Hawking Radiation and Entropy from Horizon Degrees of Freedom

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We study the thermodynamic properties of horizons using the dynamical description of the gravitational degrees of freedom at a horizon given in [4]. We use the action of the horizon degrees of freedom to calculate the horizon entropy using the Cardy formula, and obtain the expected Bekenstein-Hawking entropy. We also couple the gravitational degrees of freedom at the horizon to a classical background scalar field, and show that Hawking radiation is produced.

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Introduction. Ever since the work of Bekenstein and Hawking in the 1970s, which established that the laws of thermodynamics can be adapted to describe black holes[1, 2], there have been repeated attempts to provide a microscopic description of black hole horizons, and of horizons in general.

There have also been attempts to find an *effective* theory of horizon microstates, that can describe the degrees of freedom of the horizon without reference to the underlying theory of quantum gravity. This suggestion is particularly plausible because of the universal appearance of conformal symmetry in the neighbourhood of a horizon, which indicates that the dynamics of a horizon will be governed by a two-dimensional conformal field theory (CFT). There are hints that this CFT will be a Liouville theory: for example, the dynamical action of the diffeomorphisms that preserve asymptotically AdS_3 spaces has been shown to be a Liouville theory living at the spatial infinity of AdS_3[3].

In [4], we imposed conditions that preserved the existence and essential characteristics of a horizon, and found the diffeomorphisms that preserved these conditions. We then derived a dynamical action for these diffeomorphisms from the Einstein-Hilbert action. The resulting action was almost exactly the Liouville action in a curved background, and was found to describe a free two-dimensional conformal field in an infinitesimal neighbourhood of the horizon.

In this paper we study the thermodynamic properties of this theory. First we use the fact that the gravitational degrees of freedom at a horizon exhibit a two-dimensional conformal symmetry to calculate the entropy of the horizon. Using the Cardy formula, we find that we can reproduce the Bekenstein-Hawking entropy.

A derivation of the Bekenstein-Hawking entropy is a useful criterion for judging the validity of a theory that describes horizon microstates: however, it is far from being a proof that the theory is correct. Another valuable indicator is seeing whether or not the theory predicts the emission of Hawking radiation. Many different methods have been devised for deriving Hawking radiation[1, 5–7]. One common feature of all these works is that they analyze quantum matter fields in a classical black hole background, and derive Hawking radiation as a consequence of quantum fields living in a curved space. In order to have a complete picture of horizon thermodynamics, it is important to do the reverse: couple quantized gravitational degrees of freedom to classical matter, and produce the blackbody spectrum of Hawking radiation.

A few steps have been taken in this direction: in [8], the near-horizon region was modeled by a Liouville conformal field theory, which was used to derive the flux of Hawking radiation (though not the spectrum.) An alternative approach was taken in [9], where the Liouville theory of diffeomorphism degrees of freedom at the spatial infinity of AdS_3 was coupled to scalar field matter. It was shown that the decay rate of the BTZ black hole exactly matched the spectrum of Hawking radiation, including greybody factors. In this work we follow a similar approach, but instead of coupling a classical scalar field to a conformal field theory at the boundary, we identify a coupling in a neighbourhood of the horizon, and obtain the Hawking radiation spectrum.

Conformal Field Theory at the Horizon. We first review some of the notation and concepts used to study the diffeomorphism degrees of freedom at the horizon in [4]. We use the system of **Gaussian null coordinates** (denoted "GN coordinates")[10], which are well suited for studying horizons as they are adapted to null hypersurfaces. In the neighbourhood of any null hypersurface Δ, we can define coordinates (u, r, x^i) such that the metric takes the form:

$$\mathrm{d}s^2 = rF\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r + 2rh_i\,\mathrm{d}u\,\mathrm{d}x^i + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \quad (1)$$

We will also use the coordinate $\tilde{r} := -\ln r$, which puts the metric in the form

$$\mathrm{d}s^2 = e^{-\tilde{r}}\left(F\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}\tilde{r} + 2h_i\,\mathrm{d}u\,\mathrm{d}x^i\right) + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad (2)$$

where F, h_i, and g_{ij} are now taken to be functions of $\tilde{r}$. The horizon is now located at $\tilde{r} \to \infty$. We call these coordinates **tortoise Gaussian null coordinates**. Finally, we may define conformal coordinates (x_+, x_-) in terms of (u,r) such that the metric takes the form

$$\mathrm{d}s^2 = 2g_{+-}\left(\mathrm{d}x_+\,\mathrm{d}x_- + h_{+i}\,\mathrm{d}x^+\,\mathrm{d}x^i\right) + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad (3)$$

and g_{+-} has a simple root at $r=0$. Without loss of generality, we can impose $\partial_+ u = 0$ and $\partial_r x_+ = 0$.

Our horizon boundary conditions are based on the notion of weakly isolated horizons (WIHs)[15]. We assume that we have a weakly isolated horizon Δ in our spacetime, and that we can define GN coordinates in a neighbourhood of Δ so that the horizon lies at $r=0$. In order for Δ to be a WIH, it should have zero expansion $\theta_{(l)} = 0$ for all normal vectors l^μ, where the expansion is defined as $\theta_{(l)} = q^{ab}\nabla_a l_b$ for an inverse q^{ab} of the intrinsic metric q_{ab} on Δ. In GN coordinates, the requirement that the horizon satisfy $\theta_{(l)} = 0$ is equivalent to saying that $\partial_u g_{ij} = O(r)$. We then impose conditions that preserve the essential characteristics of the horizon by demanding that, after applying a diffeomorphism:

1. There is still a null hypersurface at $r=0$.
2. In conformal coordinates, the metric remains in the form given by Eq.(3).
3. The induced metric on the $r=0$ hypersurface is preserved. This also ensures that the null hypersurface continues to satisfy $\theta_{(l)} = 0$.

The diffeomorphisms preserving these conditions were found in [4] to have the form:

$$\begin{aligned} \xi^+ &= \xi^+(x_+) + O(r) \\ \xi^- &= \xi^-(x_-) + O(r) \\ \xi^i &= O(r) \end{aligned} \quad (4)$$

A dynamical action was derived for these diffeomorphism degrees of freedom in a neighbourhood of the horizon, by evaluating the Einstein-Hilbert action for the new metric $g'_{\mu\nu} := g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$ after applying the diffeomorphism, and taking the background metric on-shell in the near-horizon region. This allows us to isolate the gravitational fluctuations about the background metric that preserved the existence and characteristics of the horizon. Defining the field $\phi = \phi(x_+, x_-)$ by $g'_{\mu\nu} := e^\phi g_{\mu\nu} \approx (1+\phi)g_{\mu\nu}$, the action has the form:

$$I_{\mathrm{hor}} = \frac{a_\Delta}{16\pi G}\int \mathrm{d}^2x\,\sqrt{-\hat{g}}\,\left(\partial_a\phi\partial^a\phi - \phi\hat{R} + \lambda e^\phi\right) \quad (5)$$

where a_Δ is the cross-sectional area of the horizon, $\hat{g}$ is the induced metric on the (x_+, x_-) submanifold, and $\lambda = \frac{4\Lambda}{n-2}$ (except in the case $n=2$, when $\lambda = 0$). The equation of motion for ϕ becomes that of a *free* two-dimensional conformal field infinitesimally close to the horizon.

Horizon Entropy. We now calculate the entropy of a large class of horizons using the Cardy formula, a remarkable result that allows us to determine the entropy of any system with a two-dimensional conformal symmetry. The Cardy formula states that, given any unitary two-dimensional conformal field theory, the asymptotic density of states at eigenvalues Δ^+ and Δ^- is given by:

$$\ln\rho(\Delta^+, \Delta^-) \sim 2\pi\sqrt{\frac{c^+\Delta^+}{6}} + 2\pi\sqrt{\frac{c^-\Delta^-}{6}} \quad (6)$$

As we have found that the gravitational degrees of freedom at a horizon have an effective description as a 2D CFT, we can apply the Cardy formula to calculate the entropy of the horizon.

The energy-momentum tensor derived from I_{hor} is

$$\begin{aligned} T_{ab} = \frac{a_\Delta}{16\pi G}\Big[&\partial_a\phi\partial_b\phi - \hat{g}_{ab}\left(\frac{1}{2}\partial_a\phi\partial^a\phi + \frac{\lambda}{2}e^\phi\right) \\ &- \left(\hat{g}_{ab}\Box\phi - \nabla_a\nabla_b\phi\right)\Big] \end{aligned} \quad (7)$$

The first step in calculating the horizon entropy is to compute the algebra of charges corresponding to the symmetries of the CFT. The symmetries are given by the diffeomorphisms $\xi^+(x^+)$ and $\xi^-(x_-)$. In order to obtain a countable set of charges, we impose a cutoff scale l and define

$$\xi_n^\pm = \frac{l}{2\pi}e^{\frac{2\pi i}{l}n x_\pm} \quad (8)$$

The generators of the corresponding conformal transformations are:

$$\begin{aligned} L_n^\pm &= \int_{-l/2}^{l/2}\mathrm{d}x_\pm\,\xi_n^\pm T_{\pm\pm} \\ &= \frac{a_\Delta}{16\pi G}\frac{l}{2\pi}\int_{-l/2}^{l/2}\mathrm{d}x_\pm\,e^{\frac{2\pi i}{l}n x_\pm}\left[(\partial_\pm\phi)^2 + \partial_\pm^2\phi\right] \end{aligned} \quad (9)$$

Classically, we do not allow incoming modes from the horizon, so we will only consider the generators L_n^+ when calculating physical charges, and drop the superscript "+" from now on for clarity of notation.

To evaluate the algebra, we define coordinates (t,ρ) such that $x_+ = t+\rho$, $x_- = t-\rho$, and define t to be the time coordinate. We see from the action in (5) that the canonical momentum Π conjugate to the field ϕ is $\Pi := \delta\mathcal{L}/\delta(\partial_t\phi) = \frac{a_\Delta}{16\pi G}\partial_t\phi$. We thus obtain the Poisson bracket

$$\{\phi(t,\rho), \partial_t\phi(t,\rho')\} = \frac{16\pi G}{a_\Delta}\delta(\rho-\rho') \quad (10)$$

Using the Poisson bracket to evaluate the algebra of charges, we find that

$$\{L_n, L_m\} = i(n-m)L_{n+m} + \frac{ic}{12}n^3\delta_{n+m,0} \quad (11)$$

with central charge

$$c = \frac{3a_\Delta}{4G} \quad (12)$$

Now we simply need to determine the eigenvalue of L_0 for a given horizon in order to calculate its entropy. We present two different ways of calculating L_0.

Calculation 1. We would like to evaluate L_0 for the classical horizon configuration: this corresponds precisely to the case when there are no fluctuations about the horizon, so that $\phi = 0$, giving $L_0 = 0$. This is not a serious problem: it is merely due to the fact that the overall normalization of L_0 has not been fixed. One way to determine the normalization is to posit a ground state for the theory, and demand that L_0 be zero on the ground state.

In order to determine the ground state, we look at the form of static metrics with Killing horizons. A general static metric with non-extremal Killing horizon at $r = 0$ can be written in the form[16]:

$$\mathrm{d}s^2 = e^{\sigma(x,r)} \left(-\mathrm{d}t^2 + \frac{\beta_H^2 \mathrm{d}r^2}{4r^2} + \frac{g_{ij}(x,r)}{r} \, \mathrm{d}x^i \, \mathrm{d}x^j \right) \tag{13}$$

with $\sigma(x,r) = \ln r + \sigma_0(x) + O(r)$, and $g_{ij}(x,r) = g_{ij}^{(0)}(x) + O(r)$. The parameter β_H is the inverse Hawking temperature of the horizon. The metric can be put in Gaussian null form by defining $u := t + \frac{\beta_H}{2} \ln r$, and then put in conformal form by defining $x_+ := u$ and $x_- := -(u + \beta_H \ln r)$.

Extremal black holes have zero temperature. Therefore it is reasonable to assume that the ground state of the horizon theory is the *extremal* horizon configuration, when the function $F(u, r, x^i)$ in the Gaussian null coordinate system has a simple pole at $r = 0$. For static metrics with a Killing horizon, this corresponds to a fluctuation ϕ of the metric such that $e^\phi = r$, or equivalently, $\phi = -\frac{1}{\beta_H}(x_+ + x_-)$. We claim that the same conclusion holds for all other horizons satisfying our boundary conditions. Evaluating L_0 for this solution gives

$$L_0 = \frac{a_\Delta l^2}{32\pi^2 G \beta_H^2} \tag{14}$$

We can now calculate the horizon entropy using the Cardy formula, and find:

$$S_H = \frac{a_\Delta l}{8G\beta_H} \tag{15}$$

We obtain the desired result of $S_H = \frac{a_\Delta}{4G}$ with $l = 2\beta_H$.

Calculation 2. Another way to fix the normalization of L_0 is by demanding that the charges L_n satisfy the usual form of the Virasoro algebra, so that the charges L_{-1}, L_0, and L_1 form a proper $sl(2,\mathbb{R})$ sub-algebra. This can be achieved by shifting L_0 by a constant. By shifting

$$L_0 \to L_0 + \frac{c}{24}, \tag{16}$$

the algebra becomes

$$\{L_n, L_m\} = i(n-m)L_{n+m} + \frac{ic}{12}(n^3 - n)\delta_{n+m,0} \tag{17}$$

Since L_0 is originally zero evaluated on the classical configuration, we find that $L_0 = \frac{a_\Delta}{32G}$, and the horizon entropy is

$$S_H = \frac{\pi a_\Delta}{8G} \tag{18}$$

This result differs by a factor of $\frac{2}{\pi}$ from the Bekenstein-Hawking entropy, but we have at least obtained an answer that is proportional to the desired result, with the constant of proportionality being a pure number.

Hawking Radiation. We now investigate another aspect of horizon thermodynamics: Hawking radiation. We couple the gravitational degrees of freedom in the near-horizon region to a classical scalar field, and show that we can produce Hawking radiation from this coupling. Unlike most derivations of Hawking radiation, we will quantize the gravitational theory, and treat the scalar field as a classical background, as in [9].

To proceed, it is enough to know that the gravitational degrees of freedom in a neighbourhood of a horizon can be described by a 2D CFT, and to know how the metric components change under the infinitesimal conformal transformations of the CFT. As previously discussed, we eliminate incoming modes from the horizon when evaluating the charges that generate the conformal transformations. As a result, when we quantize the CFT, we obtain only a chiral half of the original theory, with infinitesimal conformal transformations $x_+ \to x'_+(x_+)$. The fields and operators of the CFT are the components of the metric and objects constructed from the metric, since the metric is the only dynamical field in our framework.

For convenience, we use the tortoise GN coordinates given in (2) as well as conformal coordinates. As $x_+ = x_+(u)$ and $u = u(x_+)$, infinitesimal transformations of u are equivalent to transformations of x_+. Conversely, since $\partial_{\tilde{r}} x_+ = 0$, we find that the coordinate $\tilde{r}$ does not transform under transformations of x_+. It follows that the conformal weight of a metric component in tortoise GN coordinates is equal to the number of lower u indices it has. For example, $g_{u\tilde{r}}$ has conformal weight 1.

In tortoise GN coordinates, the scalar field action in the near-horizon region is

$$I_s = \int_{\mathcal{M}} \sqrt{-g} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + \int_{\Delta} \sqrt{-g} g^{\tilde{r}\mu} \psi \partial_\mu \psi \tag{19}$$

The boundary term is evaluated at the horizon. Inspecting the action, we see that all of the couplings at the boundary have conformal weight 0, while the following terms in the bulk action exhibit a coupling of the classical scalar field to an operator of conformal weight 1:

$$\begin{aligned} I_{\text{int}} &= \int_{\mathcal{M}} \sqrt{-g} \left(g^{\tilde{r}\tilde{r}} \partial_{\tilde{r}} \psi \partial_{\tilde{r}} \psi + g^{ij} \partial_i \psi \partial_j \psi \right) \\ &= \int_{\mathcal{M}} g_{u\tilde{r}} \left(g^{\tilde{r}\tilde{r}} \partial_{\tilde{r}} \psi \partial_{\tilde{r}} \psi + g^{ij} \partial_i \psi \partial_j \psi \right) \end{aligned} \tag{20}$$

When we quantize the near-horizon CFT, these couplings add a perturbation to the CFT action in the form of an operator $\mathcal{O}(u)$ of conformal weight 1. This perturbation will induce transitions between closely spaced states of the CFT, resulting in the emission of radiation.

We assume that the form of the scalar field is not affected by small deformations of the metric. It is not necessary to know the specific form of ψ, but it is instructive to work out a simple example. If the background metric is a spherically symmetric, static metric with a Killing horizon that has the form

$$\mathrm{d}s^2 = -f(x)\mathrm{d}t^2 + \frac{1}{f(x)}\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2, \qquad (21)$$

with $f(x) = \frac{2}{\beta_H}(x-x_h)+O((x-x_h)^2)$, then we can write this metric in GN coordinates by defining $r := x - x_h$ and $u := t + \frac{\beta_H}{2}\ln r$. We can describe ψ as an infinite collection of 2-dimensional scalar fields $\psi_{l,m}$ of the form

$$\psi_{l,m} = e^{i(t-\tilde{r})} + e^{i(t+\tilde{r})} \qquad (22)$$

where $\tilde{r} := \frac{\beta_H}{2}\ln r$ is the radial tortoise coordinate[5]. In this simple case of a static metric with a Killing horizon, substituting the modes $\psi_{l,m}$ into (20) gives a coupling that remains finite even infinitesimally close to the horizon.

This coupling will lead to transitions between the states of the CFT, so that the horizon produces Hawking radiation. We can compute the macroscopic decay rate using standard conformal field theory methods, following the approaches of [13] and [9]. The operator $\mathcal{O}(u)$ introduced by the coupling to the scalar field will lead to a transition amplitude between initial and final states of the horizon in the presence of an external flux with frequency ω of the form:

$$\mathcal{M} \sim \int \mathrm{d}u \langle f|\mathcal{O}(u)|i\rangle e^{-i\omega u}. \qquad (23)$$

Squaring and summing over final states, we get:

$$\sum_f |\mathcal{M}|^2 \sim \int \mathrm{d}u\,\mathrm{d}u' \langle i|\mathcal{O}(u)\mathcal{O}(u')|i\rangle e^{-i\omega(u-u')} \qquad (24)$$

for the decay rate. Since we are assuming that the horizon is in thermal equilibrium and is therefore a thermal state with a well-defined temperature, we average over the intial states assuming that the distribution is given by a Boltzmann spectrum. If the temperature of the horizon is T_H, then the decay rate is given by finite temperature two-point functions, which have the form

$$\langle \mathcal{O}^\dagger(0)\mathcal{O}(u)\rangle_{T_H} \sim \left[\frac{\pi T_H}{\sinh(\pi T_H u)}\right]^2 \qquad (25)$$

In order to evaluate the integrals in (24), we use standard techniques of contour integration and assume that we are calculating emission rates. We find that the emission rate is given by

$$\Gamma \sim \frac{\pi\omega}{e^{\frac{\omega}{T_H}} - 1}, \qquad (26)$$

where we have divided by a factor of ω to account for the normalization of the outgoing scalar. Thus we obtain the familiar blackbody spectrum of Hawking radiation. This method works for any horizon that has a well-defined temperature and can therefore be regarded as a thermal state of our horizon CFT. As we have already stated, we can define the surface gravity κ (and therefore, the temperature) of any horizon that satisfies our boundary conditions as $\kappa = -\frac{1}{2}F|_{r=0}$.

Conclusion. We have investigated the thermodynamic properties of horizons by using the dynamical description of the diffeomorphism degrees of freedom obtained in [4]. Using the Cardy formula, we computed the entropy of the horizon and found it to be at least proportional to the Bekenstein-Hawking entropy. This result suggests that the classical conformal symmetry imposed by boundary conditions at a horizon is enough to determine the entropy of the horizon, without reference to the underlying theory of quantum gravity. The wide applicability of the result to many kinds of horizons, including cosmological and acceleration horizons, indicates that the universality of the Bekenstein-Hawking entropy formula is results from the fact that a two-dimensional conformal symmetry is always induced near a horizon. We also provided evidence for the validity of the effective description of horizon degrees of freedom as a 2D CFT by coupling the effective theory to a classical scalar background and showing that this produces Hawking radiation. Our result shows that although the effective theory is not the "true" theory of quantum gravity, it can provide a way of quantizing the gravitational degrees of freedom at a horizon.

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