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Tachyons in General Relativity

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Abstract

We consider the motion of tachyons (faster-than-light particles) in the framework of General Relativity. An important feature is the large contribution of low energy tachyons to the energy-momentum tensor. We also calculate the gravitational field produced by tachyons in particular geometric arrangements; and it appears that there could be self-cohering bundles of such matter. This leads us to suggest that such theoretical ideas might be relevant to major problems (dark matter and dark energy) in current cosmological models.

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1 Introduction

While there is no reliable evidence for the existence of faster-than-light particles (tachyons) some physicists have considered them a theoretical possibility. Yet it appears that many carry a prejudice against the legitimacy of tachyonic theories because of concerns about causality. This comes from a familiar "paradox" that posits communication via tachyons that appear to travel backwards in time. In Appendix A I have tried to dampen that prejudice by looking carefully at the assumptions behind the paradox. If tachyons interact only weakly with ordinary matter, yet are profusely present throughout the universe, then it may be that those objections become nugatory.

The present paper undertakes an investigation of how faster-then-light particles might be considered within the General Theory of Relativity and perhaps make some interesting contributions to current models of cosmology.

We start, in Section 2, with how they would be described in Special Relativity, noting the particular role of the energy-momentum tensor $T^{\mu\nu}$. Then we turn to General Relativity. In Sections 3 and 4 we study the motion of a tachyon in two familiar gravitational models. Then in Sections 5 and 6 we derive the gravitational field produced by a uniform flow of tachyons; and find that there is an attractive force on other nearby tachyons which increases in strength as their energy decreases. In Sections 7, 8, 9 and 10 we explore various models for gravitational fields produced by static flows of tachyons, using the linear approximation to Einstein's equation; and we find that self-cohering bundles of tachyonic matter are feasible. Section 11 has a note about how tachyons might contribute to gravitational lensing. These results, summarized in Section 12, suggest that tachyons should be given closer attention in the ongoing study of cosmology.

2 Special Relativity

In Special Relativity, we have the motion of a particle described by the four-vector coordinate $\xi^\mu(\tau),\ \mu = 0, 1, 2, 3$ and τ is a parameter called the proper time. For a free particle moving in a straight line with velocity $\mathbf{v}$ we have,

$$\xi^\mu(\tau) = (\gamma\tau, \gamma\mathbf{v}\tau). \tag{2.1}$$

Then, writing $\dot{\xi}^\mu = d\xi^\mu/d\tau$, we calculate

$$\eta_{\mu\nu}\,\dot{\xi}^\mu\,\dot{\xi}^\nu = \dot{\xi}^\mu\,\dot{\xi}_\mu = \gamma^2\,(1 - v^2), \tag{2.2}$$

where we use the metric of Special Relativity, $\eta_{\mu\nu}$ (with $\eta_{00} = +1$ and the other diagonal components equal to -1) to raise and lower indices. We define $\gamma = 1/\sqrt{|1-v^2|}$ so that this is always a real number. Then we see that for normal (slower-than-light) particles the invariant in Eq. (2.2) has the value +1 and for tachyons it is -1. (I am using units in which the velocity of light is $c = 1$.)

Trying to follow conventional habits we can define the "energy" and "momentum" of a free tachyon as follows:

$$E = m\,\gamma, \qquad \mathbf{p} = m\,\gamma\,\mathbf{v}, \qquad E^2 = p^2 - m^2 \tag{2.3}$$

where m is a (real, positive) mass parameter. This allows us to characterize the full set of free-particle tachyon states as,

$$E \geq 0, \qquad p^2 \geq m^2; \tag{2.4}$$

and the direction of the vector $\mathbf{p}$ covers the entire sphere.

Troubles occur when one regards this E and $\mathbf{p}$ as a 4-vector. For tachyons this is a space-like vector and so the sign of the first component can change under a proper Lorentz transformation. This can lead to unnecessary confusion about negative energy states. This confusion was straightened out in an earlier study [1], where it was shown how to properly use the labels for "in" and "out" states in any interaction. It was also noted there that the best way to talk about energy and momentum is by starting with the familiar second rank tensor.

The energy-momentum tensor for a classical particle can be written as

$$T^{\mu\nu}(x) = m \int d\tau\; \dot{\xi}^{\mu}\dot{\xi}^{\nu}\; \delta^4(x - \xi(\tau)) \tag{2.5}$$

and we note that the energy density, T^{00}, is always positive. For the free-particle motion noted above, we calculate that this tensor density has the diagonal elements $\rho, P_{11}, P_{22}, P_{33}$, where

$$\rho = m\gamma, \qquad P_{ii} = m\gamma v_i^2. \tag{2.6}$$

Note that for tachyons the pressure terms ($\sum_i \; P_{ii}$) come out larger than the energy term (ρ); this is exactly the opposite of the situation for normal particles.[1]

[1]It is usually so that a repeated index implies summation; but there are exceptions, as seen here.

To illuminate this unfamiliar situation, imagine the neutrino as being a tachyon. Then its contribution to the energy-momentum tensor, in the pressure term, would be larger than what one would ascribe to a massless particle by a factor of $v^2 = (p/E)^2 = (E^2 + m^2)/E^2$. If one were to assume a mass of around 0.1 electron Volts and an average energy corresponding to the cosmic background temperature of around 3 ^oK, then this would give a numerical factor of $v^2 \sim 10^5$. That would, presumably, have a significant impact on current calculations of cosmological models. [2]

For some further musings on kinematical aspects of tachyons, see Appendix B.

3 General Relativity metric for uniform cosmology

The Robertson-Walker metric, describing a homogeneous isotropic universe, is

$$\pm ds^2 = dt^2 - A(t)[B(r)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta \; d\varphi^2]. \tag{3.1}$$

where we have identified the four (spherical polar) coordinates t, r, θ, φ. The functions $A(t), B(r)$ are commonly written as:

$$A(t) = a(t)^2/c^2; \qquad B(r) = \frac{1}{1 - \epsilon r^2/R^2}. \tag{3.2}$$

With this metric we construct the following equations for the geodesic.

$$\ddot{t} + \frac{1}{2}A'[\dot{r}^2 B + \dot{\theta}^2 r^2 + \dot{\varphi}^2 r^2 sin^2\theta] = 0, \tag{3.3}$$

$$-AB\ddot{r} + \dot{r}[-\dot{t}A'B - \dot{r}AB'] + \frac{1}{2}A[\dot{r}^2 B' + \dot{\theta}^2 2r + \dot{\varphi}^2 2r \; sin^2\theta] = 0, \tag{3.4}$$

$$-Ar^2\ddot{\theta} + \dot{\theta}[-\dot{t}A'r^2 - \dot{r}2Ar] + \frac{1}{2}A[\dot{\varphi}^2 r^2 2 \; sin\theta \; cos\theta] = 0, \tag{3.5}$$

$$-Ar^2 \; sin^2\theta \; \ddot{\varphi} + \dot{\varphi}[-\dot{t}A'r^2 \; sin^2\theta - \dot{r}A2r \; sin^2\theta - \dot{\theta}Ar^2 2 \; sin\theta \; cos\theta] = 0, \tag{3.6}$$

where the "prime" on A or B means derivative with respect to its argument.

We can integrate the last two equations, yielding

$$Ar^2 \; sin^2\theta \; \dot{\varphi} = M, \tag{3.7}$$

$$A^2 r^4 \dot{\theta}^2 + M^2/sin^2\theta = L^2, \tag{3.8}$$

where M and L are some constants. Then, substituting these results back into the first two equations we also get the integrals:

$$A^2 B\dot{r}^2 + L^2/r^2 = 2\mathcal{E}, \tag{3.9}$$

$$\dot{t}^2 - 2\mathcal{E}/A = Q, \tag{3.10}$$

with two more constants, $\mathcal{E}$ and Q.

Using these results we now calculate the invariant

$$g_{\mu\nu}\ \dot{x}^\mu\ \dot{x}^\nu = Q, \tag{3.11}$$

So, for normal (slower-than-light) particles we would choose $Q = +1$; for light we would choose $Q = 0$; and for tachyons, we choose $Q = -1$. We shall see that in some instances the constant $2\mathcal{E}$ may be taken as v^2 for some initial velocity v, while L and M relate to the angular momentum for unit mass.

From Eq. (3.9) we see that motion of a tachyon will be contained in the case of a closed universe, ($\epsilon = +1$), even though its velocity may be arbitrarily large. This follows because the factor $B(r)$ becomes infinite at $r = R$, and thus the radial coordinate r must stop its increase at that point.

Another simple solution of Eq. (3.9) is a circular orbit ($\dot{r} = 0$) at a radius $r_0 = L/v$. But such orbits are not stable: for $B > 0$ they will spiral outward; and they will spiral inward for $B < 0$, which may occur for $r_0 > R$ in a closed universe.

4 Tachyon in Schwarzschild metric

Outside of a central source of gravitation, we have the Schwarzschild metric,

$$\pm ds^2 = A(r)dt^2 - [A(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta\ d\varphi^2]. \tag{4.1}$$

where $A(r) = (1 - r_s/r)$ and $r_s = 2GM$.

The solution of the geodesic equations in this metric yields the following.

$$r^2\dot{\varphi} = L, \qquad A\dot{t} = \gamma, \qquad A^{-1}(\dot{r}^2 - \gamma^2) + L^2/r^2 = -Q \tag{4.2}$$

where γ and L are constants of integration and Q is the same constant (with values +1, 0, -1) defined in Eq. (3.11).

For slower-than-light particles (Q=+1) this gives the Kepler orbits for $r > r_s$, along with small relativistic corrections. For tachyons (Q=-1) we see that this does not give localized orbits but only scattering states.

Introducing the variable, $u = 1/r(\varphi)$, and introducing the asymptotic velocity v, with $\gamma^2 = Q + v^2\gamma^2$, $L = bv\gamma$, we get the equation for the orbital,

$$u'^2 + u^2 - \frac{Qr_s u + v^2\gamma^2}{L^2} = r_s u^3. \tag{4.3}$$

From this equation, we can calculate the scattering angle to first order in the gravitational field strength and find the result

$$\Delta\phi = \frac{r_s}{b}\,\chi, \qquad \chi = [2 + \frac{Q}{v^2\gamma^2}]. \tag{4.4}$$

For light (Q=0), we have $\chi = 2$, which is the well known result. For tachyons (Q=-1), χ has the value 2 at high energies and then decreases to 1 as the energy drops. For ordinary particles (Q=+1) at very high energies (v close to 1) χ is again 2; and at much lower velocities we should not accept the first order approximation but solve the exact Kepler problem.

We can also easily look at a "head-on" collision by setting $L = 0$ in Eq. (4.2). Then we see how a tachyon will behave approaching the edge of a black hole, $r \to r_s$.

5 Metric in cylindrical symmetry

For tachyons as a gravitational source, the simplest model is a straight line of particles, flowing constantly along the z-axis, with equal velocities in both directions. With cylindrical symmetry we use the four coordinates $\xi^\mu = (t, r, \theta, z)$, where $r = \sqrt{x^2 + y^2}$. The source is then a diagonal energy-momentum tensor density, following Eq. (2.6),

$$T^{00} = \rho\gamma\delta(r)/(2\pi r), \qquad T^{11} = T^{22} = 0, \qquad T^{33} = \rho\gamma v^2\delta(r)/(2\pi r), \tag{5.1}$$

where ρ is the mass per unit length. We now make this ansatz for the metric:

$$\pm ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\theta^2 - C(r)dz^2. \tag{5.2}$$

Next, we calculate the elements of the Christoffel symbol.

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'}{2A} \tag{5.3}$$

$$\Gamma^1_{00} = \frac{A'}{2B}, \qquad \Gamma^1_{11} = \frac{B'}{2B} \tag{5.4}$$

$$\Gamma^1_{22} = -r/B, \qquad \Gamma^1_{33} = -\frac{C'}{2B} \tag{5.5}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = 1/r, \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{C'}{2C}, \tag{5.6}$$

where the prime means derivative with respect to r.

From this, we calculate the components of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$.

$$G_{00} = \frac{A}{2B}\{\frac{C''}{C} - \frac{C'^2}{2C^2} - \frac{B'C'}{2BC} + \frac{C'}{rC} - \frac{B'}{rB}\} \tag{5.7}$$

$$G_{11} = -\frac{C'}{2rC} - \frac{A'}{2rA} - \frac{A'C'}{4AC} \tag{5.8}$$

$$G_{22} = \frac{r^2}{2B}\{-\frac{A''}{A} + \frac{A'^2}{2A^2} + \frac{A'B'}{2AB} - \frac{C''}{C} + \frac{C'^2}{2C^2} - \frac{A'C'}{2AC} + \frac{B'C'}{2BC}\} \tag{5.9}$$

$$G_{33} = \frac{C}{2B}\{-\frac{A''}{A} + \frac{A'^2}{2A^2} + \frac{A'B'}{2AB} - \frac{A'}{rA} + \frac{B'}{rB}\}. \tag{5.10}$$

If we set all these components equal to zero, we find the solutions

$$A = a\, r^{\alpha}, \qquad B = b\, r^{\beta} \qquad C = c\, r^{\gamma} \tag{5.11}$$

$$\beta = \frac{\alpha^2}{\alpha+2}, \quad \gamma = \frac{-2\alpha}{\alpha+2} \tag{5.12}$$

where α is some constant ($\neq -2$).

For small α we have the ("weak field") solutions: $A = 1 + \alpha\, ln(r)$, $B = 1$, $C = 1 - \alpha\, ln(r)$.

Now, using the Einstein field equation $G_{\mu\nu} = -8\pi G T_{\mu\nu}$, with Eq. (5.1) we find $\alpha/2 = 8\pi G\rho\gamma/(2\pi)$. This leads to the result

$$Force = -\Gamma^1_{00} = -4G\rho\gamma/r \tag{5.13}$$

for the case of a slow particle being attracted by a line of mass-density ρ. This is to be compared with the result of Newtonian gravity; and there is a difference of a factor of 2. (This problem in General Relativity was studied long ago by Levi-Civita [3] and he obtained the correct Newtonian force.) I have no explanation of this discrepancy. Maybe it has to do with the line of matter extending to infinity.

Similarly, for a fast particle near a line of streaming tachyons we find the attractive force

$$Force = -\Gamma^1_{33}\dot{\xi}^3\dot{\xi}^3 = -4G\rho\gamma v^2(\dot{z}^2)/r. \tag{5.14}$$

What is novel here is that the strength of the gravitational attraction between tachyons increases as the velocity increases (without limit).

There is some question about the legitimacy of this result due to the non-linearity of the equations and the singularity at $r = 0$. Please see Appendix C for resolution of this question.

6 Circular model

To make a more realistic model of tachyons as a gravitational source, we can imagine something like a circular flow, which is limited in its spatial extent. The full solution of Einstein's field equations then becomes more difficult; but we can start out by making the weak field approximation. The general approach (see, for example, [4]) goes like this:

$$(\frac{\partial}{\partial t}^2 - \nabla^2)h_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{6.1}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \qquad h = \eta^{\mu\nu}\, h_{\mu\nu}, \tag{6.2}$$

with the side condition (gauge choice)

$$\partial^\mu h_{\mu\nu} = 0. \tag{6.3}$$

This gives us Newtonian gravity in the familiar case of mass at rest ($T_{00} = \rho(\mathbf{x})$); and we use this now for a tachyon source, following (2.5), (2.6).

If this is a large uniform and static circulating flow of tachyons and we look close to the edge of it, then we can approximate it as an infinite cylinder, along the local z-axis, and immediately see the result. From h_{33} we get an attractive force on a unit mass at rest; and it is without that previous extra factor of 2.

$$F = -2GP/r \tag{6.4}$$

where r is the distance from the center line of this cylinder and P is the "pressure" $(m\gamma v^2)$ per unit length of the source. For large values of v, which means low energy for the tachyons in the source, this force becomes large, proportional to $|v|$ (or $1/E$). This suggests remarkable new physics, if such things actually exist.

Let us see how this force works upon another tachyon travelling nearby. The geodesic equation gives is

$$\frac{d^2r}{d\tau^2} = F, \qquad \frac{d^2r}{dt^2} = F/\gamma^2 \tag{6.5}$$

where this γ describes the free-moving tachyon. So, a low energy tachyon is accelerated toward the flow of other low energy tachyons. This effect increases with the velocity of each component and suggests how such a structure - a self sustaining and accreting mass flow of tachyons held together by gravity - might evolve.

For example, if we have a total mass M of tachyons moving within a tube of radius r at large velocity V in a circle of radius R, then an additional tachyon moving with large velocity along the edge of this tube will have its trajectory bent to keep up with that circle provided that $GMV/r > \pi$.

In Section 10 we shall carry this study further to include tachyon orbits spiraling around the center line of this source.

Any such structure would also have gravitational effects upon ordinary matter and upon light beams. This is what the geometric theory of gravity requires and we now try to consider this in some generality.

7 Static gravitational fields

Now we want to follow the first-order approach of Eqs. (6.1 - 6.3) for a general distribution of ordinary matter and tachyons that is static, not changing in time. We would like to be able to specify a source $T_{\mu\nu}(\mathbf{x})$, for example, one that has a simple structure like that of Eq. (2.6), where the energy density ρ and the pressure terms P_{ii} are now limited by some spatial envelope we might choose.

This model has an obvious shortcoming in that $\partial^\mu T_{\mu\nu}$ is not zero. (It would be zero if we only had the energy density and ignored the pressure terms; but for tachyons the motion is the main thing.) At first, I thought to put that criticism aside with the following argument. The particles that comprise this source experience some acceleration in order to remain in a confined region and we assume that the requisite forces are themselves gravitational. Thus the error introduced into the Einstein equation will be of second order in the constant G and we shall ignore it. (This seems analogous to the distinction between the ordinary derivative of $T_{\mu\nu}$ and its covariant derivative.)

This argument, however, is false on two fronts. In terms of the mathematics: The Einstein equation $G_{\mu\nu} = -8\pi G T_{\mu\nu}$ has, by construction, a left hand side that automatically satisfies the conservation law. Thus, using a right hand side that fails this test may (will) lead to incorrect solutions. In terms of the physics: For non-relativistic particles moving in gravitational orbits we know that the average potential energy is just as big as the kinetic energy. Therefore, it would be wrong to ignore the former and only keep the latter in constructing the energy-momentum tensor. (Again, for non-relativistic particles these pressure terms are generally negligible, of order v^2/c^2 compared to their rest-mass; but for tachyons they dominate.)

First, we shall find a general statement about the gravitational fields produced (in first order) from a static source that is localized in space. Then we shall proceed to some particular models.

We have both $T_{\mu\nu}$ and $h_{\mu\nu}$ independent of the time variable, with the former assumed confined in some region of space. We also take the time-space components of each of these tensors to be zero. The basic equations are,

$$-\nabla^2 h_{\mu\nu}(\mathbf{x}) = -16\pi G T_{\mu\nu}(\mathbf{x}) \tag{7.1}$$

$$h_{00}(\mathbf{x}) = -4G \int d^3x'\ T_{00}(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'| \tag{7.2}$$

$$h_{ij}(\mathbf{x}) = -4G \int d^3x'\ T_{ij}(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'| \tag{7.3}$$

$$\partial_i h_{ij}(\mathbf{x}) = \partial_i T_{ij}(\mathbf{x}) = 0, \tag{7.4}$$

with $i, j = 1, 2, 3$. The asymptotic form of Eq. (7.3) is

$$h_{ij} \sim -\frac{4G}{r} \int d^3x' T_{ij}(\mathbf{x}'), \tag{7.5}$$

but from Eq. (7.4) we calculate,

$$0 = \int d^3x\ x_k\ \partial_i T_{ij}(\mathbf{x}) = -\int d^3x\ T_{kj}(\mathbf{x}). \tag{7.6}$$

This means that we get no long range force from the spatial parts of the source, as suggested by Eq. (7.5); that comes only from the T_{00} component, Eq. (7.2). Is this well-known? For ordinary (slow) matter this is no big deal (maybe it gets involved when one asks about v^2/c^2 corrections to Newtonian gravity). But for tachyons it would be the major feature of their gravitational role. If there are such static, localized configurations of tachyons as we have been speculating here, then they may make major contributions to the local gravitational field; but they produce little or no long range gravitational attraction. What a lovely way to explain both dark matter and dark energy!

Incidentally, Eq. (7.6) may be recognized as a generalization of the Virial Theorem, which is well known in non-relativistic mechanics, either the classical or the quantum variety.

What can one say about the next term, after $\sim 1/r$ in the asymptotic expansion of Eq.(7.3)? It would appear to be $\sim 1/r^2$ with a coefficient $\int d^3x\, x_k\, T_{ij}(\mathbf{x})$. This can be made to vanish, however, by choosing where we place the origin of coordinates.

In what follows we shall try to construct models of the energy-momentum tensor that obey the conservation law, are static in time, and localized in space. That means they need to involve two parts which we may call the "kinetic energy" part and the "potential energy" part.

8 Circular flows

We shall be interested in problems with 3-dimensional and also 2-dimensional symmetry. So let us do this analysis first for n-dimensional Euclidean tensors $T_{ij}(\mathbf{x})$, with $i, j = 1, \dots n$. We have the radial coordinate $r = \sqrt{\sum_i x_i x_i}$.

For circular motion we can see that the kinetic part of the nxn matrix T_{ij} should look like this

$$T_{ij}^{(kin)} = (\delta_{ij} - x_i x_j / r^2) b(r) \sim (v_i\, v_j) \tag{8.1}$$

where any multiplying envelope $b(r)$ should be non-negative.

The easiest way to verify that this does represent circular flow is to calculate $x_i T_{ij}^{(kin)}$ and see that it is zero, since there should be no velocity in the radial direction.

It is then convenient to introduce the nxn coordinate matrices I and D as follows .

$$I_{ij} = \delta_{ij}, \qquad D_{ij} = \delta_{ij} - n x_i x_j / r^2, \tag{8.2}$$

$$Trace(I) = n, \qquad Trace(D) = 0, \qquad D^2 = (n-1)I - (n-2)D. \tag{8.3}$$

Then, we construct the full tensor as

$$T_{ij} = \delta_{ij} a(r) + D_{ij} b(r) = \tag{8.4}$$
$$\delta_{ij}[a(r) - (n-1)b(r)] + nT_{ij}^{(kin)}. \tag{8.5}$$

and thus we identify $[a - (n-1)b]$ as the "potential energy" portion of T. From the conservation requirement $\partial_i T_{ij} = 0$, we get the constraint

$$r^n a'(r) = (n-1)(r^n b(r))', \qquad \frac{d}{dr}[a - (n-1)b] = n(n-1)b(r)/r \geq 0. \tag{8.6}$$

Thus the part we identify as potential energy in the tensor is diagonal and increases with r. Assuming that both the functions $a(r)$ and $b(r)$ go to zero at large r, this means that the potential energy is negative - an "attractive well." From this we also learn that $\int_0^\infty r^{n-1} dr\ a(r) = 0$.

One more general result is the following, regarding the n-dimensional Lapacian operator.

$$\nabla^2\ D\ g(r) = D[g'' + (n-1)g'/r - 2ng/r^2]. \tag{8.7}$$

Then, if we write

$$h_{ij} = \delta_{ij} f(r) + D_{ij} g(r) \tag{8.8}$$
$$r^n f'(r) = (n-1)(r^n g(r))', \tag{8.9}$$

where we have required $\partial_i h_{ij} = 0$, then we get the solution

$$g(r) = -\frac{16\pi G}{n+2}\ [r^{-n} \int_0^r s^{n+1} ds\ b(s) + r^2 \int_r^\infty s^{-1} ds\ b(s)] \tag{8.10}$$
$$f'(r) = -16\pi G(n-1)\ r \int_r^\infty s^{-1} ds\ b(s). \tag{8.11}$$

There are alternative solutions in terms of the source function $a(r)$; but they are determined by the relations stated above between f and g, on the one hand, and a and b on the other.

From the fact that we have $b(r) \geq 0$, we can describe the two functions qualitatively as follows: $g(r)$ is everywhere negative and goes to zero at both $r \to 0$ and $r \to \infty$; $f(r)$ has everywhere a negative derivative and falls off to zero as r increases, thus it must be positive at all finite r.

The asymptotic solution at large r is, as expected,

$$g(r) \sim -\frac{16\pi G}{(n+2)}\, r^{-n} \int_0^\infty s^{n+1} ds\; b(s), \tag{8.12}$$

and the function $f(r)$ falls off even faster.

From the equations above one can derive a number of identities, for example,

$$f(0) = -n(n-1)\int_0^\infty dr\; r^{-1}\; g(r) = 16\pi G \frac{n-1}{2}\int_0^\infty dr\; r\; b(r). \tag{8.13}$$

Finally, that portion of the metric which depends on the function $g(r)$ can be simplified, as follows.

$$\sum_{i,j=1}^{n} dx_i\; dx_j[\delta_{ij} - n x_i x_j / r^2]\; g(r) = \tag{8.14}$$

$$\sum_i (dx_i)^2\; g(r) - n\left(\sum_i x_i\; dx_i\right)^2 g(r)/r^2 = \tag{8.15}$$

$$\left(\sum_i (dx_i)^2 - n(dr)^2\right)\; g(r). \tag{8.16}$$

9 Spherical source

We start by applying the machinery of the previous Section to the case of a spherically symmetric source, $n = 3$. We have the following solution of the field equations, with $r = \sqrt{x^2+y^2+z^2}$,

$$g(r) = -\frac{16}{5}\pi G[r^{-3}\int_0^r s^4 ds\; b(s) + r^2\int_r^\infty s^{-1} ds\; b(s)] \sim -\frac{4G}{5r^3}\int d^3x\; r^2 b(r), \tag{9.1}$$

and the function $f(r)$ falls off even faster at large r.

We still have

$$h_{00} = -4G\int d^3x' \frac{T_{00}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \equiv 4V_0(r) \sim -r^{-1}. \tag{9.2}$$

Next, we re-write the metric, changing from Cartesian to spherical polar coordinates.

$$ds^2 = [1+A]dt^2 + [-1+B]dr^2 + [-1+C]\; r^2[d\theta^2 + sin^2\theta\; d\varphi^2] \tag{9.3}$$

$$A(r) = 2V_0 + \frac{3}{2}f, \quad B(r) = 2V_0 - \frac{1}{2}f - 2g, \quad C(r) = 2V_0 - \frac{1}{2}f + g. \tag{9.4}$$

Then, we calculate the Christoffel symbols and put them into the geodesic equations. We immediately integrate two of them,

$$\dot{t} = \gamma e^{-A}, \qquad r^2 \dot{\varphi} = L e^{C}, \tag{9.5}$$

where γ and L are constants and, as usual, we choose $\theta = \pi/2$. Now the radial equation, in strictly first order form, leads to

$$(\dot{r})^2 + \gamma^2 A(r) + L^2/r^2 = 2\mathcal{E} = \gamma^2 - Q. \tag{9.6}$$

The critical question we want to answer is whether tachyons may find bound orbits in this configuration of circulating tachyons (plus stationary matter). The potential V_0 is negative, falling off to zero at large r; and the additional component of $A(r)$, which is $f(r)$, merely adds to the familiar L^2/r^2 term. Thus, with $Q = -1$ for tachyons, we see that $\mathcal{E} > 0$ and so conclude that this equation does *not* allow bound states.

Since we are seeking a self-supporting arrangement of tachyons, held together by their own gravitational forces, we must look for some other geometry. That is where we turn now.

10 Cylindrical source

We now apply the above machinery to the case of a cylindrically symmetric source, $n = 2$. Here we have $T_{3i} = h_{3i} = 0$; and we have the following solution of the field equations, with $r = \sqrt{x^2 + y^2}$:

$$g(r) = -\frac{16}{4}\pi G[r^{-2}\int_0^r s^3 ds\, b(s) + r^2 \int_r^\infty s^{-1} ds\, b(s)] \sim -\frac{2G}{r^2}\int d^2x\, r^2 b(r), \tag{10.1}$$

and the function $f(r)$ falls off even faster at large r.

We still have

$$h_{00} = -4G\int d^3x' \frac{T_{00}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \equiv 4V_0(r) \sim ln(r/R), \tag{10.2}$$

$$h_{33} = -4G\int d^3x' \frac{T_{33}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \equiv 4V_3(r) \sim ln(r/R), \tag{10.3}$$

where R is some measure of the linear extent of this source; in other words, these potentials are negative with a positive slope.

Next, we want to calculate the Christoffel symbols and put them into the geodesic equation. Changing from Cartesian coordinates to cylindrical coordinates (t, r, θ, z), we find,

$$ds^2 = (1 + A)dt^2 + (-1 + B)dr^2 + (-1 + C)r^2 d\theta^2 + (-1 + D)dz^2 \quad (10.4)$$

$$A(r) = [2V_0 + 2V_3 + f], \qquad B(r) = [2V_0 - 2V_3 - g], \quad (10.5)$$

$$C(r) = [2V_0 - 2V_3 + g], \qquad D(r) = [2V_0 + 2V_3 - f]. \quad (10.6)$$

The non-zero Christoffel symbols, to first order, are then

$$\Gamma^t_{tr} = \Gamma^r_{tt} = \frac{1}{2}A', \quad (10.7)$$

$$\Gamma^r_{rr} = -\frac{1}{2}B', \quad (10.8)$$

$$\Gamma^\theta_{\theta r} = \frac{1}{r} - \frac{C'}{2}, \quad (10.9)$$

$$\Gamma^r_{\theta\theta} = -r(1 + B - C) + \frac{r^2 C'}{2}, \quad (10.10)$$

$$\Gamma^z_{zr} = -\Gamma^r_{zz} = -\frac{1}{2}D'. \quad (10.11)$$

Three of the four geodesic equations are immediately integrated, as follows,

$$\dot{t} = \gamma e^{-A}, \quad \dot{z} = \kappa e^{D}, \quad r^2\dot{\theta} = Le^{C}, \quad (10.12)$$

where γ, κ, L are constants, which might be named energy, linear momentum and angular momentum, respectively.

The fourth equation, for the radial coordinate, is more complicated. In a strictly first-order approximation it can be integrated to

$$(\dot{r})^2 + L^2/r^2 + \gamma^2 A + \kappa^2 D = 2\mathcal{E} = \gamma^2 - \kappa^2 - Q, \quad (10.13)$$

where, as always, $Q = -1$ for tachyons.

The question we address now is whether this equation allows bound orbits, which would imply particles moving in spirals around the z-axis. The effective potential in this radial equation is,

$$\frac{L^2}{2r^2} + U(r), \qquad U(r) = (\gamma^2 + \kappa^2)(V_0(r) + V_3(r)) + (\gamma^2 - \kappa^2)f(r)/2. \quad (10.14)$$

For bound states we would require, at least, that $U(r)$ be a function that increases with r at large r. Both potentials V_0 and V_3 are of this character;

and the function $f(r)$ falls off faster at large r. So this requirement is satisfied. (This is no trivial result. It depends on how various plus-and-minus signs have worked out in this calculation.) The next requirement for bound states is that the integration constant $\mathcal{E}$ have a value equal to or greater than the minimum point of the total effective potential.

To get an idea about this, we can say that for a free tachyon, or one in a scattering state, we would have,

$$\gamma^2 = 1/(v^2 - 1), \quad \kappa^2 = \gamma^2 \, v_z^2, \tag{10.15}$$

which leads to $2\mathcal{E} = \gamma^2(v^2 - v_z^2) > 0$.

However, the integration parameters γ and κ are free from such constraints, and we see that $\mathcal{E}$ certainly can be negative, thus allowing for bound state solutions. Note the contrast between this result and that of the previous Section, which looked at spherical symmetry.

This result expands on what we had seen in the earlier work of Sections 5 and 6, allowing for the circulatory motion in the source, along with the translational motion, relative to the z-axis. We conclude that streams of tachyons could form into coherent localized bundles held together by their own gravitational forces.

We are now in the position to invite speculation about detailed clustering of large numbers of tachyons (assuming that such things exist) in something akin to rope-like structures throughout the universe. How big or small might such ropes be - as tight bundles or wide tubes; how might they connect to themselves - in simple circles or in knots, in doughnuts or in halos; and how might they attach themselves, gravitationally, to large masses of ordinary matter? Model-making and detailed calculations are needed to work out the answers to these questions; and further expert analysis is needed to see how such models may comport with known observational data.

11 Gravitational lensing

The deflection of light beams by the gravitational field of dense matter is an important tool in the observation of galactic structures; and it has been used to impute the distribution of dark matter. If tachyonic matter actually exists, and is collected in static bundles, as has been suggested in the analysis of this paper, then one would want to ask how that might contribute to gravitational lensing.

In the standard analysis, one works from the assumption that the source of gravitational fields is slow moving massive matter; and so one starts with the Newtonian potential, which we have called $V_0(\mathbf{x})$ and carries out a sort of ray-tracing calculation to get the pattern of light beam deflection.

If tachyonic matter is a significant presence, then this calculation needs to be augmented. It turns out that one should use,

$$V_0(\mathbf{x}) + V_3(\mathbf{x}) = -G \int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}[T_{00}(\mathbf{x}') + T_{33}(\mathbf{x}')], \tag{11.1}$$

where the 3-axis denotes the original direction of the light beam. For low energy tachyons, the T_{33} could be quite large.

12 Summary

There are several new results noted in this paper about how faster-than-light particles (tachyons) would behave under the General Theory of Relativity.

First is the form of the energy-momentum tensor $T^{\mu\nu}$, where the spatial components would be larger than the time component. This is in contrast with the situation for ordinary matter; and it raises the question of how this might effect theoretical models of the overall structure of the universe, as in the Robertson-Walker metric.

Second is the attractive gravitational force between tachyons and the suggestion of a stable model for a closed stream of such particles. This leads to questions about how such structures, if they exist, might relate to astrophysical observations of significance in the context of current cosmological models. The possibility of tachyons explaining some or much of what is currently called "dark matter" is an obvious thought, which I leave to better experts for consideration.

Another significant result, noted in Section 7 , is the absence of a long range gravitational force ($\sim 1/r^2$) produced by localized static bundles of low energy tachyons. This prediction is something that may be readily subjected to observational verifiction, to prove or disprove the substantial existence of such tachyons. A recent study [5] of tidal properties of satellite galaxies may be relevant.

In Appendix B we have noted a few additional aspects of tachyon kinematics that might lead to observational tests of whether or not these things actually exist.

This paper has been about classical particles that move faster than light, in the context of General Relativity. Further theoretical study is needed to look into tachyonic fields, how they propagate and how they may be quantized.

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Appendix A: Fear of tachyons

Lets start by reciting a version of the most common paradox in considering the possibility of faster-than-light particles (tachyons) within the Special Theory of Relativity. [6]

A space Ship moves away from Earth (along the positive x-axis) with velocity $v < 1$. At time t_A, when the Ship is at coordinate x_0, a tachyon is sent out from Earth with velocity $V > 1$ in the direction of the Ship. This is event A. That tachyon reaches the Ship (this is Event B) at a later time t_B. This is the story in the Earth frame of reference; and it may be summarized with the following formula.

$$t_B - t_A = \frac{x_0}{V - v}. \tag{A.1}$$

What is the story as seen in the space Ship frame of reference? Here we use the standard formulas for a Lorentz transformation and come to the formula

$$t'_B - t'_A = \gamma(1 - vV)(t_B - t_A). \tag{A.2}$$

This says that, if $vV > 1$, then the tachyon arrives at the Ship before it was sent from Earth according to observers in the Ship frame of reference.

This is the first chapter of the paradox. It is no surprise to those of us with some sophistication with the Special Theory of Relativity. We know that if two events are separated by a space-like interval, then the actual time sequence of those two events may be different as seen in different Lorentz frames.

The full horror of anti-causal happenings appears when one adds a second chapter to this story; observers in the Ship respond to receiving the tachyon by sending another tachyon back to Earth, even faster, and this is seen to arrive back on Earth *before* the first signal was sent out.

In order to avoid jumping to the conclusion that tachyons cannot exist, we should ask what lesser restrictions upon them might be considered. The following is a list of assumptions used in the story recited above.

a) Special Relativity is correct;

b) Tachyons exist;

c) Tachyons can be created and sent out at will;

d) Tachyons can be detected reliably.

It is assumptions (c) and (d) that we should address first, before we reject assumption (b).

We can suggest that if tachyons exist, they should occur as particle-waves of the sort we describe in the quantum theory; they may be emitted and absorbed by interacting with ordinary matter but these interactions must be very weak. This could allow for a very uncertain experimental set-up regarding controlled emission and detection (absorption) of tachyons. One would want to make a quantitative theory and show that the probability of observing the sort of nasty "paradox" described above is swamped by unavoidable uncertainty or background noise.

One simple example is the following. Suppose the emission of the tachyon, as described above, was governed by a wave equation and appeared to observer A as a wave packet such as this.

$$\varphi(x,t) \sim e^{i(kx-\omega t)}\ e^{-\Gamma(t-x/V)}\ \theta(t-x/V), \tag{A.3}$$

where $V = k/\omega$ is the group velocity. This packet has a size, in the original reference frame, of $\Delta x = V/\Gamma$. If we perform the Lorentz transformation on these coordinates to see what this wave packet looks like in the frame of observer B, then we find that the size of the wave packet is $\Delta x' = V/[\gamma\Gamma(1 - vV)]$. Thus, when we approach that critical condition when $vV \approx 1$, the wave packet is so large that it encompasses both the "emitter" and the "receiver" at the same time. This gives us a view of how wave properties for tachyons can indeed erase the distinction between "emission" and "absorption" of a particle that travels faster than light. There is no paradox here.

One common idea is that neutrinos might be a candidate for tachyonity: they have weak interactions and are nevertheless plentiful throughout the universe. Against this idea, one might argue that neutrinos cannot be restricted by disallowing assumption (c) and/or (d), above, because of the possibility (in fact an established experiment) of using an accelerator to create a beam of neutrinos, which can be pulsed at the emitter and then detected reliably at a distant absorber. To refute this objection, one should notice a particular condition for the "paradoxical" situations discussed earlier. In order to get the effect of time-sequence reversal, between the emission and the absorption of a tachyon, one must have the source and the receiver moving *away* from each other. If they are moving toward each other, then there is no such effect predicted by the Lorentz transformation. (Notice what happens if you change the sign of v in Eq. (A.2).) The arrangement of an accelerator sending out a large pulse of particles in the forward direction of the high energy beam involves just this second situation: source and receiver approaching one another; so there is no paradox to be expected here.

Of course, other arguments may be advanced and need to be studied. But, at least at present, I believe this is a constructive step in clarifying this controversial topic. Tachyons may exist and should not be dismissed from consideration in either theoretical or experimental studies.

Appendix B: Kinematical musings

If we calculate the number of free tachyon states that are "on the energy shell" inside a spatial volume V, we get

$$\sum_{states} \delta(E - E_0) = \frac{V}{2\pi^2}\, E_0\, p_0 \tag{B.1}$$

which is exactly the same formula one has for normal (slower-than-light) particles.

So, if one wanted to ask whether neutrinos might actually be tachyons, then the search for neutrino mass, as carried out by looking at the end point of electron energy spectrum in beta-decay, takes on new possibilities. The phase space factor (B.1) then gives us the spectrum for electron energies E_e:

$$(E_0 - E_e)^2 \quad for\ massless\ neutrinos \tag{B.2}$$

$$(E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m^2} \quad for\ ordinary\ massive\ neutrinos \tag{B.3}$$

$$(E_0 - E_e)\sqrt{(E_0 - E_e)^2 + m^2} \quad for\ tachyon\ neutrinos \tag{B.4}$$

The second version shifts the end point of the energy spectrum by m while the third one does not shift it. But each of the three versions has its own distinctive shape near the end point: The first comes in with zero slope; the second with infinite slope; and the third with finite slope. Earlier experimental results, which did not survive, led to the hypothesis of tachyonic neutrinos.[7]

An intriguing aspect is that the Dirac equation for tachyons gives special prominence to the helicity of particle states.

Also, regarding neutrinos, there are the experiments that show the effects of mass-mixing. The standard view is based upon the expansion of the wavefunction in terms of a small mass

$$e^{i(kx-\omega t)} \sim e^{i(k(x-t)-(m^2/2k)t)}, \tag{B.5}$$

and then subsequent treatment of m^2 as a matrix that causes mixing with other neutrino states.

For tachyons there would be only the slightest difference:

$$e^{i(kx-\omega t)} \sim e^{i(\omega(x-t)+(m^2/2\omega)x)}, \tag{B.6}$$

and since these are high energy particles, $\omega \approx k$ and $x \approx t$, it would look very much the same.

One can also ask how the simplest model for cosmological expansion, with uniform distribution of matter, would look in the case of a tachyon-dominated universe. The basic equations are

$$\frac{d}{da}(\rho a^3) = -3Pa^2 \tag{B.7}$$

$$\rho = const. \times \int E\, d^3p/(e^{E/kT} \pm 1), \tag{B.8}$$

where a is the scale factor and T is the temperature, with ρ, P being the energy density and pressure as previously defined. In the case where $kT >> mc^2$, where m is the tachyonic mass, then we have the familiar results for any relativistic particle: $\rho = 3P \sim T^4 \sim a^{-4}$. Now we want to see what this looks like in the other limit, when $kT << mc^2$. Then we find that $\rho \sim T^3$ and, if we assume Fermi statistics, then the pressure P goes as T. This leads to a relation between T and a as follows:

$$\left(\frac{T}{T_1}\right)^2 = 2\left(\frac{a_1}{a}\right)^2 - 1. \tag{B.9}$$

This is a rather unusual result, predicting that the temperature goes to zero at some finite expansion. Of course, this very simple-minded calculation needs to be studied further; and it might lead to some observational comparisons that help us decide whether such tachyons exist. For some related investigations, see [8].

Appendix C: Smoothed cylindrical problem

In order to arrive at a clear solution of the field equations for the static-cylindrical problem studied in Section 5, we need to specify some structure for the source. We shall do this in a backwards manner: make up a solution for the metric and see what source it produces from the Einstein field equations. We start with a modified version of the solutions given in the earlier Section.

$$A = as^{\alpha}, \quad B = bs^{\beta}, \quad C = cs^{\gamma}, \quad s = \sqrt{r^2 + \epsilon^2} \tag{C.1}$$

$$\beta = \alpha^2/(\alpha + 2), \quad \gamma = -2\alpha/(\alpha + 2). \tag{C.2}$$

We then calculate,

$$G_{00} = -\frac{A}{2B} \frac{\alpha(\alpha + 4)}{\alpha + 2} \frac{\epsilon^2}{s^4}, \qquad G_{33} = \frac{C}{A} G_{00} \tag{C.3}$$

$$G_{11} = -\frac{1}{2} \frac{\alpha^2}{\alpha + 2} \frac{\epsilon^2}{s^4}, \qquad G_{22} = \frac{r^2}{B} G_{11}. \tag{C.4}$$

Next, we integrate these densities $\int_0^\infty r\, dr$ and find, in the limit of small α and small ϵ:

$$\int G_{00} \longrightarrow -\frac{a}{2b}\, \alpha = -4G\rho\gamma \tag{C.5}$$

$$\int G_{33} \longrightarrow -\frac{c}{2b}\, \alpha = -4G\rho\gamma v^2. \tag{C.6}$$

This verifies the results given earlier.

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