

# Second-order hyperbolic Fuchsian systems and applications

Florian Beyer<sup>1</sup> and Philippe G. LeFloch<sup>2</sup>

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## Abstract

We introduce a new class of singular partial differential equations, referred to as the *second-order hyperbolic Fuchsian systems*, and we investigate the associated initial value problem when data are imposed on the singularity. First, we establish a general existence theory of solutions with asymptotic behavior prescribed on the singularity, which relies on a new approximation scheme, suitable also for numerical purposes. Second, this theory is applied to the (vacuum) Einstein equations for Gowdy spacetimes, and allows us to recover, by more direct arguments, *well-posedness results* established earlier by Rendall and collaborators. Another main contribution in this paper is the proposed approximation scheme, which we refer to as the *Fuchsian numerical algorithm* and is shown to provide highly accurate numerical approximations to the singular initial value problem. For the class of Gowdy spacetimes, the numerical experiments presented here show the interest and efficiency of the proposed method and demonstrate the existence of a class of Gowdy spacetimes containing a smooth, incomplete, and non-compact Cauchy horizon.

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<sup>1</sup>Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin 9054, New Zealand, E-mail: fbeyer@maths.otago.ac.nz.

<sup>2</sup>Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Université Pierre et Marie Curie (Paris 6), 4 Place Jussieu, 75252 Paris, France. E-mail: pgLeFloch@gmail.com.

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## 1 Introduction

In general relativity and, more generally, in the theory of partial differential equations, singular solutions play a central role in driving the development of the theory and interpretation of the results – from the discovery of the Schwarzschild solution to the Penrose and Hawking’s celebrated singularity theorems (cf. for instance [24]), as well as in the modern efforts to understand the cosmic censorship and BKL (Belinsky-Khalatnikov-Lifshitz) conjectures [2, 40].

In fundamental and pioneer work, Choquet-Bruhat [10] established that the initial value problem associated with Einstein’s field equations is well-posed (in suitable Sobolev spaces) and, later together with Geroch [11], that for each choice of initial data consistent with the constraints there exists a unique maximal globally hyperbolic development. This theory allows to define a unique correspondence, between the space of solutions of the Einstein equations and the space of initial data set on a given (spacelike) hypersurface. In principle, solutions could be extended until a singularity forms and this theory provides the basis to tackle outstanding questions about the long-time behavior of solutions, in both future and past directions. Consider for instance the situation that singularities develop *in the past* of the given hypersurface, and let us refer to the construction based on [10, 11] as the “backward approach”. In practice, however, revealing information about singular solutions requires additional techniques of analysis going much beyond [10, 11]. On one hand, there is no a-priori information whether a given choice of initial data evolves into a “singularity” at all. On the other hand, if a singularity arises, it is not a-priori clear at “which location” it will occur, nor what kind of “singular solution” it will be. This approach requires a global-in-time control of solutions which is hopeless in many cases due to the complexity of the nonlinear field equations, the freedom of choice of gauge etc. In fact, the “backward approach” has so far led to successful conclusions only under strong symmetry assumptions; cf. Ringström [40] (and the references therein).

The so-called Fuchsian method, which we refer to here as a “forward” approach, provides an alternative to the above (backward) approach. It has the advantage of being simpler to deal with in practice (both analytically and numerically) in the class of problems it does apply, but has the disadvantage that it does not allow – for essential reasons – to handle the full solution space and the conclusions are stated in terms of “generic data”. Nevertheless it allows to study particular classes of singular solutions. The main idea is to give data “on the singularity”, that is, to prescribe the leading-order behavior of solutions in a neighborhood of the singular time and, then, to evolve forward in time away from the singularity – in contrast to prescribing data on a Cauchy surface and evolving towards the singularity. (We will make this idea precise below.)

Our main objective in the present paper is precisely to further study this **singular initial value problem** (corresponding to the forward direction).

The Fuchsian method was introduced to general relativity by Kichenassamy and Rendall [28] who covered a class of singular equations, the so-called “Fuchsian partial differential equations”, while establishing that Einstein’s field equations under Gowdy symmetry are included in this class. Under suitable analyticity conditions on the solutions they proved the well-posedness of the singular initial value problem (without imposing any hyperbolicity condition). Later, Rendall [36] took the hyperbolicity property into account and generalized the theory to the smooth solutions to the Gowdy equations. For further results about Fuchsian equations we refer to [29, 12, 13, 14]; especially, in [13], a generalization of the standard Fuchsian theory was recently introduced.

Our motivation in this paper now is threefold. First, it is of general interest to find a reliable numerical scheme for the singular initial value problem of hyperbolic Fuchsian equations, which would allow us to construct solutions to Einstein’s field equations with prescribed singular behavior. This was indeed the motivation of Amorim, Bernardi, and LeFloch in [1]. However, the approximation scheme suggested by the theoretical works above is neither suitable nor natural for the numerical treatment, and it is not easy to obtain error estimates which are necessary in practice to judge the quality of the numerical solutions. In this paper, we address this drawback by introducing a new approximation scheme.

Secondly, we want to perform these studies —both theoretically and numerically— as economically as possible. Importantly, the Einstein equations for Gowdy spacetimes are naturally expressed in a second-order form. This motivates us to develop here a theory of **second-order Fuchsian equations** which is advantageous as it saves us from the hassle of turning the system into a first-order form first, as required by the classical Fuchsian theory. In fact, there is no unique way of turning a second-order system into a first-order form, and this issue can be particularly problematic when the solutions are expected to be singular. It has also been recognized in the numerical literature that the direct discretization in second-order form leads to more accurate results [31].

Our third motivation for this paper is that the existing works make no statement about how to guess the leading-order part. This is a delicate issue and we prove here that, for a large class of systems and in way compatible with the (already mentioned) BKL conjecture, it is possible to make a “canonical guess”, as we explain below.

These three main issues are addressed in the present paper, which is organized as follows.

Sections 2 and 3 are the main theoretical part. In Section 2.1, we introduce the class of equations of interest in this work. We focus on a class of hyperbolic Fuchsian equations, the **second-order hyperbolic Fuchsian equations**, which are systems of semi-linear wave equations including certain singular terms. This section includes a rigorous definition of the singular initial value problem for this class of equations. Section 2.3 is devoted to determine the leading-order behavior of solutions and hence to make a “canonical guess”. We study the case when the principal part of the equation dominates over the source-term at  $t = 0$  in a certain sense and derive a **canonical two-term expansion**. The reasoning is based on suitable heuristics compatible with the BKL conjecture. (Later in this paper we derive precise conditions under which this heuristics is justified.)

The singular initial value problem is discussed rigorously in Section 3. We introduce our new approximation scheme which, both, yields a simple and direct proof of existence of solutions to the singular initial value problem, and a natural numerical algorithm (introduced in Section 3.4) for which practical error estimates can be obtained. The main idea is to approximate the solution of the singular initial value problem by a sequence of solutions to the standard (regular) initial value problem.

Finally, in Section 4, we turn our attention to the class of Gowdy spacetimes satisfying Einstein's field equations which is an application of particular interest, and we demonstrate the practical use of our theory. First, we recapitulate the standard heuristic arguments for Gowdy spacetimes and demonstrate that these are consistent with the heuristics introduced earlier (with some important subtleties discussed below). In Section 4.3, the now classical results by Rendall [36] are recovered while our new approach shed some new light on the Gowdy equations. In Section 5 we present numerical solutions to the singular initial value problem associated with the Gowdy equations. After some test cases, we numerically construct solutions with incomplete, non-compact Cauchy horizons.

## 2 Second-order hyperbolic Fuchsian systems

### 2.1 A class of singular equations

**Definition 2.1.** *A second-order hyperbolic Fuchsian system is a set of partial differential equations of the form*

$$D^2v + 2ADv + Bv - t^2K^2\partial_x^2v = f[v], \quad (2.1)$$

in which the function  $v : (0, \delta] \times U \rightarrow \mathbb{R}^n$  is the main unknown (defined for some  $\delta > 0$  and some interval  $U$ ), while the coefficients  $A = A(x)$ ,  $B = B(x)$ ,  $K = K(t, x)$  are diagonal  $n \times n$  matrix-valued maps and are smooth in  $x \in U$  and  $t$  in the half-open interval  $(0, \delta]$ , and  $f = f[v](t, x)$  is an  $n$ -vector-valued map of the following form

$$f[v](t, x) := f\left(t, x, v(t, x), Dv(t, x), tK(t, x)\partial_x v(t, x)\right).$$

We assume that the time variable  $t$  satisfies  $t > 0$  and use the operator  $D := t\partial_t$  to write the equations. The equation is henceforth assumed to be singular at  $t = 0$ . The assumption that  $U$  is a one-dimensional domain makes the presentation simpler, but most results below remain valid for arbitrary spatial dimensions. For definiteness and without much loss of generality, we assume throughout this paper that all functions under consideration are periodic in the spatial variable  $x$  and that  $U$  is the periodicity domain. All data and solutions are extended by periodicity outside the interval  $U$ . Moreover, we assume that the coefficients  $A$  and  $B$  do not depend on  $t$ , see below. We denote the eigenvalues of  $A$  and  $B$  by  $a^{(1)}, \dots, a^{(n)}$  and  $b^{(1)}, \dots, b^{(n)}$ , respectively. When it is not necessary to specify the superscripts, we just write  $a, b$  to denote any eigenvalues of  $A, B$ . With this convention, we introduce:

$$\lambda_1 := a + \sqrt{a^2 - b}, \quad \lambda_2 := a - \sqrt{a^2 - b}. \quad (2.2)$$

It will turn out that these coefficients, which might be complex in general, are important to describe the expected behavior at  $t = 0$  of general solutions to (2.1). Further restrictions on the coefficients and on the right-hand side will be imposed and discussed in the course of our investigation.

After a suitable reduction to a first-order form, our choice of singular equations falls into the class of hyperbolic Fuchsian equations [28, 36, 29, 13]. We make this particular choice of equations here for the following reasons. First we restrict to hyperbolic equations because this is the case of interest and allows us to control solutions in much greater detail than without this assumption. Second, the equations are kept in second-order form here because it is economic and efficient to do this for the applications we have in mind both analytically and numerically. Third, the particular class of equations allows us to simplify the presentation of the general results obtained in this paper. However, all results presented here can be generalized to a

general class of symmetric hyperbolic Fuchsian equations – compare to the discussion in Rendall [36] – for arbitrary spatial dimensions. Even beyond this it is possible to generalize the theory to equations with time-dependent coefficients  $A$  and  $B$ . Also the restriction to spatial periodicity is not essential because these equations obey the domain of dependence property just as usual non-singular hyperbolic equations under suitable assumptions on  $K$ , see below.

The eigenvalues of the matrix  $K$  are denoted by  $k^{(i)}$  and, in the scalar case (or when there is no need to specify the index), we simply write  $k$ . These quantities are interpreted as characteristic speeds. Throughout this section, we assume that they have the form

$$k^{(i)}(t, x) = t^{\beta^{(i)}(x)} \nu^{(i)}(t, x), \quad (2.3)$$

with  $\beta^{(i)} : U \rightarrow (-1, \infty)$ ,  $\nu^{(i)} : [0, \delta] \times U \rightarrow (0, \infty)$  smooth functions.

In particular, we assume that each derivative of  $\nu^{(i)}$  has a unique finite limit at  $t = 0$  for each  $x \in U$ . Note that we allow for the characteristic speeds to diverge at  $t = 0$ . At a first glance, this appears to conflict with the standard finite domain of dependence property of hyperbolic equations. A closer look at the requirement  $\beta(x) > -1$ , however, indicates that the characteristic curves are *integrable* at  $t = 0$  and, hence that the *finite* domain of dependence property is preserved under our assumptions.

The operator associated with the **principal part**<sup>1</sup> of the system is

$$L := D^2 + 2AD + B - t^2 K^2 \partial_x^2 =: \tilde{L} - t^2 K^2 \partial_x^2. \quad (2.4)$$

This is a linear wave operator for  $t > 0$  and, indeed, (2.1) is hyperbolic for all  $t > 0$ . Later on we will construct solutions where the first three terms  $\tilde{L}$  of the principal part are of the same order at  $t = 0$  and “dominant”, while the source-term of the equation as well as the second spatial derivative term are assumed to be of higher-order in  $t$  at  $t = 0$  and hence “negligible”. Note that at this level generality, there is some freedom in bringing terms from the principal part to the right-hand side of the equation, and absorbing them into the source-function  $f$  (or vice-versa). This freedom has several (interesting) consequences: roughly speaking, some normalization will be necessary later, yet at this stage, we do not fix the behavior of  $f$  at  $t = 0$ .

## 2.2 Singular initial value problem

Consider any second-order hyperbolic Fuchsian system with coefficients  $a, b, \lambda_1, \lambda_2$ , satisfying (2.2). To simplify the presentation, we restrict attention to scalar equations ( $n = 1$ ) and shortly comment on the general case in the course of the discussion.

Fix some integers  $l, m \geq 0$  and constants  $\alpha, \delta > 0$ . For  $w \in C^l((0, \delta], H^m(U))$ , we define the norm

$$\|w\|_{\delta, \alpha, l, m} := \sup_{0 < t \leq \delta} \left( \sum_{p=0}^l \sum_{q=0}^m \int_U t^{2(\Re \lambda_2(x) - \alpha)} |\partial_x^q D^p w(t, x)|^2 dx \right)^{1/2},$$

and denote by  $X_{\delta, \alpha, l, m}$  the space of all such functions with finite norm  $\|w\|_{\delta, \alpha, l, m} < \infty$ . Throughout,  $H^m(U)$  denotes the standard Sobolev space and we recall that all functions are periodic in the variable  $x$  with  $U$  being a periodicity domain. To cover a system of  $n \geq 1$  second-order Fuchsian equations, the norm above is defined by summing over all vector components with different exponents used for different components. Recall that each equation in the system will have a different root function  $\lambda_2$ . We allow that  $\alpha = (\alpha^{(1)}, \dots, \alpha^{(n)})$  is a vector of different positive constants for each equation. The constant  $\delta$ , however, is assumed to be common for all

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<sup>1</sup>Our motivation to include first-derivative terms into the principal part will be explained later.

equations in the system. With this modification, all results in the present section remain valid for systems of equations. We comment later that in fact,  $\alpha$  is not required to be a constant, but in most of the following results it will be treated like a constant for simplicity. The reason to include the quantity  $\lambda_2$  into the definition of the norms is motivated by the canonical choice of the leading-order term introduced later. Throughout it is assumed that  $\Re\lambda_2$  is continuous and it is then easy to check that  $(X_{\delta,\alpha,l,m}, \|\cdot\|_{\delta,\alpha,l,m})$  is a Banach space.

For the discussion of hyperbolic equations, it makes sense to also introduce the following Banach spaces. For each non-negative integer  $l$  and real numbers  $\delta, \alpha > 0$ , we define  $X_{\delta,\alpha,l} := \bigcap_{p=0}^l X_{\delta,\alpha,p,l-p}$ , and introduce the norm

$$\|f\|_{\delta,\alpha,l} := \left( \sum_{p=0}^l \|f\|_{\delta,\alpha,p,l-p}^2 \right)^{1/2}, \quad f \in X_{\delta,\alpha,l}.$$

As we will see in the course of the following, however, it is not possible to control solutions of our equations in the spaces  $X_{\delta,\alpha,l}$  directly. It turns out that we must use spaces  $(\tilde{X}_{\delta,\alpha,l}, \|\cdot\|_{\tilde{\delta,\alpha,l}})$  instead. These are defined as earlier, but in the norm  $\|f\|_{\tilde{\delta,\alpha,l}}$  of some function  $f$ , the highest spatial derivative term  $\partial_x^l f$  is weighted with the additional factor  $t^{\beta+1}$ . Here  $\beta$  is the exponent of the characteristic speed given by (2.3). It is easy to see under the earlier conditions that also  $(\tilde{X}_{\delta,\alpha,l}, \|\cdot\|_{\tilde{\delta,\alpha,l}})$  are Banach spaces. We also note that  $X_{\delta,\alpha,l} \subset \tilde{X}_{\delta,\alpha,l}$ . Let us also define  $X_{\delta,\alpha,\infty} := \bigcap_{l=0}^{\infty} X_{\delta,\alpha,l}$ , and note that  $X_{\delta,\alpha,\infty} = \bigcap_{l=0}^{\infty} \tilde{X}_{\delta,\alpha,l}$ .

For  $w \in \tilde{X}_{\delta,\alpha,l}$  we set

$$w_\eta(t, x) = t^{-\lambda_2(x)} \int_{-\infty}^{\infty} \int_0^{\infty} t^{\lambda_2(y)} w(s, y) k_\eta \left( \log \frac{s}{t} \right) k_\eta(x - y) \frac{1}{s} ds dy.$$

Here,  $k_\eta : \mathbb{R} \rightarrow \mathbb{R}_+$  is a smooth kernel supported in  $[-\eta, \eta]$ , satisfying  $\int_{\mathbb{R}} k_\eta(x) dx = 1$  for all positive  $\eta$ . Then  $w_\eta$  is an element of  $\tilde{X}_{\delta,\alpha-\epsilon,l} \cap C^\infty((0, \delta] \times U)$  for every  $\epsilon > 0$ . Furthermore, the sequence of such mollified functions  $w_\eta$  in the limit  $\eta \rightarrow 0$  converges to  $w$  in the norm  $\|\cdot\|_{\tilde{\delta,\alpha-\epsilon,l}}$ . Hence any element in  $\tilde{X}_{\delta,\alpha,l}$  can be approximated by smooth functions.

We mentioned earlier that we are interested in solving the “forward problem”, referred to as the **singular initial value problem** (SIVP) – in this paper. More precisely, we need to guess a leading-order term  $u$  of solutions  $v$  to (2.1) so that the **remainder**

$$w(t, x) := v(t, x) - u(t, x),$$

can be interpreted as “higher order” in  $t$  at  $t = 0$ . By this we mean that  $w$  is an element in  $X_{\delta,\alpha,l}$  for some (sufficiently large)  $\alpha > 0$  on a small time interval  $(0, \delta]$ . If for a given  $u$  such a solution  $v$  exists then we say that  $v$  **obeys the leading-order behavior given by  $u$** . Often  $u$  will be parametrized by certain free functions which we call **asymptotic data**, see below. For later convenience, we introduce the operator  $F$  as

$$F[w](t, x) := f[u + w](t, x). \tag{2.5}$$

### 2.3 Canonical leading-order term

The main first step for solving the singular initial value problem is to guess a leading-order term  $u$ . In some applications this can be very tricky, but in many situations, which we will be most interested in in this paper, one can make a canonical guess. These situations are described heuristically as follows.

### Canonical two-term expansion

Consider the principal part operator  $\tilde{L}$  in (2.4) and note that it incorporates certain lower derivative terms. The reason for writing  $\tilde{L}$  like this is that we expect in many cases that these terms are significant and of leading-order at the singularity  $t = 0$ . In contrast, the source-term and spatial derivatives can often be anticipated as negligible in some sense under suitable assumptions below. This is motivated by the BKL conjecture in general relativity. In order to make this more concrete, let us assume that the leading-order term is an exact solution of the system of ordinary differential equations (parametrized by  $x$ ), which is obtained when all terms in the equation, except for those given by  $\tilde{L}$ , are set to zero. We refer to this leading-order term  $u$  as the “canonical leading-order term” or the **canonical two-term expansion**:

$$u(t, x) = \begin{cases} u_*(x) t^{-a(x)} \log t + u_{**}(x) t^{-a(x)}, & (a(x))^2 = b(x), \\ u_*(x) t^{-\lambda_1(x)} + u_{**}(x) t^{-\lambda_2(x)}, & (a(x))^2 \neq b(x), \end{cases} \quad (2.6)$$

for some freely prescribed **asymptotic data**  $u_*$  and  $u_{**} \in H^{m'}(U)$ , where  $m'$  is some non-negative integer. We refer to this as the “Fuchsian heuristics” because the leading-order behavior will be determined by Fuchsian ordinary differential equations.

We clearly see the dependence of the expected leading-order behavior at  $t = 0$  on the coefficients of the principal part of the equation. If the roots  $\lambda_1$  and  $\lambda_2$  are real and distinct, i.e. if  $a^2 > b$ , we expect a *power-law* behavior. In the degenerate case  $\lambda_1 = \lambda_2$ , i.e. if  $a^2 = b$ , we expect a *logarithmic* behavior. Finally, when  $\lambda_1$  and  $\lambda_2$  are complex for  $a^2 < b$ , the solution is expected to have an *oscillatory* behavior at  $t = 0$  of the form

$$u(t, x) = t^{-a(x)} (\tilde{u}_* \cos(\lambda_I(x) \log t) + \tilde{u}_{**} \sin(\lambda_I(x) \log t)) + \dots$$

for some real coefficient functions  $\tilde{u}_*(x)$  and  $\tilde{u}_{**}(x)$ ; note that in this case,  $\lambda_1 = \bar{\lambda}_2 = a + i\lambda_I$  with  $\lambda_I := \sqrt{b^2 - a}$ .

If the coefficients of the equations are such that there is a continuous transition between the two cases in (2.6), then the asymptotic data functions  $u_*$  and  $u_{**}$  must be renormalized as follows. Define  $\Gamma(x) := \sqrt{a(x)^2 - b(x)}$  which might be real or imaginary dependent on the values of the coefficients. If there are points  $x_0 \in U$  so that  $\Gamma(x_0) = 0$  and other points  $x_1 \in U$  with  $\Gamma(x_1) \neq 0$ , then let us set

$$u_*(x) = \frac{\hat{u}_*(x) - \hat{u}_{**}(x)/\Gamma(x)}{2}, \quad u_{**}(x) = \frac{\hat{u}_*(x) + \hat{u}_{**}(x)/\Gamma(x)}{2}, \quad (2.7)$$

and choose  $\hat{u}_*(x)$ ,  $\hat{u}_{**}(x)$  as asymptotic data functions. This guarantees that  $u(t, x)$  given by (2.6) is smooth for all  $t > 0$ .

### Higher-order canonical expansions

For some applications we require expansions of the solutions at  $t = 0$  with more than two terms in order to describe the leading-order behavior. A particular important example is the Gowdy case in Section 4. Following Rendall [36], those can be constructed as follows, without going into the details. Consider the Fuchsian ODE case of (2.1), written here for the scalar case only,

$$D^2 v(t, x) + 2a(x) Dv(t, x) + b(x)v(t, x) = f[v](t, x),$$

where  $x$  is interpreted as a parameter. Let first  $f[v](t, x) = f_0(t, x)$  be a given function. Under suitable decay assumption<sup>2</sup> on  $f_0$  at  $t = 0$ , there exists a unique solution  $v$  of this equation

<sup>2</sup>Details can be found in [7, 8].

obeying the canonical two-term expansion  $u$  given by (2.6) for given asymptotic data functions  $u_*$  and  $u_{**}$ . Let  $H$  be the operator mapping  $f_0$  to the remainder  $w = v - u$  of the solution  $v$ . Now consider an arbitrary source-term  $f[v]$  and let the operator  $F$  be as defined in (2.5) for the given function  $u$ . Let  $w_1 \equiv 0$  and

$$w_{j+1} := H \circ F[w_j], \quad j \in \mathbb{N}.$$

Finally, set  $v_j = u + w_j$  for all  $j \in \mathbb{N}$ . Clearly,  $v_1 = u$ . One finds that the order of  $v_{j+1} - v_j$  in  $t$  at  $t = 0$  increases with  $j$ . Hence  $v_j$  can be interpreted as an expansion of the solution at  $t = 0$  whose order in  $t$  increases with  $j$ . Moreover, it turns out that the order of the residual in  $t$  at  $t = 0$ , obtained when  $v_j$  is plugged into the equation, increases with  $j$ . Thus  $v_j$  can be considered as an asymptotic solution of the Fuchsian ODE. In many situations it is thus meaningful to use  $v_j$  as the canonical leading-order term  $u$  for any given  $j \in \mathbb{N}$ .

### Limitations of this heuristics

At this stage it is of course highly unclear under which conditions there exist solutions of the equations which obey the leading-order term  $u$  given by (2.6) or any of its higher-order version  $v_j$  constructed before. The resolution of this problem will be central to this paper. In many applications in general relativity, the canonical two-term expansion is the correct guess for the leading-order term, if the asymptotic data are consistent with constraint equations implied by Einstein's field equations. However, we know of several cases when the operator  $\tilde{L}$  does not give rise to the dominant term at  $t = 0$ . For example for the Gowdy case, nonlinear terms from the source-term need to be taken into account. As we discuss there, however, the problem can be reduced to the canonical case by adding a certain term to the equation. For Gowdy solutions with spikes [6, 37, 34, 3, 4, 35, 41, 38, 40] the situation is significantly more complicated because then, other nonlinear terms and spatial derivative terms can become significant. Another important example is given by the mixmaster dynamics [19, 2, 4, 25, 41, 40]. There, one has to control a complicated interplay between nonlinear terms in the source-term in order to fix the leading-order term. In general when the equations are generalized to time-dependent coefficients in the principal part or to the quasi-linear case, the notion of canonical two-term expansions will apply only under suitable conditions.

## 3 Well-posedness theory and the Fuchsian numerical algorithm

### 3.1 An approximation scheme

We begin with some notation. For  $w \in \tilde{X}_{\delta, \alpha, 1}$ , the operator  $L$  in (2.4) is defined in the sense of distributions, only, via:

$$\begin{aligned} \langle \mathcal{L}[w], \phi \rangle &:= \int_0^\delta \int_{\mathbb{R}} t^{\Re \lambda_2(x) - \alpha} \left( \begin{aligned} &- Dw(t, x) D\phi(t, x) + (2A(x) - \Re \lambda_2(x) + \alpha - 1) Dw(t, x) \phi(t, x) \\ &+ B(x) w(t, x) \phi(t, x) + tK(t, x) \partial_x w(t, x) tK(t, x) \partial_x \phi(t, x) \\ &+ (2t \partial_x K(t, x) + \partial_x \Re \lambda_2(x) K(t, x) t \log t) tK(t, x) \partial_x w(t, x) \phi(t, x) \end{aligned} \right) dx dt, \end{aligned}$$

where  $\phi$  is any test function, i.e. a real-valued  $C^\infty$ -function on  $(0, \delta] \times \mathbb{R}$  together with some  $T \in (0, \delta)$  and a compact (i.e. closed and bounded) set  $K \in \mathbb{R}$  so that  $\phi(t, x) = 0$  for all  $t > T$



and  $x \notin K$ , and each derivative of  $\phi$  has a finite (not necessarily vanishing) limit at  $t = 0$  for every  $x \in U$ . For our later discussion, we note that for any given test function  $\phi$ , the linear functional  $\langle \mathcal{L}[\cdot], \phi \rangle : \tilde{X}_{\delta, \alpha, 1} \rightarrow \mathbb{R}$  is continuous with respect to the norm  $\|\cdot\|_{\tilde{\delta, \alpha, 1}}$ . This is the main reason to include the factor  $t^{\Re \lambda_2(x) - \alpha}$  in the definition of  $\mathcal{L}$ .

If the operator  $F$  defined by (2.5) for a given leading-order term  $u$  gives rise to a map  $\tilde{X}_{\delta, \alpha, 1} \rightarrow X_{\delta, \alpha, 0}$ , where  $w \mapsto F[w]$ , it is meaningful to define its weak form by (for all test functions  $\phi$ )

$$\langle \mathcal{F}[w], \phi \rangle := \int_0^\delta \int_{\mathbb{R}} t^{\Re \lambda_2(x) - \alpha} F[w](t, x) \phi(t, x) dx dt.$$

**Definition 3.1** (Weak solutions of second-order hyperbolic Fuchsian systems). *Let  $u$  be a given function and  $\delta, \alpha > 0$  be constants. Then, one says that  $w \in \tilde{X}_{\delta, \alpha, 1}$  is a weak solution to the second-order hyperbolic Fuchsian equation (2.1), provided*

$$\mathcal{P}[w] := \mathcal{L}[w] + \mathcal{L}[u] - \mathcal{F}[w] = 0.$$

Let us now start our discussion with the linear case of second-order hyperbolic Fuchsian equations and introduce our new approximation scheme. The following conditions are assumed:

1. Vanishing leading-order part:  $u \equiv 0$ .
2. Linear source-term:

$$F[w](t, x) = f_0(t, x) + f_1(t, x)w + f_2(t, x)Dw + f_3(t, x)tk\partial_x w, \quad (3.1)$$

with given functions  $f_0, f_1, f_2, f_3$ , so that  $f_1, f_2, f_3$  are smooth spatially periodic on  $(0, \delta] \times U$ , and near  $t = 0$

$$\sup_{x \in \tilde{U}} f_a(t, x) = O(t^\mu), \quad a = 1, 2, 3, \quad (3.2)$$

for some constant  $\mu > 0$ .

We have not made any assumptions for the function  $f_0$  yet, since in the following discussion this function will play a different role than  $f_1, f_2, f_3$ . Moreover, no loss of generality is implied by the condition  $u \equiv 0$ , since the general case can be recovered by absorbing  $L[u]$  into the function  $f_0$ .

Under these assumptions, we pose the question whether there exists a unique weak solution  $w$  in  $\tilde{X}_{\delta, \alpha, 1}$  for some  $\delta, \alpha > 0$  of the given second-order hyperbolic equation. The main idea here is our new approximation scheme. We approximate a solution of the *singular* value problem by a sequence of solutions of the *regular* initial value problem.

**Definition 3.2** (Regular initial value problem (RIVP)). *Fix  $t_0 \in (0, \delta]$  and some smooth periodic functions  $g, h : U \rightarrow \mathbb{R}$ , and suppose that the right-hand side is of the form (3.1) with given smooth spatially periodic functions  $f_0, f_1, f_2, f_3$  on  $[t_0, \delta] \times U$ . Then,  $w : [t_0, \delta] \times U \rightarrow \mathbb{R}$  is called a solution of the **regular initial value problem** associated with the “regular data”  $g, h$ , if (2.1) holds everywhere on  $(t_0, \delta] \times U$  and, moreover, the remainder  $w := v - u$ , for some function  $u$ , satisfies*

$$w(t_0, x) = g(x), \quad \partial_t w(t_0, x) = h(x).$$

For the regular initial value problem, we indeed assume that  $f_0$  is smooth, just as  $f_1, f_2$  and  $f_3$ . By the general theory of linear hyperbolic equations, the regular initial value problem is well-posed, in the sense that there exists a unique smooth solution  $w$  defined on  $[t_0, \delta]$  for any

choice of smooth regular data. Let  $u$  be a choice of leading-order term. Let  $(t_n)$  be a sequence of positive times converging to zero so that each  $t_n$  is smaller than some  $\delta > 0$ . For each  $n \in \mathbb{N}$ , we construct an approximate solutions  $w_n$  of the singular initial value problem as follows. Let  $w_n \equiv 0$  on the time interval  $(0, t_n]$ . On the time interval  $[t_n, \delta]$  we set  $w_n$  to be the solution of the regular initial value problem with initial time  $t_n$  and zero regular data. Hence,  $w_n$  is a  $C^1$ -function in time on  $(0, \delta]$ .

The central result of this section is that this sequence of approximate solutions converges to the solution of the singular initial value problem under suitable conditions.

**Proposition 3.3** (Existence of solutions of the linear singular initial value problem in  $\tilde{X}_{\delta, \alpha, 1}$ ). *Under the assumptions (3.1) and (3.2), the sequence  $(w_n)$  converges to the unique solution  $w \in \tilde{X}_{\delta, \alpha, 1}$  of the singular initial value problem for given  $\delta, \alpha > 0$  provided:*

1. *The matrix*

$$N := \begin{pmatrix} \Re(\lambda_1 - \lambda_2) + \alpha & ((\Im\lambda_1)^2/\eta - \eta)/2 & 0 \\ ((\Im\lambda_1)^2/\eta - \eta)/2 & \alpha & t\partial_x k - \partial_x \Re(\lambda_1 - \lambda_2)(tk \log t) \\ 0 & t\partial_x k - \partial_x \Re(\lambda_1 - \lambda_2)(tk \log t) & \Re(\lambda_1 - \lambda_2) + \alpha - 1 - Dk/k \end{pmatrix} \quad (3.3)$$

*is positive semidefinite at each  $(t, x) \in (0, \delta) \times U$  for a constant  $\eta > 0$ .*

2. *The source-term function  $f_0$  is in  $X_{\delta, \alpha + \epsilon, 0}$  for some  $\epsilon > 0$ .*

*Then, the solution operator  $\mathbb{H} : X_{\delta, \alpha + \epsilon, 0} \rightarrow \tilde{X}_{\delta, \alpha, 1}$ ,  $f_0 \mapsto w$ , is continuous and there exists a finite constant  $C_\epsilon > 0$  so that*

$$\|\mathbb{H}[f_0]\|_{\tilde{\delta}, \alpha, 1} \leq \delta^\epsilon C_\epsilon \|f_0\|_{\delta, \alpha + \epsilon, 0},$$

*for all such  $f_0$ . The constant  $C_\epsilon$  can depend on  $\delta$ , but is bounded for all small  $\delta$ . The approximate solutions satisfy the following error estimate for all  $n, m \in \mathbb{N}$ :*

$$\|w_m - w_n\|_{\tilde{\delta}, \alpha, 1} \leq C |G(\tau_n) - G(\tau_m)|,$$

*where  $C > 0$  is a constant and*

$$G(t) := \int_0^t s^{-1} \|s^{\Re\lambda_2 - \alpha} f_0(s, \cdot)\|_{L^2(U)} ds.$$

We call  $N$  the **energy dissipation matrix**. We have assumed that  $\alpha$  is a positive constant. If, however,  $\alpha$  is a positive spatially periodic function in  $C^1(U)$ , the definition of the spaces  $X_{\delta, \alpha, k}$  and  $\tilde{X}_{\delta, \alpha, k}$  remains the same, and only the (2, 3)- and (3, 2)-components of the energy dissipation matrix  $N$  change to  $t\partial_x k - \partial_x (\Re(\lambda_1 - \lambda_2) + \alpha)(tk \log t)$ . In the following, we continue to assume that  $\alpha$  is a constant in order to keep the presentation as simple as possible, but we stress that all following results hold (with this slight change of  $N$ ) if  $\alpha$  is a function, and hence no new difficulties arise.

The proof is based on controlling the energy of the approximate solutions. Let us restrict our presentation here to the scalar case  $n = 1$  for this whole section; the general case can be obtained with the same ideas. Choose  $\delta, \alpha > 0$  and let  $w \in C^\infty((0, \delta] \times U)$  be a spatially periodic function. Then, we define its **energy** at the time  $t \in (0, \delta]$  by

$$\begin{aligned} E[w](t) &:= e^{-\kappa t^\gamma} \int_U t^{2(\lambda_2(x) - \alpha)} e[w](t, x) dx, \\ e[w](t, x) &:= \frac{1}{2} ((\eta w(t, x))^2 + (Dw(t, x))^2 + (tk(t, x)\partial_x w(t, x))^2), \end{aligned} \quad (3.4)$$

for some constants  $\kappa \geq 0$ ,  $\gamma > 0$  and  $\eta > 0$ . For convenience, we also introduce the following notation. For any scalar-valued function  $w$ , we define the vector-valued function

$$\widehat{w}(t, x) := t^{\Re\lambda_2(x)-\alpha}(\eta w(t, x), Dw(t, x), tk(t, x)\partial_x w(t, x)), \quad (3.5)$$

involving the same constants as in the energy. Then, we can write

$$E[w](t) = \frac{1}{2}e^{-\kappa t^\gamma} \|\widehat{w}(t, \cdot)\|_{L^2(U)}^2,$$

the norm here being the Euclidean  $L^2$ -norm for vector-valued functions in  $x$ . It is important to realize that, provided  $\eta > 0$ , the expression  $\sup_{0 < t \leq \delta} \|\widehat{w}(t, \cdot)\|_{L^2(U)}$  for functions of the form (3.5) yields a norm which is equivalent to  $\|\cdot\|_{\widetilde{X}_{\delta, \alpha, 1}}$ , thanks to (2.3). Therefore, the energy (3.4) is of relevance for the space  $\widetilde{X}_{\delta, \alpha, 1}$ .

Of central importance for the proof of Proposition 3.3 are energy estimates for the regular initial value problem.

**Lemma 3.4** (Fundamental energy estimate for the regular initial value problem). *Fix  $t_0 \in (0, \delta]$  and suppose that the source-term is of the form (3.1) with the conditions (3.2) and that the energy dissipation matrix (3.3) is positive semidefinite on  $(0, \delta] \times U$  for given constants  $\alpha, \eta > 0$ . Then, there exist constants  $C, \kappa, \gamma > 0$ , independent of  $t_0$ , so that for all solutions  $w$  of the regular initial value problem with smooth regular data at  $t = t_0$ , we have*

$$\|\widehat{w}(t, \cdot)\|_{L^2(U)} \leq C e^{\frac{1}{2}\kappa(t^\gamma - t_0^\gamma)} \left( \|\widehat{w}(t_0, \cdot)\|_{L^2(U)} + \int_{t_0}^t s^{-1} \|s^{\Re\lambda_2 - \alpha} f_0(s, \cdot)\|_{L^2(U)} ds \right),$$

for all  $t \in [t_0, \delta]$ .

The proof of this lemma is standard. However, one has to confirm that the energy stays uniformly finite at  $t = 0$ , and this is guaranteed by the positivity condition for  $N$  in (3.3). Moreover, this result demonstrates the importance of the assumption  $\beta(x) > -1$  in (2.3). Namely, if  $\beta(x) \leq -1$  at a point  $x \in U$ , then for any choice of  $\alpha$  and  $\eta$ , the matrix  $N$  would not be positive semidefinite for small  $t$  at  $x$ . While the energy estimate would still be true for a given  $t_0$ , we would nevertheless lose uniformity of the constants in the estimates with respect to  $t_0$ . We stress that this uniformity is crucial in the proof of Proposition 3.3. All proofs and more details can be found in [7].

## 3.2 Nonlinear theory

It turns out that the space  $\widetilde{X}_{\delta, \alpha, 1}$  is too large for our nonlinear theory. Namely, we will need to require a Lipschitz property of the source-term for which this space rules out natural nonlinearities, for instance quadratic ones. Moreover, the statement that the solution of the Fuchsian equation  $w$  is an element of  $\widetilde{X}_{\delta, \alpha, 1}$  yields only weak information about the behavior of the first spatial derivative at  $t = 0$ , which is indeed not sufficient to interpret  $w$  and *all* its first derivatives as the *remainder* of the solution. We resolve these problems by going to  $\widetilde{X}_{\delta, \alpha, k}$  for some  $k > 1$ . The first issue above disappears for  $k = 2$  in one spatial dimension. In some applications, the spaces  $\widetilde{X}_{\delta, \alpha, k}$  still impose a too strong restriction due to the weak control of the highest spatial derivative. In such a case we are required to formulate the theory in the space  $X_{\delta, \alpha, \infty}$ .

The first step is to reconsider the linear case and derive a result analogous to Proposition 3.3 in the spaces  $\widetilde{X}_{\delta, \alpha, k}$  for arbitrary  $k$  by making suitable stronger assumptions on  $f_0$ . With this at hand, the central result of this section, which we write for  $k = 2$  for definiteness, is the following.

**Proposition 3.5** (Existence of solutions of the nonlinear singular initial value problem in  $\tilde{X}_{\delta,\alpha,2}$ ). *Suppose that we can choose  $\alpha > 0$  so that the energy dissipation matrix (3.3) is positive semidefinite at each  $(t, x) \in (0, \delta) \times U$  for a constant  $\eta > 0$ . Suppose that  $u \equiv 0$  and that the operator  $F$  has the following Lipschitz continuity property: For a constant  $\epsilon > 0$  and all sufficiently small  $\delta$ , the operator  $F$  maps  $\tilde{X}_{\delta,\alpha,2}$  into  $X_{\delta,\alpha+\epsilon,1}$  and, moreover, for each  $r > 0$  there exists  $\hat{C} > 0$  (independent of  $\delta$ ) so that*

$$\|F[w] - F[\tilde{w}]\|_{\delta,\alpha+\epsilon,1} \leq \hat{C} \|w - \tilde{w}\|_{\delta,\alpha,2}^{\sim} \quad (3.6)$$

for all  $w, \tilde{w} \in \overline{B_r(0)} \subset \tilde{X}_{\delta,\alpha,2}$ . Then, there exists a unique solution  $w \in \tilde{X}_{\delta,\alpha,2}$  of the singular initial value problem.

*Proof of Proposition 3.5:* We define the operator  $\mathbb{G} := \mathbb{H} \circ F$ . Here  $\mathbb{H}$  is the solution operator as in Proposition 3.3 and  $F$  is the source-term operator as before. We find that under the hypothesis, the operator  $\mathbb{G}$  is a contraction on compact subsets of  $\tilde{X}_{\delta,\alpha,2}$  if  $\delta$  is a sufficiently small. Hence the iteration sequence defined by  $w_{j+1} = \mathbb{G}[w_j]$  for  $j \geq 1$  and, say,  $w_1 = 0$  converges to a fixed point  $w \in \tilde{X}_{\delta,\alpha,2}$  with respect to the norm  $\|\cdot\|_{\delta,\alpha,2}^{\sim}$ . Because of the properties of  $\mathbb{H}$ , a fixed point of  $\mathbb{G}$  is a solution of the SIVP. Hence, we have shown existence of solutions. Uniqueness can be shown as follows. Given any other solution  $\tilde{w}$  in  $\tilde{X}_{\delta,\alpha,2}$ , it is a fixed point of the iteration  $w_{j+1} = \mathbb{G}[w_j]$ . Because  $\mathbb{G}$  is a contraction, there, however, only exists one fixed point, and hence  $\tilde{w} = w$ .  $\square$

For the analogous result of infinite differentiability, we only need to substitute the Lipschitz continuity property in the previous result as follows. For a constant  $\epsilon > 0$ , every sufficiently small  $\delta > 0$  and every non-negative integer  $k$ , the operator  $F$  maps  $X_{\delta,\alpha,k+1}$  into  $X_{\delta,\alpha+\epsilon,k}$  and, moreover, for each  $r > 0$ , there exists  $\hat{C} > 0$  (independent of  $\delta$ ) so that

$$\|F[w] - F[\tilde{w}]\|_{\delta,\alpha+\epsilon,k} \leq \hat{C} \|w - \tilde{w}\|_{\delta,\alpha,k+1}^{\sim} \quad (3.7)$$

for all  $w, \tilde{w} \in \overline{B_r(0)} \cap X_{\delta,\alpha,k+1} \subset \tilde{X}_{\delta,\alpha,k+1}$ . Then, there exists a unique solution  $w \in X_{\delta,\alpha,\infty}$  of the singular initial value problem. In order to prove this result, we first proceed as in the previous proposition for finitely many derivatives. The only remaining task is to show that for  $k \rightarrow \infty$ , we are allowed to choose some non-vanishing  $\delta$ . This can be done with a standard argument for hyperbolic equations. The set  $\overline{B_r(0)}$  is defined with respect to the norm  $\|\cdot\|_{\delta,\alpha,k+1}^{\sim}$ . We note that the constant  $\hat{C}$  is allowed to depend on  $k$  and is not required to be bounded for  $k \rightarrow \infty$ . Note that the Lipschitz estimate involves the norm  $\|\cdot\|_{\delta,\alpha,k+1}^{\sim}$ , while the elements for which this estimates needs to be satisfied are required to be only in the subspace  $X_{\delta,\alpha,k+1}$  of  $\tilde{X}_{\delta,\alpha,k+1}$ . The main advantage of the  $C^\infty$  result over the finite differentiability case is that we only need to check that  $F$  maps  $X_{\delta,\alpha,k+1}$  into  $X_{\delta,\alpha+\epsilon,k}$ , instead of  $\tilde{X}_{\delta,\alpha,k+1}$  into  $X_{\delta,\alpha+\epsilon,k}$ .

### 3.3 Standard singular initial value problem

The following discussion is devoted to the case when the function  $u$  is given by the canonical two-term expansion (2.6). In this case, we speak of the **standard singular initial value problem**.

**Theorem 3.6** (Well-posedness of the standard singular initial value problem in  $\tilde{X}_{\delta,\alpha,2}$ ). *Given arbitrary asymptotic data  $u_*, u_{**} \in H^3(U)$ , the standard singular initial value problem admits a unique solution  $w \in \tilde{X}_{\delta,\alpha,2}$  for  $\alpha, \delta > 0$ , provided  $\delta$  is sufficiently small and the following conditions hold:*

1. *Positivity condition.* Suppose that we can choose  $\alpha > 0$  so that the energy dissipation matrix (3.3) is positive semidefinite at each  $(t, x) \in (0, \delta) \times U$  for a constant  $\eta > 0$ .
2. *Lipschitz continuity property.* For the given  $\alpha > 0$ , the operator  $F$  satisfies the Lipschitz continuity property stated in Proposition 3.5 for all asymptotic data  $u_*, u_{**} \in H^3(U)$  for some  $\epsilon > 0$ .
3. *Integrability condition.* The constants  $\alpha$  and  $\epsilon$  satisfy

$$\alpha + \epsilon < 2(\beta(x) + 1) - \Re(\lambda_1(x) - \lambda_2(x)), \quad x \in U. \quad (3.8)$$

We note that the regularity assumptions on the asymptotic data can certainly be improved. An analogous theorem can be formulated for the  $C^\infty$ -case.

*Proof.* We can apply Proposition 3.5 if we are able to control the additional contribution of the term  $L[u]$  which has to be considered as part of the source-term. It has no contribution to the Lipschitz estimate (3.6), but we have to guarantee that under these hypotheses,  $L[u] \in X_{\delta, \alpha + \epsilon, 1}$  for the given constant  $\epsilon$ . This is indeed the case if (3.8) holds.  $\square$

**Example 3.7.** Consider the second-order hyperbolic Fuchsian equation

$$D^2v - \lambda Dv - t^2 \partial_x^2 v = 0,$$

with a constant  $\lambda$ . This is the Euler-Poisson-Darboux equation. In the standard notation it is

$$\partial_t^2 v - \partial_x^2 v = \frac{1}{t}(\lambda - 1)\partial_t v.$$

Note that  $\lambda = 1$  is the standard wave equation, and in this case, the standard singular initial value problem reduces to the standard Cauchy problem.

1. *Case  $\lambda \geq 0$ .* With our notation, we have  $\lambda_1 = 0$ ,  $\lambda_2 = -\lambda$ ,  $\beta \equiv 0$ ,  $\nu \equiv 1$  and  $f \equiv 0$ . Thus the leading-order term for the standard singular initial value problem is

$$u(t, x) = \begin{cases} u_*(x) + u_{**}(x)t^\lambda & \lambda > 0, \\ u_*(x) \log t + u_{**}(x) & \lambda = 0. \end{cases}$$

The positivity condition of the energy dissipation matrix (3.3) is satisfied precisely for  $\alpha \geq 1 - \lambda$  and all sufficiently small  $\eta > 0$ . The integrability condition (3.8) is satisfied precisely for  $\lambda < 2 - \alpha$ . Hence, our previous proposition implies that the singular initial value problem is well-posed, provided

$$0 \leq \lambda < 2. \quad (3.9)$$

Namely, in this case there exists a solution  $w$  in  $\tilde{X}_{\delta, \alpha, 2}$  for some  $\alpha > 0$  for arbitrary asymptotic data in  $H^3(U)$ .

2. *Case  $\lambda < 0$ .* With our notation, we have  $\lambda_1 = |\lambda|$ ,  $\lambda_2 = 0$ ,  $\beta \equiv 0$ ,  $\nu \equiv 1$  and  $f \equiv 0$ . The positivity condition of the energy dissipation matrix (3.3) is satisfied precisely for  $\alpha \geq 1 - |\lambda|$  and all sufficiently small  $\eta > 0$ . The integrability condition (3.8) is satisfied precisely for  $|\lambda| < 2 - \alpha$ . Hence, our previous proposition implies that the singular initial value problem is well-posed, provided

$$-2 < \lambda < 0.$$

Namely, in this case there exists a solution  $w$  in  $\tilde{X}_{\delta, \alpha, 2}$  for some  $\alpha > 0$  for arbitrary asymptotic data in  $H^3(U)$ .

It turns out that general smooth solutions to the Euler-Poisson-Darboux equation can be expressed explicitly by a Fourier ansatz in  $x$  and by Bessel functions in  $t$ . It is then easy to check that (3.9) (and similarly for  $\lambda < 0$ ) is sharp: While for  $0 \leq \lambda < 2$ , all solutions of the equation behave consistently with the two-term expansion at  $t = 0$ , this is not the case for  $\lambda \geq 2$  for general asymptotic data. Hence the standard singular initial value problem is not well-posed for  $\lambda \geq 2$ . This is completely consistent with our heuristic discussion in Section 2.3 underlying the canonical guess for the leading-order term. If e.g.  $\lambda = 2$ , the assumption that the source-term  $t^2 \partial_x^2 v$  is negligible at  $t = 0$  fails since it is of the same order in  $t$  at  $t = 0$  as the second leading-order term. However, we can see in the proof of Theorem 3.6 that in the special case  $u_* = 0$  (and arbitrary  $u_{**}$ ), the integrability condition (3.8) can be relaxed. For this special choice of data, solutions to the singular initial value problem exist even for  $\lambda \geq 2$ .

It turns out that, often in applications, the three conditions in Theorem 3.6 cannot be satisfied simultaneously. While it is often possible to find constants  $\alpha$  and  $\epsilon$  in accordance with the second and third condition, it can turn out that the corresponding choice of  $\alpha$  is too small to make the energy dissipation matrix positive semidefinite. In order to circumvent this problem, the trick is to choose canonical expansions of higher order  $v_j$ , see Section 2.3, as the leading-order term  $u$  for sufficiently large  $j$ . We refer to the singular initial value problem based on this choice of leading-order term as **singular initial value problem of order  $j$** . For  $j = 1$ , it reduces to the standard singular initial value problem; hence we will focus on the case  $j \geq 2$  in the following. Note that, if  $w$  is a solution of the singular initial value problem of order  $j$ , it is also a solution of the standard singular initial value problem. However, if there is only one solution  $w$  of the singular initial value problem of order  $j$  for given asymptotic data, it does not necessarily mean that  $w$  is the only solution of the standard initial value problem for the same asymptotic data.

For the statement of the following theorem, we need the following notation. For all  $w \in X_{\delta, \alpha, k}$  (or  $w \in \tilde{X}_{\delta, \alpha, k}$  respectively), we introduce the functions  $E_{\delta, \alpha, k}[w] : (0, \delta] \rightarrow \mathbb{R}$  (or  $\tilde{E}_{\delta, \alpha, k}[w] : (0, \delta] \rightarrow \mathbb{R}$  respectively) which are defined in the same way as the respective norms, but the supremum in  $t$  has not been evaluated yet. In particular, this means that  $E_{\delta, \alpha, k}[w]$  (or  $\tilde{E}_{\delta, \alpha, k}[w]$ ) is a bounded continuous function on  $(0, \delta]$ .

**Theorem 3.8** (Well-posedness of the singular initial value problem of order  $j$  in  $\tilde{X}_{\delta, \alpha, 2}$ ). *Given any integer  $j \geq 2$  and any asymptotic data  $u_*, u_{**} \in H^{m_1}(U)$  with  $m_1 = 2j + 1$ , there exists a unique solution  $w \in \tilde{X}_{\delta, \alpha, 2}$  of the singular initial value problem of order  $j$  for some  $\alpha > 0$ , provided*

1.  $F$  maps  $\tilde{X}_{\delta, \tilde{\alpha}, m_1}$  into  $X_{\delta, \tilde{\alpha} + \epsilon, m_1 - 1}$  for all asymptotic data  $u_*, u_{**} \in H^{m_1}(U)$  for some  $\epsilon > 0$  and  $\tilde{\alpha}$  given by  $\alpha := \tilde{\alpha} + (j - 2)\kappa\epsilon$ . for an arbitrary  $\kappa < 1$ .
2. The characteristic speed satisfies

$$2(\beta(x) + 1) > \kappa\epsilon \quad \text{for all } x \in U$$

for the same constant  $\kappa$  chosen earlier.

3.  $F$  satisfies the following Lipschitz condition: for each  $r > 0$  there exists a constant  $C > 0$  (independent of  $\delta$ ) so that

$$E_{\delta, \tilde{\alpha} + \epsilon, 1}[F[w] - F[\tilde{w}]](t) \leq C \tilde{E}_{\delta, \tilde{\alpha}, 2}[w - \tilde{w}](t)$$

for all  $t \in (0, \delta]$  and for all  $w, \tilde{w} \in \overline{B_r(0)} \subset \tilde{X}_{\delta, \tilde{\alpha}, 2}$ .

4. The energy dissipation matrix (3.3) (evaluated with  $\alpha$ ) is positive semidefinite at each  $(t, x) \in (0, \delta) \times U$  for a constant  $\eta > 0$ .

The third condition above is meaningful since both sides of the inequality are continuous and bounded functions on  $(0, \delta]$ . Note that this theorem can be formulated without difficulty for the  $C^\infty$ -case and indeed leads to a simpler statement.

In effect we have obtained a value of  $\alpha$  which increases with  $j$  and henceforth improves the positivity of the energy dissipation matrix. The main prize to pay here is that the asymptotic data must be sufficiently regular and that we must live with a loss of regularity in space.

### 3.4 The Fuchsian numerical algorithm

We proceed with the numerical implementation of our approximation scheme. For linear source-terms we have shown that the solution of the singular initial value problem can be approximated by solutions to the regular initial value problem. We have established an explicit estimate for these approximate solutions. For the nonlinear case, an additional fixed point argument was necessary for the proof, but the Lipschitz continuity condition should allow us to extend the error estimates to nonlinear source-terms.

The regular initial value problem for second-order hyperbolic equations corresponds to the standard initial value problem of a system of (nonlinear) wave equations with initial time  $t_0 > 0$ , and there exists a huge amount of numerical techniques for computing solutions [30, 33]. However, a second-order Fuchsian equation written out with the standard time-derivative  $\partial_t$  (instead of  $D$ ) clearly involves factors  $1/t$  or  $1/t^2$ . Although these are finite for the regular initial value problem, they still can cause severe numerical problems when the initial time  $t_0$  approaches zero, due to the finite representation of numbers in a computer. In order to solve this problem, we introduce a new time coordinate  $\tau := \log t$ , and observe that  $D = \partial_\tau$ . For instance, the Euler-Poisson-Darboux equation becomes

$$\partial_\tau^2 v - \lambda \partial_\tau v - e^{2\tau} \partial_x^2 v = 0,$$

where  $v$  is the unknown and  $\lambda$  is a constant. We have achieved that there is no singular term in this equation; the main price to pay, however, is that the singularity  $t = 0$  has been “shifted to”  $\tau = -\infty$ . Another disadvantage is that the characteristic speed of this equation (defined with respect to the  $\tau$ -coordinate) is  $e^\tau$  and hence increases exponentially with time. For any explicit discretization scheme, we can thus expect that the CFL-condition<sup>3</sup> is always violated from some time on. We must either adapt the time step to the increasing characteristic speeds in  $\tau$ , or, when we decide to work with a fixed time resolution, accept the fact that the numerical solution will eventually become unstable. However, this is not expected to be a severe problem since one can compute the numerical solution with respect to the  $\tau$ -variable until some finite positive time when the numerical solution is still stable and then, if necessary, switch to a discretization scheme based on the original  $t$ -variable. For all the numerical solutions presented in this paper, however, this was not necessary.

We can simplify the following discussion slightly by writing (and implementing numerically) the equation not for the function  $v$  but for the remainder  $w = v - u$ . Henceforth we assume that  $u$  is the canonical two-term expansion determined by given asymptotic data. We have to solve the equation for  $w$  on a time interval  $[\tau_0, \delta]$  for some  $\tau_0 \in \mathbb{R}$  successively going to  $-\infty$  with regular data

$$w(\tau_0, x) = 0, \quad \partial_\tau w(\tau_0, x) = 0, \quad x \in U.$$

Inspired by Kreiss et al. in [31] and by the general idea of the “method of lines”, see [30], we proceed as follows to discretize the equation. First we consider second-order Fuchsian ordinary

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<sup>3</sup>The Courant-Friedrichs-Lewy (CFL) condition for the discretization of hyperbolic equations with explicit schemes is discussed, for instance, in [30].

differential equations (written for a scalar equation now for simplicity)

$$\partial_\tau^2 w + 2a \partial_\tau w + bw = f(\tau),$$

where  $f$  is a given function and the coefficients  $a$  and  $b$  are constants. We discretize the time variable  $\tau$  so that  $\tau_n := \tau_0 + n\Delta\tau$ ,  $w_n := w(\tau_n)$  and  $f_n := f(\tau_n)$  for some time step  $\Delta\tau > 0$  and  $n \in \mathbb{N}$ . Then the equation is discretized in second-order accuracy as

$$\frac{w_{n+1} - 2w_n + w_{n-1}}{(\Delta\tau)^2} + 2a \frac{w_{n+1} - w_{n-1}}{2\Delta\tau} + bw_n = f_n. \quad (3.10)$$

Solving this for  $w_{n+1}$  allows to compute the solution  $w$  at the time  $\tau_{n+1}$  from the solution at the given and previous time  $\tau_n$  and  $\tau_{n-1}$ , respectively. At the initial two time steps  $\tau_0$  and  $\tau_1$ , we set, consistently with the initial data for  $w$  at  $\tau_0$  above,

$$w_0 = 0, \quad w_1 = \frac{1}{2}(\Delta\tau)^2 f(\tau_0). \quad (3.11)$$

We will refer to this scheme as the *Fuchsian ODE solver*.

The idea of the method of lines for Fuchsian partial differential equations is to discretize also the spatial domain with the spatial grid spacing  $\Delta x$  and to use our Fuchsian ODE solver to integrate one step forward in time at each spatial grid point. The source-term function  $f$ , which might now depend on the unknown itself and its first derivatives, is then computed from the data on the current or the previous time levels. Here we understand that spatial derivatives are part of the source-term and are discretized by means of the standard second-order centered stencil using periodic boundary conditions. A problem is that  $f$ , besides spatial derivatives, can also involve time derivatives of the unknown  $w$  (in fact this can also be the case for Fuchsian ordinary differential equations when the source term depends on the time derivative of the unknown). In order to compute those time derivatives in second-order accuracy without changing the stencil of the Fuchsian ODE solver, we made the following choice. In the code we store the numerical solution not only on two time levels, as it is necessary up to now for the scheme given by (3.10) and (3.11), but on a further third past time level. The time derivatives in the source-term can then be computed in second-order accuracy from data exclusively at the present and previous time steps as follows

$$\partial_\tau w(\tau_n) = \frac{3w_n - 4w_{n-1} + w_{n-2}}{2\Delta\tau} + O((\Delta\tau)^2).$$

For this, we need to initialize three time levels at  $\tau = \tau_0$  and hence we set  $w_2 = 2(\Delta\tau)^2 f(\tau_0)$ , in addition to (3.11).

### 3.5 An example: the Euler-Poisson-Darboux equation

We present numerical test results for the Euler-Poisson-Darboux equation now. Recall from Example 3.7 that the singular initial value problem with two-term asymptotic data for this equation is well-posed in particular for  $0 \leq \lambda < 2$ , and in general becomes ill-posed for  $\lambda \geq 2$ . The singular initial value problem for  $\lambda > 0$  considers solutions of the form

$$v(t, x) = u_*(x) + u_{**}(x)t^\lambda + w(t, x),$$

with remainder  $w$ . For the purposes of this test, we choose the asymptotic data  $u_* = \cos x$ ,  $u_{**} = 0$ . Note that in this case, this leading-order behavior is consistent even with the case



$\lambda = 0$ . But according to our previous discussion, it is not consistent with  $\lambda = 2$ , and we expect that this becomes visible in the numerical solutions. For  $u_{**} = 0$  and  $0 < \lambda < 2$ , we can show that the leading-order behavior of the remainder at  $t = 0$  is

$$w(t, x) = u_*(x) \left( -\frac{1}{2(2-\lambda)} t^2 + \frac{1}{8(2-\lambda)(4-\lambda)} t^4 + \dots \right). \quad (3.12)$$

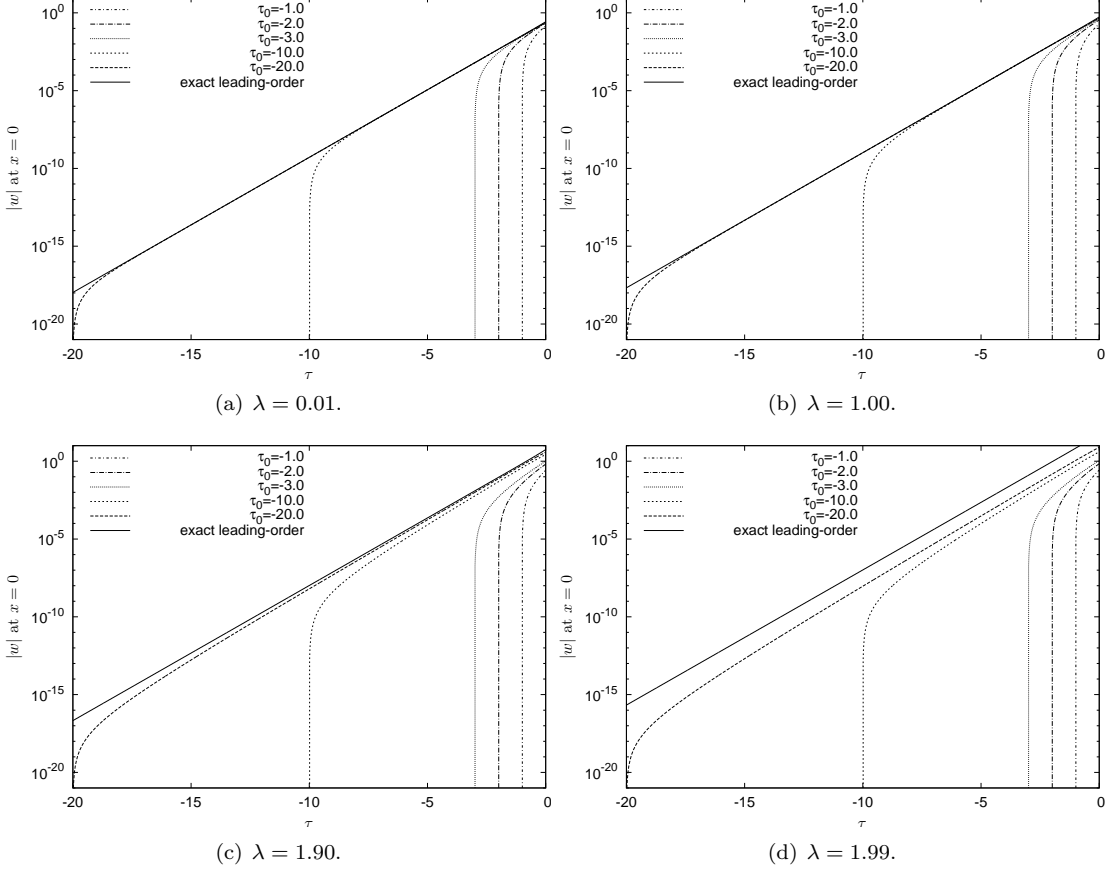


Figure 1: Numerical solutions Euler-Poisson-Darboux equation (as explained in the text).

First we confirm that the numerical solutions converge in second-order when  $\Delta\tau$  and  $\Delta x$  are changed proportionally to each other for a given fixed choice of initial time  $\tau_0 > 0$ . In the following, we choose the resolution so that discretization errors are negligible relative to other errors. In Figure 1 now, we show the following results obtained with  $N = 20$ ,  $\Delta\tau = 0.003$ . Here,  $N$  is the number of spatial grid points, i.e. one has  $\Delta x = 2\pi/N$ . The CFL-parameter is  $\Delta t/\Delta x \approx 0.01$ , and we find that the runs are stable for all  $\tau < 5$ . For each of the plots of Figure 1, we fix a value of  $\lambda$  and study the convergence of the approximate solutions to the (leading-order of the) exact solution (3.12) for various values of the initial time  $\tau_0$ . We plot the value at one spatial point  $x = 0$  only. The convergence rate for  $\tau_0 \rightarrow -\infty$  is fast if  $\lambda = 1$  or  $\lambda = 0.01$ , but becomes lower, the more  $\lambda$  approaches the value 2, where it becomes zero. This is in exact agreement with our expectations and consistent with the error estimates in Proposition 3.3. Hence the numerical results are very promising and confirm the analytic expectations.

Let us comment on numerical round-off errors. All numerical runs in this paper were done with double-precision (binary64 of IEEE 754-2008), where the real numbers are accurate for 16 decimal digits. However, for the case  $\tau_0 = -20$  for instance, the second spatial derivative of the unknown in the equation is multiplied by  $\exp(-40) \approx 10^{-18}$  at the initial time which is not resolved numerically and hence could possibly lead to a significant error. This, however, does not seem to be the case since we obtained virtually the same numerical solution with quadruple precision (binary128 of IEEE 754-2008), i.e. when the numbers in the computer are represented with 34 significant decimal digits.

## 4 Application to Gowdy spacetimes

### 4.1 Background material

Let us provide some background material on Gowdy spacetimes [23, 15]. Introduce coordinates  $(t, x, y, z)$  such that  $(x, y, z)$  describe spatial sections diffeomorphic to  $T^3$  while  $t$  is a timelike variable. We can arrange that the Killing fields associated with the Gowdy symmetry coincide with the coordinate vector fields  $\partial_y, \partial_z$  in a global manner so that the spacetime metric reads

$$g = \frac{1}{\sqrt{t}} e^{\Lambda/2} (-dt^2 + dx^2) + t(e^P(dy + Qdz)^2 + e^{-P}dz^2), \quad t > 0.$$

Hence, the metric depends on three coefficients  $P = P(t, x)$ ,  $Q = Q(t, x)$ , and  $\Lambda = \Lambda(t, x)$ . We also assume spatial periodicity with periodicity domain  $U := [0, 2\pi)$ .

In the chosen gauge, the Einstein's vacuum equations imply the following second-order wave equations for  $P, Q$ ,

$$\begin{aligned} P_{tt} + \frac{P_t}{t} - P_{xx} &= e^{2P}(Q_t^2 - Q_x^2), \\ Q_{tt} + \frac{Q_t}{t} - Q_{xx} &= -2(P_t Q_t - P_x Q_x), \end{aligned} \tag{4.1}$$

which are decoupled from the wave equation satisfied by the third coefficient  $\Lambda$ :

$$\Lambda_{tt} - \Lambda_{xx} = P_x^2 - P_t^2 + e^{2P}(Q_x^2 - Q_t^2). \tag{4.2}$$

Moreover, the Einstein equations also imply constraint equations, which read

$$\Lambda_x = 2t(P_x P_t + e^{2P} Q_x Q_t), \tag{4.3a}$$

$$\Lambda_t = t(P_x^2 + t e^{2P} Q_x^2 + P_t^2 + e^{2P} Q_t^2). \tag{4.3b}$$

It turns out that (4.2) can sometimes be ignored in the following sense. Given a time  $t_0 > 0$ , we can prescribe initial data  $(P, Q)|_{t_0}$  for the system (4.1) while assuming the condition

$$\int_0^{2\pi} (P_x P_t + e^{2P} Q_x Q_t) dx = 0 \quad \text{at } t = t_0.$$

Then, the first constraint (4.3a) determines the function  $\Lambda$  at the initial time, up to a constant which we henceforth fix. Next, one easily checks that the solution  $(P, Q)$  of (4.1) corresponding to these initial data does satisfy the compatibility condition associated with (4.3) and, hence, (4.3) determines  $\Lambda$  uniquely for *all* times of the evolution. Moreover, one checks that (4.2) is satisfied identically by the constructed solution  $(P, Q, \Lambda)$ . One can also consider the alternative viewpoint which follows from the natural 3+1-splitting and treats the three equations (4.1)-(4.2)

as an evolution system for the unknowns  $(P, Q, \Lambda)$ , and (4.3) as constraints that propagate if they hold on an initial hypersurface. In any case, equations (4.1) represent the essential set of Einstein's field equations for Gowdy spacetimes. We refer to (4.1) as the *Gowdy equations* and focus our attention on them in most of what follows. An alternative, more geometrical formulation of Einstein's field equation for Gowdy symmetry has been introduced in [3].

## 4.2 Heuristics about singular solutions of the Gowdy equations

We provide here a formal discussion which motivates the (rigorous) analysis in subsequent sections. Based on extensive numerical experiments [6, 5, 3], it was first conjectured (and later established rigorously [39, 40]) that as one approaches the singularity the spatial derivative of solutions  $(P, Q)$  to (4.1) becomes negligible and  $(P, Q)$  should approach a solution of the *ordinary* differential equations

$$P_{tt} + \frac{P_t}{t} = e^{2P} Q_t^2, \quad Q_{tt} + \frac{Q_t}{t} = -2 P_t Q_t. \quad (4.4)$$

These equations are referred to, in the literature<sup>4</sup>, as the *velocity term dominated* (VTD) equations. Interestingly enough, they admit solutions given explicitly by

$$P(t, x) = \log(\alpha t^k (1 + \zeta^2 t^{-2k})), \quad Q(t, x) = \xi - \frac{\zeta t^{-2k}}{\alpha (1 + \zeta^2 t^{-2k})}, \quad (4.5)$$

where  $x$  plays simply the role of a parameter and  $\alpha > 0, \zeta, \xi, k$  are arbitrary  $2\pi$ -periodic functions of  $x$ .

Based on (4.5), it is a simple matter to determine the first terms in the expansion of the function  $P$  near  $t = 0$ , that is

$$\lim_{t \rightarrow 0} \frac{P(t, x)}{\log t} = \lim_{t \rightarrow 0} t P_t(t, x) = -|k|,$$

hence

$$\lim_{t \rightarrow 0} (P(t, x) + |k(x)| \log t) = \varphi(x), \quad \varphi := \begin{cases} \log \alpha, & k < 0, \\ \log(\alpha(1 + \zeta^2)), & k = 0, \\ \log(\alpha \zeta^2), & k > 0. \end{cases}$$

Similarly, for the function  $Q$  we obtain if  $\zeta \neq 0$ ,

$$\lim_{t \rightarrow 0} Q(t, x) = q(x), \quad q := \begin{cases} \xi, & k < 0, \\ \xi - \frac{\zeta}{\alpha(1 + \zeta^2)}, & k = 0, \\ \xi - \frac{1}{\alpha \zeta}, & k > 0, \end{cases}$$

$$\lim_{t \rightarrow 0} t^{-2|k|} (Q(t, x) - q(x)) = \psi(x), \quad \psi := \begin{cases} -\frac{\zeta}{\alpha}, & k < 0, \\ 0, & k = 0, \\ \frac{1}{\alpha \zeta^3}, & k > 0. \end{cases}$$

If  $\zeta = 0$ , then  $Q \equiv \xi$ .

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<sup>4</sup>To our knowledge, this terminology has been introduced in [20] and, in the context of Gowdy spacetimes, was first used in [27].

From (4.5), we thus have the expansion

$$P = -|k| \log t + \varphi + o(1), \quad Q = q + t^{2|k|} \psi + o(t^{2|k|}), \quad (4.6)$$

in which  $k, \varphi, q, \psi$  are functions of  $x$ . In general,  $P$  *blows-up* to  $+\infty$  when one approaches the singularity, while  $Q$  remains *bounded*. Observe that the sign of  $k$  is irrelevant as far the asymptotic expansion is concerned, and we are allowed to restrict attention to  $k \geq 0$ .

By plugging the explicit solution into the nonlinear terms arising in (4.4) one sees that  $e^{2P} Q_t^2$  is of order  $t^{2(|k|-1)}$  which is negligible since the left-hand side of the  $P$ -equation is of order  $t^{-2}$ , at least when  $k \neq 0$ . On the other hand, the nonlinear term  $P_t Q_t$  is of order  $t^{2(|k|-1)}$ , which is the same order as the left-hand side of the  $Q$ -equation. It is not negligible, but we observe that  $P_t Q_t$  has the same behavior as  $-(|k|/t) Q_t$ .

In fact, observe that the homogeneous system deduced from (4.4):

$$P_{tt} + \frac{P_t}{t} = 0, \quad Q_{tt} + \frac{1-2k}{t} Q_t = 0, \quad (4.7)$$

is solved precisely by the leading-order terms in (4.6). This tells us that, as  $t \rightarrow 0$ , the term  $e^{2P} Q_t^2$  is negligible in the first equation in (4.4), while  $P_t Q_t + (|k|/t) Q_t$  is negligible at  $t = 0$ . This discussion hence allows us to conclude that as far as the behavior at the coordinate singularity  $t = 0$  is concerned, the nonlinear VTD equations (4.4) are well approximated by the system (4.7).

We return now to the nonlinear terms which were not included in the VTD equations, but yet are present in the full model (4.1). Allowing ourselves to differentiate the expansion (4.6), we get the following leading-order terms at  $t = 0$ :

$$e^{2P} Q_x^2 = \begin{cases} t^{-2|k|} e^{2\varphi} q_x^2 + \dots, & q_x \neq 0, \\ 2e^{2\varphi} |k|_x \psi \log t + \dots, & q_x = 0, \quad |k|_x \neq 0, \\ e^{2\varphi} \psi_x^2 + \dots, & q_x = 0, \quad |k|_x = 0, \quad \psi_x \neq 0, \end{cases}$$

$$P_x Q_x = \begin{cases} -\log t |k|_x q_x + \dots, & |k|_x, q_x \neq 0, \\ \varphi_x q_x + \dots, & |k|_x = 0, \quad \varphi_x, q_x \neq 0, \\ -2 \log^2 t t^{2|k|} (|k|_x)^2 \psi + \dots, & |k|_x \neq 0, \quad q_x = 0 \\ t^{2|k|} \varphi_x \psi_x + \dots, & |k|_x = q_x = 0, \quad \varphi_x, \psi_x \neq 0. \end{cases}$$

To check (formally) the validity of the expansion (4.6) we now return to the full system. Consider the nonlinear term  $e^{2P} Q_x^2$  in (4.1), and observe the following:

- Case  $q_x \neq 0$  everywhere on an open subinterval of  $[0, 2\pi]$ . Then, on the one hand, the left-hand side of the first equation in (4.1) is of order  $t^{-2}$ , at most. On the other hand, the term  $e^{2P} Q_x^2$  is negligible with respect to  $t^{-2}$  if and only if  $|k| < 1$  and is of the same order if  $|k| = 1$ .
- Case  $q_x = 0$  on an open subinterval of  $[0, 2\pi]$ . Then,  $e^{2P} Q_x^2$  is negligible with respect to  $t^{-2}$ , and no condition on the velocity  $k$  is required on that interval.
- Case  $q_x(x_0) = 0$  at some isolated point  $x_0$ . Then, no definite conclusion can be obtained and a “competition” between  $|k|$  (which may approach the interval  $[0, 1]$ ) and  $q_x$  (which approaches zero) is expected.

Similarly, at least when  $|k|_x q_x \neq 0$ , the nonlinear term  $P_x Q_x$  is of order  $\log t$  and, therefore, negligible with respect to  $t^{2(|k|-1)}$  (given by the left-hand side of the second equation in (4.1)) if and only if  $|k| \leq 1$ . Points where  $|k|_x$  or  $q_x$  vanish lead to a less singular behavior and condition on the velocity can also be relaxed.

The formal derivation above strongly suggests that we seek solutions to the full nonlinear equations admitting an asymptotic expansion of the form (4.6), that is

$$P = -k \log t + \varphi + o(1), \quad Q = q + t^{2k} (\psi + o(1)),$$

where  $k \geq 0$  and  $\varphi, q, \psi$  are prescribed. In other words, these solutions asymptotically approach a solution of the VTD equations and, in consequence, such solutions will be referred to as *asymptotically velocity term dominated* (AVTD) solutions [27].

Based on this analysis and extensive numerical experiments, it has been conjectured that asymptotically as one approaches the coordinate singularity  $t = 0$  the function  $P(t, x)/\log t$  should approach some limit  $k = k(x)$ , referred to as the *asymptotic velocity*, and that  $k(x)$  should belong to  $[0, 1)$  with the exception of a zero measure set of “exceptional values”. The reason for this name of  $k$  is the following. Based on the work by Geroch [21, 22], it was noted by Moncrief [32] that the evolution equations (4.1) for  $P$  and  $Q$  can be considered as wave map equations with the hyperbolic space as the target space. If a solution of these equations has an expansion of the form (4.6) at  $t = 0$ , then the velocity of the image points of this map, which must be defined with the correct convention of the sign and must be measured with respect to the hyperbolic metric, approaches  $k$  as  $t \rightarrow 0$ .

It was demonstrated in [3] that solutions of the Gowdy equations which are compatible with (4.6) approach certain Kasner solutions at  $t = 0$ , with possibly different parameters along each timeline to the singularity.

### 4.3 Gowdy equations as a second-order hyperbolic Fuchsian system

The first step in our (rigorous) analysis of the Gowdy equations (4.1) now is to write them as a system of second-order hyperbolic Fuchsian equations. After multiplication by  $t^2$ , the equations (4.1) immediately take the second-order hyperbolic Fuchsian form

$$\begin{aligned} D^2 P &= t^2 \partial_x^2 P + e^{2P} (DQ)^2 - t^2 e^{2P} (\partial_x Q)^2, \\ D^2 Q &= t^2 \partial_x^2 Q - 2DPDQ + 2t^2 \partial_x P \partial_x Q. \end{aligned}$$

The general canonical two-term expansion then reads

$$P(t, x) = P_*(x) \log t + P_{**}(x) + \dots$$

for the function  $P$  and, similarly, an expansion  $Q_*(x) \log t + Q_{**}(x) + \dots$  for the function  $Q$  with prescribed data  $Q_*, Q_{**}$ . At this stage, we do not make precise statements about the (higher-order) remainders, yet. In any case, the Fuchsian theory does not apply to this system due to the presence of the term  $-2DPDQ$  (with the exception of the cases  $P_* = 0$  or  $Q_* = 0$ ). Namely, this term does not behave as a positive power of  $t$  at  $t = 0$  when we substitute  $P$  and  $Q$  by their canonical two-term expansions, but this is required by the theory. The reason for this problem is the significance of the nonlinear term in the source-term as found in Section 4.2, cf. (4.7).

We propose to add the term  $-2kDQ$  to the equation for  $Q$  where  $k$  is a prescribed (smooth, spatially periodic) function depending on  $x$ , only. The function  $k$  will play the role of the asymptotic velocity mentioned before. This yields the system of equations

$$\begin{aligned} D^2 P &= t^2 \partial_x^2 P + e^{2P} (DQ)^2 - t^2 e^{2P} (\partial_x Q)^2, \\ D^2 Q - 2kDQ &= t^2 \partial_x^2 Q - 2(k + DP)DQ + 2t^2 \partial_x P \partial_x Q. \end{aligned} \tag{4.8}$$

The resulting system is of second-order hyperbolic Fuchsian form with two equations, corresponding to

$$\lambda_1^{(1)} = \lambda_2^{(1)} = 0, \quad \lambda_1^{(2)} = 0, \quad \lambda_2^{(2)} = -2k.$$

Here, the superscript determines the respective equation of the system (4.8). If we assume that  $k$  is a strictly positive function, as we will do in all of what follows, the expected leading-order behavior at  $t = 0$  given by the canonical two-term expansions is

$$\begin{aligned} P(t, x) &= P_*(x) \log t + P_{**}(x) + \dots, \\ Q(t, x) &= Q_*(x) + Q_{**}(x)t^{2k(x)} + \dots \end{aligned} \tag{4.9}$$

One checks easily that the problem associated with the term  $-2DPDQ$  before does not arise if  $P_* = -k$ . Indeed, the canonical two-term expansion (4.9) is consistent with the heuristics of the Gowdy equations above and we recover the singular initial value problem studied rigorously in [28, 36] and numerically in [1]. We only mention here without further notice that the case  $k \equiv 0$  with the logarithmic canonical two-term expansion for  $Q$  is covered by the following discussion. Furthermore, the case of  $k$  vanishing at only certain points may be also included via a suitable renormalization of the asymptotic data, see (2.7).

When  $P_* = -k$ , the function  $k$  plays a two-fold role in (4.8). On the one hand, it is an asymptotic data for the function  $P$  and, on the other hand, it is a coefficient of the principal part of the second equation. In order to keep these two roles of  $k$  separated in a first stage, we consider the system

$$\begin{aligned} D^2 P &= t^2 \partial_x^2 P + e^{2P} (DQ)^2 - t^2 e^{2P} (\partial_x Q)^2, \\ D^2 Q - 2kDQ &= t^2 \partial_x^2 Q - 2(-P_* + DP)DQ + 2t^2 \partial_x P \partial_x Q, \end{aligned} \tag{4.10}$$

instead of (4.8). Studying the singular initial value problem with two-term asymptotic data means that we search for solutions to (4.10) of the form (as  $t \rightarrow 0$ )

$$\begin{aligned} P(t, x) &= P_*(x) \log t + P_{**}(x) + w^{(1)}(t, x), \\ Q(t, x) &= Q_*(x) + Q_{**}(x)t^{2k(x)} + w^{(2)}(t, x), \end{aligned} \tag{4.11}$$

for general asymptotic data  $P_*$ ,  $P_{**}$ ,  $Q_*$ ,  $Q_{**}$ , and remainders  $w^{(1)}$ ,  $w^{(2)}$ . After studying the well-posedness for this problem, we can always choose  $P_*$  to coincide with  $-k$  and, therefore, recover our original Gowdy problem (4.8)-(4.9). For simplicity in the presentation, we always assume that  $k$  is a  $C^\infty$  function.

In the following discussion, we write the vector-valued remainder as  $w := (w^{(1)}, w^{(2)})$ , and we fix some asymptotic data  $P_*$ ,  $P_{**}$ ,  $Q_*$ , and  $Q_{**}$  and choose the leading-order term  $u$  according to (4.11). The source-term operator  $F[w](t, x) := (F_1[w](t, x), F_2[w](t, x))$  reads

$$\begin{aligned} F_1[w] &= \left( t^{P_*} e^{P_{**}} e^{w^{(1)}} (2k t^{2k} Q_{**} + Dw^{(2)}) \right)^2 \\ &\quad - \left( t^{P_*} e^{P_{**}} e^{w^{(1)}} (t \partial_x Q_* + 2 \partial_x k t^{2k} t \log t Q_{**} + t^{2k} t \partial_x Q_{**} + t \partial_x w^{(2)}) \right)^2, \end{aligned}$$

and

$$\begin{aligned} F_2[w] &= -2Dw^{(1)} (2k t^{2k} Q_{**} + Dw^{(2)}) \\ &\quad + 2 \left( t \partial_x P_* \log t + t \partial_x P_{**} + t \partial_x w^{(1)} \right) \left( (t \partial_x Q_* + 2 \partial_x k t^{2k} t \log t Q_{**} + t^{2k} t \partial_x Q_{**} + t \partial_x w^{(2)}) \right). \end{aligned}$$

#### 4.4 Properties of the source-term operator

To establish the well-posedness of the singular initial value problem for the Gowdy equations, we need first to derive certain decay properties of the source-term operator  $F$  consistent with Section 3. Let us introduce some notation specific to the Gowdy equations. Let  $X_{\delta,\alpha_1,k}^{(1)}$  be the space defined as above based on the coefficients of the first equation in (4.10) and, similarly, let  $X_{\delta,\alpha_2,k}^{(2)}$  be the space associated with the second equation. By definition, a vector-valued map  $w = (w^{(1)}, w^{(2)})$  belongs to  $X_{\delta,\alpha,k}$  precisely if  $w^{(1)} \in X_{\delta,\alpha_1,k}^{(1)}$  and  $w^{(2)} \in X_{\delta,\alpha_2,k}^{(2)}$ , with  $\alpha := (\alpha_1, \alpha_2)$ . An analogous notation is used for the spaces  $\tilde{X}_{\delta,\alpha_1,k}^{(1)}$ ,  $\tilde{X}_{\delta,\alpha_2,k}^{(2)}$  and  $\tilde{X}_{\delta,\alpha,k}$ .

Now we are ready to state a first result about the source-term of (4.10).

**Lemma 4.1** (Operator  $F$  in the finite differentiability class). *Fix any  $\delta > 0$  and any asymptotic data  $P_*, P_{**}, Q_*, Q_{**} \in H^m(U)$ ,  $m \geq 2$ . Suppose there exist  $\epsilon > 0$  and a continuous function  $\alpha = (\alpha_1, \alpha_2) : U \rightarrow (0, \infty)^2$  so that, at each  $x \in U$ ,*

$$\alpha_1(x) + \epsilon < \min(2(P_*(x) + 2k(x)), 2(P_*(x) + 1)), \quad (4.12a)$$

$$\alpha_2(x) + \epsilon < 2(1 - k(x)), \quad (4.12b)$$

$$\alpha_1(x) - \alpha_2(x) > \epsilon + \min(0, 2k(x) - 1), \quad (4.12c)$$

$$\epsilon < 1. \quad (4.12d)$$

*Then, the operator  $F$  associated with the system (4.10) and the given asymptotic data maps  $\tilde{X}_{\delta,\alpha,m}$  into  $X_{\delta,\alpha+\epsilon,m-1}$  and satisfies the following Lipschitz continuity condition: For each  $r > 0$  and for some constant  $C > 0$  (independent of  $\delta$ ),*

$$E_{\delta,\alpha+\epsilon,m-1} [F[w] - F[\hat{w}]](t) \leq C \tilde{E}_{\delta,\alpha,m} [w - \hat{w}](t), \quad t \in (0, \delta]$$

*for all  $w, \hat{w} \in B_r \subset \tilde{X}_{\delta,\alpha,m}$ , where  $B_r$  denotes the closed ball centered at the origin.*

In this lemma, since  $P_* \in H^1(U)$ , in particular, a standard Sobolev inequality implies that  $P_*$  can be identified with a unique bounded continuous periodic function on  $U$ , and the inequality (4.12a) makes sense pointwise.

*Proof.* Consider the expression of  $F$  given before. Let  $w \in \tilde{X}_{\delta,\alpha,m}$  for some (so far unspecified) positive spatially dependent functions  $\alpha_1, \alpha_2$ , hence  $w^{(1)} \in \tilde{X}_{\delta,\alpha_1,m}^{(1)}$  and  $w^{(2)} \in \tilde{X}_{\delta,\alpha_2,m}^{(2)}$ . By a standard Sobolev inequality (since  $m \geq 2$  and the spatial dimension is 1), we get that  $F[w](t, \cdot) \in H^{m-1}(U)$  for all  $t \in (0, \delta]$ . Namely, if  $m \geq 2$  we can control the nonlinear terms of  $F[w](t, \cdot)$  in all generality for a given  $t > 0$  if any factor in any term of  $F[w](t, \cdot)$ , after applying up to  $m - 1$  spatial derivatives, is an element in  $L^\infty(U)$  – with the exception of the  $m$ th spatial derivative of  $w$  which is only required to be in  $L^2(U)$ . This is guaranteed by the Sobolev inequalities. Having found that  $F[w](t, \cdot) \in H^{m-1}(U)$  for all  $t \in (0, \delta]$ , it is easy to check that  $F_1[w] \in X_{\delta,\alpha_1+\epsilon,0}^{(1)}$  if

$$\alpha_1(x) + \epsilon \leq \min(2(P_*(x) + 2k(x)), 2(P_*(x) + 1)), \quad x \in U. \quad (4.13)$$

Even more, condition (4.13) implies that  $D^l F_1[w] \in X_{\delta,\alpha_1+\epsilon,0}^{(1)}$  for all  $l \leq m - 1$ .

Considering now spatial derivatives, we have to deal with two difficulties. The first one is that logarithmic terms arise with each spatial derivative. We find  $\partial_x^k D^l F_1[w] \in X_{\delta,\alpha_1+\epsilon,0}^{(1)}$  for all  $l \leq m - 1$  and  $k \leq m - 2$  and  $k + l \leq m - 1$  (excluding first the case  $k = m - 1, l = 0$ ) provided

$$\alpha_1(x) + \epsilon < \min(2(P_*(x) + 2k(x)), 2(P_*(x) + 1)), \quad x \in U. \quad (4.14)$$

A second difficulty arises in the case  $k = m - 1$ ,  $l = 0$ . Namely, since  $w \in \tilde{X}_{\delta, \alpha, m}$  (and not in  $X_{\delta, \alpha, m}$ ), it follows that in particular  $t\partial_x^m w^{(2)} \sim t^{2k+\alpha_2}$  (and not  $t^{1+2k+\alpha_2}$ ); note that the function  $\beta$  which determines the behavior of the characteristic speeds at  $t = 0$  is identically zero in the case of the Gowdy equations. The potentially problematic term is hence of the form  $AB$  with

$$\begin{aligned} A &:= t^{P_*} e^{P_{**}} e^{w^{(1)}} (t\partial_x Q_* + 2\partial_x k t^{2k} t \log t Q_{**} + t^{2k} t \partial_x Q_{**} + t\partial_x w^{(2)}), \\ B &:= t^{P_*} e^{P_{**}} e^{w^{(1)}} (\partial_x^{m-1} (t\partial_x Q_* + 2\partial_x k t^{2k} t \log t Q_{**} + t^{2k} t \partial_x Q_{**}) + t\partial_x^m w^{(2)}), \end{aligned}$$

originating from taking  $m - 1$  spatial derivatives of  $F_1[w]$ . To ensure  $\partial_x^{m-1} F_1[w] \in X_{\delta, \alpha_1 + \epsilon, 0}^{(1)}$ , we need

$$\alpha_1(x) + \epsilon < (P_*(x) + 1) + (P_*(x) + 2k(x) + \alpha_2(x)), \quad x \in U. \quad (4.15)$$

If (4.14) is satisfied, we have (for all  $x$ )

$$\begin{aligned} \alpha_1(x) + \epsilon &< \min \left( 2(P_*(x) + 2k(x)), 2(P_*(x) + 1) \right) \\ &\leq (P_*(x) + 1) + (P_*(x) + 2k(x)) \end{aligned}$$

and, thus, (4.15) follows from (4.14). In conclusion, (4.14) is sufficient to guarantee that  $F_1[w] \in X_{\delta, \alpha_1 + \epsilon, m-1}^{(1)}$ .

Let us proceed next with the analysis of the term  $F_2[w]$ . If

$$\alpha_1(x) - \alpha_2(x) \geq \epsilon, \quad \alpha_2(x) + \epsilon < 2(1 - k(x)), \quad x \in U, \quad (4.16)$$

then  $F_2[w] \in X_{\delta, \alpha_2 + \epsilon, 0}^{(2)}$ . This inequality also implies that all time derivatives are in  $X_{\delta, \alpha_2 + \epsilon, 0}^{(2)}$  as before. We have to deal with the same two difficulties as before when we consider spatial derivatives of  $F_2[w]$ . On the one hand, equality in (4.16) cannot occur due to additional logarithmic terms. On the other hand, we must be careful with the  $(m - 1)$ -th spatial derivative of  $F_2[w]$ . Here, the two problematic terms are of the form  $AB$  with either

$$\begin{aligned} A &:= \partial_x^{m-1} (t\partial_x P_* \log t + t\partial_x P_{**}) + t\partial_x^m w^{(1)}, \\ B &:= t\partial_x Q_* + 2\partial_x k t^{2k} t \log t Q_{**} + t^{2k} t \partial_x Q_{**} + t\partial_x w^{(2)}, \end{aligned}$$

or else

$$\begin{aligned} A &:= t\partial_x P_* \log t + t\partial_x P_{**} + t\partial_x w^{(1)}, \\ B &:= \partial_x^{m-1} (t\partial_x Q_* + 2\partial_x k t^{2k} t \log t Q_{**} + t^{2k} t \partial_x Q_{**}) + t\partial_x^m w^{(2)}. \end{aligned}$$

The first one is under control provided  $\alpha_1(x) + 1 > 2k(x) + \alpha_2(x) + \epsilon$ , for all  $x \in U$ , while for the second one it is sufficient to require  $\epsilon < 1$ . The claimed Lipschitz continuity condition follows from the above arguments.  $\square$

Positive functions  $\alpha_1$  and  $\alpha_2$  and constants  $\epsilon > 0$  satisfying the hypothesis of Lemma 4.1 can obviously exist only if  $k(x) < 1$  for all  $x \in U$  (due to (4.12b)). In Lemma 4.3 below we identify a special case where this limitation is avoided. Hence, we make the assumption that  $0 < k(x) < 1$  for all  $x$ , which is consistent with our formal analysis in Section 4.2. As a consistency check for the case of interest  $P_* = -k$ , let us determine under which conditions the inequalities (4.12) can be hoped to be satisfied at all. For this, consider (4.12a) and (4.12c) in the borderline case  $\alpha_2 = \epsilon = 0$ . This leads to the condition  $0 < k < 3/4$ , which shows that Lemma 4.1 does not



apply within the full interval  $0 < k < 1$ . It is interesting to note that Rendall was led to the same restriction in [36], but its origin stayed unclear in his approach. Here, we find that this is caused by the presence of the condition (4.12c) in particular which reflects the fact that  $w$  is an element of the space  $\tilde{X}_{\delta,\alpha,m}$  rather than of the smaller space  $X_{\delta,\alpha,m}$ . Interestingly, we can eliminate this condition and, hence, retain the full interval  $0 < k < 1$ , when we consider the  $C^\infty$ -case, instead of finite differentiability. See also [3] for a detailed discussion of the different intervals of  $k$ .

**Lemma 4.2** (Operator  $F$  in the  $C^\infty$  class. General theory). *Fix any  $\delta > 0$  and any asymptotic data  $P_*, P_{**}, Q_*, Q_{**} \in C^\infty(U)$ . Suppose there exist a constant  $\epsilon > 0$  and a continuous functions  $\alpha = (\alpha_1, \alpha_2) : U \rightarrow (0, \infty)^2$  such that, at each  $x \in U$ ,*

$$\alpha_1(x) + \epsilon < \min(2(P_*(x) + 2k(x)), 2(P_*(x) + 1)), \quad (4.17a)$$

$$\alpha_2(x) + \epsilon < 2(1 - k(x)), \quad (4.17b)$$

$$\alpha_1(x) - \alpha_2(x) > \epsilon. \quad (4.17c)$$

*Then, for each integer  $m \geq 1$ , the operator  $F$  maps  $X_{\delta,\alpha,m}$  into  $X_{\delta,\alpha+\epsilon,m-1}$  and satisfies the following Lipschitz continuity property: for each  $r > 0$  and some constant  $C > 0$  (independent of  $\delta$ ),*

$$E_{\delta,\alpha+\epsilon,m-1}[F[w] - F[\hat{w}]](t) \leq C\tilde{E}_{\delta,\alpha,m}[w - \hat{w}](t), \quad t \in (0, \delta],$$

*for all  $w, \hat{w} \in B_r \cap X_{\delta,\alpha+\epsilon,m} \subset \tilde{X}_{\delta,\alpha+\epsilon,m}$ .*

The proof is completely analogous to that of Lemma 4.1. Since only spaces  $X_{\delta,\alpha,k}$  need to be checked (i.e. without the tilde) in the  $C^\infty$ -case we obtain stronger control than in the finite differentiability case. In consequence, the  $C^\infty$ -case does not require the condition (4.12c). Thus  $k$  can have values in the whole interval  $(0, 1)$  as we show in detail later. In a special case, which will be of interest for the later discussion, however, we can relax the constraints for  $k$  even in the finite differentiability case.

**Lemma 4.3** (Operator  $F$  in the finite differentiability class. A special case). *Fix any  $\delta > 0$  and any asymptotic data  $P_*, P_{**}, Q_{**} \in H^m(U)$ ,  $Q_* = \text{const}$ ,  $m \geq 2$ . Suppose there exist  $\epsilon > 0$  and a continuous function  $\alpha = (\alpha_1, \alpha_2) : U \rightarrow (0, \infty)^2$  such that, at each  $x \in U$ ,*

$$\alpha_1(x) + \epsilon < 2(P_*(x) + 2k(x)),$$

$$\alpha_2(x) + \epsilon < 2, \quad \alpha_1(x) - \alpha_2(x) > \epsilon - 1, \quad \epsilon < 1.$$

*Then, the operator  $F$  satisfies the conclusions of Lemma 4.1.*

In the special case  $Q_* = \text{const}$ , we have hence characterized the map  $F$  for  $k$  being any positive function in the finite differentiability case. The analogous result for the  $C^\infty$ -case can also be derived.

## 4.5 Well-posedness theory

Relying on Theorem 3.6 and the results in the previous sections, we now determine conditions that ensure that the singular initial value problem for the Gowdy equations is well-posed. Besides the properties of the source-operator  $F$  already discussed, we have to check the positivity of the energy dissipation matrix. This leads us to the matrix

$$N^{(1)} := \begin{pmatrix} \alpha_1 & -\eta/2 & 0 \\ -\eta/2 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 - 1 \end{pmatrix}$$

for the first component and to the matrix

$$N^{(2)} := \begin{pmatrix} 2k + \alpha_2 & -\eta/2 & 0 \\ -\eta/2 & \alpha_2 & -2\partial_x k(t \log t) \\ 0 & -2\partial_x k(t \log t) & 2k + \alpha_2 - 1 \end{pmatrix}$$

for the second component. For the matrix  $N^{(1)}$  to be positive, it is necessary that  $\alpha_1(x) > 1$  for all  $x \in U$ . However, if  $P_* = -k$ , then condition (4.12a) in Lemma 4.1 in the finite differentiability case (or the corresponding one in Lemma 4.2 in the  $C^\infty$ -case) implies that  $\alpha_1(x) < 1$ . Hence, in the same way as in Rendall [36], one does not arrive at a well-posedness result for the singular initial value problem yet. However, since the positivity of the energy dissipation matrix is the only part of the hypothesis in Theorem 3.6 which is violated, we can use instead Theorem 3.8 to prove well-posedness of the singular initial value problem with asymptotic solutions of sufficiently high order  $j$ .

Let us be specific about what we mean by  $j$  being “sufficiently large”, and we now make some choice for the parameters  $\alpha_1$ ,  $\alpha_2$  and  $\epsilon$ , consistent with Lemma 4.2, which will allow us to estimate the required size of  $j$ . We make no particular effort to choose these quantities optimally, but still the goal is to choose  $j$  “reasonably” small. Henceforth, we restrict to the  $C^\infty$ -case and  $P_*(x) = -k(x)$  with  $0 < k(x) < 1$  for all  $x \in U$ . We introduce positive constants  $\mu_1$  and  $\mu_2$  (with further restrictions later) and the function  $\chi(x) := 1 - 2|x - 1/2|$ . The condition (4.17a) states that we must choose  $\alpha_1(x)$  and  $\epsilon$  so that  $\alpha_1(x) + \epsilon < \chi(k(x))$ . We set

$$\alpha_1(x) := 1 - \sqrt{4(k(x) - 1/2)^2 + \mu_1^2}, \quad (4.18)$$

and find  $\chi(k(x)) - \alpha_1(x) > \sqrt{1 + \mu_1^2} - 1$  for all  $x \in U$ , provided  $0 < k(x) < 1$ . Similarly, we set

$$\alpha_2(x) := 1 - \sqrt{4(k(x) - 1/2)^2 + \mu_2^2}, \quad (4.19)$$

and it follows that  $\alpha_1(x) - \alpha_2(x) > \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2}$  for  $\mu_2 > \mu_1$ . For the conditions (4.17a) and (4.17c) to hold true, we have to choose

$$0 < \mu_1 < \mu_2, \quad \text{and} \quad 0 < \epsilon \leq \min\left(\sqrt{1 + \mu_1^2} - 1, \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2}\right).$$

Condition (4.17b) is then satisfied automatically.

Now, assume in what follows that  $k(x) \in (1/2 - \Delta k, 1/2 + \Delta k)$  for all  $x \in U$  for a constant  $\Delta k \in (0, 1/2)$ . Then it is clear that both functions  $\alpha_1$  and  $\alpha_2$  are positive for all such  $k(x)$  if and only if

$$\mu_1 < \mu_2 < \sqrt{1 - 4(\Delta k)^2}.$$

This assumption will be made in the following. In Theorem 3.8, we could choose  $j$  as small as possible if we pick the maximal allowed value for  $\epsilon$ . Hence, we set

$$\epsilon := \min\left(\sqrt{1 + \mu_1^2} - 1, \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2}\right).$$

We find easily that

$$\sqrt{1 + \mu_1^2} - 1 \leq \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2},$$

provided

$$\mu_1^2 \leq \frac{1}{4}(\mu_2^2 + 2\sqrt{1 + \mu_2^2} - 2),$$

and check that this is consistent with the condition  $0 < \mu_1 < \mu_2$  made before. In order to make a specific choice, we assume this inequality for  $\mu_1$  and hence obtain that

$$\epsilon = \sqrt{1 + \mu_1^2} - 1. \quad (4.20)$$

Now, in order to make the energy dissipation matrix positive, we must choose  $j$  so that for all  $x \in U$ ,

$$\begin{aligned} \tilde{\alpha}_1(x) &:= \alpha_1(x) + (j - 2)\kappa\epsilon > 1, \\ \tilde{\alpha}_2(x) &:= \alpha_2(x) + (j - 2)\kappa\epsilon > 1 - 2k(x); \end{aligned}$$

cf. Theorem 3.8. These two inequalities are satisfied for all functions  $k$  under our assumptions if in particular

$$j > 2 + \frac{\sqrt{4(\Delta k)^2 + \mu_2^2}}{\kappa(\sqrt{1 + \mu_1^2} - 1)}. \quad (4.21)$$

In any case, we choose the maximal value for  $\mu_1$

$$\mu_1 := \frac{1}{2}\sqrt{\mu_2^2 + 2\sqrt{1 + \mu_2^2} - 2}, \quad (4.22)$$

since this minimizes the value on the right side of (4.21). We find that for this value of  $\mu_1$ , the right side of (4.21) is monotonically decreasing in  $\mu_2$  and diverges to  $+\infty$  for  $\mu_2 \rightarrow 0$  for all values of  $\Delta k$ .

**Theorem 4.4** (Well-posedness theory for the Gowdy equations). *Consider some asymptotic data  $P_* = -k$ ,  $P_{**}$ ,  $Q_*$ ,  $Q_{**} \in C^\infty(U)$ , where  $k$  is a smooth function  $U \rightarrow (1/2 - \Delta k, 1/2 + \Delta k)$  for a constant  $\Delta k \in (0, 1/2)$ . Then, for the Gowdy equations with these prescribed data are well-posed in certain weighted Sobolev spaces.*

*More precisely, the singular initial value problem with asymptotic solutions of order  $j$  has a unique solution with remainder  $w \in X_{\delta, \alpha + (j-2)\kappa\epsilon, \infty}$  for some sufficiently small  $\delta > 0$  and some  $\kappa < 1$ . Here, the exponents  $\alpha = (\alpha_1, \alpha_2)$  and  $\epsilon$  are given in (4.18), (4.19), and (4.20) explicitly in terms of the data and parameters  $\mu_1, \mu_2$  chosen such that  $\mu_1$  is an explicit expression in  $\mu_2$  given in (4.22) while  $\mu_2$  is a sufficiently close to (but smaller than)  $\sqrt{1 - 4(\Delta k)^2}$ , and the order of differentiation  $j$  satisfies*

$$j > 2 + \frac{2}{\sqrt{3 - 4(\Delta k)^2 + 2\sqrt{2 - 4(\Delta k)^2} - 2}}.$$

The above condition implies that to reach  $\Delta k \rightarrow 0$  we need  $j > 7$ , while  $\Delta k \rightarrow 1/2$  requires  $j \rightarrow \infty$ . Although our estimates may not be quite optimal, the latter implication cannot be avoided.

## 4.6 Fuchsian analysis for the function $\Lambda$

So far we have considered the equations (4.1) for  $P$  and  $Q$ . We can henceforth assume that these equations are solved identically for all  $t > 0$  (and  $t \leq \delta$  for some  $\delta > 0$ ) and that hence  $P$  and  $Q$  are given functions with leading-order behavior (4.9) and remainders in a given  $X_{\delta, \alpha, k}$ . The equations which remain to be solved in order to obtain a solution of the full Einstein's

field equations are (4.2) and (4.3). In particular we are interested in the function  $\Lambda$  in order to obtain the full geometrical information. We compute  $\Lambda$  from a singular initial value problem with “data” on the singularity analogously to  $P$  and  $Q$ . The following discussion resembles the previous one and we only discuss new aspects now.

Clearly, the three remaining equations (4.2) and (4.3) for  $\Lambda$  are overdetermined, and hence solutions will exist only under certain conditions. Let us define the following “constraint quantities” from (4.3)

$$\begin{aligned} C_1(t, x) &:= -\partial_t \Lambda + t(P_x)^2 + e^{2P} t(Q_x)^2 + t(\partial_t P)^2 + e^{2P} t(\partial_t Q)^2, \\ C_2(t, x) &:= -\Lambda_x + 2P_x DP + 2e^{2P} Q_x DQ. \end{aligned}$$

Moreover, we define

$$H(t, x) := -\Lambda_{tt} + \Lambda_{xx} + P_x^2 - P_t^2 + e^{2P}(Q_x^2 - Q_t^2)$$

from (4.2). From the evolution equations for  $P$  and  $Q$ , we find the subsidiary system

$$\partial_t C_1 = \partial_x C_2 + H, \quad \partial_t C_2 = \partial_x C_1. \quad (4.23)$$

These equations have the following consequences. Suppose that we use (4.3b) as an evolution equation for  $\Lambda$ . This implies that  $C_1 \equiv 0$  for all  $t > 0$ . Moreover, suppose that we prescribe data at some  $t_0 > 0$  (indeed  $t_0$  is allowed to be zero later) so that  $C_2(t_0, x) = 0$  for all  $x \in U$ . Then the equations imply that  $H \equiv 0$  and  $C_2 \equiv 0$  for all  $t > 0$  and thus we have constructed a solution of the full set of field equations. Alternatively, let us use (4.2) as the evolution equation for  $\Lambda$ , i.e.  $H \equiv 0$ . Suppose that we prescribe data so that  $C_1(t_0, x) = C_2(t_0, x) = 0$  at some  $t_0$ . It follows that  $C_1 \equiv C_2 \equiv 0$  for all  $t > 0$  because the evolution system (4.23) for  $C_1$  and  $C_2$  is symmetric hyperbolic. Again, Einstein’s field equations are solved.

Now, we want to consider the case  $t_0 = 0$ . First note that (4.23) is regular even at  $t = 0$ . Suppose that  $P$  and  $Q$  are functions with leading-order behavior (4.9) and remainders in a given  $X_{\delta, \alpha, k}$  with  $k \geq 1$ . If there exists a function  $w_3$  so that

$$\Lambda(t, x) = \Lambda_*(x) \log t + \Lambda_{**}(x) + w_3(t, x) \quad (4.24)$$

with  $w_3$  converging to zero in a suitable norm at  $t = 0$  and

$$\Lambda_*(x) = k^2(x), \quad \Lambda_{**}(x) = \Lambda_0 + 2 \int_0^x k(\tilde{x}) (-\partial_{\tilde{x}} P_{**}(\tilde{x}) + 2e^{2P_{**}(\tilde{x})} Q_{**}(\tilde{x}) \partial_{\tilde{x}} Q_*(\tilde{x})) d\tilde{x}, \quad (4.25)$$

where  $\Lambda_0$  is an arbitrary real constant, then, in particular,

$$\lim_{t \rightarrow 0} C_2 = 0.$$

It also follows that  $\lim_{t \rightarrow 0} tC_1 = 0$ . Let us first use (4.3b) as a singular evolution equation for  $\Lambda$ . Since this is “only” a singular ODE, one can show easily that there exists a unique solution for  $\Lambda$  for  $t > 0$  which obeys the two-term expansion above, and hence  $C_1 \equiv 0$ . Our discussion before implies that  $H, C_2 \equiv 0$ . Hence we obtain a solution of the full Einstein’s field equation for all  $t > 0$ . Alternative, choose (4.2) as the evolution equation for  $\Lambda$  now. This equation can be written in second-order hyperbolic Fuchsian form

$$D^2 \Lambda - t^2 \partial_x^2 \Lambda = (t \partial_x P)^2 + (D\Lambda - (DP)^2) + e^{2P} ((t \partial_x Q)^2 - (DQ)^2).$$

Indeed this equation is compatible with the leading-order expansion (4.24) at  $t = 0$  and we can show well-posed of this singular initial value problem in the same way as we did for the functions

$P$  and  $Q$  before (also going to sufficiently high-order in  $j$ ). In particular, for any asymptotic data  $\Lambda_*$  and  $\Lambda_{**}$ , not necessarily those given by (4.25), there exists a unique solution of this equation  $\Lambda$  with remainder  $w_3$  in a certain space  $X_{\delta,\alpha,k}$ . By means of uniqueness we find that the solution  $\Lambda$  of this equation coincides with the solution for  $\Lambda$  obtained using (4.3b) as the evolution equation. Hence, we must have  $H, C_1, C_2 \equiv 0$  for all  $t > 0$ , and thus also this method yields a solution of the full Einstein's field equations.

Note that periodicity and (4.25) implies that the asymptotic data for  $P$  and  $Q$  must satisfy the relation

$$\int_0^{2\pi} k(\tilde{x})(-\partial_{\tilde{x}}P_{**}(\tilde{x}) + 2e^{2P_{**}(\tilde{x})}Q_{**}(\tilde{x})\partial_{\tilde{x}}Q_*(\tilde{x}))d\tilde{x} = 0$$

for smooth solutions.

## 5 Numerical experiments

### 5.1 Test 1. Homogeneous pseudo-polarized solutions

We continue our discussion with the singular initial value problem for the Gowdy equations. In all of what follows we consider the singular initial value problem with two-term asymptotic data for the Gowdy equations. As we show this works very well and we get good convergence. This is a strong indication that the standard singular initial value problem is well-posed. In contrast, recall from Theorem 4.4 that our analytical techniques are only sufficient to show that the initial value problem with asymptotic solutions of sufficiently high order is well-posed for the Gowdy equations.

Before we proceed with “interesting” solutions of the Gowdy equations, let us start with a case for which we can construct an explicit solution and hence test the numerical implementation. Let  $\tilde{P}$  and  $\tilde{Q}$  be solutions of the polarized equations in the homogeneous case, i.e. set  $\tilde{Q} = 0$  and  $\tilde{P}(t, x) = \tilde{P}(t)$ . In this case, it follows directly that the exact solution of the Gowdy equations is

$$\tilde{P}(t) = -k \log t + \tilde{P}_{**},$$

where both  $k$  and  $\tilde{P}_{**}$  are arbitrary constants. The corresponding full solutions of Einstein's equations are Kasner solutions whose parameters<sup>5</sup> are determined by  $k$  exclusively ( $\tilde{P}_{**}$  is just a gauge quantity). By a reparametrization of the Killing orbits of the form

$$\tilde{x}_2 = x_2/\sqrt{2} + x_3/\sqrt{2}, \quad \tilde{x}_3 = -x_2/\sqrt{2} + x_3/\sqrt{2},$$

where  $\tilde{x}_2$ , and  $\tilde{x}_3$  are the coordinates used to represent the orbits of the polarized solution above, the same solution gets re-expressed in terms of functions

$$P = \log \cosh(-k \log t + \tilde{P}_{**}), \quad Q = \tanh(-k \log t + \tilde{P}_{**}). \quad (5.1)$$

These functions  $(P, Q)$  are again solutions of (4.1). Asymptotically at  $t = 0$ , they satisfy

$$P = -k \log t + (\tilde{P}_{**} - \log 2) + \dots, \quad Q = 1 - 2e^{-2\tilde{P}_{**}}t^{2k} + \dots,$$

from which we can read off the corresponding asymptotic data.

Now we compute the solutions corresponding to these asymptotic data numerically and compare them to the exact solution (5.1). We pick  $\tilde{P}_{**} = 1$ , so that  $P_{**} = 1 - \log 2$ ,  $Q_* = 1$  and

<sup>5</sup>In the conventions of [42], we have  $p_1 = (k^2 - 1)/(k^2 + 3)$ ,  $p_2 = 2(1 - k)/(k^2 + 3)$ ,  $p_3 = 2(1 + k)/(k^2 + 3)$ , and the three flat cases are realized by  $k = 1$ ,  $k = -1$  and  $|k| \rightarrow \infty$ .

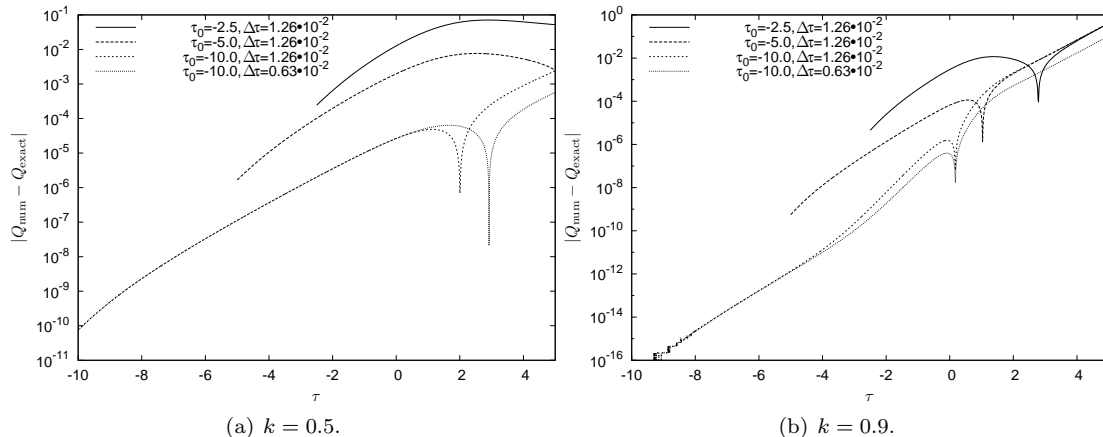


Figure 2: Convergence of numerical solutions of the test case 1 (as explained in the text).

$Q_{**} = -2e^{-2}$ . Since the solution is spatially homogeneous – in fact this is an ODE problem – we only need to do the comparison at one spatial point.

The results are presented in Figure 2 where we plot the difference of the numerical and the exact value of  $Q$  versus time for various values of  $\tau_0$ . In the first plot, this is done for  $k = 0.5$  and in the second plot for  $k = 0.9$ . The plots confirm nice convergence of the approximate solutions to the exact solution. The fact that each approximate solution diverges from the exact solution almost exponentially in time is a feature of the approximate solutions themselves and not of the numerical discretization, as is checked by comparing two different values of  $\Delta\tau$  in these plots. From our experience with the Euler-Poisson-Darboux equation, we could have expected that the convergence rate is lower in the case  $k = 0.9$  than in the case  $k = 0.5$  (note that  $k$  plays the same role  $\lambda/2$ ). In the case of the Euler-Poisson-Darboux equation, the rate of convergence decreases when  $\lambda$  approaches 2, due to the influence of the second-spatial derivative term in the equation. In the spatially homogeneous case here, however, this term is zero and hence this phenomenon is not present. The “spikes” in Figure 2 are just a consequence of the logarithmic scale of the horizontal axes and the fact that the numerical and exact solutions equal for some instances of time.

## 5.2 Test 2. General Gowdy equations

Now we want to study the convergence for a “generic” inhomogeneous Gowdy case (still ignoring the equation for the quantity  $\Lambda$ ). Here we choose the following asymptotic data

$$\begin{aligned} k(x) &= 1/2 + A \cos(x), & Q_* &= 1.0 + \sin(x), \\ P_{**} &= 1 - \log 2 + \cos(x), & Q_{**} &= -2e^{-2}, \end{aligned}$$

with a constant  $A \in (-1/2, 1/2)$ . We do not know of an explicit solution in this case. In Figure 3, we show the following numerical results for  $A = 0.2$  and  $A = 0.4$ , respectively. For the given value of  $A$ , we compute five approximate solutions with initial times  $\tau_0 = -30, -35, -40, -45, -50$  numerically, each with the same resolution  $\Delta\tau = 0.01$  and  $N = 80$ . The resolution parameters have been chosen so that the numerical discretization errors are negligible in the plots of Figure 3. Then, for each time step for  $\tau \geq -30$ , we compute the supremum norm in space of the difference of the remainders  $w^{(1)}$  of the two approximate solutions given by  $\tau_0 = -30$  and  $\tau_0 = -35$ . In

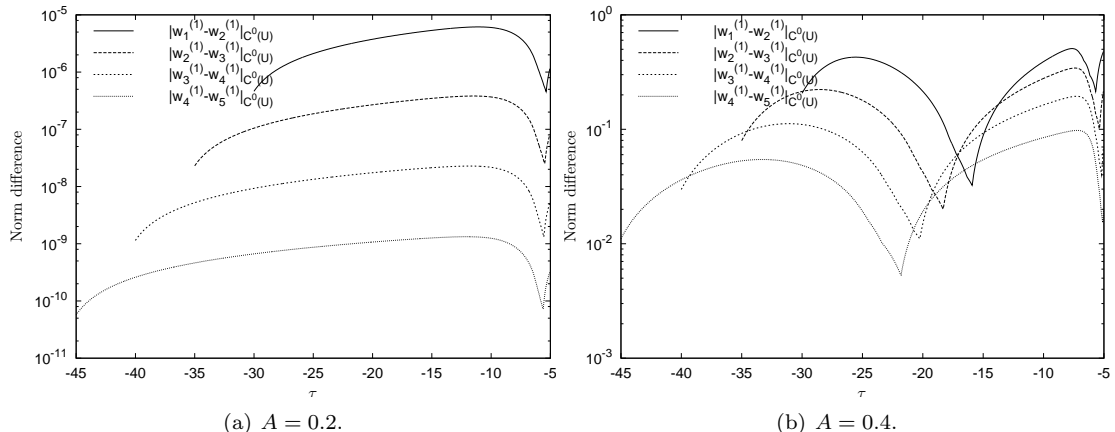


Figure 3: Convergence of numerical solutions of the test case 2 as explained in the text.

this way we obtain the first curve in each of the plots of Figure 3. The same is done for the difference between the cases  $\tau_0 = -35$  and  $\tau_0 = -40$  for all  $\tau \geq -35$  to obtain the second curve etc. Hence these curves yield a measure of the convergence rate of the approximation scheme (without referring to the exact solution). In agreement with our observation for the Euler-Poisson-Darboux equation, the convergence rate is high if  $k$  is close to  $1/2$  and becomes lower, the more  $k$  touches the “extreme” values  $k = 0$  and  $k = 1$ .

Much in the same way as for the Euler-Poisson-Darboux equation we find that double precision is sufficient for these computations despite of the fact that  $\exp(2\tau)$  is  $10^{-44}$  for  $\tau = -50$ .

### 5.3 Test 3. Gowdy spacetimes containing a Cauchy horizon

The papers [16, 17, 18, 26, 27] were devoted to the construction and characterization of Gowdy solutions with Cauchy horizons in order to prove the strong cosmic censorship conjecture in this class of spacetimes. Spacetimes with Cauchy horizons are expected to have saddle and physically “undesired” properties, in particular they often allow various inequivalent smooth extensions. This has the undesired consequence that the Cauchy problem of Einstein’s field equations does not select one of them uniquely. Some explicit examples are known, but most of the analysis is on the level of existence proofs and asymptotic expansions.

Hence, it is of interest to construct such solutions numerically and analyze them in much greater detail than possible with purely analytic methods. Constructing these solutions numerically, however, is delicate since the strong cosmic censorship conjecture suggests that they are unstable under generic perturbations. It can hence often be expected that numerical errors would most likely “destroy the Cauchy horizon”. This is so, in particular, when the singular time at  $t = 0$  is approached backwards in time from some regular Cauchy surface at  $t > 0$ , i.e. for the “backward approach”.

In the Gowdy case, where the strong cosmic censorship conjecture has been proven [40], however, there are clear criteria for the asymptotic data so that the corresponding solution of the singular initial value problem has a Cauchy horizon (or only pieces thereof; cf. below) at  $t = 0$ , as discussed in [16] for the polarized case and in [18] for the general case. Our novel method here allows us to construct such solutions with arbitrary accuracy and it can hence be expected that this allows us to study the saddle properties of such solutions. Our main aim so far is to compute such a solution and hence to demonstrate the feasibility of our approach. A

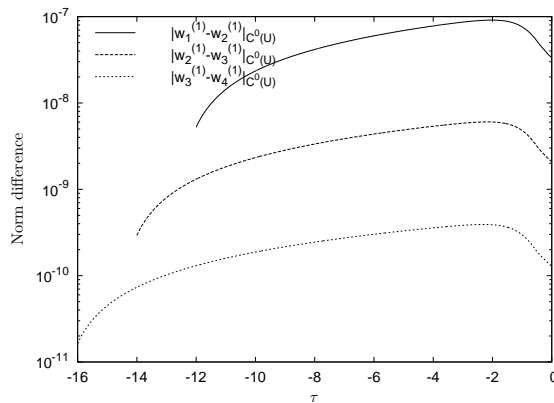


Figure 4: Convergence of numerical solutions corresponding to asymptotic data in Section 5.3.

follow-up work will be devoted to the numerical construction and detailed analysis of relevant classes of such solutions.

Motivated by the results in [16], we choose the asymptotic data as follows

$$\begin{aligned}
 k(x) &= \begin{cases} 1, & x \in [\pi, 2\pi], \\ 1 - e^{-1/x} e^{-1/(\pi-x)}, & x \in (0, \pi), \end{cases} & P_{**}(x) &= 1/2, \\
 Q_*(x) &= 0, & Q_{**}(x) &= \begin{cases} 0, & x \in [\pi, 2\pi], \\ e^{-1/x} e^{-1/(\pi-x)}, & x \in (0, \pi), \end{cases} \\
 \Lambda_*(x) &= k^2(x), & \Lambda_{**}(x) &= 2.
 \end{aligned}$$

With these asymptotic data, the corresponding solution has a smooth Cauchy horizon at  $(t, x) \in \{0\} \times (\pi, 2\pi)$  (namely where  $k \equiv 1$ ), and a curvature singularity at  $(t, x) \in \{0\} \times (0, \pi)$  (namely where  $0 < k < 1$ ). Note that the function  $k$  is smooth everywhere (but not analytic). Our analysis in Section 4.3 shows that we are allowed to set  $k = 1$  at some points since  $\partial_x Q_* = 0$ . This motivates our choice of  $Q_*$ . With this, our choice of  $Q_{**}$  implies that the solution is polarized on the “domain of dependence”<sup>6</sup> of the “initial data” interval  $(\pi, 2\pi)$ . All data were chosen as simple as possible to be consistent with the constraints.

First we repeated the same error analysis as for the previous Gowdy case, see Figure 4. For all the runs in the plots, we choose  $N = 500$ ,  $\Delta\tau = 0.005$  which guarantees that discretization errors are negligible in the plot. We find that our numerical method allows us to compute the Gowdy solution very accurately. Here, we solve the full system for  $(P, Q, \Lambda)$ .

In Figure 5, we show the numerical solution obtained from  $N = 1000$ ,  $\Delta\tau = 0.0025$  and  $\tau_0 = -18$ . We plot the Kretschmann scalar at two times  $\tau = -10$  and  $\tau = 0$ . Hence, near the time  $t = 0$  (corresponding to  $\tau = -\infty$ ), the Kretschmann scalar is large on the spatial interval  $(0, \pi)$  while it stays bounded at  $(\pi, 2\pi)$ . At the later time, the curvature becomes smaller as expected. We also plot the remainders  $w^{(1)}$  and  $w^{(2)}$  of  $P$  and  $Q$ , respectively. It is instructive to study how the polarized region inside  $(\pi, 2\pi)$  gets “displaced” by the non-polarized solution.

<sup>6</sup>The notion of “domain of dependence” for the singular initial value problem follows from the energy estimate.



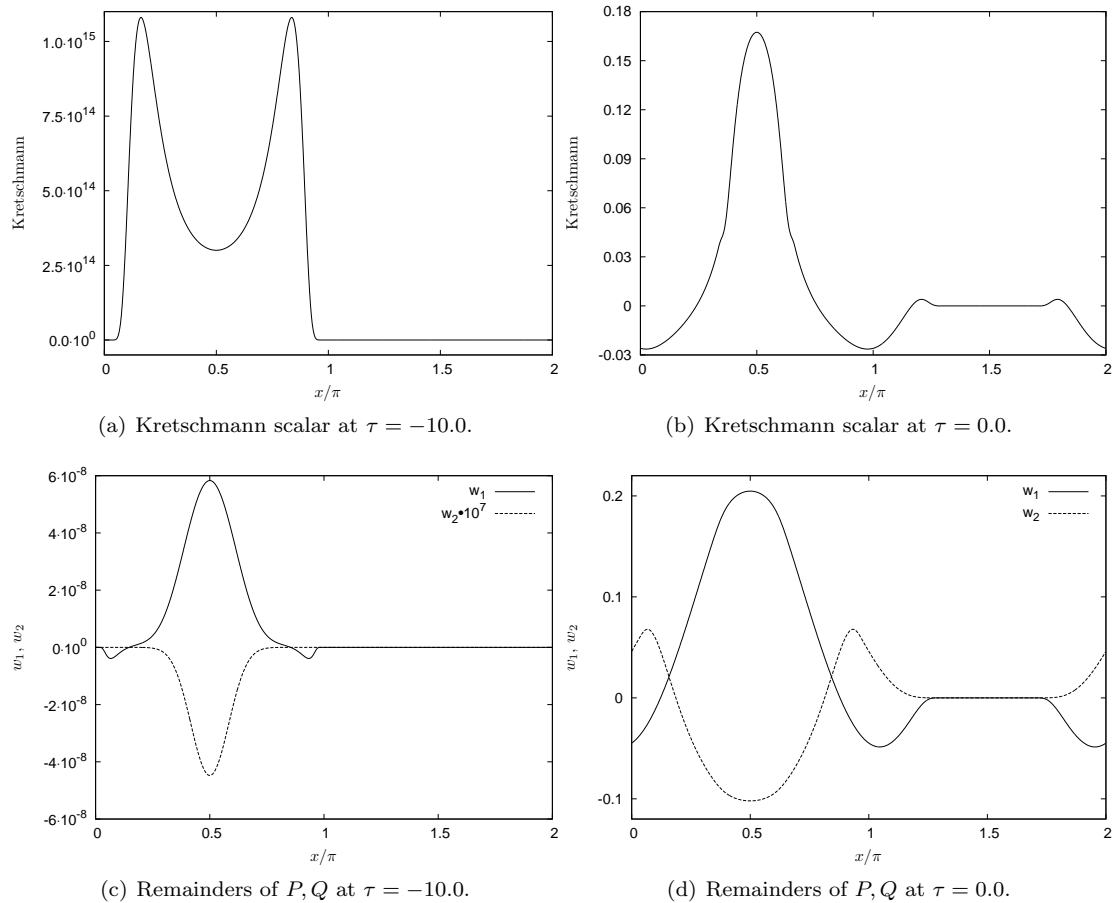


Figure 5: Numerical solutions corresponding to asymptotic data in Section 5.3.

## 6 Concluding remarks

This paper presented a new approach to the singular initial value problem for second-order Fuchsian-type equations. Our original motivation was to find a reliable and accurate numerical scheme which, following [1], allowed us to evolve data numerically from the singularity. It turned out that the classical Fuchsian theory –despite its major successes otherwise– was not directly applicable to our purpose. This led us, on one hand, to develop a new approximation scheme, which is particularly natural to handle hyperbolic Fuchsian equations, and, on the other hand, to revisit the classical theory and deduce a direct existence proof for the singular initial value problem. Our scheme yields a reliable and accurate numerical method, referred to here as the *Fuchsian numerical algorithm*. Importantly, we demonstrated that our method applies to Gowdy-symmetric solutions to Einstein’s field equations.

Our method should allow us to contribute to the understanding of strong gravitational fields. In this direction, a particularly interesting and outstanding problem in general relativity is the strong cosmic censorship conjecture. Our approach allows to numerically construct, in particular, exceptional spacetimes, for example solutions with Cauchy horizons. This is of interest for two reasons, at least. First, we can learn more about the “solution space” of Einstein’s field equations

and, consequently, about the validity of general relativity as a physical theory, due to the unusual and sometimes physically “undesired” properties of these exceptional solutions. It is currently work in progress [9] to investigate the geometry of the solutions obtained in Section 5.3 in greater detail.

Second, our method allows one to study the stability of such exceptional solutions, which was checked here, as a first step, by perturbing the induced data on a spacelike hypersurface and computing the corresponding solution via the standard (backward) approach. This allows a systematic numerical study of the strong cosmic censorship conjecture. Note, however, that the strong cosmic censorship conjecture is known for Gowdy-symmetric solutions [39, 40]. Hence, in order to obtain new interesting results about this conjecture we need to apply our theory to more general classes of solutions; see [9].

Our Fuchsian heuristics enables us to distinguish between “dominant” and “negligible” terms in the equations as one approaches the “singularity”. Sometimes these terms can be interpreted as physically interesting quantities like “kinetic” or “potential energy”. Importantly, we discovered in the present paper that the principal part of the partial differential system need not, by itself, determines the singular behavior of the solutions. Instead, nonlinear terms (classically treated as lower-order source-terms) often also play an important role. This is so for Gowdy-symmetric solutions, but it is even more important for general solutions where mixmaster behavior is expected according to the BKL conjecture. According to this conjecture and more recent investigations [34, 35], spatial derivative terms are expected to be insignificant except for exceptional points where spikes occur.

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