Dynamics of Diffeomorphism Degrees of Freedom at a Horizon

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We define a set of boundary conditions that ensure the presence of a null hypersurface with the essential characteristics of a horizon, using the formalism of weakly isolated horizons as a guide. We then determine the diffeomorphisms that preserve these boundary conditions, and derive a dynamical action for these diffeomorphisms in a neighbourhood of the horizon. The action is almost identical to that of Liouville theory, and the equation of motion of the gravitational degrees of freedom approaches that of a free two-dimensional conformal field in the near-horizon region.

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Introduction. Ever since the work of Bekenstein[1], who posited that black hole horizons have entropy, and Hawking[2], who showed that horizons can emit radiation, it has generally been accepted that black holes are thermodynamic objects. It was later shown that these characteristics are shared by other types of horizons, such as cosmological deSitter horizons and even acceleration horizons[3].

A central problem in studying the thermodynamic properties of horizons has been to determine the microscopic theory that accounts for the entropy and temperature of horizons, as well as the phenomenon of Hawking radiation. There have been many proposals, based on string theory[4], loop quantum gravity[5], and other formalisms, but none of them is wholly satisfactory. The derivation of the Bekenstein-Hawking entropy is generally taken to be a criterion for testing the validity of a theory of quantum gravity, and in recent years the well-known formula for the entropy of a horizon has been obtained using a wide variety of techniques (see [6] for a review).

The very fact that so many approaches to the problem of calculating the horizon entropy converge to the same answer, raises the possibility that there is an effective description of the gravitational degrees of freedom at a horizon that does not depend on the details of quantum gravity. The work of Strominger and Carlip, among others, in using the Cardy formula to calculate horizon entropy is strong evidence in favour of this idea[7, 8]. The applicablity of the Cardy formula relies on determining the symmetries that govern the gravitational degrees of freedom in a spacetime with a horizon. The striking feature of this method is that a *classical* conformal symmetry can be enough to determine the entropy, without knowing anything about the underlying quantum theory, lending support to the notion that there is an effective description of the horizon degrees of freedom that is independent of the true theory of quantum gravity. Moreover, the appearance of the Virasoro algebra suggests that this effective theory is a two-dimensional conformal field theory (CFT).

It has been suggested by Carlip that the degrees of freedom of this effective theory are "would-be gauge" degrees of freedom: that is, diffeomorphisms that become dynamical at the horizon due to the presence of boundaries or constraints. Dynamical actions have been derived for such would-be gauge degrees of freedom at spatial infinity in AdS_3 and AdS_5, with the former yielding a Liouville field theory[9], and the latter a theory of four-dimensional conformal gravity[10].

In this work we identify similar dynamical degrees of freedom that arise at a horizon, by restricting to diffeomorphisms that preserve the characteristics of the horizon. Most work on horizon entropy has concentrated on black hole horizons, but there is good reason to believe that all horizons have thermodynamic properties[11]. We therefore use a formalism that covers a very wide class of horizons. We find that the dynamical action for the gravitational degrees of freedom describes a two-dimensional theory that becomes a free 2D CFT in an infinitesimal neighbourhood of the horizon. The action is almost identical to the Liouville action.

Earlier works have found a Liouville action for the horizon degrees of freedom, either choosing the dilaton field[12], or the conformal factor[13] to be the Liouville field. Both derivations assume that the metric is spherically symmetric, and begin with an *ad hoc* dimensional reduction to the $r-t$ plane. Furthermore, in [13] the final action is not obtained from the Einstein-Hilbert action, but rather chosen so that it will lead to the equation of motion for the conformal field. This work directly relates the gravitational degrees of freedom at the horizon to the diffeomorphisms that preserve horizon boundary conditions, and we only integrate over the spatial coordinates as a final step, after determining that the leading order dynamics are in the $r-t$ plane. Lastly, we do not require the metrics to be spherically symmetric.

Isolated Horizons. We must first decide on a definition of a horizon that we can use to set up boundary conditions on the metric. As we are concerned with the local degrees of freedom in a neighbourhood of the horizon, we will use the formalism of **weakly isolated horizons** as a guide. This formalism describes horizons that are in equilibrium, and has several features that make it suitable for our purposes. We largely follow the approach of [14] and [15].

Isolated horizons are null sub-manifolds Δ of spacetime, with an intrinsic metric q_{ab} that is the pull-back of the spacetime metric to Δ. A tensor q^{ab} on Δ is defined to be an inverse of q_{ab} if it satisfies $q_{am}q_{bn}q^{mn} = q_{ab}$. The inverse is not unique, but all of the definitions and constructions in the isolated horizon formalism are independent of the choice of inverse. Given a null normal l^μ to Δ, the *expansion* $\theta_{(l)}$ of l^μ is defined to be

$$\theta_{(l)} := q^{ab}\nabla_a l_b. \tag{1}$$

A weakly isolated horizon (WIH) is a sub-manifold Δ of a spacetime that satisfies the following conditions:

1. Δ is topologically $S^2 \times \mathbb{R}$ and null,
2. Any null normal l^μ of Δ has vanishing expansion, $\theta_{(l)} = 0$, and
3. All equations of motion hold at Δ and the stress energy tensor $T_{\mu\nu}$ is such that $-T^\mu_\nu l^\nu$ is future-causal for any future directed null normal l^μ.

Note that if Condition 2 holds for one null normal to Δ, then it holds for all. WIHs generalize the definitions of Killing horizons and apparent horizons, with the normal vector l^μ being analogous to a Killing vector, and the requirement that $\theta_{(l)} = 0$ clearly being inspired by the notion of trapped surfaces. However, the definition of a WIH is given only *at* the horizon, and does not require a Killing vector to exist even within an infinitesimal neighborhood. Thus, WIHs allow for much greater freedom in the dynamics of the matter and spacetime outside the horizon, while preserving the essential characteristics of the horizon itself.

The surface gravity of an isolated horizon is not uniquely defined. However, given a normal vector l^μ to Δ, there is a function κ_l such that

$$l^\mu \nabla_\mu l^\nu = \kappa_l l^\nu. \tag{2}$$

It is always possible to choose a normal vector l^μ such that the corresponding κ_l is constant everywhere on Δ. Therefore, κ_l may be interpreted as the surface gravity of the horizon corresponding to l^μ.

Gaussian Null Coordinates and Conformal Coordinates. We now introduce the coordinate systems that we use in this work. The first set of coordinates are known as **Gaussian null coordinates** (denoted "GN coordinates"), and are analogous to Eddington-Finkelstein coordinates in Schwarzschild spacetime[17]. In the neighbourhood of any smooth null hypersurface Δ, it is possible to define GN coordinates (u, r, x^i) such that the metric takes the form

$$\mathrm{d}s^2 = rF\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r + 2rh_i\,\mathrm{d}u\,\mathrm{d}x^i + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{3}$$

where g_{ij} is positive definite, and F, h_i, and g_{ij} are smooth functions of (u, r, x^i). The null hypersurface is defined by $r = 0$, and we have chosen a smooth, non-vanishing vector field l^μ that is normal to Δ, so that the integral curves of l^μ are the null geodesic generators of Δ and we have $l^\mu = (\partial/\partial u)^\mu$ on Δ.

Since an isolated horizon is a null hypersurface, we can construct such a coordinate system in a neighbourhood of any isolated horizon. In fact, in the neighbourhood of the event horizon of a stationary black hole, or a stationary Killing horizon, we can define these coordinates so that all the metric components are independent of u. Extremal Killing horizons correspond to the case where F has a simple root at $r = 0$, so that it has the form $F = rf(u, r, x^i)$. We will consider only *non-extremal* horizons, such that $F|_{r=0} \neq 0$. We can then define a non-zero surface gravity for the horizon, using the definition (2) and taking the normal vector to the $r = 0$ hypersurface to be $l^\mu := g^{r\mu}$. The surface gravity κ associated with this normal vector is $-\frac{1}{2}F|_{r=0}$.

For clarity of presentation, from now on we restrict ourselves to the case when $h_i = 0$, and $F = F(u, r)$, so that the metric takes the form

$$\mathrm{d}s^2 = rF\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{4}$$

Later we will return to study the most general case. We will find that our results can be easily generalized, with only minor modifications. Considering this simpler class of metrics, we can define conformal coordinates (x_+, x_-) in terms of the coordinates (u, r) such that the metric takes the form

$$\mathrm{d}s^2 = 2g_{+-}\,\mathrm{d}x_+\,\mathrm{d}x_- + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{5}$$

where $g_{+-} = e^{\sigma(u,r)}$ with $\sigma(u, r) = \ln r + \sigma_0(u) + O(r)$. Without loss of generality, we can require the GN coordinates and the conformal coordinates to satisfy:

$$\partial_- u = 0, \quad \partial_r x_+ = 0. \tag{6}$$

We can also determine the useful relations:

$$\partial_+ r = O(r), \quad \partial_- r = O(r). \tag{7}$$

Horizon boundary conditions. We use the notion of WIHs as a guide to formulate boundary conditions that ensure the existence of a horizon. We assume that we have a weakly isolated horizon Δ in our spacetime, and that we can define GN coordinates in a neighbourhood of Δ so that the horizon lies at $r = 0$ and the metric takes the form in Eq.(4). The requirement that the horizon satisfy $\theta_{(l)} = 0$ is equivalent to saying that $\partial_u g_{ij} = O(r)$. We now apply a diffeomorphism ξ, and obtain a new metric $g'_{\mu\nu} := g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$. We then impose conditions that preserve the essential characteristics of the horizon by demanding that, after the diffeomorphism:

1. There is still a null hypersurface at $r = 0$. This is equivalent to saying that g'_{uu} (or, in conformal coordinates, g'_{+-}) has a simple root at $r = 0$.

2. In conformal coordinates, the metric remains in the form given by Eq.(5).

3. The induced metric on the $r = 0$ hypersurface is be preserved, so that $g'_{ij} = g_{ij} + O(r)$. This also ensures that the null hypersurface continues to satisfy $\theta_{(l)} = 0$.

Working in conformal coordinates, we determine the diffeomorphisms that satisfy the above conditions, and find that they have the form:

$$\begin{aligned} \xi^+ &= \xi^+(x_+) + O(r) \\ \xi^- &= \xi^-(x_-) + O(r) \\ \xi^i &= O(r) \end{aligned} \tag{8}$$

This form of ξ is extremely suggestive: the (x_+, x_-) coordinates define a natural two-dimensional submanifold where our CFT will live, with infinitesimal conformal transformations being given by $x_+ \to x_+ + \xi^+(x_+)$, $x_- \to x_- + \xi^+(x_-)$. We also see that the coordinate $x_+ + x_-$ transforms like the Liouville field. In the case of a simple static horizon, this is the radial tortoise coordinate. Thus the radial degree of freedom may play the part of a Liouville-like field in the CFT at the horizon. Under this diffeomorphism, the metric transforms as:

$$\begin{aligned} g'_{+-} &= g_{+-}(1 + \partial_+\xi^+ + \partial_-\xi^- + \xi^+\partial_+\sigma + \xi^-\partial_-\sigma) \\ g'_{ij} &= g_{ij} + \xi^+\partial_+ g_{ij} + \xi^-\partial_- g_{ij} \end{aligned} \tag{9}$$

Boundary terms in the action. Before we derive an action for the horizon degrees of freedom, we need to make sure that the boundary conditions given above allow us to define a gravitational action with a well defined variational principle. The boundary conditions at $r = 0$ are different from the usual boundary conditions imposed at spatial infinity, and therefore the Einstein-Hilbert action may gain a new boundary term at Δ that is different from the usual Gibbons-Hawking term. We can determine the boundary term by varying the Einstein-Hilbert action[16]. The horizon is a null hypersurface, but we can still apply Stokes's theorem as long as the normal vector l^α satisfies

$$\frac{1}{n}\epsilon_{\alpha_1\ldots\alpha_n} = l_{[\alpha_1}\tilde{\epsilon}_{\alpha_2\ldots\alpha_n]} \tag{10}$$

where $\tilde{\epsilon}$ is an induced volume form defined on Δ and ϵ is the natural volume form in the bulk. Upon varying the Einstein-Hilbert action, we obtain the following term:

$$\int_{\mathcal{M}} \nabla_\alpha v^\alpha = \int_\Delta v_\alpha l^\alpha, \tag{11}$$

where we have

$$v_\alpha l^\alpha = l^\alpha g^{\beta\gamma}\left[\nabla_\gamma\left(\delta g_{\alpha\beta}\right) - \nabla_\alpha\left(\delta g_{\beta\gamma}\right)\right] \tag{12}$$

We can evaluate the boundary term in conformal coordinates for an arbitrary variation of the metric that is allowed under our horizon boundary conditions. We obtain:

$$\int_\Delta v_\alpha l^\alpha = \int_\Delta -l^+ g^{+-}\nabla_+(\delta g_{+-}) - l^- g^{+-}\nabla_-(\delta g_{+-}) \tag{13}$$

$$= -\partial_u\left[g^{+-}\delta g_{+-}\right] = -\frac{1}{2}\partial_u \mathrm{Tr}(\hat{g}), \tag{14}$$

where $\hat{g}$ is the induced metric on the (x_+, x_-) submanifold. But $\mathrm{Tr}(\hat{g})$ is a constant for the form of the metric in (5), so we can discard the boundary term altogether.

The Dynamical Action. We are now ready to derive a dynamical action for the gravitational degrees of freedom at the horizon. These degrees of freedom are the diffeomorphisms that are permitted by the horizon boundary conditions. In order to derive the action, we first apply a diffeomorphism ξ of the form (8) to the background metric $g_{\mu\nu}$, obtaining a new metric $g'_{\mu\nu} := g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$. We then evaluate the Einstein-Hilbert action with the background metric on-shell in the near-horizon region, thus isolating the gravitational fluctuations about this background that preserve the horizon. When evaluating the Einstein-Hilbert action, everything is calculated from the new metric, including the inverse metric and the metric determinant, as the metric itself is the only dynamical field in the problem. The final form of the action determines the dynamics of the horizon degrees of freedom. Note that we do not restrict the form of the background metric in the bulk. Our conditions only apply in the neighbourhood where the GN coordinates are defined, and are therefore formulated as fall-off constraints near the horizon.

We begin with the Einstein-Hilbert action

$$I_{EH} = \frac{1}{16\pi G}\int \mathrm{d}^n x\,\sqrt{-g}\,(R - 2\Lambda), \tag{15}$$

and evaluate the action for the new metric $g'_{\mu\nu}$. As we are working with infinitesimal transformations, we can make the approximation $g'_{+-} = e^\phi g_{+-} \approx (1 + \phi)g_{+-}$, with ϕ given by

$$\phi = \partial_+\xi^+ + \partial_-\xi^- + \xi^+\partial_+\sigma + \xi^-\partial_-\sigma. \tag{16}$$

We work to $O(r, \phi)$, and require that the background metric $g_{\mu\nu}$ is on-shell in the near-horizon region. Considering the Einstein equations for $g_{\mu\nu}$ order by order in r, this imposes:

$$\begin{aligned} R_{+-} &\hat{=} O(r) \\ R_{++} &\hat{=} 0 \\ R &\hat{=} \frac{2n\Lambda}{n-2}, \end{aligned} \tag{17}$$

where "$\hat{=}$" indicates that the equality holds on-shell. These conditions give:

$$\begin{aligned} \partial_+\partial_-\sigma &\hat{=} O(r) \qquad (18)\\ -g^{ik}\partial_+^2 g_{ik} + g^{+-}g^{ik}\partial_+ g_{+-}\partial_+ g_{ik} &\hat{=} O(r^2). \end{aligned}$$

Keeping only the leading order terms in the Einstein-Hilbert action, we find that all dependence on the coordinates x^i disappears, so that we can simply integrate over them. We finally obtain

$$I_{\rm hor} = \frac{a_\Delta}{16\pi G}\int \mathrm{d}^2x\,\sqrt{-\hat{g}}\,\left(\partial_a\phi\partial^a\phi - \phi\hat{R} + \lambda e^\phi\right) \quad (19)$$

where a_Δ is the cross-sectional area of the horizon, $\hat{g}$ is the induced metric on the (x_+, x_-) submanifold, and $\lambda := \frac{4\Lambda}{n-2}$. (In the case $n = 2$ we have $\Lambda = 0$.) We note that the action I_{CFT} is almost identical to the familiar Liouville action with a background metric $\hat{g}_{ab}$. The only difference is the coefficient of the $\phi\hat{R}$ coupling.

The dynamics described by $I_{\rm hor}$ are very simple. The non-kinetic terms in the action, although we have included them in the Lagrangian, are $O(r)$ and therefore become irrelevant in the near-horizon region. The equation of motion for ϕ has the form

$$\partial_+\partial_-\phi + O(r) = 0, \quad (20)$$

so that ϕ becomes a free two-dimensional conformal field in an infinitesimal neighbourhood of the horizon.

This is our main result: we have isolated the diffeomorphism degrees of freedom in a neighbourhood of a horizon that preserve the essential characteristics of the horizon, and derived a dynamical action for these degrees of freedom. We find that the gravitational fluctuations in the neighbourhood of a horizon are described by a two-dimensional field theory similar to Liouville theory that becomes conformal in the near-horizon region.

More General Horizons. We now return to the more general case where the GN coordinates in the near-horizon region yield a metric of the form Eq.(3), with $h_i = h_i(u, r, x^i) \neq 0$, and $F = F(u, r, x^i)$. This allows us to consider a much wider class of horizons. Once again we consider non-extremal horizons, with $F|_{r=0} \neq 0$.

As before, we can define coordinates (x_+, x_-) in terms of the coordinates (u, r) so that the metric takes the form

$$\mathrm{d}s^2 = 2g_{+-}\left(\mathrm{d}x_+\,\mathrm{d}x_- + h_{+i}\,\mathrm{d}x^+\,\mathrm{d}x^i\right) + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad (21)$$

where $g_{+-} = e^{\sigma(u,r,x^i)}$, and $\sigma(u, r, x^i) = \ln r + \sigma_0(u, x^i) + O(r)$. We still have a well-defined variational principle, with a constant boundary term in the Einstein-Hilbert action that can be discarded as before. The boundary conditions assuring the existence of a horizon at $r = 0$ are almost the same as before, except that now the metric is required to remain in the form given by Eq.(21) rather than Eq.(5). We find that the allowed diffeomorphisms have the same form as in Eq.(8). This may seem surprising, as the form of the metric is now much less restricted. And indeed, almost all of the boundary conditions may be satisfied by more general diffeomorphisms of the form $\{\xi^+(x_+), \xi^-(x_-, x^i), 0, 0\}$. However, if the metric must remain in the form (21), then the factor g_{+-} must be of the form $g_{+-} = e^{\ln r + \sigma_0(u,x^i) + O(r)}$, so that $\partial_i\partial_-\sigma = O(r)$. The need to preserve this condition imposes $\partial_i\xi^- = O(r)$.

We derive the dynamical action for the horizon degrees of freedom in the same way as before, and require the background metric to be on-shell in the near-horizon region. In order to obtain a simple Liouville-type action, we have to impose one more condition on the form of the metric: we require that $\partial_i\partial_+\sigma = O(r)$, so that $\sigma = \ln r + \sigma_0(u) + \sigma_1(x^i) + O(r)$. As the form of the functions h_{+i} are left unrestricted, this still allows us to describe a very wide variety of horizon metrics. We find that once again all the dynamics in the x^i coordinates disappear to $O(r^2)$, so we can integrate over these coordinates. We obtain the same action $I_{\rm hor}$ for the same field ϕ defined by $g'_{+-} = e^\phi g_{+-} \approx (1+\phi)g_{+-}$. Although σ is now a function of x^i, and ϕ depends on σ, the requirement that $\partial_i\partial_+\sigma = O(r)$ means that to leading order ϕ is independent of x^i and may be considered as a field on the (x_+, x_-) submanifold.

Conclusion. We have considered a very large class of horizons, including cosmological and acceleration horizons, and have explicitly derived the diffeomorphisms that preserve the characteristics of a horizon in some neighbourhood. We then determined the dynamical action of these diffeomorphisms, and found that the microstates of a horizon can be described by an effective two-dimensional Liouville-like theory that becomes conformal infinitesimally close to the horizon.

We can now use this theory to study the thermodynamic properties of horizons. The form of $I_{\rm hor}$ is very suggestive: the central charge of the theory will be proportional to a_Δ, indicating that applying the Cardy formula may yield the Bekenstein-Hawking entropy. Now that we have an explicit action for the horizon degrees of freedom, it should also be possible to couple these degrees of freedom to scalar field matter and reproduce Hawking radiation. Both of these problems are addressed in [19].

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