Turk J Math
34 (2010) , 515 – 521.

doi:10.3906/mat-0809-22

On construction of coherent states associated with homogeneous spaces

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Abstract

In this article, assume that $G = H \times_\tau K$ is the semidirect product of two locally compact groups H and K, respectively and consider the quasi regular representation on G. Then for some closed subgroups of G we investigate an admissible condition to generate the Gilmore-Perelomov coherent states. The construction yields a wide variety of coherent states, labelled by a homogeneous space of G.

Key Words: Locally compact abelian group, Semidirect product, Fourier transform, Square integrable representation, Coherent states.

1. Introduction

Wavelet transforms are often studied in the general framework of square-integrable representations [7, 13]. The coherent states, as a general form of wavelet transform, have become a widely used in mathematics and physics during the last decade. This type of coherent states introduced by Gilmore [10] and Perelomov [14] could be reformulated as a problem in group representation theory. The construction of coherent states on the Galilean group analyzed in [4] also one can find analogous results in the earlier papers [2, 3] for the Poincaré group. The study of coherent states for some semidirect product groups has been continued by Ali et al. [1].

The present paper extends the concept of coherent states to a general semidirect product group $H \times_\tau K$, where H and K are locally compact groups and K is also abelian. More precisely, the natural action H on K (i.e. $(h,k) \mapsto \tau_h(k)$) induces a dual action from H on $\widehat{K}$, the dual group of K, which is given by $(h,\gamma) \mapsto \gamma \circ \tau_h$. Fix $\omega \in \widehat{K}$ and assume that O_ω and H^ω are the orbit and stabilizer subgroup of ω, respectively. Take $X = G/(H^\omega \times \{1_K\})$. Then there exists a one to one correspondence between X and $O_\omega \times K$. A case in point is precisely that $H^\omega = H$, analyzed in [6]. In [11] it is shown that X is topological isomorphic to $O_\omega \times K$ if O_ω is an open orbit. Hence, we can transfer the (Haar) measure of $O_\omega \times K \subseteq \widehat{K} \times K$ to X. This is a G-invariant measure on X (section 3). Section 2 presents some basic facts about the continuous wavelet transform, with an introduction to the general theory of coherent states. Section 3 is devoted to introduce a condition to generate coherent states associated to the quasi regular representation of G.

2000 *AMS Mathematics Subject Classification:* 43A65; Secondary 81R30..

2. Preliminaries and notations

Let G be a locally compact topological group with the left Haar measure μ_G and modular function Δ_G. We review the basic definitions and properties of coherent states based on square integrable group representation associated to a homogeneous space of underlying group.

By a *homogeneous space* we mean a transitive G-space X that is homeomorphic to a quotient space G/H, for a closed subgroup H of G. Finding a G*-invariant* measure under the natural action $x \longmapsto gx$ is impossible in general. However, it is well-known that the quasi-invariant measures exist on an arbitrary homogeneous space [9]. In fact, for a Radon measure ν on X and $g \in G$ the translation ν_g of ν is given by

$$d\nu_g(x) = d\nu(g^{-1}x).$$

The measure ν is called *quasi-invariant* if the measures ν_g are all equivalent.

Definition 2.1 *A Borel section on the homogeneous space X is a Borel map $\sigma : X \longrightarrow G$, satisfying $q(\sigma(x)) = x$, for all $x \in X$, where $q : G \longrightarrow X$ is the canonical quotient map.*

Now assume that ν is a quasi-invariant measure on X and σ is a Borel section. In order to construct coherent states we require another quasi-invariant measure ν_σ which is given by

$$d\nu_\sigma(x) = \lambda(\sigma(x), x)d\nu(x).$$

The Borel measures ν_σ is independent of the choice of the quasi-invariant measure ν used to define it. Moreover if X admits a G-invariant measure m then ν_σ is a scalar multiple of m, for every quasi-invariant measure ν, see [1].

Let π be a square integrable unitary representation of G on a separable Hilbert space $\mathcal{H}$. Then the continuous wavelet transform (CWT) on G is defined by

$$W_\psi : \mathcal{H} \longrightarrow L^2(G), \qquad (W_\psi \phi)(g) = C_\psi^{-1} < \pi(g)\psi, \phi >, \qquad \text{for } \phi \in \mathcal{H},\ g \in G,$$

where ψ is a nonzero (admissible) vector in $\mathcal{H}$ and

$$C_\psi^2 := \frac{1}{\|\psi\|^2} \int_G | < \pi(g)\psi, \psi > |^2 d\mu_G(g) < \infty.$$

The CWT is a linear isometry and its adjoint is W_ψ^{-1} on ImW_ψ. Hence a vector $\phi \in \mathcal{H}$ can be reconstructed uniquely by

$$\phi = W_\psi^\star(W_\psi \phi) = \frac{1}{C_\psi} \int_G (W_\psi \phi)(g)\pi(g)\psi \ d\mu_G(g). \tag{1}$$

To develop the notion of square integrability, we use the following rank-one operators on $\mathcal{H}$; $|\xi >< \eta| : \phi \longmapsto < \phi, \eta > \xi$, for all $\xi, \eta \in \mathcal{H}$. It is easy to see that $|\xi >< \eta|$ is a bounded linear operator and $\| \, |\xi >< \eta| \, \| = \|\xi\| \, \|\eta\|$.

Definition 2.2 *([4]) Suppose $(\pi, \mathcal{H})$ is a unitary representation on G and H is a closed subgroup of G. Consider a quasi-invariant measure ν on $X := G/H$ and fix a Borel section $\sigma : X \longrightarrow G$. Then we say* *that π is square integrable mod(H,σ) for the vector ψ if the integral*

$$\int_X \pi(\sigma(x))\ |\psi><\psi|\ \pi(\sigma(x))^\star d\nu_\sigma(x)$$

converges weakly to a bounded positive invertible operator A_σ on $\mathcal{H}$, i.e.

$$\int_X |<\pi(\sigma(x))\psi,\eta>|^2 d\nu_\sigma(x) =<\eta, A_\sigma\eta>, \quad \forall\eta\in\mathcal{H}.$$

We also say that the vector ψ is admissible mod(H,σ) or that the section σ is admissible for (π,η). Now we define the family of covariant coherent states, indexed by points $x\in X$, as the orbit of ψ under G, through the representation U and the section σ:

$$\mathcal{H}_{\psi,\sigma} = \{\pi(\sigma(x))\psi;\ x\in X\}.$$

In other words, one has the resolution

$$\int_X |\pi(\sigma(x))\psi><\pi(\sigma(x))\psi|\ d\nu_\sigma(x) = A_\sigma$$

(the integral interpreted in the weak sense).

It may happen that A_σ^{-1} is unbounded. In fact, $\mathcal{H}_{\psi,\sigma}$ constructs a frame if A_σ^{-1} is bounded. Moreover $A_\sigma = \lambda I, \lambda > 0$ if and only if $\mathcal{H}_{\psi,\sigma}$ is a tight frame [8].

Notice that $\mathcal{H}_{\psi,\sigma}$ is total in $\mathcal{H}$ and if we define

$$W_{\psi,\sigma}:\mathcal{H}\longrightarrow L^2(X,d\nu), \qquad (W_{\psi,\sigma}\phi)(x) = C_{\psi,\sigma}^{-1} <\pi(\sigma(x))\psi,\phi>,$$

where

$$C^2_{\psi,\sigma} = \frac{1}{\|\psi\|^2}\int_X |<\pi(\sigma(x))\psi,\psi>|^2\ d\nu_\sigma(x) < \infty, \tag{2}$$

then $W_{\psi,\sigma}$ that is an isometry can be considered as the generalized continuous wavelet transform on homogeneous space X, hence $W_{\psi,\sigma}^{-1} = W_{\psi,\sigma}^\star$ on $ImW_{\psi,\sigma}$ and so we can obtain the reconstruction formula similar to (1), for more details see [4];

$$\phi = \frac{1}{C_{\psi,\sigma}}\int_X (W_{\psi,\sigma}\phi)(x)\ A_\sigma^{-1}\pi(\sigma(x))\psi\ d\nu_\sigma(x), \qquad \forall\ \phi\in\mathcal{H}.$$

3. Main results

Throughout this section we assume that H and K are two locally compact topological groups and K is also abelian. Let $G = H\times_\tau K$ be the semi direct product group of H and K where $h\longmapsto\tau_h$ is a homomorphism of H into the group of automorphisms of K such that the mapping $(h,k)\longmapsto\tau_h(k)$ from $H\times K$ onto K is continuous.

Moreover, the left Haar measure of G is $d\mu_G(h,k) = \delta(h)d\mu_H(h)d\mu_K(k)$ and $\Delta_G(h,k) = \delta(h)\Delta_H(h)\Delta_K(k)$ is its modular function, in which the positive continuous homomorphism δ on H is given by

$$\mu_K(E) = \delta(h)\mu_K(\tau_h(E)), \tag{3}$$

for all measurable subsets E of K (15.29 of [12]).

As before, fix a $\omega \in \widehat{K}$ with open orbit and take $X = G/\widetilde{H}$ where $\widetilde{H} = H^\omega \times \{1_K\}$ and H^ω is the ω-stabilizer subgroup of the action $H \times \widehat{K} \mapsto \widehat{K}; (h,\gamma) \mapsto \gamma \circ \tau_h$. Then it is obvious that

$$\rho : X \longrightarrow O_\omega \times K$$

$$(h,k)\widetilde{H} \mapsto (\omega \circ \tau_{h^{-1}}, k)$$

is a bijection. In fact, it is a topological isomorphism [11].

Lemma 3.1 *Let $\tau : H \to Aut(K)$ be the homomorphism used in the definition of $H \times_\tau K$. For every $h \in H$ and $\gamma \in \widehat{K}$ we have*

$$(f \circ \tau_h)\widehat{\ }(\gamma) = \delta(h)\widehat{f}(\gamma \circ \tau_{h^{-1}}), \tag{4}$$

$$d\mu_{\widehat{K}}(\gamma \circ \tau_{h^{-1}}) = \delta(h)d\mu_{\widehat{K}}(\gamma), \tag{5}$$

in which $f \in L^1(K)\bigcap L^2(K)$.

Proof. Let $f \in L^1(K)$ then there exists a sequence $\{f_n\}$ in $C_c(K)$, the space of all continuous and compact supported functions on K, such that $f_n \longrightarrow f$ in $L^1(K)$. It is clear that $f_n \circ \tau_h \in C_c(K)$ for all $h \in H$ and $n \in I\!N$. Moreover by (3) we have

$$\|f_n \circ \tau_h - f \circ \tau_h\|_1 = \delta(h)\|f_n - f\|_1.$$

That is, $f \circ \tau_h \in L^1(K)$. Now a straightforward calculation gives (4). To obtain (5), note that $d\mu_{\widehat{K}}(\gamma \circ \tau_h)$ is a translation invariant measure on $\widehat{K}$ and by the Plancherel theorem (4.25 of [9]) for all $f \in L^1(K)\bigcap L^2(K)$ we have

$$\begin{aligned}\int_{\widehat{K}} |\widehat{f}(\gamma)|^2 d\mu_{\widehat{K}}(\gamma) &= \int_K |f(x)|^2 d\mu_K(x) = \delta(h^{-1})\int_K |f(\tau_h(x))|^2 d\mu_K(x)\\ &= \delta(h^{-1})\int_{\widehat{K}} |(f\circ\tau_h)\widehat{\ }(\gamma)|^2 d\mu_{\widehat{K}}(\gamma) = \delta(h)\int_{\widehat{K}} |\widehat{f}(\gamma\circ\tau_{h^{-1}})|^2 d\mu_{\widehat{K}}(\gamma).\end{aligned}$$

□

Now we can construct a measure on X, in fact for every Borel set $\mathcal{B}$ of X define $\nu(\mathcal{B}) = \mu_{\widehat{K}} \times \mu_K(\rho(\mathcal{B}))$. Then by using (3) and (5) for each $g = (h,k) \in G$ and $x = (h_0,k_0)\widetilde{H} \in X$ we have

$$
\begin{aligned}
d\nu_g(x) &= d(\mu_{\widehat{K}} \times \mu_K)(\rho(g^{-1}x)) \\
&= d\mu_{\widehat{K}}(\gamma \circ \tau_{h_0^{-1}} \circ \tau_h)\ d\mu_K(\tau_{h^{-1}}(k^{-1}k_0)) \\
&= \delta(h^{-1})\ d\mu_{\widehat{K}}(\gamma \circ \tau_{h_0^{-1}})\ \delta(h)\ d\mu_K(k_0) \\
&= d(\mu_{\widehat{K}} \times \mu_K)(\gamma \circ \tau_{h_0^{-1}}, k_0) \\
&= d\nu(x)
\end{aligned}
$$

i.e. ν is a G-invariant measure on X.

Therefore, ν_σ is also a G-invariant measure on X, for every Borel section σ. In other words, such a measure is unique up to constant multiple (see §4.1 of [1]). In the sequel, we denote this measure again by ν.

The general form of a Borel section for a semidirect product group has shown in the following theorem;

Theorem 3.2 *Let $G = H \times_\tau K$ be the semidirect product of H and K and $X = G/\widetilde{H}$. Then every Borel section $\sigma : X \longrightarrow G$ of G can be expressed as $\sigma = (\sigma_1, \sigma_2)$ such that*

$$\omega \circ \tau_{\sigma_1(x)^{-1}} = \omega \circ \tau_{h^{-1}} \tag{6}$$

$$\sigma_2(x) = k, \tag{7}$$

for all $x = (h,k)\widetilde{H} \in X$.

Proof. Let $\sigma = (\sigma_1, \sigma_2)$. Then it is easy to see that $q(\sigma(x)) = x$ if and only if

$$(h,k)^{-1}(\sigma_1(x), \sigma_2(x)) \in \widetilde{H}.$$

So $h^{-1}\sigma_1(x) \in H^\omega$ and $\tau_h(k^{-1}\sigma_2(x)) = 1$. This proves (6). Moreover, (7) immediately follows the fact that τ_h is an automorphism on K, for each $h \in H$. □

We are now ready to state our main result. In fact, we aim to simplify (2) to establish coherent states on a semidirect product group. In this way, we can develop the notion of continuous wavelet transform on G. The same idea was exploited to a certain extent in [6].

Definition 3.3 *The quasi regular representation $(U, L^2(K))$ associated to the semidirect product group $G = H \times_\tau K$ is defined by*

$$U(h,k)f(y) = \delta(h)^{\frac{1}{2}} f(\tau_{h^{-1}}(yk^{-1})),$$

for all $f \in L^2(K)$, $(h,k) \in G$ and $y \in K$.

This representation is not irreducible in general (e.g. Affine group $G = (0, +\infty) \times_\tau \mathbb{R}$). However, a characterization of irreducible subrepresentations of U can be found in [5].

Theorem 3.4 *Let $(U, L^2(K))$ be the quasi regular representation on $G = H \times_\tau K$. Put $X = G/\widetilde{H}$ and fix a Borel section σ. Then $\psi \in L^2(K)$ is an admissible mod$(\widetilde{H}, \sigma)$ vector for U if*

$$\int_X \delta(\sigma_1(x)) \, \|\psi \circ \tau_{\sigma_1(x)^{-1}}\|_2^2 \, d\nu(x) < \infty. \tag{8}$$

Proof. For any $\eta \in L^2(K)$ let $\eta^\bullet(k) = \overline{\eta(k^{-1})}$ then $\widehat{\eta^\bullet} = \overline{\widehat{\eta}}$. Hence by using the Plancherel theorem we have;

$$\begin{aligned} < U(\sigma(x))\psi, \eta > &= \int_{\widehat{K}} [U(\sigma(x))\psi]\widehat{\ }(\gamma) \, \overline{\widehat{\eta}(\gamma)} \, d\mu_{\widehat{K}}(\gamma) \\ &= \delta(\sigma_1(x))^{\frac{1}{2}} \int_{\widehat{K}} (\psi \circ \tau_{\sigma_1(x)^{-1}})\widehat{\ }(\gamma) \, \overline{\widehat{\eta}(\gamma)} \, \overline{\gamma}(\sigma_2(x)) \, d\mu_{\widehat{K}}(\gamma) \\ &= \delta(\sigma_1(x))^{\frac{1}{2}} \int_{\widehat{K}} \widehat{\xi_x}(\gamma) \, \overline{\gamma}(\sigma_2(x)) \, d\mu_{\widehat{K}}(\gamma), \end{aligned}$$

in which $\xi_x = (\psi \circ \tau_{\sigma_1(x)^{-1}}) \star \eta^\bullet$ and $\star$ denotes the convolution on $L^2(K)$. Note that $\xi_x \in C_0(K)$ and $\|\xi_x\|_\infty \leq \|\psi \circ \tau_{\sigma_1(x)^{-1}}\|_2 \, \|\eta\|_2$ by Theorem 2.40 of [9]. Hence, by the Fourier inversion theorem (4.32 of [9]) we obtain:

$$\begin{aligned} < \eta, A_\sigma \eta > &= \int_X < U(\sigma(x))\psi, \eta > \overline{< U(\sigma(x))\psi, \eta >} \, d\nu_\sigma(x) \\ &= \int_X |< U(\sigma(x))\psi, \eta >|^2 \, d\nu(x) \\ &= \int_X \delta(\sigma_1(x)) \, | \int_{\widehat{K}} \widehat{\xi_x}(\gamma) \, \overline{\gamma}(\sigma_2(x)) \, d\mu_{\widehat{K}}(\gamma) \, |^2 \, d\nu(x) \\ &= \int_X \delta(\sigma_1(x)) \, | \, \xi_x(\sigma_2(x)) \, |^2 \, d\nu(x) \\ &\leq \|\eta\|_2^2 \int_X \delta(\sigma_1(x)) \, \|\psi \circ \tau_{\sigma_1(x)^{-1}}\|_2^2 \, d\nu(x). \end{aligned}$$

□

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Received 02.09.2008