{\phi(r)}$ and in the derivatives the indices $i,\ j, \cdots$ are raised using δ^{ij} i.e. $\partial^i = \delta^{ij}\partial_j$. The normalization of the coupling is assumed to be $\phi(r_h) = 0$. We shall solve these equations of motion along with the Bianchi identities

$$\begin{aligned}
&\partial_\tau F_{ri} + \partial_r F_{i\tau} + \partial_i F_{\tau r} = 0,\\
&\partial_\tau F_{ij} + \partial_i F_{j\tau} + \partial_j F_{\tau i} = 0,\\
&\partial_r F_{ij} + \partial_i F_{jr} + \partial_j F_{ri} = 0,
\end{aligned} \tag{136}$$

with the in falling boundary condition at the horizon, which means the momentum flux tangent to the horizon vanishes i.e. $T_{rr}(r_h) = 0$ [14], which means $F_{ir}(r_h) = 0$.

The current and charge density at the horizon are

$$J_i(\tau,\ x_i, r_h) = F_{i\tau}(\tau,\ x_i, r_h) \quad q(\tau,\ x_i, r_h) = F_{r\tau}(\tau,\ x_i, r_h), \tag{137}$$

which obey the continuity equation $\partial_\tau q + \partial^i J_i = 0$, courtesy the first equation of eq(135) after setting the condition $t(r_h) = 0$.

8.1 Exact Solutions

In this subsection, we shall find the exact solution to the Maxwell system first for $3+1$ dimensional space time and then for any spacetime.

3+1 dimension:

Let us denote the spacetime coordinate as $\tau,\ x,\ y$ and r. The solution for which the coupling is constant i.e. $\phi' = 0$ with a non-trivial electric field and a constant magnetic field

$$F_{xr} = 0, \quad F_{yr} = 0, \quad F_{r\tau} = 0, \quad F_{y\tau} = E_y(\tau), \quad F_{xy} = \text{constant} \equiv B, \quad F_{x\tau} = E_x(\tau), \tag{138}$$

for some functions $E_x(\tau)$ and $E_y(\tau)$ whose functional form is not fixed by the equations of motion or the Bianchi identity.

For $p = 2$, there exists another exact solution for which the the coupling is constant i.e. $\phi' = 0$ and the rest of the components of field strength are

$$F_{xr} = 0, \quad F_{yr} = 0, \quad F_{r\tau} = q\ e^{2t(r)}, \quad F_{y\tau} = E_y(\tau), \quad F_{xy} = \text{constant} \equiv B, \quad F_{x\tau} = E_x(\tau), \tag{139}$$

where q is a constant and the functions $E_x(\tau)$ and $E_y(\tau)$ whose functional form is not fixed by the equations of motion or the Bianchi identity.

Any arbitrary dimension:

There exists exact solution to the Maxwell system in any arbitrary dimension but unfortunately with zero electric field. In fact all the components of field strength vanishes except

$$F_{r\tau} = qe^{-pt(r)-\phi(r)}, \tag{140}$$

where q is a constant. One can find another solution with a non-trivial electric field provided the inverse coupling goes as

$$1/g^2_{YM} = e^{-(p-2)t(r)+\text{constant}}, \tag{141}$$

with $F_{r\tau} = qe^{-2t(r)}$ and the other non-vanishing component to field strength is

$$F_{i\tau} = E_i(\tau) \tag{142}$$

for some functions $E_i(\tau)$, again whose functional form is not fixed by the equations of motion or the Bianchi identity.

9 Appendix B: Solution to DBI action

Looking at the existence of an exact solution to Maxwell system for $3+1$ dimensions suggest there should be an exact solution to the non-linearly generalized Maxwell system that is the DBI action. The DBI action is

$$S_{DBI} = -T\int e^{-\phi}\sqrt{-det([g]_{ab}+F_{ab})} \equiv -T\int e^{-\phi}\sqrt{-det(M_{ab})}, \tag{143}$$

where [ ] is used to denote the pull back of the bulk metric onto the world volume of the brane. Let us assume the following structure to metric and U(1) gauge field strength

$$\begin{aligned} ds_4^2 &= -h(r)d\tau^2 + 2d\tau dr + e^{2t(r)}(dx^2+dy^2), \\ F_2 &= F_{r\tau}dr\wedge d\tau + F_{x\tau}dx\wedge d\tau + F_{y\tau}dy\wedge d\tau + F_{xy}dx\wedge dy + F_{xr}dx\wedge dr + F_{yr}dy\wedge dr \end{aligned} \tag{144}$$

The equation of motion to gauge field and the current associated to it are

$$\partial_K[T\ e^{-\phi}\sqrt{-det(M_{AB})}\quad \theta^{KL}] = 0, \quad J^\mu = -T\ e^{-\phi}\sqrt{-det(M_{AB})}\quad \theta^{r\mu}, \tag{145}$$

where the indices $N,\ K,\ L$ etc run over the entire bulk spacetime whereas $\mu,\ \nu,\ \rho$ etc run over only the field theory directions, $\tau,\ x,\ y$. The function $\theta^{KL} = \frac{M^{KL}-M^{LK}}{2}$ and $M^{KL}M_{LP} = \delta^K_P$. The explicit form of the spatial components of the current are

$$\begin{aligned} \sqrt{-det(M_{AB})}\quad J^x &= -T\ e^{-\phi}\Big[F_{y\tau}(F_{r\tau}F_{xy} + F_{x\tau}F_{yr} - F_{xr}F_{y\tau}) + e^{2t}(F_{x\tau} + hF_{xr})\Big], \\ \sqrt{-det(M_{AB})}\quad J^y &= -T\ e^{-\phi}\Big[F_{x\tau}(-F_{r\tau}F_{xy} + F_{y\tau}F_{xr} - F_{yr}F_{x\tau}) + e^{2t}(F_{y\tau} + hF_{yr})\Big] \end{aligned} \tag{146}$$

Let us assume that the non-vanishing components of field strength are $F_{x\tau},\ F_{y\tau}$ and F_{xy} following the previous example, in which case there occurs a lot of simplification to both the currents and equations of motion

$$\begin{aligned} & J^x = -T\ e^{-\phi}\frac{F_{x\tau}}{\sqrt{1+e^{-4t}F_{xy}^2}}, \quad J^y = -T\ e^{-\phi}\frac{F_{y\tau}}{\sqrt{1+e^{-4t}F_{xy}^2}}, \\ & \partial_y\Big[T\frac{e^{-\phi}\ F_{xy}}{\sqrt{e^{4t}+F_{xy}^2}}\Big] + \partial_r\Big[T\frac{e^{-\phi+2t}\ F_{x\tau}}{\sqrt{e^{4t}+F_{xy}^2}}\Big] = 0, \quad \partial_x\Big[-T\frac{e^{-\phi}\ F_{xy}}{\sqrt{e^{4t}+F_{xy}^2}}\Big] + \partial_r\Big[T\frac{e^{-\phi+2t}\ F_{y\tau}}{\sqrt{e^{4t}+F_{xy}^2}}\Big] = 0, \\ & \partial_x\Big[Te^{-\phi+2t}\frac{F_{x\tau}}{\sqrt{e^{4t}+F_{xy}^2}}\Big] + \partial_y\Big[Te^{-\phi+2t}\frac{F_{y\tau}}{\sqrt{e^{4t}+F_{xy}^2}}\Big] = 0 \end{aligned} \tag{147}$$

and the solution for $\phi = \text{constant} = \phi_0$ becomes

$$F_{y\tau} = E_y(\tau,\ r), \quad F_{xy} = \text{constant} \equiv B, \quad F_{x\tau} = E_x(\tau,\ r), \tag{148}$$

for some functions $E_x(\tau,\ r)$ and $E_y(\tau,\ r)$, whose functional form is

$$E_x(\tau,\ r) = f_2(\tau)e^{-2t}\sqrt{e^{4t}+B^2}, \quad E_y(\tau,\ r) = f_3(\tau)e^{-2t}\sqrt{e^{4t}+B^2} \tag{149}$$

determined in terms of two unknown functions $f_2(\tau)$ and $f_3(\tau)$. The Bianchi identity sets the condition on $F_{x\tau}$ and $F_{y\tau}$ that these components should not depend on r and can happen only when $B = 0$. This solution indeed is an exact solution to the complete equation of motion with only non-vanishing components of field strength are $F_{x\tau}$ and $F_{y\tau}$ [12].

There exists another exact solution but unfortunately with zero electric field and the non-vanishing components to field strength reads

$$F_{xy} = B = \text{constant}, \quad F_{r\tau} = \frac{f_1}{\sqrt{f_1^2+e^{4t}+B^2}}, \tag{150}$$

where f_1 is a constant.

10 Appendix C: Energy Minimization

After extremizing eq(26), it follows that the extremum occurs when the following equation is satisfied

$$\begin{aligned}&\left[g'_{rr}(g_{tt}g_{xx}-E^2)+g_{rr}(g'_{tt}g_{xx}+g_{tt}g'_{xx})\right]\left[N^2(\prod g_{y_a y_a})-\frac{c^2}{g_{tt}}\right]+\\&\left[g_{rr}(g_{tt}g_{xx}-E^2)\right]\left[N^2(\prod g_{y_a y_a})'+\frac{c^2 g'_{tt}}{g_{tt}^2}\right]=0.\end{aligned} \tag{151}$$

Let us denote it as r_m which is a function of $(T_H,\ E,\ J^x)$. But recall that vanishing and reality of energy, H_L, implies that it occurs at a scale $r_\star$, which is a function of T_H and E only. So one can ask the question: Can $r_\star$ be same as r_m i.e. $r_\star(T_H,\ E) = r_m(T_H,\ E,\ J^x)$? The answer to this question is: It can happen only if the current J^x is a function of T_H and E. If we take the case as in eq(24) then it gives us a solution to eq(151). In fact for this solution the energy extremization at $r_\star$ again is in the indeterminate form i.e $\left(\frac{dH_L}{dr}\right)_{r_\star} = \frac{0}{0}$ because in this case both H_L and the numerator of $\frac{dH_L}{dr}$ vanishes. So we shall take the physical reason of choosing a scale $r_\star$ is the condition of reality and vanishing of energy, H_L. In general, a priori, it is not clear what other value of r one should choose so as to find the current as a function of temperature and electric field that solves eq(151) for which $r_\star = r_m$.

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