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Higher Spins and Open Strings: Quartic Interactions

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Abstract

We analyze quartic gauge-invariant interactions of massless higher spin fields by using the vertex operators constructed in our previous works and computing their four-point amplitudes in superstring theory. The behaviour of the amplitudes is quite different from the standard Veneziano structure, due to their nonstandard ghost coupling. The kinematic part of the quartic interactions of the higher spins is determined by the matter structure of their vertex operators while nonlocality of the interactions is the consequence of the ghost structure of these operators. We compute explicitly the four-point amplitude describing the complete gauge-invariant $1-1-3-3$ quartic interaction (two massless spin 3 particles interacting with two photons) and comment on more general $1-1-s-s$ cases, particularly pointing out the structure of $1-1-5-5$ coupling.

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1. Introduction

Constructing consistent gauge-invariant field theories of interacting higher spins is an important and fascinating problem that has attracted deep interest. Despite many efforts by the leading experts in the field and some remarkable results over recent years (for an incomplete list of references, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [11], [24], [25], [26],[27], [28], [29], [30], [31], [32], [33], [34], [35], [31],[33], [28], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45])

the entire subject is well known to be difficult to approach. In particular, while there has been some progress in formulating free higher spin field theories as well as those with cubic interactions, our understanding of higher order interactions (such as quartic) is still very limited.

There are many reasons why the field theories of spins greater than 2 are of interest and importance. To mention some of them, while it may not seem plausible that higher spin particles could ever be observed in four-dimensional world, objects such as higher spins are likely to be present in higher dimensional physics . Higher spin fields in AdS space are known to be important ingredient in AdS/CFT correspondence [46]; in addition, constructing gauge invariant interactions of higher spins is by itself an interesting and challenging mathematical problem. String theory appears to be a particularly efficient framework to approach the problem of higher spins. One reason for this is that the vertex operators describing the emissions of higher spins by a string, appear very naturally in the massive sector of string theory (although the mass to spin relations for such operators are usually quite rigid, with m^2 roughly proportional to the spin value s.) One could then consider the tensionless limit $\alpha' \to \infty$ in which the higher spin operators formally become massless. There are several difficulties one faces in this approach. Firstly, the space-time fields coupling to the massive operators usually would lack the gauge symmetries necessary to ensure the consistency of the interactions, and it is not clear how to recover these symmetries in the tensionless limit. Secondly, to recover the gauge-invariant interactions of the higher spins from string theory correlators, one generally has to consider the low energy limit of string theory, which of course is different from the tensionless limit. In our previous works [47], [48] we have constructed the open string vertex operators that describe the higher spin fields with spin values from 3 to 9, which are massless at an arbitrary tension due to their nontrivial couplings to the $\beta-\gamma$ ghost system. The explicit expressions for these vertex operators are given by:

$$
\begin{aligned}
V_{s=3}(p) &= H_{a_1a_2a_3}(p)ce^{-3\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}\\
V_{s=4}(p) &= H_{a_1...a_4}(p)c\eta e^{-4\phi}\partial X^{a_1}\partial X^{a_2}\partial\psi^{a_3}\psi^{a_4}e^{i\vec{p}\vec{X}}\\
V_{s=5}(p) &= H_{a_1...a_5}(p)ce^{-4\phi}\partial X^{a_1}...\partial X^{a_3}\partial\psi^{a_4}\psi^{a_5}e^{i\vec{p}\vec{X}}\\
V_{s=6}(p) &= H_{a_1...a_6}(p)c\eta e^{-5\phi}\partial X^{a_1}...\partial X^{a_3}\partial^2\psi^{a_4}\partial\psi^{a_5}\psi^{a_6}e^{i\vec{p}\vec{X}}\\
V_{s=7}(p) &= H_{a_1...a_7}(p)ce^{-5\phi}\partial X^{a_1}...\partial X^{a_4}\partial^2\psi^{a_5}\partial\psi^{a_6}\psi^{a_7}e^{i\vec{p}\vec{X}}\\
V_{s=8}(p) &= H_{a_1...a_8}(p)c\eta e^{-5\phi}\partial X^{a_1}...\partial X^{a_7}\psi^{a_8}e^{i\vec{p}\vec{X}}\\
V_{s=9}(p) &= H_{a_1...a_9}(p)ce^{-5\phi}\partial X^{a_1}...\partial X^{a_8}\psi^{a_9}e^{i\vec{p}\vec{X}}
\end{aligned}
\tag{1}
$$

where X^a and ψ^a are the RNS worldsheet bosons and fermions ($a = 0, ..., d-1$), the ghost fields are bosonized according to

$$
\begin{aligned}
b = e^{-\sigma}&, c = e^{\sigma}\\
\gamma &= e^{\phi-\chi} \equiv e^{\phi}\eta\\
\beta = e^{\chi-\phi}\partial\chi &\equiv \partial\xi e^{-\phi}
\end{aligned}
\tag{2}
$$

The operators (1) are picture inequivalent and are the elements of ghost cohomologies H_{-3},H_{-4} and H_{-5}. All the expressions for the operators (1) are given at their minimal negative superconformal ghost pictures (e.g. -3 for $s = 3$ and -5 for $s = 9$) at which they are annihilated by the direct picture-changing transformation. The symmetric tensors $H_{a_1...a_s}(p)$ describe massless higher spin fields in space-time, with the spin values $3 \leq s \leq 9$. The equations of motion and the gauge symmetry transformations follow from the BRST constraints on the operators (1) [47]. Namely, the on-shell Fierz-Pauli constraints:

$$
\begin{aligned}
H^{a_1}_{a_1a_3...a_S}(p) &= 0\\
p^{a_1}H_{a_1...a_S}(p) &= 0\\
p^2H_{a_1...a_s}(p) &= 0
\end{aligned}
\tag{3}
$$

follow from the invariance condition $\{Q, V_s\} = 0$ where

$$
Q{=}Q_1 + Q_2 + Q_3 \tag{4}
$$

is the BRST operator with

$$Q_1 = \oint \frac{dz}{2i\pi}\{cT - bc\partial c\}$$
$$Q_2 = -\frac{1}{2}\oint \frac{dz}{2i\pi}\gamma\psi_a\partial X^a \qquad (5)$$
$$Q_3 = -\frac{1}{4}\oint \frac{dz}{2i\pi}b\gamma^2$$

The BRST nontriviality conditions, in turn, entail the gauge symmetry transformations for the higher spins [48]. For the symmetric tensors, the transformations are given by

$$H_{a_1...a_s}(p) \to H_{a_1...a_s}(p) + p_{(a_1}\Lambda_{a_2...a_s)}(p) \qquad (6)$$

(where Λ is also traceless and symmetric) as under the the shift of symmetric H-tensors by symmetrized derivatives of Λ the vertex operators (1) are shifted by the terms not contributing to correlation functions. Therefore the gauge invariance of the interaction terms for the higher spins, obtained in the field theory limit of string theory, is ensured by construction, since the structure of these terms is entirely determined by the correlation functions in string theory. For detailed BRST analysis of the operators (1) see [47]; below, we shall briefly review the relation between BRST constraints, equations of motion and gauge symmetries on the example of the $s = 3$ operator (the $s > 3$ cases are treated similarly). The vertex operator for $s = 3$ is given by:

$$V_{s=3}(p) = H_{abc}(p)ce^{-3\phi}\partial X^a\partial X^b\psi^c e^{i\vec{p}\vec{X}} \qquad (7)$$

This operator commutes with Q_2 and Q_3 of the BRST charge. To commute with Q_1 it has to be a dimension 0 primary, i.e. its OPE with stress-energy tensor must not contain singularities stronger than a simple pole. This entails constraints on the rank 3 H-tensor. For general H, the OPE contains singularities up to quartic pole, so to ensure the commutation with Q_1 the coefficients in front of quartic, triple and double poles must vanish separately. This leads to tracelessness, transversality and masslessness conditions respectively, i.e. to the Fierz-Pauli constraints (3) on H. At the same time, the shift (6) shifts the operator (7) by terms not contributing to correlation functions. To see this, consider the general (not necessarily symmetric) tensor $H_{a|bc}$ (note that the form of constraints (3) following from BRST-invariance arguments does not depend on the symmetric propertirs of H and remains the same). Under the shift $H_{a|bc}(p) \to H_{a|bc}(p) +$

$p_c\Lambda_{ab}(p)$ where Λ is symmetric and traceless, the operator (7) is shifted by the BRST-exact part

$$\begin{aligned}\sim ce^{-3\phi}(\vec{p}\vec{\psi})\Lambda_{ab}\partial X^a\partial X^b e^{i\vec{p}\vec{X}} \sim \\ \{Q, ce^{\chi-4\phi}\partial\chi(\vec{p}\vec{\psi})(\vec{\psi}\partial\vec{X})\Lambda_{ab}\partial X^a\partial X^b e^{i\vec{p}\vec{X}}\}\end{aligned} \tag{8}$$

which insertion to any correlator is zero. On the other hand, the tensor $p_c\Lambda_{ab}$ can be decomposed as

$$p_c\Lambda_{ab} = \frac{1}{2}(p_{(c}\Lambda_{ab)} + p_{[c}\Lambda_{a]b}) \tag{9}$$

and insertions of operators corresponding to different Young tableau to correlation functions vanish separately. As a matter of fact, vanishing of $p_{[c}\Lambda_{a]b}$-type insertions to correlators is a just a particular example of a general property of S-matrix elements of $s=3$ vertex operators coupling 3-tensors with hook-like Young diagrams - it can be shown that such operators do not contribute to S-matrices, which is reminiscent of what happens in the frame-like approach [49], [50], [51], [42], [11] where contributions with hook-like symmetries are eliminated by algebraic constraints.

Therefore the correlators are invariant under shifting symmetric tensor H_{abc} by symmetrized derivative of Λ , implying the gauge symmetry (6) in the field theory limit. In order to compute the correlation functions involving the operators (1), one also needs their representations in dual positive ghost pictures. In order to obtain the positive picture presentation for elements of H_{-n-2} (physical operators existing at minimal negative picture $-n-2$ and below;$n=1,2,...$) one has to replace $e^{-(n+2)\phi}$ with $e^{n\phi}$ (without changing the matter part) and perform the homotopy transformation using the K-operator [52]. Namely, if a higher spin vertex at minimal negative picture $-n-2$ has the structure

$$V_{-n-2} = ce^{-(n+2)\phi}F_{\frac{n^2}{2}+n+1}(X,\psi) \tag{10}$$

where $F_{\frac{n^2}{2}+n+1}(X,\psi)$ the is matter primary field of conformal dimension $\frac{n^2}{2}+n+1$, one starts with the operator

$$\oint V_n \equiv \oint dz e^{n\phi}F_{\frac{n^2}{2}+n+1}(X,\psi) \tag{11}$$

This charge commutes with Q_1 since it is a worldsheet integral of dimension 1 and $b-c$ ghost number zero but doesn't commute with Q_2 and Q_3. To make it BRST-invariant, one has to add the correction terms by using the following procedure [53], [52]. We write

$$[Q_{brst}, V_n(z)] = \partial U(z) + W_1(z) + W_2(z) \tag{12}$$

and therefore

$$[Q_{brst}, \oint dzV_n] = \oint dz(W_1(z) + W_2(z)) \tag{13}$$

where

$$\begin{aligned} U(z) &\equiv cV_n(z) \\ [Q_1, V_n] &= \partial U \\ W_1 &= [Q_2, V_n] \\ W_2 &= [Q_3, V_n] \end{aligned} \tag{14}$$

Introduce the dimension 0 K-operator:

$$K(z) = -4ce^{2\chi-2\phi}(z) \equiv \xi\Gamma^{-1}(z) \tag{15}$$

satisfying

$$\{Q_{brst}, K\} = 1 \tag{16}$$

It is easy to check that this operator has a non-singular operator product with W_1:

$$K(z_1)W_1(z_2) \sim (z_1 - z_2)^{2n}Y(z_2) + O((z_1 - z_2)^{2n+1}) \tag{17}$$

where Y is some operator of dimension $2n+1$. Then the complete BRST-invariant operator can be obtained from $\oint dzV_n(z)$ by the following transformation:

$$\begin{aligned} \oint dzV_n(z) \to A_n(w) = \oint dzV_n(z) &+ \frac{1}{(2n)!}\oint dz(z-w)^{2n} : K\partial^{2n}(W_1 + W_2) : (z) \\ &+ \frac{1}{(2n)!}\oint dz\partial_z^{2n+1}[(z-w)^{2n}K(z)]K\{Q_{brst}, U\} \end{aligned} \tag{18}$$

where w is some arbitrary point on the worldsheet. It is then straightforward to check the invariance of A_n by using some partial integration along with the relation (34) as well as the obvious identity

$$\{Q_{brst}, W_1(z) + W_2(z)\} = -\partial(\{Q_{brst}, U(z)\}) \tag{19}$$

Although the invariant operators $A_n(w)$ depend on an arbitrary point w on the worldsheet, this dependence is irrelevant in the correlators since all the w derivatives of A_n are BRST exact - the triviality of the derivatives ensures that there will be no w-dependence in any correlation functions involving A_n. Alternative (yet technically more complicated) method to obtain the positive picture representations for the higher spin operators is to use sequences of Z-transformations combined with picture changing [52] Namely, introduce the Z-operator, transforming the $b-c$ pictures (in particular, mapping integrated vertices to unintegrated) given by [54]

$$Z(w) = b\delta(T)(w) = \oint dz(z-w)^3(bT + 4c\partial\xi\xi e^{-2\phi}T^2)(z) \tag{20}$$

where T is the full stress-energy tensor in RNS theory. The usual picture-changing operator, transforming the $\beta-\gamma$ ghost pictures, is given by $\Gamma(w) =: \delta(\beta)G : (w) =: e^{\phi}G : (w)$. Introduce the *integrated* picture-changing operators $R_n(w)$ according to

$$R_n(w) = Z(w) : \Gamma^n : (w) \tag{21}$$

where $:\Gamma^n:$ is the nth power of the standard picture-changing operator:

$$\begin{aligned} :\Gamma^n : (w) &=: e^{n\phi}\partial^{n-1}G...\partial GG : (w) \\ &\equiv: \partial^{n-1}\delta(\beta)...\partial\delta(\beta)\delta(\beta) : \end{aligned} \tag{22}$$

Then the positive picture representations for the higher spin operators A_n can be obtained from the negative ones V_{-n-2} (1) by the transformation:

$$A_n(w) = (R_2)^{n+1}(w)V_{-n-2}(w) \tag{23}$$

Since both Z and Γ are BRST-invariant and nontrivial, the A_n-operators by construction satisfy the BRST-invariance and non-triviality conditions identical to those satisfied by their negative picture counterparts V_{-2n-2} and therefore lead to the same Pauli-Fierz on-shell conditions (3) and the gauge symmetries (6) for the higher spin fields. For the $s=3$ operator the above procedure gives

$$\begin{aligned} V_{s=3} = ce^{-3\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}H_{a_1a_2a_3}(p) &\to \oint dz V_1 \\ = H_{a_1a_2a_3}(p)\oint e^{\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}} \\ [Q_1, V_1] = \partial U = H_{a_1a_2a_3}(p)\partial(ce^{\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}) \\ [Q_2, V_1] = W_1 = \frac{1}{2}H_{a_1a_2a_3}(p)e^{2\phi-\chi}\{(-(\vec{\psi}\partial\vec{X}) + i(\vec{p}\vec{\psi})P^{(1)}_{\phi-\chi} + i(\vec{p}\partial\vec{\psi}))\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}} \\ +\partial X^{a_1}(\partial^2\psi^{a_2} + 2\partial\psi^{a_2}P^{(1)}_{\phi-\chi})\psi^{a_3} - \partial X^{a_1}\partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3}P^{(1)}_{\phi-\chi})\}e^{i\vec{p}\vec{X}} \\ [Q_3, V_1] = W_2 = -\frac{1}{4}H_{a_1a_2a_3}(p)e^{3\phi-2\chi}P^{(1)}_{2\phi-2\chi-\sigma}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}} \end{aligned} \tag{24}$$

where the conformal weight n polynomials in the derivatives of the ghost fields ϕ,χ,σ are defined according to [53], [52]:

$$P^{(n)}_{f(\phi,\chi,\sigma)} = e^{-f(\phi(z),\chi(z),\sigma(z))}\frac{\partial^n}{\partial z^n}e^{f(\phi(z),\chi(z),\sigma(z))} \tag{25}$$

where f is some linear function in ϕ,χ,σ. For example, $P^{(1)}_{\phi-\chi} = \partial\phi - \partial\chi$, etc. Note that the product (43) is defined in the algebraic sense (not as an operator product).

Accordingly,

$$\begin{aligned}
:K\partial^2 W_1 &:= 4H_{a_1a_2a_3}(p)c\xi\{(-(\vec{\psi}\partial\vec{X}) + i(\vec{p}\vec{\psi})P^{(1)}_{\phi-\chi} + i(\vec{p}\partial\vec{\psi}))\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}\\
&+\partial X^{a_1}(\partial^2\psi^{a_2} + 2\partial\psi^{a_2}P^{(1)}_{\phi-\chi})\psi^{a_3} - \partial X^{a_1}\partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3}P^{(1)}_{\phi-\chi})\}e^{i\vec{p}\vec{X}}\\
:K\partial^2 W_2 &:= H_{a_1a_2a_3}(p)\{-\partial^2(e^{\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}) + P^{(2)}_{2\phi-2\chi-\sigma}e^{\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}\}
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
:\partial^{2n+1}KK\{Q_{brst},U\} &:= -24H_{a_1a_2a_3}(p)\partial cc\partial\xi\xi e^{-\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}\\
:\partial^m KK\{Q_{brst},U\} &:= 0(m < 2n+1)
\end{aligned} \tag{27}$$

and therefore, upon integrating out total derivatives, the complete BRST-invariant expression for the $s=3$ operator at picture 1 is

$$\begin{aligned}
A_{s=3}(w) &= H_{a_1a_2a_3}(p)\oint dz(z-w)^2\{\frac{1}{2}P^{(2)}_{2\phi-2\chi-\sigma}e^{\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}\\
&+2c\xi[(-(\vec{\psi}\partial\vec{X}) + i(\vec{p}\vec{\psi})P^{(1)}_{\phi-\chi} + i(\vec{p}\partial\vec{\psi}))\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}\\
&+\partial X^{a_1}(\partial^2\psi^{a_2} + 2\partial\psi^{a_2}P^{(1)}_{\phi-\chi})\psi^{a_3} - \partial X^{a_1}\partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3}P^{(1)}_{\phi-\chi})]\\
&-12\partial cc\partial\xi\xi e^{-\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}\}e^{i\vec{p}\vec{X}}
\end{aligned} \tag{28}$$

To abbreviate notations for our calculations of the correlation functions in the following sections, it is convenient to write the vertex operator $A_{s=3}$ (46) as a sum

$$A_{s=3} = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \tag{29}$$

where

$$A_0(w) = \frac{1}{2}H_{a_1a_2a_3}(p)\oint dz(z-w)^2 P^{(2)}_{2\phi-2\chi-\sigma}e^{\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}(z) \tag{30}$$

and

$$A_6(w) = -12H_{a_1a_2a_3}(p)\oint dz(z-w)^2\partial cc\partial\xi\xi e^{-\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}\}e^{i\vec{p}\vec{X}}(z) \tag{31}$$

have ghost factors proportional to e^{ϕ} and $\partial c c \partial\xi\xi e^{-\phi}$ respectively and the rest of the terms carry ghost factor proportional to $c\xi$:

$$
\begin{aligned}
A_1(w) &= -2H_{a_1a_2a_3}(p)\oint dz(z-w)^2 c\xi(\vec{\psi}\partial\vec{X})\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}(z)\\
A_2(w) &= 2H_{a_1a_2a_3}(p)\oint dz(z-w)^2 c\xi(\partial^2\psi^{a_2}+2\partial\psi^{a_2}P^{(1)}_{\phi-\chi})\psi^{a_3}e^{i\vec{p}\vec{X}}(z)\\
A_3(w) &= -2H_{a_1a_2a_3}(p)\oint dz(z-w)^2 c\xi\partial X^{a_1}\partial X^{a_2}(\partial^2 X^{a_3}+\partial X^{a_3}P^{(1)}_{\phi-\chi})e^{i\vec{p}\vec{X}}(z)\\
A_4(w) &= 2iH_{a_1a_2a_3}(p)\oint dz(z-w)^2 c\xi(\vec{p}\vec{\psi})P^{(1)}_{\phi-\chi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}(z)\\
A_5(w) &= 2iH_{a_1a_2a_3}(p)\oint dz(z-w)^2 c\xi(\vec{p}\partial\vec{\psi})\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}(z)
\end{aligned}
\tag{32}
$$

We are now prepared to analyze the 4-point $1-1-3-3$ amplitude (leading to the gauge-invariant quartic interaction of spin 3 and spin 1 particles), which will be computed in the next sections.

$1-1-3-3$ Quartic Potential - preliminaries

The goal of next two sections is to compute the 4-point function of two $s=3$ vertex operators with 2-photons, describing the gauge-invariant $1-1-3-3$ interactions in the low energy limit of string theory. The photon vertex operators are the standard ones, and it is convenient to take them unintegrated at superconformal pictures -1 and -2 :

$$
\begin{aligned}
V^{(-1)}_{s=1}(p) &= ce^{-\phi}\psi^m e^{i\vec{p}\vec{X}}A_m(p)\\
V^{(-2)}_{s=1}(p) &= ce^{-2\phi}\partial X^m e^{i\vec{p}\vec{X}}A_m(p)
\end{aligned}
\tag{33}
$$

To cancel the background charges, the operators in the 4-point 3-3-1-1 amplitude must be chosen to have total $b-c$ ghost number $+3$, ϕ-ghost number -2 and χ-ghost number $+1$. Therefore, with the picture choice (33) for the photons it is clear that both of the $s=3$ operators have to be taken at their positive picture $+1$ representation (32). It is furthermore clear that the amplitude $A(1-1-3-3)(p_1,...,p_4)$ will only be contributed by the terms:

$$
\begin{aligned}
A(1-1-3-3)(p_1,...,p_4) &= S(1-1-3-3)(p_1,...,p_4)+(p_3{\leftrightarrow}p_4)\\
S(1-1-3-3) &\equiv < V_{s=1}(p_1)V_{s=1}(p_2)V_{s=3}(p_3)V_{s=3}(p_4) >\\
&= \sum_{j=1}^{5} < V_{s=1}(p_1)V_{s=1}(p_2)A_j(p_3)A_0(p_4) > +(p_3\leftrightarrow p_4)
\end{aligned}
\tag{34}
$$

withA_0, A_j given in (32). Note that, with the picture choice (33) for the $s=1$ operators, the A_6-part of $V_{s=3}$ at positive picture does not contribute to the correlator at all due to the ghost balance constraint. The structure of the amplitude (34) is remarkably different from the standard Veneziano form. Recall that the standard Veneziano expression for 4-point amplitude in string theory arises as a result of 3 out of 4 operators taken unintegrated (multiplied by the c-ghosts) and one integrated (with the $b-c$ ghost number 0), in order to ensure the $b-c$ ghost anomaly cancellation (this choice is related to fixing the $SL(2,R)$ global symmetry with the ghost part of the correlator producing the standard Koba-Nielsen's determinant). The single integration then leads to the Veneziano structures $\sim \frac{\Gamma\Gamma}{\Gamma}$ in the open string case or $\sim \frac{\Gamma\Gamma\Gamma}{\Gamma\Gamma\Gamma}$ for closed strings where Γ are the gamma-functions of Mandelstam variables. With the $s=3$ vertex operators the structure of amplitudes is different, as their ghost couplings (both $b-c$ and $\beta-\gamma$) are nonstandard. For example, the $s=3$ operators at positive pictures exist in the integrated form only (unlike the standard operators that can be taken integrated or unintegrated); at the same time, their integrands contain terms with $b-c$ ghost numbers 1 and 2 (as opposed to the standard integrated vertices which integrands have ghost number zero). As it is clear from (32)-(34) the ghost number balance of the $1-1-3-3$ four-point function requires both of the $s=3$ operators to be taken integrated at positive pictures. Therefore the 4-point amplitude involves the double worldsheet integration and its form is quite different from Veneziano type. In particular, it leads to nonlocalities appearing in the quartic interactions involving the higher spins. Our goal now is to analyze the $< VVA_jA_0 >$-correlators contributing to the 4-point amplitude (34) one by one. The first step is to fix the points u_1, u_2, w_1, w_2 in the amplitude $< V_{s=1}(u_1)V_{s=1}(u_2)A_{s=3}(w_1)A_{s=3}(w_2) >$ by using the remnant gauge symmetry on the worldsheet. Note that, while u_1, u_2 are the actual points of the unintegrated $s=1$ vertices, w_1 and w_2 are the points defining the contours for the integrated $s=3$ vertices at positive pictures (corresponding to the w-points in the expression (32) for the $A_{s=3}$-vertex). In the standard N-point amplitude case (involving 3 unintegrated vertices and $N-3$ integrated) the remnant gauge symmetry is well-known to be given by $SL(2,R)$ subgroup of conformal symmetry, allowing to fix the locations of the unintegrated operators at 3 particular points (with the standard choice $0, 1$ and ∞). In the operator language, the $SL(2,R)$ symmetry simply reflects the fact that, translating an unintegrated vertex operator of the form $\sim cV(z_1)$ to some new point z_2 changes it by BRST-exact terms not contributing to correlation functions (since all the z-derivatives of the unintegrated vertices are BRST-exact, e.g. $\partial(cV)(z) = [Q, V(z)]$

etc. In our case, because of the nonstandard ghost structure of the spin 3 operators, the situation is different and the actual remnant gauge symmetry is bigger than $SL(2,R)$. Namely, all w-derivatives of the $A_{s=3}(w)$ operators are BRST-exact, so the w-points can be chosen arbitrarily. So in case of the 4-point amplitude (34) the remnant gauge symmetry on the worldsheet allows to fix 4 rather than 3 points, i.e. contains an extra generator in addition to the standard $SL(2,R)$ part. As it has been pointed out in [52], the extra gauge symmetries on the worldsheet are closely related to the *global* space-time α-symmetries that are realized nonlinearly and stem from hidden space-time dimensions in string theory. Just like the higher spin vertices, the α-symmetry generators are essentially mixed with the ghosts, being the elements of nontrivial ghost cohomologies $H_{-3} \sim H_1, H_{-4} \sim H_2$ and $H_{-5} \sim H_3$ with each cohomology essentially contributing an extra space-time dimension. In this context, as the higher spin vertex operators and the α-symmetries have similar ghost cohomology structures, the appearance of extra gauge symmetries on the worldsheet is not surprising. Therefore using the $SL(2,R)$ symmetries plus the extra symmetry it is convenient to set

$$\begin{aligned} z_1 = 0, z_2 = \infty \\ w_1 = w_2 = 0 \end{aligned} \tag{35}$$

Such a choice may appear somewhat unusual; indeed, in the standard case the unintegrated vertices are set at three different points (e.g. such as $0, 1, \infty$), since, if one formally fixes two operators at coincident (or infinitely close) points, one faces the normal ordering issue (although the $SL(2,R)$ symmetry in principle allows to fix the operators at 3 infinitely close points) In case of the higher spin operators, however, fixing the w-points is merely related to the choice of their integration contours,thus the gauge choice (35) is appropriate.

$3-3-1-1$ Amplitude: the calculation

It is convenient to start with evaluating the ghost part of the 4-point function, common for all the terms in (34). We get

$$\begin{aligned} F_{gh}(u, z_1, z_2) = lim_{u\to\infty} < ce^{-2\phi}(0)ce^{-\phi}(u)ce^{\chi}(z_1)P^{(2)}_{2\phi-2\chi-\sigma}e^{\phi}(z_2) > \\ = \frac{6uz_1(z_1^2+z_2^2)}{(z_1-z_2)^2} \end{aligned} \tag{36}$$

The first contribution is given by

$$\begin{aligned} S_1(1-1-3-3) =< V_{s=1}(p_1;0)V_{s=1}(p_2;\infty)A_1(p_3;0)A_0(p_4,0) > \\ = A_m(p_1)A_n(p_2)H_{a_2a_3a_4}(p_3)H_{b_1b_2b_3}(p_4)lim_{u\to\infty}\int_0^1 dz_2\int_0^{z_2} dz_1 z_1^2 z_2^2 F_{gh}(u,z_1,z_2) \\ \times < \partial X^m e^{i\vec{p_1}\vec{X}}(0)\psi^n e^{i\vec{p_2}\vec{X}}(u)(\psi_{a_1}\partial X^{a_1})\psi^{a_4}\partial X^{a_2}\partial X^{a_3}e^{i\vec{p_3}\vec{X}}\partial X^{b_1}\partial X^{b_2}\psi^{b_3}e^{i\vec{p_4}\vec{X}} > \end{aligned} \tag{37}$$

The ψ-correlator gives

$$lim_{u\to\infty} < \psi^n(u)\psi^{a_1}\psi^{a_4}(z_1)\psi^{b_3}(z_2) >= \frac{\eta^{na_1}\eta^{a_4b_3} - \eta^{na_4}\eta^{a_1b_3}}{u(z_1 - z_2)} \tag{38}$$

so the ψ-correlator multiplied by $F_{gh}(u, z_1, z_2)$ gives $\frac{6(\eta^{na_1}\eta^{a_4b_3}-\eta^{na_4}\eta^{a_1b_3})z_1(z_1+z_2)^2}{(z_1-z_2)^3}$ with the u-factor cancelled. Due to conformal invariance, it is clear that the remaining X-correlator will contribute terms of the order of u^0 to the overall correlator, with all other terms vanishing on-shell - in other words, no pairings of ∂X's with $e^{\vec{p_2}\vec{X}}$ contribute to the overall 4-point amplitude. For this reason, the relevant contributions from the X-correlator are reduced to the three-point function

$$S_X =< \partial X^m e^{\vec{p_1}\vec{X}}(0)\partial X^{a_1}\partial X^{a_2}\partial X^{a_3}e^{\vec{p_3}\vec{X}}(z_1)\partial X^{b_1}\partial X^{b_2}\partial X^{b_3}e^{\vec{p_4}\vec{X}}(z_2) > \tag{39}$$

This function is not difficult to evaluate. To keep our expressions as compact as possible for the subsequent integrations in z_1, z_2, it is convenient to use the following notations for computing the X-correlators.

Namely, each term contributing to the correlator (39) can be classified in terms of numbers of pairings between ∂X's with the exponents and between each other. That is, let M_1, M_2 be pairing numbers between $\partial X_m(0)$ and $e^{\vec{p_3}\vec{X}}(z_1), e^{\vec{p_4}\vec{X}}(z_2)$ respectively with the obvious constraint $0 \leq M_1, M_2 = 1$ (since there is only one ∂X in the expression for the photon. Next, let N_1, N_2 be pairing numbers of ∂X's in the $s = 3$ operator at z_1 with $e^{\vec{p_1}\vec{X}}(0)$ and $e^{\vec{p_4}\vec{X}}(z_2)$ with $0\leq N_1, N_2\leq 3$. Finally P_1, P_2 satisfying $0\leq P_1, P_2\leq 2$ shall stand for the pairings between ∂X's of the second $s = 3$ vertex at z_2 with $e^{\vec{p_1}\vec{X}}(0)$ and $e^{\vec{p_3}\vec{X}}(z_1)$ It is then straightforward to show that the correlator is contributed by two types of terms. The first type includes the kinematic factors sextic in momentum (accordingly, leading to six derivative interactions in the low energy limit). These terms appear when all ∂X's in the correlator (39) (total number 6) are contracted with the exponents. The second type ivolves the kinematic factors quartic in momentum, appearing when 4 out of 6 ∂X's are contracted with the exponents, while the remaining two are contracted with each other. These terms lead to four derivative quartic interactions in space-time. Given the Pauli-Fierz conditions (3) on the $s = 3$ fields, there are no terms quadratic in momentum or momentum-independent.

Computing the X-correlator (39) and multiplying by the ψ-ghost factor (38), we obtain the six-derivative part of the correlator $S_1^{6-der}(1 - 1 - 3 - 3)$ (37), given by

$$S_1^{6-der}(1-1-3-3) = 72A_m(p_1)A_n(p_2)H_{a_2a_3a_4}(p_3)H_{b_1b_2b_3}(p_4)$$
$$\times(\eta^{na_1}\eta^{a_4b_3} - \eta^{na_4}\eta^{a_1b_3})\sum_{M_1=0}^{1}\sum_{N_1=0}^{3}\sum_{P_1=0}^{2}\frac{(-1)^{P_1}}{N_1!(3-N_1)!P_1!(2-P_1)!}$$
$$\times\prod_{\alpha=1}^{n_1}\prod_{\beta=N_1+1}^{3}\prod_{\gamma=1}^{P_1}\prod_{\lambda=P_1+1}^{3}(ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_1)^{b_\gamma}(ip_3)^{b_\lambda}(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}$$
$$\times\int_0^1 dz_2\int_0^{z_2} dz_1 z_1^2 z_2^2(z_1^2+z_2^2)z_1^{1+\vec{p}_1\vec{p}_3-M_1-N_1}z_2^{\vec{p}_1\vec{p}_4-1+M_1-P_1}(z_1-z_2)^{\vec{p}_3\vec{p}_4-8+N_1+P_1} \tag{40}$$

Few comments should be made to explain our notations here and below. Firstly, regarding the products appearing in (40): for example, $\prod_{\alpha=1}^{N_1}(ip_1)^{a_\alpha}$ stands for the usual product $(ip_1^{a_1})...(ip_1^{a_{N_1}})$ for $1 \leq N_1 \leq 3$, but is set to 1 if $N_1 = 0$. Similarly, $\prod_{\beta=N_1+1}^{3}(ip_4^{a_\beta})$ stands for the product $ip_4^{a_{N_1+1}}...ip_4^{a_3}$ if $N_1 = 0, 1, 2$ but is set to 1 if $N_1 = 3$. Similarly for all other products of that type. The product $(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}$ obviously stands for ip_4^m for $M_1 = 0$ and ip_3^m for $M_1 = 1$ and similarly for all other products of that type. The next step is to perform the integration of (37) in z_1 and z_2. This can be done by using

$$\int_0^1 dz_2\int_0^{z_2} dz_1 z_1^a z_2^b(z_1-z_2)^c = \frac{\Gamma(a+1)\Gamma(c+1)}{(a+b+c+2)\Gamma(a+c+2)} \tag{41}$$

Integrating (37) then gives the following answer for $S_1^{6-der}(1-1-3-3)$:

$$S_1^{6-der}(1-1-3-3) = 72A_m(p_1)A_n(p_2)H_{a_2a_3a_4}(p_3)H_{b_1b_2b_3}(p_4)$$
$$\times(\eta^{na_1}\eta^{a_4b_3} - \eta^{na_4}\eta^{a_1b_3})\sum_{M_1=0}^{1}\sum_{N_1=0}^{3}\sum_{P_1=0}^{2}\frac{(-1)^{P_1}}{N_1!(3-N_1)!P_1!(2-P_1)!}$$
$$\times\prod_{\alpha=1}^{n_1}\prod_{\beta=N_1+1}^{3}\prod_{\gamma=1}^{P_1}\prod_{\lambda=P_1+1}^{3}(ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_1)^{b_\gamma}(ip_3)^{b_\lambda}(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(\vec{p}_3\vec{p}_4-\vec{p}_1\vec{p}_2)^{-1}$$
$$\times\Gamma(\vec{p}_3\vec{p}_4+N_1+P_1-7)[\frac{\Gamma(\vec{p}_1\vec{p}_3-M_1-N_1+6)}{\Gamma(-\vec{p}_2\vec{p}_3-M_1+P_1-1)}+\frac{\Gamma(\vec{p}_1\vec{p}_3-M_1-N_1+4)}{\Gamma(-\vec{p}_2\vec{p}_3-M_1+P_1-3)}] \tag{42}$$

We find that the expression (42) contains the factor $G(p_1, p_2, p_3, p_4) = (\vec{p}_3\vec{p}_4 - \vec{p}_1\vec{p}_2)^{-1}$ (this factor will actually appear in all the terms in the 4-point amplitude (34)). If we are on-shell, the denominator in this expression is zero and the correlator (42) diverges. It must be stressed, however, that terms in the low-energy effective action, appearing in the field theory limit of string theory, are determined by appropriate terms in conformal beta-functions on the worldsheet, rather than by the on-shell correlators. The conformal beta-function, in turn, is determined by the structure constants that are essentially taken off-shell (the on-shell limit then corresponds to the constraint $\beta = 0$). For example, if Φ is a scalar massless space-time field, to obtain linear term in its β-function proportional to $\sim \Delta\Phi = -p^2\Phi$ (corresponding to the free field part of its low energy effective action), one has to take the dilaton's vertex operator initially off-shell (so that $p^2 \neq 0$) and perform the internal normal ordering in this vertex operator leading to the flow $\sim p^2\Phi log\Lambda$ where Λ is the worldsheet cutoff. Similarly, the denominator of $G(p_1, ..., p_4)$ is nonzero in the off-shell limit relevant to the β-function computations, so the corresponding quartic terms in the low-energy effective action include the factor

$$G(p_1, p_2, p_3, p_4) = (p_1^2 + p_2^2 - p_3^2 - p_4^2)^{-1} \tag{43}$$

where we used $(p_1 + p_2)^2 = (p_3 + p_4)^2$. This is the factor reflecting the nonlocality of the quartic couplings of the higher spin fields in the position space. We find that, from the string theory point of view, this nonlocality is the consequence of the specific ghost structure of the higher spin vertex operators, as we already noted above. The calculation of the $4 - derivative$ part of the correlator (37) (quartic in momentum) is similar. The result is given by

$$S_1^{4-der}(1 - 1 - 3 - 3) = D_1 + D_2 + D_3 \tag{44}$$

where

$$\begin{aligned} D_1 = &-72 A_m(p_1) A_n(p_2) H_{a_1 a_2 a_4}(p_3) H_{b_1 b_2 b_3}(p_4)(\eta^{n a_3}\eta^{a_4 b_3} - \eta^{n a_4}\eta^{a_3 b_3})\eta^{m a_3} \\ &\times \sum_{N_1=0}^{2}\sum_{P_1=0}^{2} \frac{(-1)^{P_1}}{N_1!(2-N_1)!P_1!(2-P_1)!} \prod_{\alpha=1}^{n_1}\prod_{\beta=N_1+1}^{3}\prod_{\gamma=1}^{P_1}\prod_{\lambda=P_1+1}^{3} (ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_1)^{b_\gamma}(ip_3)^{b_\lambda} \\ &\times G(p_1, p_2, p_3, p_4)\Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 6)\left[\frac{\Gamma(\vec{p_1}\vec{p_3} - N_1 + 4)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - 2)} + \frac{\Gamma(\vec{p_1}\vec{p_3} - N_1 + 2)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - 4)}\right] \end{aligned} \tag{45}$$

$$\begin{aligned} D_2 = &-72 A_m(p_1) A_n(p_2) H_{a_1 a_2 a_4}(p_3) H_{b_1 b_2 b_3}(p_4)(\eta^{n a_3}\eta^{a_4 b_3} - \eta^{n a_4}\eta^{a_3 b_3})\eta^{m b_2} \\ &\sum_{N_1=0}^{3}\sum_{P_1=0}^{1} \frac{(-1)^{P_1}}{N_1!(3-N_1)!} \prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{3} (ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1} \\ &\times G(p_1, p_2, p_3, p_4)\Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 6)\left[\frac{\Gamma(\vec{p_1}\vec{p_3} - N_1 + 6)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1)} + \frac{\Gamma(\vec{p_1}\vec{p_3} - N_1 + 4)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - 2)}\right] \end{aligned} \tag{46}$$

$$D_3 = -72 A_m(p_1) A_n(p_2) H_{a_1 a_2 a_4}(p_3) H_{b_1 b_2 b_3}(p_4)$$
$$\times(\eta^{na_3}\eta^{a_4 b_3} - \eta^{na_4}\eta^{a_3 b_3})\eta^{mb_2} \sum_{M_1=0}^{1}\sum_{N_1=0}^{2}\sum_{P_1=0}^{1} \frac{(-1)^{P_1}}{N_1!(2-N_1)!}$$
$$\times \prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{3} (ip_1)^{a_\alpha}(ip_4)^{a_\beta}(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1} \quad (47)$$
$$\times G(p_1,p_2,p_3,p_4)\Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 7)[\frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 6)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 1)}$$
$$+\frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 4)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - M_1 - 3)}]$$

This concludes the computation of $S_1(1-1-3-3)$ contribution to the quartic interaction of $1-1-3-3$. The next contribution, $S_2(1-1-3-3)$, is given by

$$S_2(1-1-3-3) =< V_{s=1}(p_1;0) V_{s=1}(p_2;\infty) A_2(p_3;0) A_0(p_4,0) >$$
$$= A_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4) lim_{u\to\infty}\int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^2 z_2^2 \quad (48)$$
$$\times < ce^{-2\phi}\partial X^m e^{i\vec{p_1}\vec{X}}(0) ce^{-\phi}\psi^n e^{i\vec{p_2}\vec{X}}(u) ce^{\chi}\partial X^{a_1}\psi^{a_2}(\partial^2\psi^{a_3}$$
$$+2\partial\psi^{a_3}P^{(1)}_{\phi-\chi})e^{i\vec{p_3}\vec{X}}(z_3) P^{(2)}_{2\phi-2\chi-\sigma} e^{\phi}\partial X^{b_1}\partial X^{b_2}\psi^{b_3} e^{i\vec{p_4}\vec{X}}(z_2) >$$

The $<\psi\times$ghost$>$ factor of this contribution is:

$$lim_{u\to\infty} < ce^{-2\phi}(0) ce^{-\phi}\psi^n(u) ce^{\chi}\psi^{a_2}(\partial^2\psi^{a_3} + 2\partial\psi^{a_3}P^{(1)}_{\phi-\chi})(z_1) e^{\phi} P^{(2)}_{2\phi-2\chi-\sigma}\psi^{b_3}(z_2) >$$
$$= \frac{24\eta^{na_2}\eta^{a_3 b_3} z_1 z_2^3 (z_1^2 + z_2^2)}{(z_1 - z_2)^3} + O(u^{-1}) \quad (49)$$

Computing the $< X >$ part using the same conventions as above and integrating the overall correlator over z_1 and z_2 we obtain

$$S_2(1-1-3-3)$$
$$= 48 A_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4)\eta^{na_2}\eta^{a_3 b_3} \sum_{M_1=0}^{1}\sum_{N_1=0}^{1}\sum_{P_1=0}^{2} \frac{(-1)^{P_1}}{P_1!(2-P_1)!}$$
$$\times \prod_{\alpha=1}^{P_1}\prod_{\beta=P_1+1}^{2} (ip_1^{b_\alpha})(ip_3^{b_\beta}) G(p_1,p_2,p_3,p_4)(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1} \quad (50)$$
$$\times\Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 7)[\frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 5)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - M_1 - 2)} + \frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 3)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - M_1 - 4)}]$$

This contribution is quartic in momentum. The next contribtion is given by

$$
\begin{aligned}
S_3(1-1-3-3) =& < V_{s=1}(p_1;0)V_{s=1}(p_2;\infty)A_2(p_3;0)A_0(p_4,0) > \\
=& A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) lim_{u\to\infty}\int_0^1 dz_2 \int_0^{z_2} dz_1 z_1^2 z_2^2 \\
\times& < ce^{-2\phi}\partial X^m e^{i\vec{p_1}\vec{X}}(0) ce^{-\phi}\psi^n e^{i\vec{p_2}\vec{X}}(u) ce^{\chi}\partial X^{a_1}\partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3} P^{(1)}_{\phi-\chi}) e^{i\vec{p_3}\vec{X}}(z_1) \\
& P^{(2)}_{2\phi-2\chi-\sigma} e^{\phi}\partial X^{b_1}\partial X^{b_2}\psi^{b_3} e^{i\vec{p_4}\vec{X}}(z_2) >
\end{aligned}
\tag{51}
$$

The computation gives:

$$
S_3(1-1-3-3) = S_3^{(1)} + S_3^{(2)} + S_3^{(3)} \tag{52}
$$

where $S_3^{(1)}$ and $S_3^{(2)}$ are the contributions quartic in momentum while $S_3^{(3)} = S_3^{(3)4-der} + S_3^{(3)6-der}$ contains both 4 and 6 derivative terms. These contributions are given by, accordingly:

$$
\begin{aligned}
S_3^{(1)} =& 24 A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) \\
&\times \eta^{nb_3}\eta^{ma_3} \sum_{N_1=0}^{2}\sum_{P_1=0}^{2} \frac{(-1)^{P_1}}{P_1!(2-P_1)!N_1!(2-N_1)!} \\
&\times \prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{2}\prod_{\gamma=1}^{P_1}\prod_{\lambda=P_1+1}^{2} (ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_\gamma})(ip_3^{b_\lambda}) G(p_1,p_2,p_3,p_4) \\
&\times \Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 6)\left[\frac{\Gamma(\vec{p_1}\vec{p_3} - N_1 + 4)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - 2)} + \frac{\Gamma(\vec{p_1}\vec{p_3} - N_1 + 2)}{\Gamma(-\vec{p_2}\vec{p_3} + P_1 - 4)}\right]
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
S_3^{(2)} =& 24 A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{ma_3} \sum_{M_1=0}^{1}\sum_{N_1=0}^{2}\sum_{P_1=0}^{1} \frac{(-1)^{P_1}}{N_1!(2-N_1)!} \\
&\times \prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{2}\prod_{\gamma=1}^{P_1}\prod_{\lambda=P_1+1}^{2} (ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_3^{m})^{M_1}(ip_4^{m})^{1-M_1}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1} G(p_1,p_2,p_3,p_4) \\
&\times \Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 7)\left[\frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 6)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 1)} + \frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 2)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 3)}\right. \\
&\left. + \frac{2\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 5)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 2)} + \frac{2\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 3)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 4)}\right]
\end{aligned}
\tag{54}
$$

The 6-derivative part of $S_3^{(3)}$ is given by

$$
\begin{aligned}
S_3^{(3)6-der} &= 24 A_m(p_1) A_n(p_2) H_{a_1 a_2 a_3}(p_3) H_{b_1 b_2 b_3}(p_4) \\
&\times \eta^{n b_3} \sum_{M_1=0}^{1} \sum_{N_1=0}^{2} \sum_{P_1=0}^{2} \{ \frac{(-1)^{P_1}}{P_1!(2-P_1)!N_1!(2-N_1)!} \\
&\times \prod_{\alpha=1}^{N_1} \prod_{\beta=N_1+1}^{2} \prod_{\gamma=1}^{P_1} \prod_{\lambda=P_1+1}^{2} (ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_\gamma})(ip_3^{b_\lambda}) \\
&\times (ip_3^m)^{M_1} (ip_4^m)^{1-M_1} G(p_1,p_2,p_3,p_4) \{ \\
&\times (ip_1)^{a_3} \Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 6) [\frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 4)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 2)} \\
&+ \frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 2)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 4)}] \\
&+ (2ip_4^{a_3}) \Gamma(\vec{p_3}\vec{p_4} + N_1 + P_1 - 7) [\frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 5)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 2)} \\
&+ \frac{\Gamma(\vec{p_1}\vec{p_3} - M_1 - N_1 + 3)}{\Gamma(-\vec{p_2}\vec{p_3} - M_1 + P_1 - 4)}] \}
\end{aligned}
\tag{55}
$$

The 4-derivative part of $S_3^{(3)}$ is given by

$$S_3^{(3)4-der} = -24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{ma_3}\sum_{N_1=0}^{1}\sum_{P_1=0}^{2}\frac{(-1)^{P_1}}{P_1!(2-P_1)!}$$
$$\times\prod_{\alpha=1}^{P_1}\prod_{\beta=P_1+1}^{2}(ip_1^{b_\alpha})(ip_3^{b_\beta})(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1}G(p_1,p_2,p_3,p_4)$$
$$\times\{ip_1^{a_3}\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-5)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+2)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-3)}+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-5)}]$$
$$+ip_4^{a_3}\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-6)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+3)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-3)}+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+1)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-5)}]\}$$
$$-24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{mb_2}\sum_{N_1=0}^{2}\sum_{P_1=0}^{1}\{\frac{(-1)^{P_1}}{P_1!(2-P_1)!}$$
$$\times\prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{2}(ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1}G(p_1,p_2,p_3,p_4)$$
$$\times\{ip_1^{a_3}\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-5)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+4)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-1)}+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+2)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-3)}$$
$$+ip_4^{a_3}\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-6)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+5)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-1)}+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+3)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-3)}]\}$$
$$-24A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{nb_3}\eta^{a_2b_2}\sum_{M_1=0}^{1}\sum_{N_1=0}^{1}\sum_{P_1=0}^{1}$$
$$\times(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1}G(p_1,p_2,p_3,p_4)$$
$$\times\{ip_1^{a_3}\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-6)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1-M_1+4)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-M_1-2)}+\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+2)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-M_1-4)}$$
$$+ip_4^{a_3}\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-7)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1-M_1+5)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-M_1-2)}+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+3)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-M_1-4)}]\} \tag{56}$$

This concludes the computation of $S_3^{(3)}$ and of $S_3(1-1-3-3)$. The final contribution to the amplitude, $S_4(1-1-3-3)$, is given by

$$S_4(1-1-3-3) =< V_{s=1}(p_1;0)V_{s=1}(p_2;\infty)(A_4+A_5)(p_3;0)A_0(p_4,0) >$$
$$= iA_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)lim_{u\to\infty}\int_0^1 dz_2\int_0^{z_2}dz_1 z_1^2z_2^2$$
$$\times < ce^{-2\phi}\partial X^m e^{i\vec{p_1}\vec{X}}(0)ce^{-\phi}(\psi^n e^{i\vec{p_2}\vec{X}}(u)c\xi(\vec{p}\vec{\psi})P^{(1)}_{\phi-\chi}+\vec{p}\partial\vec{\psi})\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p_3}\vec{X}}(z_1)$$
$$P^{(2)}_{2\phi-2\chi-\sigma}e^{\phi}\partial X^{b_1}\partial X^{b_2}\psi^{b_3}e^{i\vec{p_4}\vec{X}}(z_2)> \tag{57}$$

As previously, it is convenient to split this contribution into 6 and 4 derivative parts:

$$S_4(1-1-3-3) = S_4^{6-der} + S_4^{4-der} \tag{58}$$

The 6-derivative part is computed to give

$$\begin{aligned}
S_4^{6-der} &= -24iA_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4) \\
&\times \sum_{M_1=0}^{1}\sum_{N_1=0}^{2}\sum_{P_1=0}^{2}\frac{(-1)^{P_1}}{N_1!(2-N_1)!P_1!(2-P_1)!} \\
\times \prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{2}\prod_{\gamma=1}^{P_1}\prod_{\lambda=P_1+1}^{2} & (ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_\gamma})(ip_3^{b_\lambda})(ip_3^m)^{M_1}(ip_3^m)^{1-M_1}G(p_1,p_2,p_3,p_4) \\
&\times\{2(\eta^{na_3}p_3^{b_3}+\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-7)[\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+6)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-1)} \\
&+\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+4)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-3)}] \\
&+(2\eta^{na_3}p_3^{b_3}-\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-6)[\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+5)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-1)} \\
&+\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+3)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-3)}]\}
\end{aligned} \tag{59}$$

Finally, the 4-derivative part of S_4^{4-der} contributes

$$S_4^{4-der} = S_4^{(1)4-der} + S_4^{(2)4-der} + S_4^{(3)4-der} \tag{60}$$

where

$$\begin{aligned}
S_4^{(1)4-der} &= 24iA_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\sum_{N_1=0}^{1}\sum_{P_1=0}^{2}\frac{(-1)^{P_1}}{P_1!(2-P_1)!} \\
&\times\prod_{\alpha=1}^{P_1}\prod_{\beta=P_1+1}^{2}(ip_1^{b_\alpha})(ip_3^{b_\beta})(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1}G(p_1,p_2,p_3,p_4) \\
&\times\{2\eta^{a_2m}(\eta^{na_3}p_3^{b_3}+\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-5)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+3)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-2)} \\
&+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+1)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-4)}] \\
&+\eta^{a_2m}(2\eta^{na_3}p_3^{b_3}-\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-6)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+4)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-2)} \\
&+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+2)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-4)}]\}
\end{aligned} \tag{61}$$

$$S_4^{(2)4-der} = 24iA_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\sum_{N_1=0}^{2}\sum_{P_1=0}^{1}\frac{(-1)^{P_1}}{N_1!(2-N_1)!}$$
$$\times\prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{2}(ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1}G(p_1,p_2,p_3,p_4)$$
$$\times\{2\eta^{b_2m}(\eta^{na_3}p_3^{b_3}+\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-5)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+5)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1)} \tag{62}$$
$$+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+3)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-2)}]$$
$$+\eta^{b_2m}(2\eta^{na_3}p_3^{b_3}-\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-6)[\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+6)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1)}$$
$$+\frac{\Gamma(\vec{p_1}\vec{p_3}-N_1+4)}{\Gamma(-\vec{p_2}\vec{p_3}+P_1-2)}]\}$$

and

$$S_4^{(3)4-der} = 24iA_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\sum_{M_1=0}^{1}\sum_{N_1=0}^{1}\sum_{P_1=0}^{1}(-1)^{P_1}$$
$$\times(ip_3^m)^{M_1}(ip_4^m)^{1-M_1}(ip_1^{a_1})^{N_1}(ip_4^{a_1})^{1-N_1}(ip_1^{b_1})^{P_1}(ip_3^{b_1})^{1-P_1}G(p_1,p_2,p_3,p_4)$$
$$\times\{2\eta^{a_2b_2}(\eta^{na_3}p_3^{b_3}+\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-6)[\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+5)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-1)} \tag{63}$$
$$+\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+3)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-3)}]$$
$$+\eta^{a_2b_2}(2\eta^{na_3}p_3^{b_3}-\eta^{a_3b_3}p_3^n)\Gamma(\vec{p_3}\vec{p_4}+N_1+P_1-7)[\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+6)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-1)}$$
$$+\frac{\Gamma(\vec{p_1}\vec{p_3}-M_1-N_1+4)}{\Gamma(-\vec{p_2}\vec{p_3}-M_1+P_1-3)}]\}$$

This concludes the computation of $S_4(1{-}1{-}3{-}3)$ and of $S(1{-}1{-}3{-}3)(p_1,...,p_4)$ in general. The overall 4-point amplitude $A(1-1-3-3)(p_1,...,p_4)$ describing the $1-1-3-3$ quartic interaction is obtained from $S(1{-}1{-}3{-}3)(p_1,...,p_4)$ by adding $A(1{-}1{-}3{-}3)(p_1,...,p_4) = S(1-1-3-3)(p_1,...,p_4)+(p_3\leftrightarrow p_4)$, according to (34).

4-point Amplitude and $1-1-3-3$ Quartic Interaction

Now that our computation of the $1-1-3-3$ point amplitude is complete, the concluding step is to deduce the related quartic interaction from the structure of $A(1-1-3-3)$. The momentum factors of $ip_J(J=1,...4)$ translate into derivatives of the space-time fields $A^m, A^n, H^{a_1a_2a_3}$ and $H^{b_1b_2b_3}$ in the position space, while the common

$f(p_1,p_2,p_3,p_4)$ factor reflects the nonlocality of the interaction. In addition, each of the terms in the amplitude (34) contains γ-function factor with the structure

$$\Xi(M_1,N_1,P_1)\sim\Gamma(\vec{p_3}\vec{p_4}-a(M_1,N_1,P_1))$$
$$\times[\frac{\Gamma(\vec{p_1}\vec{p_3}+b(M_1,N_1,P_1))}{\Gamma(-\vec{p_2}\vec{p_3}-c(M_1,N_1,P_1))}+\frac{\Gamma(\vec{p_1}\vec{p_3}+b(M_1,N_1,P_1)-2)}{\Gamma(-\vec{p_2}\vec{p_3}-c(M_1,N_1,P_1)-2)}] \quad (64)$$

where a,b and c are the numbers appearing in summations over M_1,N_1 and P_1. The γ function factor is of some subtlety. While the numbers a,b and c differ from term to term , it is easy to see that in general $a>0,b>0$ and $b\geq 2$ for each term in the amplitude. For this reason, in the field theory limit $\vec{p_I}\vec{p_J}\to 0$ that we are interested in, $\Xi(M_1,N_1,P_1)$ generally includes singular part, with simple poles in $\vec{p_3}\vec{p_4}$ and $\vec{p_1}\vec{p_3}$ (the latter only appear in terms with $b=2$), as well as the part regular in $\vec{p_I}\vec{p_J}$. The singular part is actually related to the flow of the cubic part of the effective action, rather than to the genuine quartic interaction we are looking for. That is, the singular terms in the γ-function factors are related to two types of exchanges: the first is the $s=1$ field exchange between two $s=3$ vertices, while the second is the $s=3$ field exchange between $s=3$ and $s=1$ operators. These are the exchanges that induce the RG flows on the worldsheet for $s=1$ and $s=3$ fields, resulting in the leading (cubic) order terms in the low energy effective action. Schematically, the β-function of the $s=1$ field in the $s=3$ background is given by $\beta_A\sim-\Delta A+CH^2$ where C are the structure constants appearing in the $1-3-3$ 3-point amplitude (expressed in the position space). This particularly leads to cubic terms of the type $\sim CAH^2$ in the low-energy effective action. At the same time, the low-energy effective equations of motion for the $s=3$ gauge field in the presence of $s=1$ background are given by, in the leading order, $\beta_H{\sim}\Delta H-CAH=0$ which, if substituted into cubic terms, lead to "nonlocal" quartic terms of the type $\sim\frac{C^2A^2H^2}{\Delta}$ which structurally coincide with the contribution of the singular part of the Γ-function factor to the 4-point amplitude. To obtain the genuine quartic $1-1-3-3$ interaction from the 4-point amplitude (34), one has to subtract the singularities from each of the Γ-function factors appearing in the expressions (42)-(63), similarly to the procedure explained in [55]. The Γ-function factors with $b(M_1,N_1,P_1) > 2$ can be expanded in $\vec{p}_I\vec{p}_J$ with the leading order terms given by

$$\begin{aligned} &\Xi(a(M_1,N_1,P_1),b(M_1,N_1,P_1),c(M_1,N_1,P_1)) \\ &= \Gamma(\vec{p_3}\vec{p_4}-a)[\frac{\Gamma(\vec{p_1}\vec{p_3}+b)}{\Gamma(-\vec{p_2}\vec{p_3}-c)}+\frac{\Gamma(\vec{p_1}\vec{p_3}+b-2)}{\Gamma(-\vec{p_2}\vec{p_3}-c-2)}] \\ &\approx \frac{(-1)^{a+c}(b-3)!(c-2)!\vec{p_2}\vec{p_3}}{a!\vec{p_3}\vec{p_4}} \\ \times\{1+(b-2)(b-1)(c-1)c+(\vec{p_1}\vec{p_3})&[(b-2)(b-1)(c-1)cL(b-1)+L(b-3)] \\ &+(\vec{p_2}\vec{p_3})[(b-2)(b-1)(c-1)cL(c)+L(c-2)] \\ &-(\vec{p_3}\vec{p_4})[(b-2)(b-1)(c-1)c+1]+... \end{aligned} \tag{65}$$

Then the related factors in the quartic terms in the low energy effective action are given by replacing each of the $\Xi(a,b,c)$ factors in the amplitude according to

$$\begin{aligned} &\Xi(a,b,c)\rightarrow\tilde{\Xi}(a,b,c) \\ = \Xi(a(M_1,N_1,P_1),b(M_1,N_1,P_1),c(M_1,N_1,P_1)) &- \frac{(-1)^{a+c}(b-3)!(c-2)!\vec{p_2}\vec{p_3}}{a!\vec{p_3}\vec{p_4}} \\ \times\{1+(b-2)(b-1)(c-1)c+\vec{p_1}\vec{p_3}&[(b-2)(b-1)(c-1)cL(b-1)+L(b-3)] \\ &+\vec{p_2}\vec{p_3}[(b-2)(b-1)(c-1)cL(c)+L(c-2)] \\ &-\vec{p_3}\vec{p_4}[(b-2)(b-1)(c-1)c+1] \end{aligned} \tag{66}$$

where

$$L(n)=\sum_{m=1}^{n}\frac{1}{m} \tag{67}$$

Each of $\tilde{\Xi}$ contributes, in the leading order in $\vec{p_I}\vec{p_J}$, the factor given by

$$\tilde{\Xi}(a,b,c)\approx(-1)^{a+c+1}\vec{p_2}\vec{p_3}\frac{(b-3)!(c-2)!}{a!}L(a)[1+(b-2)(b-1)(c-1)c] \tag{68}$$

Similarly, in the special case of $b=2$ the $\Xi\rightarrow\tilde{\Xi}$ replacement for the quartic term in the low energy effective action is given by

$$\begin{aligned} &\Xi(a,2,c)\rightarrow\tilde{\Xi}(a,2,c) \\ &= \Xi(a,2,c)+\frac{(-1)^{a+c+1}a!c!(\vec{p_2}\vec{p_3})}{(\vec{p_3}\vec{p_4})}[1+\vec{p_1}\vec{p_3}+L(c)\vec{p_2}\vec{p_3}] \\ &+(-1)^{a+c}a!(c-2)!(\frac{1}{\vec{p_3}\vec{p_4}}+\frac{1}{\vec{p_1}\vec{p_3}})(1+L(c-2)\vec{p_2}\vec{p_3}) \end{aligned} \tag{69}$$

with the leading order contribution

$$\tilde{\Xi}(a,2,c)\approx(-1)^{a+c+1}a!c!L(a)\vec{p}_2\vec{p}_3+(-1)^{a+c+1}a!(c-2)!(L(a)-2L(c-2)) \qquad (70)$$

to the amplitude. This concludes the evaluation of the gauge-invariant $1-1-3-3$ quartic interaction in the low energy effective action.

$1-1-5-5$ Structure. Conclusion and Discussion

In this paper we have obtained gauge-invariant quartic interaction of massless higher spin fields in string theory approach. Although we have concentrated on $1-1-3-3$ case, with the structure of higher spin vertex operators basic properties of amplitudes discussed in this paper (such as nonlocality and derivative structure of the kinematic part of the amplitude) will also hold for more general $1-1-s-s$ cases. The nonlocality structure of the 4-point amplitude calculated in this paper is the consequence of specific the ghost structure of the vertex operators for the massless $s=3$ fields. In particular, nonstandard ghost coupling of $s=3$ vertices leads to two integrated vertices appearing in the 4-point amplitude (contrary to one out of 4 integrated vertex in the standard Veneziano case) producing the factor that diverges on-shell but leads to nonlocalities in the β-function equations (which essentially are the off-shell equations). As for the local part of the $3-3-1-1$ interaction terms, it is structurally reminiscent of the 3-point $3-3-2$ amplitude on the disc describing *cubic* gravitational couplings of massless spin 3 field, that can be expressed in terms of linearized Weyl tensor [41]. In particular, the minimal number of derivatives in the kinematic part of $1-1-3-3$ is equal to $2\times 3-2=4$, similar to the $3-3-2$ case. While it is known that cubic $s-s-2$ gauge-invariant couplings always contain a minimum of $2s-2$ space-time derivatives [41] it looks plausible that similar derivative rule applies to the kinematic part quartic $1-1-s-s$ couplings as well, as the disc $2-s-s$ amplidude (with two spin s operators on the boundary and the spin 2 operator in the bulk) is structurally similar to special limit ($p_3=p_4$) of the kinematic part of the $1-1-s-s$ point amplitude in open string theory (where p_3 and p_4 are the momenta of the spin s particles). So at least string theory appears to predict that the minimal derivative rule for the quartic $1-1-s-s$ couplings should be similar to the one established for the $2-s-s$ case. This certainly is the case for $s=3$ and it would be interesting to check if this rule also works for spins higher than 3. Conceptually, the calculation performed in this paper for the $1-1-3-3$ case, should be quite similar for $1-1-s-s$ amplitudes with higher values of s as well. In any case, the derivative/momentum structure of the amplitudes is tightly controlled by the ghost structure of the vertex operators and by the overall ghost number balance. For example,the $1-1-5-5$ amplitude $A(1-1-5-5)(p_1,...,p_4)$ is structurally

$$A(1-1-5-5)(p_1,...,p_4) = S(1-1-5-5)(p_1,...,p_4) + (p_3 \leftrightarrow p_4)$$
$$S(1-1-5-5) = < V_{s=1}^{(-2)} V_{s=1}^{(-2)}(0) : \Gamma^{-1} V_{s=1}^{(-2)} : (\infty) A_{s=3}^{(0)}(z_1) A_{s=3}^{(0)}(z_2) > \quad (71)$$

where, as previously $V_{s=1}^{(-2)} = ce^{-2\phi}\partial X^n e^{ipX} A_n(p)$ are unintegrated photon operators at picture -2, $\Gamma^{-1} = -4ce^{\chi-2\phi}\partial\chi$ is the inverse picture changing, so the photon at picture -3 has the overall ghost structure $\sim e^{\chi-4\phi}$, while

$$A_{s=3}^{(0)}(z) \sim H_{a_1...a_5} \oint dw (z-w)^4 e^{2\phi} P^{(4)}_{2\phi-2\chi-\sigma} \partial X^{a_1} \partial X^{a_2} \partial X^{a_3} \partial\psi^{a_4} \psi^{a_5} e^{ipX}(w) \quad (72)$$

Note that, although the full BRST-invariant expression for spin 5 operators contains, apart from $A^{(0)}$ terms with ghost structures $\sim ce^{\chi+\phi}$ and $\sim \partial cce^{2\chi}$, the ghost balance condition only allows the contributions from $A^{(0)}$-part with the ghost structure $\sim e^{2\phi}$ (provided, of course, that the photons are chosen at pictures -2 and -3). Again, we see that, first of all, the amplitude contains a double worldsheet integration (as in the $1-1-3-3$ case) leading to the nonlocality of the interaction. While the computation of the matter/kinematic part of this amplitude is relatively straightforward and similar to the $1-1-3-3$ case described above, the evaluation of the ghost part of this amplitude is quite tedious due to lengthy operator products of the ghost polynomials $P^{(4)}_{2\phi-2\chi-\sigma}$ with the ghost exponents and between themselves. Below we present the expression for the $1-1-5-5$ amplitude up to numerical coefficients which can be fixed by explicit evaluation of these operator products. Evaluating the correlator (71) and integrating in z_1, z_2 we get the answer

$$S_{1-1-5-5} = A_m(p_1)A_n(p_2)H_{a_1a_2a_3}(p_3)H_{b_1b_2b_3}(p_4)\eta^{a_4b_4}\eta^{a_5b_5}\eta^{a_3n}$$
$$\times \sum_{L=0}^{4}\sum_{Q=0}^{2L}\sum_{Q_1=0}^{Q}\sum_{Q_2=0}^{2L-Q}\sum_{R_1=0}^{4-Q_2}\sum_{M_1=0}^{1}\sum_{N_1=0}^{2}\sum_{P_1=0}^{3}\frac{(-1)^{P_1}}{N_1!(2-N_1)!P_1!(3-P_1)!}\alpha_{L,Q,Q_1,Q_2,R_1}$$
$$\times \prod_{\alpha=1}^{N_1}\prod_{\beta=N_1+1}^{2}\prod_{\gamma=1}^{P_1}\prod_{\lambda=P_1+1}^{2}(ip_1^{a_\alpha})(ip_4^{a_\beta})(ip_1^{b_\gamma})(ip_3^{b_\lambda})(ip_3^m)^{M_1}(ip_3^m)^{1-M_1}$$
$$\times G(p_1,p_2,p_3,p_4)\frac{\Gamma(9+Q_1-Q+R_1-M_1-N_1+\vec{p}_1\vec{p}_3)\Gamma(2L-20+N_1+P_1+\vec{p}_3\vec{p}_4)}{\Gamma(P_1-M_1+R_1+Q_1-Q-11-\vec{p}_2\vec{p}_3)}$$
$$+(m \leftrightarrow n, N_1 \leftrightarrow P_1) \times (-1)^{N_1+P_1} \quad (73)$$

where α_{L,Q,Q_1,Q_2,R_1} are the numerical coefficients to be extracted from the ghost OPEs. The overall amplitude $A(1-1-5-5)(p_1,...,p_4)$ is again obtained from $S(1-1-5-5)(p_1,...,p_4)$ by adding $A(1-1-5-5) = S(1-1-5-5) + (p_3 \leftrightarrow p_4)$ according to (71). The kinematic part of this amplitude contains minimum number of 6 space-time derivatives. At the same time, all the Γ-functions in the denominator of (73) are proportional to $\sim (\vec{p}_2\vec{p}_3)^{-1}$ in the field theory limit, for all the values of M_1, P_1, Q_1, R_1 and Q. For this reason, the local part of the amplitude (73) is of at least 8 powers in momentum, again in according to the $2s-2$-rule conjectured above. One has to check explicitly, however, whether this rule holds in each separate case for different values of s.

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