PROCEEDINGS OF SCIENCE

Large-N Wilsonian beta function in $SU(N)$ Yang-Mills theory by localization on the fixed points of a semigroup contracting the functional measure

Marco Bochicchio*
INFN-Roma1 and SNS-Pisa
Dipartimento di Fisica, Universita' di Roma 'La Sapienza'
Piazzale Aldo Moro 2 , 00185 Roma
E-mail: marco.bochicchio@roma1.infn.it

In a certain (non-commutative) version of large-N $SU(N)$ Yang-Mills theory there are special Wilson loops, called twistor Wilson loops for geometrical reasons, whose v.e.v. is independent on the parameter λ that occurs in their operator definition. There is a semigroup that acts on the parameter λ by rescaling and on the functional measure, resolved into anti-selfdual orbits by a non-supersymmetric version of the Nicolai map, by contracting the support of the measure. As a consequence the twistor Wilson loops are localized on the fixed points of the semigroup of contractions. This localization is a non-supersymmetric analogue of the localization that occurs in the Nekrasov partition function of the $\mathcal{N}=2$ $SUSY$ YM theory on the fixed points of a certain torus action on the moduli space of (non-commutative) instantons. One main consequence of the localization in the large-N YM case, as in the $\mathcal{N}=2$ $SUSY$ YM case, is that the beta function of the Wilsonian coupling constant in the anti-selfdual variables is one-loop exact. Consequently the large-N YM canonical beta function has a $NSVZ$ form that reproduces the first two universal perturbative coefficients.

The XXVIII International Symposium on Lattice Field Theory, Lattice2010
June 14-19, 2010
Villasimius, Italy

*Speaker.

1. Introduction

Localization in quantum field theory is a procedure by which certain (not necessarily gaussian) functional integrals are computed exactly by the saddle-point method.

As a mathematical theory localization was discovered in the finite dimensional setting of integrals over a compact manifold without boundary by Duistermaat-Heckman [1]. In the Duistermaat-Heckman approach there is a torus action on a compact symplectic manifold and the integral of the exponential of the hamiltonian for the torus action with respect to the symplectic measure is localized exactly on the fixed points of the torus action. A cohomological interpretation was given by Atiyah-Bott [2] and the infinite dimensional version for one dimensional functional integrals (i.e. for quantum mechanics) was worked out by Bismut [3].

Localization as a technique for computing exactly certain functional integrals of quantum field theory was introduced by Witten [4]. In four-dimensional supersymmetric gauge theories Witten twist of supersymmetry [5] provides the differential, Q, that defines the cohomology needed for localization.

Indeed the integral of a closed form, of the kind $O\exp(-S_{SUSY})$, with $QO = 0$ and $QS_{SUSY} = 0$, where S_{SUSY} is the action of a supersymmetric ($SUSY$) gauge theory, can be deformed by a coboundary, $Q\alpha$, without changing its value:

$$\int O\exp(-S_{SUSY} - tQ\alpha), \tag{1.1}$$

because $Q^2 = 0$ and $\int Q\alpha = 0$. Taking the limit $t \to +\infty$, the integral localizes on the set of critical points of the coboundary. Thus Witten localization in quantum field theory is a cohomology theory in which certain functional integrals are viewed as cohomology classes and they are computed choosing suitable representatives.

A remarkable application of localization is the computation of the partition function of the $\mathcal{N} = 2$ $SUSY$ YM theory, whose logarithm is the prepotential. The prepotential was found independently on localization arguments by Seiberg-Witten [6]. Later Nekrasov [7] reproduced the Seiberg-Witten solution using cohomological localization.

According to Nekrasov in the $\mathcal{N} = 2$ $SUSY$ YM theory the partition function reduces by localization to the evaluation of a sum of finite dimensional integrals over the moduli space of instantons. A remarkable feature of Nekrasov work is that also the finite dimensional integrals over the instantons moduli space can be evaluated by localization methods provided a compactification of the moduli space of instantons is chosen. The compactification is absolutely necessary to assure that the integral of a coboundary vanishes, i.e. to define the integrals over instantons moduli as cohomology classes. Nekrasov choice for the compactification was a deformation of the instantons moduli space that amounts to introduce the moduli space of instantons defined on a non-commutative space-time. In a $SU(N)$ gauge theory on ordinary commutative space-time there is an action of the group $SU(N) \times SO(4)$ on the instantons moduli space. The first factor is the gauge group at infinity and the second factor is the group of Euclidean rotations. The group $SU(N) \times SO(4)$ has a diagonal Cartan subgroup that still acts on the non-commutative deformation of the instantons moduli space. This is a torus action and in this case the finite dimensional theory of Duistermaat-Heckman [1] applies. As a result, according to Nekrasov, in the $\mathcal{N} = 2$ $SUSY$

YM case the partition function reduces to a sum of contributions of (non-commutative) Abelian instantons that are the fixed points of the mentioned torus action and it can be performed explicitly [7].

Nekrasov localization has the interesting consequence that the beta function for the Wilsonian coupling constant of the $\mathcal{N}=2$ *SUSY YM* theory is one-loop exact, a result already known (see for example [9]). Indeed localization implies that the saddle-point approximation is exact and therefore the only sources of divergences are the one-loop functional determinants and the powers of the Pauli-Villars regulator that occurs in the integration over the zero modes, due to the instantons moduli, of the operators in the functional deteminants [1].

It is clear that in the pure $SU(N)$ *YM* theory there is no natural cohomology because of the lack of supersymmetry. However, we may wonder as to whether a different kind of localization holds, perhaps linked to the large-N limit. It is also clear that such localization may exist only for special observables, since this is already the case in the supersymmetric theory.

The aim of this paper is to show that a new kind of localization holds in the large-N limit of the pure *YM* theory. One of the most striking conclusions that follow from this exact localization is that the large-N Wilsonian beta function of the *YM* theory, in certain new variables of anti-selfdual (*ASD*) type that are defined through a non-supersymmetric version [8] of the Nicolai map, is one-loop exact as in *SUSY* gauge theories. By Wilsonian coupling we mean the coupling that occurs in the Wilsonian normalization of the action, before rescaling the fields in such a way that the kinetic term is in canonical form and independent of the coupling. This is the rescaling that occurs in perturbation theory, and in this case the gauge coupling is referred to as the canonical coupling. It has been known for some time that the two definitions of the coupling have different beta functions in general (see for example [9]). In absence of supersymmetry (and thus of a natural cohomology) we developed new localization techniques in the pure *YM* theory based on homology theory and on a new holomorphic version of the loop equation for special Wilson loops [8], whose v.e.v. is invariant at quantum level for deformations that are vanishing boundaries in homology (i.e. backtracking arcs).

From the localization of the loop equation it follows that the beta function for the 't Hooft Wilsonian coupling in the *ASD* variables is exactly one-loop at large-N [8]:

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3, \tag{1.2}$$

while the canonical 't Hooft coupling, g, renormalizes according to the following exact large-N beta function [8]:

$$\frac{\partial g}{\partial \log \Lambda} = \frac{-\beta_0 g^3 + \frac{\beta_J}{4} g^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1-\beta_J g^2}, \tag{1.3}$$

with:

$$\beta_0 = \frac{1}{(4\pi)^2}\frac{11}{3}, \beta_J = \frac{4}{(4\pi)^2}, \tag{1.4}$$

[1] In general *SUSY* gauge theories the contribution of non-zero modes in such determinants cancels around an instanton background. Thus only the divergences due to zero modes survive. In the $\mathcal{N}=4$ *SUSY YM* theory also the divergences due to zero modes cancel out and the theory has zero beta function.

where $\frac{\partial \log Z}{\partial \log \Lambda}$ is computed to all orders in the Wilsonian coupling constant, g_W, by:

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{\frac{1}{(4\pi)^2}\frac{10}{3}g_W^2}{1+cg_W^2}, \tag{1.5}$$

with c a scheme dependent arbitrary constant. As a check, once the result for $\frac{\partial \log Z}{\partial \log \Lambda}$ to the lowest order in the canonical coupling,

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{1}{(4\pi)^2}\frac{10}{3}g^2 + ..., \tag{1.6}$$

is inserted in Eq.(1.3), it implies the correct value of the first and second perturbative coefficients of the beta function:

$$\frac{\partial g}{\partial \log \Lambda} = -\beta_0 g^3 + (\frac{\beta_J}{4}\frac{1}{(4\pi)^2}\frac{10}{3} - \beta_0\beta_J)g^5 + ... = -\frac{1}{(4\pi)^2}\frac{11}{3}g^3 - \frac{1}{(4\pi)^4}\frac{34}{3}g^5 + ..., \tag{1.7}$$

which are known to be universal, i.e. scheme independent.

However, the localization of the holomorphic loop equation by homology admits a simpler interpretation directly in terms of the functional integral, as localization on the fixed points of a semigroup contracting the functional measure and leaving invariant the v.e.v. of our special Wilson loops. This new localization has many analogies with Nekrasov localization on the fixed points of the torus action on the moduli of (non-commutative) instantons. Thus it is this simpler theory that we describe in the following section.

2. Localization in pure large-N YM theory on the fixed points of a semigroup of contractions

We define twistor [2] Wilson loops in the YM theory with gauge group $U(N)$ on $R^2 \times R^2_\theta$ with complex coordinates $(z = x_0 + ix_1, \bar{z} = x_0 - ix_1, u = x_2 + ix_3, \bar{u} = x_2 - ix_3)$ and non-commutative parameter θ, satisfying $[\partial_u, \partial_{\bar{u}}] = \theta^{-1}1$, (see [11, 12] for a review of non-commutative gauge theories) as follows:

$$Tr\Psi_\lambda(L_{ww}) = TrP\exp i\int_{L_{ww}} (A_z + \lambda D_u)dz + (A_{\bar{z}} + \lambda^{-1}D_{\bar{u}})d\bar{z}, \tag{2.1}$$

where $D_u = \partial_u + iA_u$ is the covariant derivative along the non-commutative direction u. The plane $(z,\bar{z})$ is instead commutative. The loop, L_{ww}, starts and ends at the marked point, w. The trace in Eq.(2.1) is over the tensor product of the $U(N)$ Lie algebra and of the infinite dimensional Fock space that defines the Hilbert space representation of the non-commutative plane $(u,\bar{u})$ [11, 12]. The limit of infinite non-commutativity in the plane $(u,\bar{u})$ is understood, being equivalent to the large-N limit of the commutative gauge theory [11, 12]. Therefore non-commutativity is for us just a mean to define the large-N limit as well as for Nekrasov it is just a mean to compactify the moduli space of instantons.

[2] The name has a geometrical origin that we do not discuss here. Twistor Wilson loops occur in a twistorial topological string theory related to the large-N limit of the YM theory [10].

It easy to prove that the v.e.v. of the twistor Wilson loops is independent on the parameter λ. The proof is obtained changing variables, rescaling functional derivatives in the usual definition of the functional integral of the non-commutative YM theory [11, 12]. The formal non-commutative integration measure is invariant under such rescaling because of the pairwise cancellation of the powers of λ and λ^{-1}. The non-commutative YM action, proportional to $Tr(-i[D_\alpha, D_\beta] - \theta^{-1}_{\alpha\beta})^2$, is invariant because of rotational invariance in the non-commutative plane. The only possibly dangerous terms couple the non-commutative parameter to the commutator $[D_u, D_{\bar{u}}]$, while all the other mixed terms are zero in our case. But the commutator is invariant under λ-rescaling. Thus the v.e.v. of twistor Wilson loops is λ-independent.

In fact the twistor Wilson loops are trivially 1 at large-θ to all orders in g [3]. We do not give a diagrammatic proof of the triviality in this paper, but we show that indeed triviality holds to the lowest non-trivial order in perturbation theory. We have in the Feynman gauge in the large-θ limit:

$$\begin{aligned} < \int_{L_{ww}} (A_z + \lambda D_u) dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}}) d\bar{z} \int_{L_{ww}} (A_z + \lambda D_u) dz + (A_{\bar{z}} + \lambda^{-1} D_{\bar{u}}) d\bar{z} > \\ = 2 \int_{L_{ww}} dz \int_{L_{ww}} d\bar{z} (< A_z A_{\bar{z}} > + i^2 < A_u A_{\bar{u}} >) \\ = 0. \end{aligned} \tag{2.2}$$

We use the λ-independence to prove that the v.e.v. of twistor Wilson loops is localized on the fixed points of the semigroup rescaling λ. It is convenient to choose our twistor Wilson loops in the adjoint representation and to use the fact that in the large-N limit their v.e.v. factorizes in the product of the v.e.v. of the fundamental representation and of its conjugate. Then, for the factor in the fundamental representation, localization proceeds as follows. We write the YM partition function by means of a non-supersymmetric analogue [8] of the Nicolai map of $\mathcal{N} = 1$ $SUSY$ YM theory [14], introducing in the functional integral the appropriate resolution of identity:

$$1 = \int \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta}) \delta\mu^-_{\alpha\beta}, \tag{2.3}$$

$$Z = \int \exp(-\frac{N8\pi^2}{g^2} Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr_f(\mu^{-2}_{\alpha\beta}) d^4x) \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta}) \delta\mu^-_{\alpha\beta} \delta A_\alpha. \tag{2.4}$$

Here Q is the second Chern class (the topological charge), not to be confused with the differential of the cohomology of the previous section, and $\mu^-_{\alpha\beta}$ is a field of ASD type. The equations of ASD type in the resolution of identity:

$$\begin{aligned} F_{01} - F_{23} = \mu^-_{01} \\ F_{02} - F_{31} = \mu^-_{02} \\ F_{03} - F_{12} = \mu^-_{03}, \end{aligned} \tag{2.5}$$

[3] We use the triviality at the end of this section, to justify the use of Morita equivalent subsequences to define the large-N limit as the limit of infinite non-commutativity with rational (dimensionless) parameter on a non-commutative torus [13]. With this definition of the large-θ limit the local holonomy of B_λ in the adjoint representation turns out to be trivial at the fixed-points. There are other kinds of twistor Wilson loops [8] whose v.e.v. is still λ independent but non-trivial. We do not discuss their localization in this paper.

can be rewritten in the form of a Hitchin system:

$$-iF_A + [D,\bar{D}] - \theta^{-1}1 = \mu^0 = \frac{1}{2}\mu_{01}^{-}$$
$$-i\partial_A \bar{D} = n = \frac{1}{4}(\mu_{02}^{-} + i\mu_{03}^{-})$$
$$-i\bar{\partial}_A D = \bar{n} = \frac{1}{4}(\mu_{02}^{-} - i\mu_{03}^{-}) \tag{2.6}$$

or equivalently in terms of the non-hermitian connection whose holonomy is computed by the twistor Wilson loop with parameter ρ, $B_\rho = A + \rho D + \rho^{-1}\bar{D} = (A_z + \rho D_u)dz + (A_{\bar{z}} + \rho^{-1}D_{\bar{u}})d\bar{z}$:

$$-iF_{B_\rho} - \theta^{-1}1 = \mu_\rho = \mu^0 + \rho^{-1}n - \rho\bar{n}$$
$$-i\partial_A \bar{D} = n$$
$$-i\bar{\partial}_A D = \bar{n}. \tag{2.7}$$

The resolution of identity in the functional integral then reads:

$$1 = \int_{C_\rho} \delta(-iF_{B_\rho} - \mu_\rho - \theta^{-1}1)\delta(-i\partial_A\bar{D} - n)\delta(-i\bar{\partial}_A D - \bar{n})\delta\mu_\rho\delta n\delta\bar{n}, \tag{2.8}$$

where the measure, $\delta\mu_\rho$, along the path, C_ρ, is interpreted in the sense of holomorphic matrix models [15], employed in the study of the chiral ring of $\mathcal{N}=1$ $SUSY$ gauge theories [16, 12]. In particular the resolution of identity is independent, as ρ varies, on the complex path of integration C_ρ.

The complex resolution of identity is convenient for the following argument about localization on fixed points and also to work out the holomorphic loop equation and the associated theory of homological localization [8].

Let us consider the v.e.v. of twistor Wilson loops:

$$\int_{C_\rho} \delta n\delta\bar{n}\delta\mu_\rho \exp(-\frac{N8\pi^2}{g^2}Q - \frac{N4}{g^2}\int Tr_f(\mu^0)^2 + 4Tr_f(n\bar{n})d^4x)$$
$$Tr_f P\exp i\int_{L_{ww}} (A_z + \lambda D_u)dz + (A_{\bar{z}} + \lambda^{-1}D_{\bar{u}})d\bar{z}$$
$$\delta(-iF_{B_\rho} - \mu_\rho - \theta^{-1}1)\delta(-i\partial_A\bar{D} - n)\delta(-i\bar{\partial}_A D - \bar{n})\delta A\delta\bar{A}\delta D\delta\bar{D} \tag{2.9}$$

and let us change variables in the functional integral rescaling the non-commutative covariant derivatives:

$$\int_{C_\rho} \delta n\delta\bar{n}\delta\mu_\rho \exp(-\frac{N8\pi^2}{g^2}Q - \frac{N4}{g^2}\int Tr_f(\mu^0)^2 + 4Tr_f(n\bar{n})d^4x)$$
$$Tr_f P\exp i\int_{L_{ww}} (A_z + D'_u)dz + (A_{\bar{z}} + D'_{\bar{u}})d\bar{z}$$
$$\delta(-iF_A + [D',\bar{D}'] - \theta^{-1}1 - \mu^0 - i\frac{\lambda}{\rho}\partial_A\bar{D}' + i\frac{\rho}{\lambda}\bar{\partial}_A D' - \rho^{-1}n + \rho\bar{n})$$
$$\delta(-i\lambda\partial_A\bar{D}' - n)\delta(-i\lambda^{-1}\bar{\partial}_A D' - \bar{n})\delta A\delta\bar{A}\delta D'\delta\bar{D}'. \tag{2.10}$$

Taking the limit $\lambda \to 0$ inside the functional integral, up to a rescaling anomaly in the functional measure, the last line implies localization on $n = 0$ and $\bar{\partial}_A D' = 0$ while the independence on the path C_ρ in the neighborhood of $\rho = 0$, that we denote, choosing $\rho = \lambda$, C_{0^+}, implies $\partial_A\bar{D}' = 0$ as well. Indeed on C_{0^+} the argument of the remaining delta function contains the combination of a hermitian $-iF_A + [D', \bar{D}'] - \theta^{-1}1 - \mu^0$ and a anti-hermitian $-i\partial_A\bar{D}' + i\bar{\partial}_A D'$ part, whose sum can be zero only if the two terms are zero separately. Therefore $-i\partial_A\bar{D}' + i\bar{\partial}_A D' = 0$ on C_{0^+} and because $\bar{\partial}_A D' = 0$ also $\partial_A\bar{D}' = 0$. Had the hermitian rather than the complex resolution of identity been used, the complex conjugate constraint would have been lost. Introducing the new connection, $B'_\rho = (A_z + \rho D'_u)dz + (A_{\bar{z}} + \rho^{-1}D'_{\bar{u}})d\bar{z}$, that is is a function of the rescaled covariant derivatives, the v.e.v. of the twistor Wilson loop is localized on:

$$\int_{C_{0^+}} \delta n \delta \bar{n} \delta \mu_{0^+} \exp\left(-\frac{N8\pi^2}{g^2}Q - \frac{N4}{g^2}\int Tr_f(\mu_{0^+}\bar{\mu}_{0^+})d^4x\right)$$
$$Tr_f P \exp i\int_{L_{ww}} (A_z + D'_u)dz + (A_{\bar{z}} + D'_{\bar{u}})d\bar{z}$$
$$\delta(-iF_{B'_1} - \theta^{-1}1 - \mu_{0^+})\delta(n)\delta(\partial_A\bar{D}')\delta A\delta\bar{A}\delta D'\delta\bar{D}', \tag{2.11}$$

where now the λ-independent rescaled functional derivatives occur both in the twistor Wilson loop and in the functional measure. The δn integral is performed by means of the delta function, the $\delta\bar{n}$ integral decouples. We notice that the localized density has a holomorphic ambiguity, since we can represent the same measure using a different density making holomorphic transformations without spoiling the localization: $\delta\mu_{0^+} = \frac{\delta\mu_{0^+}}{\delta\mu'_{0^+}}\delta\mu'_{0^+}$. This holomorphic ambiguity (and the associated holomorphic anomaly) can be resolved only through the more refined theory of the homological localization of the loop equation [17] that will not be discussed here.

A delicate point arises about the meaning of the residual holomorphic integral at the fixed points. We show below that there is a dense set in function space where we can reduce in the large-N limit the formal functional integral on a continuos product measure to a product over a lattice of points and at the same time we can give a differential geometric meaning to the corresponding connections [8]. We need such differential geometric meaning to get a structure theorem about the solutions of the localized Hitchin system and for many other related technical issues.

In a neighborhood of the fixed points this dense set corresponds to a differential geometric description of the gauge orbits of the connection as hyper-Kahler reduction on the space of connections [18, 19] (and references therein). Mathematically it corresponds to performing functional integration on parabolic sheaves [20, 18, 19] and physically on a lattice of surface operators (see below) carrying magnetic singularities.

On this dense set generically the connections have moduli, so that the functional measure truly contains the powers of the Pauli-Villars regulator needed to regularize, in the functional determinants that arise integrating the gauge connection, the zero modes associated to the moduli (this is essential to get the correct beta function [8]).

However, anticipating the result, as we reach the fixed points the moduli disappear [4]. This is the interesting case for localization because the functional measure collapses to a sum over fixed

[4] The correct statement is that we can define the large-N limit in such a way that the moduli disappear at the fixed points. This is obtained defining the large-N limit on a non-commutative torus through a diverging sequence of rational (dimensionless) non-commutative parameters [13] and using the Morita equivalence to the large-N limit of the theory on a commutative torus (see [13] for a review of Morita equivalence), for which there is an understanding of the moduli of surface operators by the work of [20, 18, 19].

points rather than to an integral over a manifold. This has an analogue in the localization of the $\mathcal{N} = 2$ *SUSY YM* partition function, where generically instantons have moduli (this is essential to get the correct beta function in that case too), but the instantons at the fixed points of the torus action introduced by Nekrasov have not. The subtle point about localization in both cases, not usually stressed in the literature, is that first we have to regularize and renormalize the functional measure in a neighborhood of the localized locus and only then localize at the fixed points.

Thus at the fixed points the contour integral over μ_{0^+} collapses to a discrete sum. The final result for the localized effective measure, up to a rescaling anomaly that is a local counterterm, and including the holomorphic ambiguity is:

$$[\int_{C_{0^+}} \delta\mu'_{0^+} \frac{\delta\mu_{0^+}}{\delta\mu'_{0^+}} \exp(-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2}\sum_{\alpha\neq\beta}\int Tr_f(\mu^{-2}_{\alpha\beta})d^4x)\delta(F^-_{\alpha\beta}-\mu^-_{\alpha\beta})]_{n=\bar{n}=0}\delta A_\alpha, \quad (2.12)$$

where we have reintroduced the covariant notation. Now the integration on the gauge connection can be explicitly performed to obtain:

$$[\int_{C_{0^+}} \delta\mu'_{0^+} \exp(-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2}\sum_{\alpha\neq\beta}\int Tr_f(\mu^{-2}_{\alpha\beta})d^4x)$$
$$Det'^{-\frac{1}{2}}(-\Delta_A\delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu^-_{\alpha\beta}})(\frac{\Lambda}{2\pi})^{n_b} Det^{\frac{1}{2}}\omega\frac{\delta\mu_{0^+}}{\delta\mu'_{0^+}}]_{n=\bar{n}=0}, \quad (2.13)$$

where, by an abuse of notation, the connection A in the determinants denotes the solution of the equation $[F^-_{\alpha\beta} - \mu^-_{\alpha\beta} = 0]_{n=\bar{n}=0}$. The $'$ superscript requires projecting away from the determinants the zero modes due to gauge invariance, since gauge fixing and the corresponding Faddeev-Popov determinant is understood but not explicitly displayed. $Det^{\frac{1}{2}}\omega$ is the contribution of the possible n_b zero modes due to the (holomorphic) moduli and Λ the corresponding Pauli-Villars regulator:

$$\omega = \int |dz|^2 Tr_f(\frac{\delta B_{1z}}{\delta m_i}\delta m_i \wedge \frac{\delta B_{1\bar{z}}}{\delta m_k}\delta m_k). \quad (2.14)$$

We refer to the functional determinant in Eq.(2.13) as the localization determinant, because it arises localizing the gauge connection on a given level, $\mu^-_{\alpha\beta}$, of the *ASD* curvature. Let us notice the unusual spin term, $iad_{F^-_{\alpha\beta}}$, as opposed to the one that arises in the background field method, $2iad_{F_{\alpha\beta}}$, by expanding the classical action $\frac{N}{2g^2}\sum_{\alpha\neq\beta}\int d^4x Tr_f(F_{\alpha\beta})^2$ around a solution of the equation of motion.

From Eq.(2.13) we read the effective action, Γ. We found convenient to choose the original Wilson loop in the adjoint representation. Since in this case the v.e.v. of the Wilson loop in the large-N limit factorizes in the product of the v.e.v. in the fundamental and in the conjugate representation we get a hermitian version of Γ, up to the irrelevant topological term:

$$\Gamma = [\frac{N4}{g^2}\int Tr_f(\mu_{0^+}\bar{\mu}_{0^+})d^4x - \log Det'^{-\frac{1}{2}}(-\Delta_A\delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu^-_{\alpha\beta}})$$
$$-n_b\log(\frac{\Lambda}{2\pi}) - \log(Det^{\frac{1}{2}}\omega\frac{\delta\mu_{0^+}}{\delta\mu'_{0^+}}) + c.c.]_{n=\bar{n}=0}. \quad (2.15)$$

We would like to give a precise mathematical meaning to the previous formal manipulations of the functional measure by introducing a lattice regularization of the functional integral according to

Wilson. However, the usual Wilsonian regularization on the links of a lattice would spoil the whole geometrical structure. Therefore we introduce a new regularization of the YM functional integral that allows us to keep the differential geometric structure. The differential geometric structure is crucial to get a structure theory of the locus of the fixed points of the functional measure and to understand the zero modes of the determinants, that in turn affect the beta function of the theory [5].

Our new regularization of the YM theory in the large-N limit is performed in two steps. In the first step, that is still formal, the resolution of identity in the Nicolai map is represented as an elliptic fibration of parabolic bundles, as suggested long ago in [18, 19]:

$$1 = \int \delta(F^{-}_{\alpha\beta}(A) - \sum_{p_{(u,\bar{u})}} \mu^{-}_{\alpha\beta}(p_{(u,\bar{u})})\delta^{(2)}(z - z_{p_{(u,\bar{u})}})) \prod_{p_{(u,\bar{u})}} \delta\mu^{-}_{\alpha\beta}(p_{(u,\bar{u})}). \tag{2.16}$$

The term elliptic implies that the base of the fibration is most conveniently chosen as a two-dimensional torus. The torus allows a simple non-commutative deformation and the corresponding non-commutative gauge theory enjoys, for rational values of the dimensionless non-commutative parameter [6], Morita duality to the commutative theory (see for example [13]). Sometimes it may be convenient to choose as base (non-commutative) R^2 or its compactification to S^2. Thus the resolution of identity at this stage is still represented by a functional integral. In the second step, taking advantage of the non-commutative partial large-N Eguchi-Kawai (EK) reduction [11, 12], the large-N functional integral is reduced to an integral over point-like parabolic singularities [8]:

$$1 = \int \delta(F^{-}_{\alpha\beta}(A) - \sum_{p} \mu^{-}_{\alpha\beta}(p)\delta^{(2)}(z - z_p)) \prod_{p} \delta\mu^{-}_{\alpha\beta}(p). \tag{2.17}$$

At this stage the integral over the parabolic residues that occur in the Nicolai map is finite dimensional at finite N. Both the representations are useful.

The aforementioned point-like parabolic singularities of the non-commutative partial large-N EK reduction are daughters of codimension-two singularities of the four-dimensional parent gauge theory. Codimension-two singularities of this kind were introduced many years ago in [17] in the pure YM theory as an "elliptic fibration of parabolic bundles" for the purpose of getting control over the large-N limit of the pure YM theory exploiting the integrability of the Hitchin fibration. Later, in [21], they were introduced in the $\mathcal{N} = 4$ $SUSY$ YM theory for the study of the geometric Langlands correspondence, under the name of "surface operators", and this is now the name universally used in the physical literature. In fact they have been studied originally in the mathematical literature at classical level in [20] as singular instantons.

In a mathematical sense we can think of parabolic bundles in two slightly different ways. Either we can think that they are defined on a space-time with no boundary and with a divisor and a parabolic structure that belong to the space-time. This is the point of view in this paper.

Or we can think that they arise on a space-time with boundaries, where the boundaries are the singular locus of the surface operators. This is the point of view of [21]. In this case the insertion

[5] In $SUSY$ gauge theories the beta function is completely determined by the zero modes, because of cancellations of functional determinants. In the pure large-N Yang-Mills case the beta function has contributions from both zero and non-zero modes associated to the non-supersymmetric Nicolai map. We would like to thank Gabriele Veneziano for a discussion on this point.

[6] The dimensionless ratio: θ divided by the area of the torus.

of a surface operator keeps the finiteness of the action, since the singular locus is not included in the space-time integral that computes the action. This justifies also the term operators, since their occurrence is the analogue of operator insertions[7].

However, our point of view is that the surface operators are dynamical objects and therefore their singular divisor is included in the space-time integral.

As a consequence the classical YM action is quadratically divergent on the singular divisor, with a divergence proportional to the area of each surface operator. We need therefore a way to handle this classical divergence. It turns out that when the codimension-two surface is non-commutative, as in our case, the YM action of the corresponding non-commutative reduced EK model is rescaled by a power of the inverse cut-off ([11] p.6 and [12] p.21) that cancels precisely [8] the quadratic divergence that occurs evaluating the classical YM action on surface operators. Thus the effective action, Γ, of the non-commutative reduced EK theory in the ASD variables is given by:

$$\Gamma = \frac{N8\pi^2}{N_2 g^2} Q + \frac{N}{g^2}\frac{2\pi\theta}{N_2}\int Tr_f(\mu_{01}^{-2} + \mu_{02}^{-2} + \mu_{03}^{-2}) d^2x \\ - \log Det'^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu^-_{\alpha\beta}}), \tag{2.18}$$

where $N_2 = (\frac{\Lambda}{2\pi})^2 Area$ is the infinite factor that arises in the partial EK reduction and the trace in the functional determinants has to be interpreted coherently with the partial EK reduction. In particular the classical part of the reduced YM action is finite on a lattice of surface operators also when we include the singular divisor. For brevity we have displayed in the reduced version of Γ only the contribution of the fundamental representation and we have omitted the contribution of the zero modes and of the holomorphic change of variables.

Once the classical quadratic divergence has been tamed by the EK reduction we need to understand the logarithmic divergences that lead to a non-trivial beta function. Therefore an important issue is the regularization of the logarithmic divergences arising in the localization determinant. We would like to find a regularization for which the loop expansion of the localization determinant satisfies the usual power counting as in the background-field computation of the beta function. This regularization of the effective action is a point-splitting regularization of the propagator in the background of the lattice of surface operators. A typical example is the following one-loop logarithmic contribution to the beta function in Euclidean configuration space:

$$\frac{1}{(4\pi^2)^2}\sum_{p\neq p'}\int d^2u d^2v \frac{NTr(\mu_p \bar{\mu}_{p'})}{(|z_p - z_{p'}|^2 + |u-v|^2)^2}, \tag{2.19}$$

where the sum over p, p' runs over the planar lattice of the parabolic divisors of the surface operators. Had the contribution with $p = p'$ been included, there would appear a quadratic divergence, thus spoiling the usual power counting in higher order terms of the loop expansion. This lattice point-spitting regularization [8] is followed by Epstein-Glaser renormalization in Euclidean configuration space (see [22] for references) and it is a possible starting point of a new constructive approach to the large-N YM theory.

[7]We would like to thank Edward Witten for a discussion about this point.

[8]This regularization has been found during joint work with Arthur Jaffe.

We now describe the moduli space of surface operators in the case of commutative space-time [20] that arises by Morita duality. We are interested only in local moduli, i.e. in the moduli that occur because of the non-trivial holonomy of the gauge connection around the parabolic divisor [20], since to compute the beta function we need only local counterterms. There might exist other moduli, associated for example to holonomies around global cycles [20], but they are irrelevant for the beta function. There is a geometric way of understanding the (local) moduli of the connections that satisfy the self-duality equations with singularities in a neighborhood of the fixed points by means of the hyper-Kahler reduction on the space of connections, due essentially to Hitchin (see [18, 19] and references therein). In our setting the role of the complex Higgs field in the Hitchin equations is played by the covariant derivatives in the $(u,\bar{u})$ plane and therefore the residues of the Higgs field vanish at the fixed points [9]. According to Hitchin the three constraints occurring in Eq.(2.5), with $\mu^0 = \sum_p g_p \lambda_p g_p^{-1} \delta^{(2)}(z - z_{p_{(u,\bar{u})}}), n = \sum_p n_p \delta^{(2)}(z - z_{p_{(u,\bar{u})}}), \bar{n} = \sum_p \bar{n}_p \delta^{(2)}(z - z_{p_{(u,\bar{u})}})$, define the levels of the three hamiltonians, the hermitian and the complex ones respectively, that are needed for the hyper-Kahler reduction. $g_p \lambda_p g_p^{-1}$ is an adjoint orbit with fixed eigenvalues, λ_p (modulo shifts of 2π) , while n_p labels complex coordinates tangent to the adjoint orbit at p. When the residues,$(n_p, \bar{n}_p)$, of the levels of the complex hamiltonians vanish, the adjoint orbits, $g_p \lambda_p g_p^{-1}$, occurring as residues of the levels of the hermitian hamiltonian have to degenerate to a point since the reduction is hyper-Kahler. Another way to get the same result is to observe that, since at the fixed points $n_p = \bar{n}_p = 0$, the Higgs field is smooth on the parabolic divisor. Therefore the only singularities of the connection A may occur at the zeroes of the Higgs field and the residues of the singularities of the connection A are necessarily quantized because they are determined by the zeroes. Thus the only possibility by general principles of the quantization of the magnetic charge in gauge theories (see for example [23]) is that the local holonomy of A [10] is in Z_N and since it is central there are no moduli [20]. Thus there are no moduli at the fixed points. However, if we perform an infinitesimal deformation of the eigenvalues at the fixed points, that preserves their multiplicity, non-trivial moduli of the corresponding adjoint orbit occur.

Then the computation of the beta function for the Wilsonian coupling rapidly reduces to result of [8] obtained by localization of the holomorphc loop equation, that is Eq.(1.2). The beta function for the Wilsonian coupling of the large-N YM theory in the ASD variables is exactly one-loop and coincides with the result of one-loop perturbation theory. However, the one-loop result for $\beta_0 = \frac{1}{(4\pi)^2}\frac{11}{3}$ is obtained as the sum $\frac{1}{(4\pi)^2}(\frac{5}{3}+2)$. The first term is the contribution of the localization determinant due to the non-supersymmetric Nicolai map that gives rise to the non-trivial multiplicative renormalization factor, $Z^{\frac{1}{2}}$, of the ASD variables. The second term is due to zero modes arising from moduli in a neighborhood of the surfaces operators with Z_N holonomy.

At the same time, in the regularization scheme of [8], if the fields are rescaled in canonical form the following relation is obtained between the canonical and the Wilsonian coupling constant:

$$\frac{1}{2g_W^2} = \frac{1}{2g_c^2} + \beta_J \log g_c + \frac{\beta_J}{4} \log Z. \tag{2.20}$$

Differentiating this relation it follows Eq.(1.3) for the canonical beta function [8].

[9] The hyper-Kahler reduction requires that the residues of the Higgs field be nilpotent [18, 19] (and references therein), but this is not restrictive since at the fixed points the residues vanish.

[10] We assume that A is irreducible.

Acknowledgments

We thank the organizers of the Lattice 2010 conference, and in particular Guido Martinelli and Giancarlo Rossi, for the fruitful atmosphere, the interesting discussions and for inviting us to give this talk.

References

[1] J. J. Duistermaat, G. J. Heckman, *Invent. Math.* **69** (1982) 259.

[2] M. F. Atiyah and R. Bott, *Phil. Trans. Roy. Soc. London* **A308** (1982) 523.

[3] J.-M. Bismut, *Comm. Math. Phys.* **103** (1986) 127.

[4] E. Witten, *J. Geom. Phys.* **9** (1992) 303 [hep-th/9204083].

[5] E. Witten, *J. Math. Phys.* **35** (1994) 5101 [hep-th/9403195].

[6] N. Seiberg, E. Witten, *Nucl. Phys.* **B426** (1994) 19; *Erratum-ibid.* **B430** (1994) 485, [hep-th/9407087].

[7] N. A. Nekrasov, proceedings of the *ICM, Beijing*, **3** (2003) 477 [hep-th/0306211].

[8] M. Bochicchio, *JHEP* **0905** (2009) 116 [hep-th/0809.4662].

[9] N. Arkani-Hamed and H. Murayama, *JHEP* **06** (2000) 030 [hep-th/9707133].

[10] M. Bochicchio, *The Yang-Mills String as the A-Model on the Twistor Space of the Complex Two-Dimensional Projective Space with Fluxes and Wilson Loops: the Beta Function*, hep-th/0811.2547.

[11] Yu. M. Makeenko, *The first thirty years of large-N gauge theory*, hep-th/0407028.

[12] H. Kawai, T. Kuroki, T. Morita, *Nucl. Phys.* **B 664** (2003) 185 [hep-th/0303210].

[13] L. Alvarez-Gaume', J. L. F. Barbon, *Nucl. Phys.* **B623** (2002) 165 [hep-th/0109176].

[14] V. De Alfaro, S. Fubini, G. Furlan, G. Veneziano, *Phys. Lett.* **B 142** (1984) 1; *Nucl. Phys.* **B 255** (1985) 399.

[15] C. I. Lazaroiu, *JHEP* **05** (2003) 044 [hep-th/0303008].

[16] R. Dijkgraaf, C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[17] M. Bochicchio, *PoS EPS-HEP2009*:**075** (2009) [hep-th/0910.0776].

[18] M. Bochicchio, *JHEP* **9901** (1999) 006 [hep-th/9810015].

[19] M. Bochicchio, *JHEP* **0306** (2003) 026 [hep-th/0305088].

[20] P. B. Kronheimer, T. S. Mrowka, *Topology* **32** (1993); **34** (1995).

[21] S. Gukov, E. Witten, *Gauge theory, ramification and the geometric Langlands program*, hep-th/0612073.

[22] C. Bergbauer, R. Brunetti, D. Kreimer, *Renormalization and resolution of singularities*, hep-th/0908.0633.

[23] R. Donagi, E. Witten, *Nucl. Phys.* **B460** (1996) 299 [hep-th/9510101].