A Series Solution to the Luminosity Distance in a Flat ΛCDM Universe

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Abstract

Cosmological observations indicate that our universe is flat and dark energy (DE) dominated at present. The luminosity distance plays an important role in the investigation of the evolution and structure of the universe. Nevertheless, the evaluation of the luminosity distance d_L is associated computationally heavy numerical quadratures in practice. In this Letter we find a series solution of the luminosity distance in a spatially flat ΛCDM cosmological model. And it is further shown that the series solution has a relative error of less than 0.36% for any relative parameter β ($\beta \equiv \frac{\Omega_m}{\Omega_\Lambda}$) from zero to four, i.e. $0.2 < \Omega_\Lambda < 1$ and redshift $z > 0.1$ when the order of the series is $n = 100$.

Keywords: Cosmological distances, Numerical methods

1. Introduction

The computation and numerical evaluation of distances is frequently encountered in the research of cosmological phenomena. In practice, it is common to compute the various cosmological distances as a function of the redshift z under certain cosmological models. Current cosmological observations indicate that the universe is expanding, has a matter content $\Omega_{\rm m} \sim 0.3$ and is spatially flat. Further, a ΛCDM model fits the data well, and is frequently used as a fiducial or background model. In the ΛCDM model, various cosmological distances can be expressed in terms of the elliptic integrals [1, 2].

Some authors have focused on the luminosity distance d_L in the ΛCDM model and derived numerical approximations for the efficient and accurate evaluation of $d_L(z)$ giving the cosmological parameters $\Omega_{\rm m}$ and Ω_Λ [3, 4]. The computation of d_L is useful in the analysis of distance-redshift relations of type Ia supernovae, and the approximation for d_L can be directly used in the evaluation of other distances, for instance the angular diameter distance or the comoving distance [1]. In this Letter we present another series approximation that is of considerable accuracy.

Our approximation to d_L can be expressed as follows:

$$d_L(z) = \frac{c}{H_0\sqrt{\Omega_\Lambda}}(1+z)l(z),$$

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where

$$l(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{1+\beta}{2} \right)^{-\frac{2n+1}{2}} \left[\sum_{i=0}^{n} \binom{n}{i} \left(\frac{\beta - 1}{2} \right)^{n-i} \frac{1 - (1+z)^{-3i-1/2}}{3i + 1/2} \right]. \quad (1)$$

where the parameter β is defined as $\beta \equiv \frac{\Omega_\mathrm{m}}{\Omega_\Lambda}$, H_0 is the Hubble constant and n is the order of the series.

The rest of this Letter we will derive equation (1) in Section 2, and discuss its application in numerical computation in Section 3.

2. Series Approximation

The luminosity distance d_L is related to the comoving distance $r(z)$ by $d_L = a_0 r(z)(1+z)$. The comoving distance r enters Friedmann-Lemaître-Robertson-Walker (FLRW) metric as the radial component of the spatial coordinates:

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - a^2(t) \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2 \theta \mathrm{d}\varphi^2 \right], \quad (2)$$

where $a(t)$ is the scale factor characterizing the cosmic expansion and k is the sign of the spatial Gaussian curvature. This is the metric characterizing the homogeneous, isotropic, and expanding universe. In the spatially flat case, we have $k = 0$. For the photon propagating through the expanding universe along the null geodesic in the line-of-sight direction, we set $\mathrm{d}s^2$ to zero, obtaining

$$r = \int_t^{t_0} \frac{c\mathrm{d}t}{a(t)} = \int_a^{a_0} \frac{c\mathrm{d}a}{a^2 H}, \quad (3)$$

where H is the Hubble parameter $H = \dot{a}/a$. For the ΛCDM universe, the solution to Friedmann's equations gives the expansion rate or Hubble parameter

$$H = H_0 \sqrt{\Omega_\mathrm{m} \left(\frac{a_0}{a} \right)^3 + \Omega_\Lambda}. \quad (4)$$

Equations (3) and (4) thus relates r to a.

The scale factor $a(t)$ is related to the cosmological redshift z: $a_0/a = 1 + z$, and the differentials therefore satisfies

$$\mathrm{d}a = -\frac{a_0}{(1+z)^2} \mathrm{d}z. \quad (5)$$

Substituting Equations (4) and (5) into Equation (3), it is straightforward to derive

$$r(z) = \frac{c}{a_0 H_0 \sqrt{\Omega_\Lambda}} \int_{\frac{1}{1+z}}^{1} \frac{\mathrm{d}x}{\sqrt{x^4 + \beta x}} \quad (6)$$

using a change-of-variable $x = 1/(1+z)$.

The integral factor in Equation (6),

$$l(z) = \int_{\frac{1}{1+z}}^{1} \frac{\mathrm{d}x}{\sqrt{x^4 + \beta x}}, \tag{7}$$

is a special case of the elliptic integral. Performing Taylor series expansion of the integrand around $x_0 = \sqrt[3]{(1-\beta)/2}$ and integrate it termwise, we find

$$l(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{1+\beta}{2}\right)^{-\frac{2n+1}{2}} \left[\sum_{i=0}^{n} \binom{n}{i} \left(\frac{\beta-1}{2}\right)^{n-i} \frac{1-(1+z)^{-3i-1/2}}{3i+1/2}\right]$$

which is the result presented in Equation (1) (see Appendix A for the proof of convergence).

3. Analysis and Conclusion

In practice, it is not possible to calculate Equation (1) up to infinite terms. If we cut off the order $n = N_{\max}$, we effectively obtain an approximation $\hat{d}_L$ for the exact $d_L(z)$, and also consider its error by analyzing the relative percentage error

$$\varepsilon = \left|\frac{\hat{d}_L - d_L}{d_L}\right| \times 100\%. \tag{8}$$

For $N_{\max} = 100$, we plot the error ε as a function of z and cosmological parameters in Figure 1 with logarithmic coordinates. In the rectangular area defined by $(z, \beta) \in (0.1, 10) \times [0, 4]$, we find that the error is maximized at $z = 0.1$ and $\Omega_\Lambda = 0.2$, with $\varepsilon_{\max} \approx 0.36\%$. As z and Ω_Λ increase, the error decreases rapidly.

We can further compare the method outlined in this Letter to earlier results in Refs. [3, 4]. In Figure 2, we plot the error ε of our method and those in Refs. [3, 4] for $\Omega_\Lambda = 0.7$. Evidently, our method is not well optimized for very small redshift ranges where $z \sim 0.01$. However, with the error less than 0.096%, it is comparatively more accurate for $z > 0.1$, where most of the type Ia supernova data lies. As z increases, the error of our method quickly diminishes.

We have adopted $N_{\max} = 100$ in the above analysis of the method's accuracy. A larger value of $N_{\max}$ could further reduce the error, however in practice not much can be gained by adding such higher-order corrections at the cost of more computing power needed. The optimal value of $N_{\max}$ should be estimated on a cost-benefit basis depending on the nature of the work and the required accuracy of numerical evaluation.

And once we get the approximative expression of d_L, the angular diameter distance in a flat universe with Ω_Λ follows, i.e. $d_A = d_L/(1+z)^2$.

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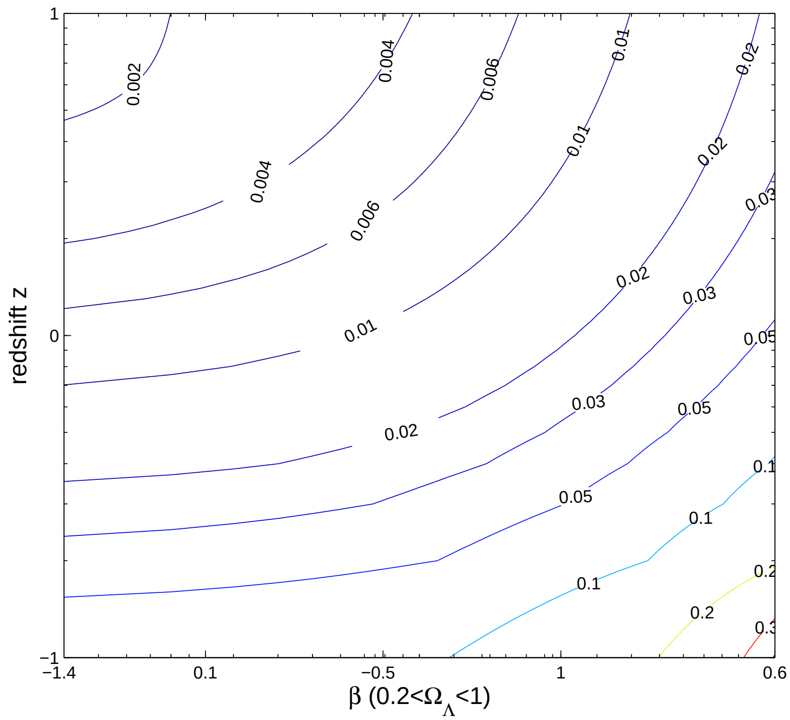

Figure 1: Contour plot of the percentage error ε. When computed up to $N_{\max} = 100$, the error of our method is well controlled within 0.36%.

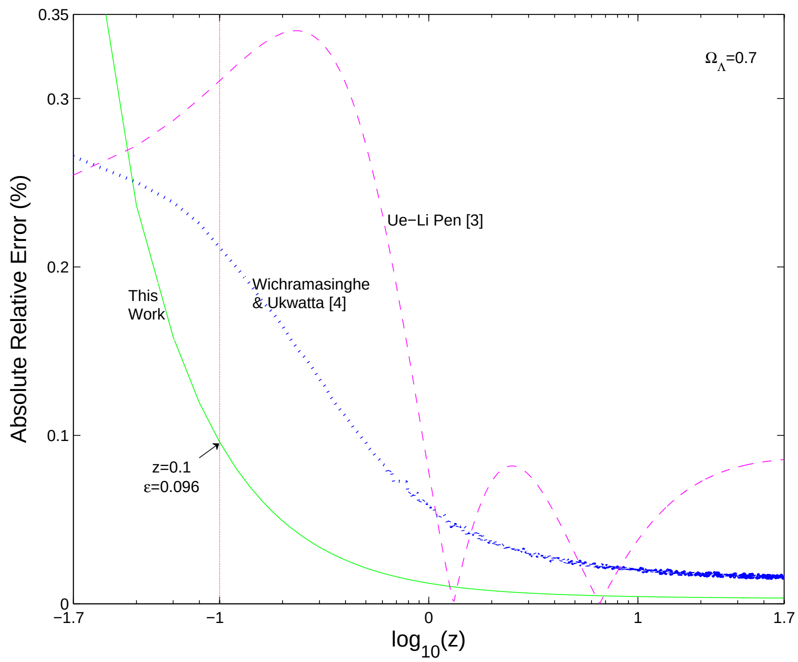

Figure 2: Comparison of the accuracy of our methods and those found in Refs. [3, 4]. The dash, dotted and solid curves represent the relative error of Ref. [3], Ref. [4], and ours respectively. The cosmological parameter Ω_Λ is fixed at 0.7. Our method gives a larger relative error in the low redshift range $z < 0.03$, but as z increases, the error approaches zero faster than those of the other methods.

Appendix A. Derivation of the Equation (1)

We define a function f: $f(x) \equiv (x^3+\beta)^{-\frac{1}{2}}$, then Taylor expanding f around $x_0^3 = \frac{1-\beta}{2}$:

$$f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!}\left(\frac{1+\beta}{2}\right)^{-\frac{2n+1}{2}}\left(x^3+\frac{\beta-1}{2}\right)^n,$$

Then we get

$$f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!}\left(\frac{1+\beta}{2}\right)^{-\frac{2n+1}{2}}\left[\sum_{i=0}^{n}\binom{n}{i}\left(\frac{\beta-1}{2}\right)^{n-i} x^{3i}\right].$$

From the expanding equation of $f(x)$ above, hence we can get equation (1) with $l(z) = \int_{\frac{1}{1+z}}^{1} x^{-1/2} f(x)dx$ easily.

Next let's prove that the function $x^{-1/2}f(x)$ can be integrated for finite redshift z, namely, it satisfies the condition of uniform convergence for any $0 \le z < \infty$ and $0 < \Omega_\Lambda < 1$. For verifying the result, we only need to consider the properties of $f(x)$. At first, we set a function sequence $\{S_n\}$, which the general form is

$$S_n = (-1)^n \frac{(2n-1)!!}{(2n)!!}\left(\frac{1+\beta}{2}\right)^{-\frac{2n+1}{2}}\left(x^3+\frac{\beta-1}{2}\right)^n,$$

then $f(x) = \sum_{n=0}^{\infty} S_n$. A noticeable character about x is $x \in (0,1]$, we should guarantee that the domain of convergence of $f(x)$ must cover the interval $(-1,1)$, namely the radius of convergence needs to satisfy $R \ge 1$. If expanding $f(x)$ around $x_0^3 = \frac{1-\beta}{2}$, and assuming $R = 1$, namely:

$$\lim_{n\to\infty}\left|\frac{S_{n+1}}{S_n}\right| = \left|\left(\frac{1+\beta}{2}\right)^{-1}\left(x^3+\frac{\beta-1}{2}\right)\right| \le 1,$$

we find,

$$-\beta \le x^3 \le 1,$$

or

$$0 \le x \le 1.$$

The result is just compatible with the domain of x. In addition, with straightforward verification, we can find that the series is also converged when $x = 1$. Hence we get a conclusion that the expanding function of $f(x)$ is uniform convergence.

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