The virial relation for compact Q-balls in the complex signum-Gordon model

Huawen Wang Hongbo Cheng*
Department of Physics, East China University of Science and Technology,
Shanghai 200237, China
The Shanghai Key Laboratory of Astrophysics,
Shanghai 200234, China

Abstract

In this work the properties of Q-balls in the complex signum-Gordon model in d spatial dimensions is studied. We obtain a general virial relation for this kind of Q-ball in the higher-dimensional spacetime. We compute the energy and radii of Q-ball with V-shaped field potential as a function of spatial dimensionality and a parameter defining the model potential energy density.

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*E-mail address: hbcheng@sh163.net

I. Introduction

The solitons have attracted much attentions from a lot of areas of physics. They are always denoted as classical solutions to nonlinear field equations and have their own structures with finite energy while these solutions are globally regular. The nontopological solitons possess a conserved Noether charge because of a symmetry of the Lagrangian of system in contrast to the topological charge resulting from the spontaneous symmetry breaking in the case of topological defects [1, 2]. Q-balls as nontopological solitons appear in extended localized solutions of models with certain self-interacting complex scalar field [3]. The stability of these nontopological solitons is associated with their charge and that their mass is smaller than the mass of a collection of scalar fields. They are spherically symmetric with the conserved charge Q. The unbroken continuous globally symmetry like $U(1)$ is also possessed. The standard models have the smooth potentials near their absolute minima, so they hold exponential tails and interact with each other. Such solitons are physically universal and have been studied in several subjects like dark matter [4-6], condensed matter physics [7] and so on. The Q-balls have also attracted much attentions in cosmology [8-11]. The Boson stars can be considered as solutions of complex scalar field models couple to gravity [12-15]. Further the Boson stars with self-interactions was considered as candidates of dark matter [16]. The Boson stars can be in the form of Q-balls as flat spacetime limits [17-19]. The general nontopological solitons have also been explored in de Sitter and anti de Sitter spacetimes respectively and the relations between the model and background were shown [20, 21]. The topic about the rotating boson stars that interact with nonrotating boson stars was discussed and the phenomenon that the rotating ones tend to absorb the nonrotating ones as well as the system's stability are presented [19].

There is plenty of motivation to research on the nontopological soliton models in the higher-dimensional spacetimes. The issue of higher-dimensional spacetime can help us to unify the interactions in nature with extra compactified dimensions and to solve the hierarchy problem with an additional warped dimension [22-25]. The signatures of these extra dimensions may be explored in various directions. The properties of high-dimensional world dominate the structure and the stability of various kinds of nontopological soliton models including Q-balls. Studying the Q-balls in the Universe with more than four dimensions certainly help us to probe the background with additional spatial dimensions. It should be pointed out that it is difficult to solve and discuss the field equations in the higher-dimensional spacetimes.

A new kind of interesting nontopological solitons was put forward [26] and some important efforts have been contributed [27]. The so-called complex signum-Gordon model has the field potential with sharp bottom rather than wholely smooth one. Further the V-sharped field potential can be denoted as $U(\phi) = \lambda|\phi|$, where λ is positive constant and ϕ is a complex scalar field. The inverted-conical-shaped potentials are certainly different from the case of other nontopological solitons with potential involving higher order of fields. There exist the Q-balls in the complex signum-Gordon model, which are stationary and whose energy is finite while their scalar field, energy density and charge density approach to the zero outside the ball region. It should be pointed out that these Q-ball solutions are simpler than the others, but they possess the general properties like Q-balls. The boson stars and black holes in scalar electrodynamics with a V-shaped scalar potential are considered [28]. It is interesting that the spacetime around this kind of boson stars consists of a Schwarzschild-type black hole in the interior and a Reissner-Nordstrom-type spacetime in the exterior appears.

It is significant to study the compact Q-balls in the complex signum-Gordon model in higher-dimensional spacetimes by means of virial theorem. The task for exploring the Q-balls with V-shaped field potential in the background with dimensions which are not more than four has been finished well [26-28], but the properties of this kind of Q-balls in the higher-dimensional spacetimes have not studied in detail because it is difficult to find the analytical solution to the field equation in this background. A generalized virial relation for Q-balls involving general smooth potentials like $V(\phi\phi^{+}) = \sum_{n=1}^{\infty} a_n (\phi\phi^{+})^n$ with constants a_n and integer n in d spatial dimensions was obtained and how spatial dimensionality affects some of the key properties of Q-balls such as their energy, minimal charge and size was also declared [29]. We will study the same kind of Q-balls in the high-dimensional spaacetime by means of the method from Ref. [29]. We should utilize the virial theorem to bring about the necessary conditions on the complex signum-Gordon model while find the model's conserved charge Q and energy associated with the dimensionality and the parameters in the potential.

In this paper we discuss the Q-balls in the complex signum-Gordon model in the world with arbitrary dimensions carefully. We present a d-dimensional virial relation for this kind of Q-balls and the relation will recover to be the Derrick's theorem when $Q = 0$. We study these Q-balls to estimate their properties in the case of large charge and radius and small ones respectively. Our results are listed in the end.

II. The varial relation for compact Q-balls in the signum-Gordon model

Here we consider the complex signum-Gordon model with Lagrangian density as follow,

$$\mathcal{L} = \partial_\mu \phi^{+} \partial^\mu \phi - V(\phi\phi^{+}) \tag{1}$$

where ϕ is a complex scalar field in $(d+1)$-dimensional Minkowski spacetime. The index $\mu = 0, 1, 2, \cdots, d$ and the signature is $(+, -, -, \cdots)$. The potential is assumed to be $V(\phi\phi^{+}) = \lambda|\phi|$ with a global minimum at $\phi = 0$ and the coupling constant $\lambda > 0$. As Q-balls this model is nonperturbative excitation about this global vacuum state carring a net particle number named charge Q which is conserved. The condition that the energy of the Q-ball E_Q is smaller than Qm_ϕ with $m_\phi^2 =$

$V''(0)$ will keep its own stability although that is energetically preferred. The Lagrangian of the complex signum-Gordon model has a conserved $U(1)$ symmetry under the global transformation $\phi(x) \longrightarrow e^{i\alpha}\phi(x)$. The associated conserved current should be $j^\mu = -i(\phi^+\partial^\mu\phi - \phi\partial^\mu\phi^+)$ and the corresponding conserved charge is given by $Q = \int d^dxj^0$. We introduce the ansatz for configurations with lowest energy,

$$\phi(x) = \frac{1}{\sqrt{2}}\Phi(\mathbf{x})e^{i\omega t} \tag{2}$$

Here the field $\Phi(\mathbf{x})$ can be taken to be spherically symmetry and $\{\mathbf{x}\}$ represent the spatial components of coordinates. The field equation can be obtained,

$$\nabla_d^2 + \omega^2\Phi - \frac{\lambda}{2}\frac{\Phi}{|\Phi|} = 0 \tag{3}$$

According to Ref. [12], the virial relation is a generalization of Derrick's theorem for Q-balls in a spacetime with arbitrary dimensionality and written as,

$$d\langle V\rangle = (2-d)\langle\frac{1}{2}(\nabla_d\Phi)^2\rangle + \frac{d}{2}\frac{Q^2}{\langle\Phi^2\rangle} \tag{4}$$

Further the absolute lower bound for Q-balls to be a preferred energy state is shown as $Q^2 \geq \frac{2(d-2)}{d}\langle\Phi^2\rangle\langle\frac{1}{2}(\nabla_d\Phi)^2\rangle$ because of $\langle V\rangle \geq 0$. The energy density can be expressed as,

$$\frac{E}{Q} = \omega(1 + \frac{1}{d-2+d\frac{\langle V\rangle}{\langle\frac{1}{2}(\nabla_d\Phi)^2\rangle}}) \leq m_\phi \tag{5}$$

where $\langle\cdots\rangle = \int\cdots d^dx$ and

$$\langle V\rangle = \frac{\lambda}{\sqrt{2}}\int d^dx|\Phi| \tag{6}$$

Only the Q-balls satisfying the condition (5) are stable.

III. The variational approach for large compact Q-balls in the signum-Gordon model

Now in $(d+1)$-dimensional spacetimes we work on the complex-signum-Gordon-type Q-balls which are characterized by large charge and radius. At first we utilize the Coleman issue [3] to explore the large Q-balls. We choose the field to be a step function which is equal to be a constant denoted as Φ_c within the ball and vanishes out side the ball's volume v. The model energy is

$$E = \frac{1}{2}\frac{Q^2}{\Phi_c^2 v} + \frac{\lambda}{\sqrt{2}}|\Phi_c|v \tag{7}$$

Having extremized the expression (7) with respect to the volume v, we obtain the minimum energy and the condition for Q-ball's stability as follow,

$$\frac{E_{\min}}{Q} = 2^{\frac{1}{4}}(\frac{\lambda}{\Phi_c})^{\frac{1}{2}} < m_\phi \tag{8}$$

The complex signum-Gordon Q-ball can exist if the model parameters are adjusted reasonably.

In order to describe the large Q-balls in the complex signum-Gordon model, we introduce the field profile,

$$\Phi(r) = \{ \begin{array}{cc} \Phi_c & r < R \\ \Phi_c e^{-\alpha(r-R)} & r \geq R \end{array} \tag{9}$$

where α is a variational parameter. According to large-Q-ball ansatz (9), the energy of model reads,

$$\begin{aligned} E &= \frac{1}{2}\frac{Q^2}{\langle\Phi^2\rangle} + \frac{1}{2}\langle(\nabla_d\Phi)^2\rangle + \frac{\lambda}{\sqrt{2}}\langle|\Phi|\rangle \\ &= \frac{1}{2}\omega^2\frac{c_d}{d}\Phi_c^2 R^d + \frac{1}{2}\omega^2 c_d\Phi_c^2(d-1)!\sum_{k=0}^{d-1}\frac{1}{k!}\frac{1}{(2\alpha)^{d-k}}R^k \\ &\quad + \frac{1}{2}c_d\Phi_c^2\alpha^2(d-1)!\sum_{k=0}^{d-1}\frac{1}{k!}\frac{1}{(2\alpha)^{d-k}}R^k \\ &\quad + \frac{\lambda}{\sqrt{2}}\frac{c_d}{d}\Phi_c R^d + \frac{\lambda}{\sqrt{2}}c_d\Phi_c(d-1)!\sum_{k=0}^{d-1}\frac{1}{k!}\frac{1}{\alpha^{d-k}}R^k \end{aligned} \tag{10}$$

while the conserved charge is obtained,

$$\begin{aligned} Q &= \omega\int d^dx\Phi^2 \\ &= \omega\frac{c_d}{d}\Phi_c^2 R^d + \omega c_d\Phi_c^2(d-1)!\sum_{k=0}^{d-1}\frac{1}{k!}\frac{1}{(2\alpha)^{d-k}}R^k \end{aligned} \tag{11}$$

which is the same as that of Ref. [29], here

$$c_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{12}$$

We just keep the dominant terms in the expressions for the energy and this approximation is accurate enough for large Q-balls. Combining the expression for Q in Eq. (11), we find

$$E_Q \leq \frac{Q^2 d}{2c_d\Phi_c^2}R^{-d} + \frac{c_d b}{\sqrt{2}d}R^d + (\frac{c_d\alpha\Phi_c^2}{4} + \frac{c_d b}{\sqrt{2}\alpha})R^{d-1} \tag{13}$$

where

$$b = \lambda\Phi_c \tag{14}$$

It is clear that the model energy depends on the variables R and α not belonging to the model parameters. We extremise the energy expression (13) with respect to R and α respectively. By means of performance $\frac{\partial E}{\partial R}|_{R=R_c}=0$ we find the equation that the critical radius R_c satisfies,

$$-\frac{Q^2d^2}{2c_d\Phi_c^2}+\frac{c_db}{\sqrt{2}}R^{2d}+(d-1)(\frac{c_d\Phi_c^2\alpha}{4}+\frac{c_db}{\sqrt{2}\alpha})R^{2d-1}=0 \tag{15}$$

The approximate solution to Eq. (15) is,

$$R_c \approx 2^{-\frac{1}{4d}}(\lambda\Phi_c)^{-\frac{1}{2d}}(\frac{Qd}{c_d\Phi_c})^{\frac{1}{d}}-\frac{d-1}{\sqrt{2}d}(\frac{\alpha\Phi_c}{4\lambda}+\frac{1}{\sqrt{2}\alpha}) \tag{16}$$

and the solution is valid for large Q-balls. We can also impose the condition $\frac{\partial E_Q}{\partial\alpha}|_{\alpha=\alpha_c}=0$ into Eq. (13) to find,

$$\alpha_c=2^{\frac{3}{4}}(\frac{\lambda}{\Phi_c})^{\frac{1}{2}} \tag{17}$$

Substituting the results (16) and (17) into Eq. (13), we obtain the extremised expression for the energy of large Q-ball,

$$\frac{E_Q[\Phi_c]|_{R_c,\alpha_c}}{Q}\approx 2^{\frac{1}{4}}(\frac{\lambda}{\Phi_c})^{\frac{1}{2}}(1+\xi_cQ^{-\frac{1}{d}}) \tag{18}$$

where

$$\xi_c=2^{\frac{1}{4}(\frac{1}{d}-3)}c_d^{\frac{1}{d}}(\frac{d}{\sqrt{\lambda}})^{1-\frac{1}{d}}\Phi_c^{\frac{1}{2}(1+\frac{3}{d})} \tag{19}$$

Having compared the results in Eq. (18) with those in Eq. (8), we also declare that the lower bound on the energy per one particle in large Q-balls is larger than that of Coleman approach. It should be pointed out that $\frac{E_Q[\Phi_c]|_{R_c,\alpha_c}}{Q}$ is finite within the context of large radius and conserved charge. We plot the dependence of the minimum energy per unit charge of complex-signum-Gordon-type Q-ball on the charge for different spatial dimensions in Fig. 1 and show that the minimum energy per unit charge is a decreasing function of the charge. As the charge Q becomes extremely large, $\frac{E_Q[\Phi_c]|_{R_c,\alpha_c}}{Q}$ will approach to the value which is just associated with the model parameters no matter how high the dimensionality is. The Q-ball will possess the larger $\frac{E_Q[\Phi_c]|_{R_c,\alpha_c}}{Q}$ in the higher-dimensional world.

IV. The variational approach for small compact Q-balls in the signum-Gordon model

We start to discuss the small Q-balls in the $(d+1)$-dimensional spacetimes. The small Q-balls with radii $R\geq m_\phi^{-1}$ can not be described well by means of thin-wall approximation. We introduce a Gaussian ansatz in order to consider the small Q-balls in the complex signum-Gordon model,

$$\Phi(\mathbf{x}) = \Phi_c e^{-\frac{r^2}{R^2}} \tag{20}$$

The small Q-ball energy just containing the dominant terms is

$$\begin{aligned} E &= \int d^d x[\frac{1}{2}\omega^2\Phi^2 + \frac{1}{2}(\nabla_d \Phi)^2 + V(\Phi^2)] \\ &= (\frac{\pi}{2})^{\frac{d}{2}}[\frac{1}{2}(\frac{2}{\pi})^d \frac{Q^2}{\Phi_c^2} R^{-d} + \frac{d}{2}\Phi_c^2 R^{d-2} + 2^{\frac{d-2}{2}}\lambda|\Phi_c|R^d] \end{aligned} \tag{21}$$

and it keeps enough validity. By extremizing the expression for the energy with respect to R like $\frac{\partial E_Q}{\partial R}|_{R=R_c} = 0$ we obtain the equation for the critical radius R_c as,

$$R_c^{2d} + \frac{d-2}{d}\frac{d}{2^{\frac{d+1}{2}}}\frac{\Phi_c}{\lambda}R_c^{2d-2} - \frac{2^{\frac{d-1}{2}}}{\pi^d}\frac{Q^2}{\lambda}\frac{1}{|\Phi_c|^3} = 0 \tag{22}$$

Similarly the approximate solution reads,

$$R_c \approx R_0(1 - \frac{d-2}{d}\frac{1}{2^{\frac{d+3}{2}}}\frac{\Phi_c}{\lambda}\frac{1}{R_0^2}) \tag{23}$$

where

$$R_0 = (\frac{2^{\frac{d-1}{2}}}{\pi^d}\frac{Q^2}{\lambda|\Phi_c|^3})^{\frac{1}{2d}} \tag{24}$$

According to the critical radius the energy can be expressed as

$$\frac{E[\Phi_c]|_{R=R_c}}{Q} = 2^{\frac{d+1}{4}}(\frac{\lambda}{\Phi_c})^{\frac{1}{2}}[1 + \frac{\eta_c Q^{-\frac{2}{d}}}{2} - \frac{(d-2)^2}{4d^2}\eta_c^2 Q^{-\frac{4}{d}}] \tag{25}$$

where

$$\eta_c = \frac{d}{2^{\frac{d}{2}-\frac{1}{2d}+1}}\pi\lambda^{\frac{1-d}{d}}\Phi_c^{\frac{3}{d}+1} \tag{26}$$

The stability of small Q-balls can be held according to Eq. (25). The energy over charge is finite while the model parameters can be adjusted. In Fig. 2, we show that the minimum energy per unit charge $\frac{E[\Phi_c]|_{R=R_c}}{Q}$ also decreases as the charge increases in the case of small charge and radius in the background with different spatial dimensions. The higher dimensionality will lead the larger $\frac{E[\Phi_c]|_{R=R_c}}{Q}$. Certainly here the energy over charge is smaller than that in the case of large ones.

V. Summary

In this work we investigate the compact Q-balls in the complex signum-Gordon model in the spacetime with arbitrary dimensionality. We discuss their general properties although it is impossible to find the analytical solution to the field equation in the higher-dimensional background. We find the virial relation for these kinds of Q-balls. We study the model with V-shaped field potential in the case of large Q-balls and small ones. We obtain the approximate analytical expressions to show the Q-balls' energy and radii depending on the spatial dimensionality and the parameters consisting of complex signum-Gordon model. The minimum energy per unit charge of the system decreases to a quantity depending on the system parameters and dimensionality, which are different from those of Ref. [27]. It should be pointed out that the accuracy of the approximate analytical expressions can be acceptable. It is interesting that the compact Q-balls in the complex signum-Gordon model in general flat spacetimes obey the necessary conditions which keep the Q-balls to be stable.

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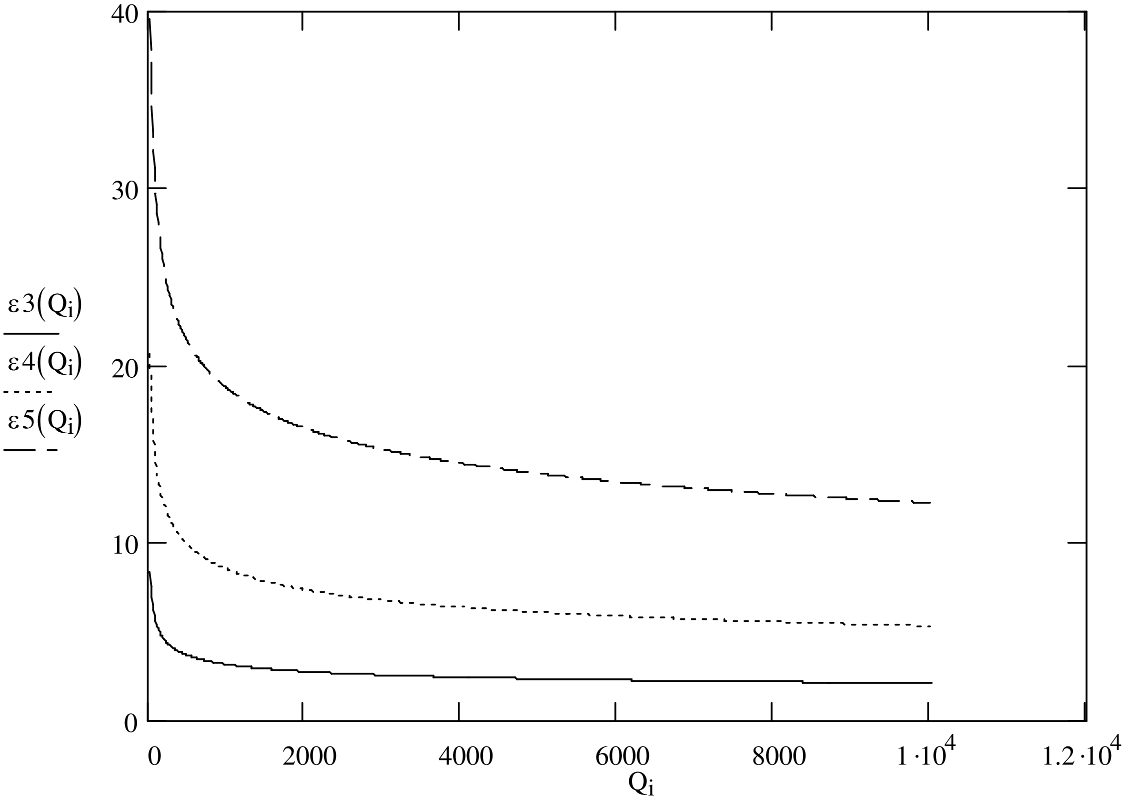

Figure 1: The solid, dot and dashed curves of the minimum energy per unit charge of large Q-balls in the complex signum-Gordon model as functions of charge Q for spatial dimensions $d = 3, 4, 5$ respectively.

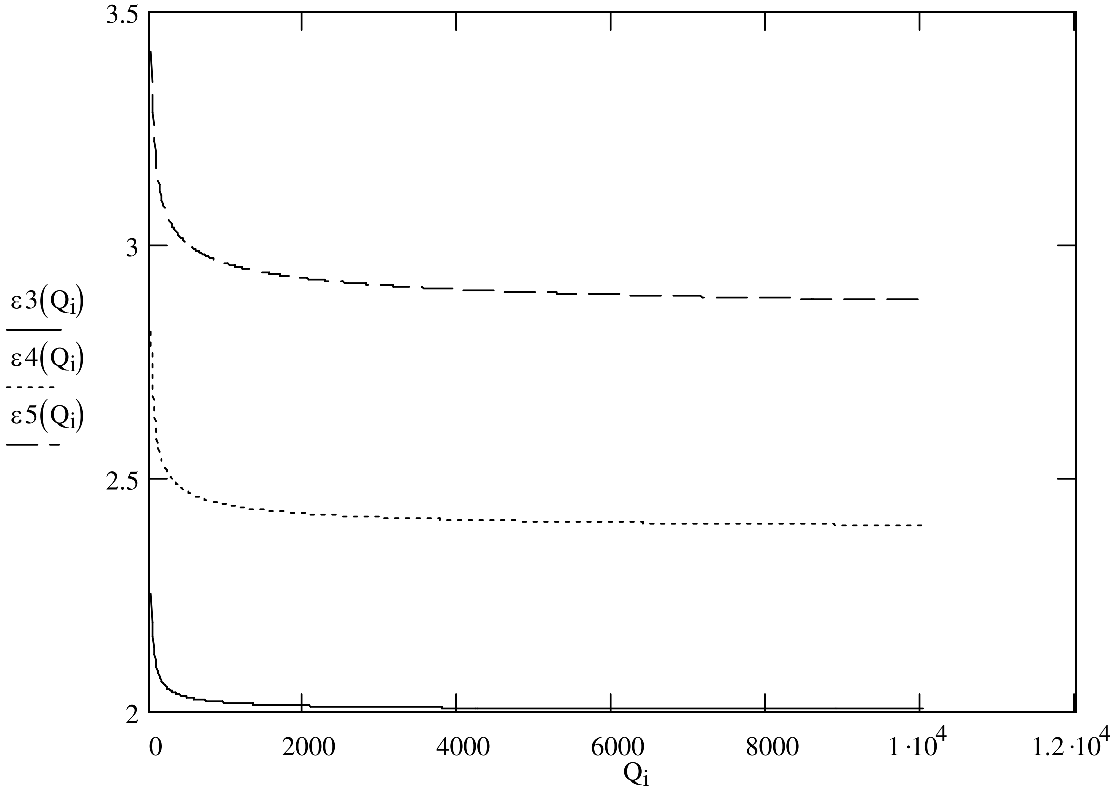

Figure 2: The solid, dot and dashed curves of the minimum energy per unit charge of small Q-balls in the complex signum-Gordon model as functions of charge Q for spatial dimensions $d = 3, 4, 5$ respectively.