

Surprises in the Evaporation of 2-Dimensional Black Holes

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Quantum evaporation of Callen-Giddings-Harvey-Strominger (CGHS) black holes is analyzed in the mean field approximation. The resulting semi-classical theory incorporates back reaction. Detailed analytical and numerical calculations show that, while some of the assumptions underlying the standard evaporation paradigm are borne out, several are not. Furthermore, if the black hole is initially macroscopic, the evaporation process exhibits remarkable universal properties. Although the literature on CGHS black holes is quite rich, these features had escaped previous analyses, in part because of lack of required numerical precision, and in part because certain properties and symmetries of the model were not recognized. Finally, our results provide support for the full quantum scenario recently developed by Ashtekar, Taveras and Varadarajan.

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I. Introduction. Many important questions remain unanswered about the quantum nature of black holes, in particular the dynamics of their evaporation. This is true even for simplified 2-dimensional (2D) models, the study of which can provide insights into the more realistic higher dimensional systems. In this letter, we present key results from a new analysis of 2D, Callen-Giddings-Harvey-Strominger (CGHS) black holes [1] within the mean-field or semi-classical approximation. Despite that this model has been studied extensively (one might have even argued exhaustively) in the past [2], we find that several features of the standard paradigm are not realized. In addition, black holes resulting from a prompt collapse of a large Arnowitt-Deser-Misner (ADM) mass exhibit rather remarkable behavior: as they evaporate, after an initial transient phase, dynamics of various physically interesting quantities at right future null infinity \mathcal{I}_R^+ flow to *universal curves*, independent of the details of the initial collapsing matter distribution. This strongly suggests information in the collapsing matter on \mathcal{I}_R^- can *not* in general be recovered at \mathcal{I}_R^+ . However, we also find strong evidence supporting the scenario of [3] in which the S -matrix from (left past infinity) \mathcal{I}_L^- to \mathcal{I}_R^+ is unitary. This distinction between unitarity and information recovery is a peculiarity of 2D.

In this letter we summarize the main results. An extensive treatment will appear in an upcoming paper [4]; details about the numerics can be found in [5]; and a more thorough investigation of the full quantum issues, in particular that of information loss, in [6].

II. Model. In the CGHS model, geometry is encoded in a physical metric g and a dilaton field ϕ , and coupled to N massless scalar fields f_i . Since we are in 2D with \mathbb{R}^2 topology, we can fix a fiducial flat metric η and write g as $g^{ab} = \Omega \eta^{ab}$. Then it is convenient to describe geometry through $\Phi := e^{-2\phi}$ and $\Theta := \Omega^{-1}\Phi$. The model has 2 constants, κ with dimensions $[L]^{-1}$ and G with dimensions $[ML]^{-1}$. We will set the speed of light $c=1$, but keep Newton's constant G and Planck's \hbar free in analytical considerations. (Since $G\hbar$ is a *Planck number* in 2D, setting both of them to 1 is a physical restriction).

Our investigation is carried out within the mean field approximation (MFA) of [3, 6] in which one ignores quantum fluctuations of geometry but not of matter. To ensure a suf-

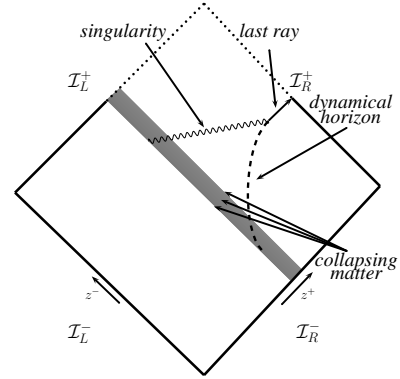


Figure 1. A Penrose diagram of an evaporating CGHS black hole in the mean field approximation (MFA). The incoming state is the vacuum on \mathcal{I}_L^- , and left moving matter distribution on \mathcal{I}_R^- . The collapse creates a generalized dynamical horizon (GDH), which subsequently evaporates. Quantum radiation fills the spacetime to the causal future of matter. Inside the GDH, a singularity forms in the geometry. It meets the GDH when the latter shrinks to zero area. The “last ray” emanating from this meeting point is a future Cauchy horizon.

ficiently large domain of validity, we must have large N and we assume that each scalar field f_i has the same profile. Black hole formation and evaporation is described entirely in terms of non-linear partial differential equations. As in 4D general relativity there are constraints which are preserved by the evolution equations. Denote by z^\pm the advanced and retarded null coordinates of η so that $\eta_{ab} = 2\partial_{(a}z^+\partial_{b)}z^-$. We will set $\partial_\pm \equiv \partial/\partial z^\pm$. Then we have the evolution equations

$$\square_{(\eta)} f_i = 0 \quad \Leftrightarrow \quad \square_{(g)} f_i = 0. \quad (1)$$

for matter fields, and

$$\begin{aligned} \partial_+ \partial_- \Phi + \kappa^2 \Theta &= G \langle \hat{T}_{+-} \rangle \equiv \bar{N} G \hbar \partial_+ \partial_- \ln(\Phi \Theta^{-1}) \\ \Phi \partial_+ \partial_- \ln \Theta &= -G \langle \hat{T}_{+-} \rangle \equiv -\bar{N} G \hbar \partial_+ \partial_- \ln(\Phi \Theta^{-1}) \end{aligned} \quad (2)$$

for geometric fields Θ, Φ . The terms on the right side are quantum corrections to the classical equations due to conformal anomaly and encode the back reaction of quantum radia-

tion. Constraint equations are

$$\begin{aligned} -\partial_-^2 \Phi + \partial_- \Phi \partial_- \ln \Theta &= G \langle \hat{T}_{--} \rangle \\ -\partial_+^2 \Phi + \partial_+ \Phi \partial_+ \ln \Theta &= G \langle \hat{T}_{++} \rangle \end{aligned} \quad (3)$$

Here, $\bar{N} := N/24$ and $\langle \hat{T}_{ab} \rangle$ denotes the expectation value of the stress-energy tensor of the N fields f_i .

We solve this system of equations as follows. As is standard in the CGHS literature, we assume that prior to $z^+ = 0$ the space-time is given by the classical vacuum solution and matter falls in from \mathcal{I}_R^- after that (see Fig. 1). To specify consistent initial data, it then suffices to choose a matter profile $f_i(z^+, z^- = -\infty) =: f_+(z^+)$, and then solve for the initial (Θ, Φ) using (3). We then evolve (Θ, Φ) to the future of the initial data surfaces using (2). Trivially, $f_i(z^+, z^-) = f_+(z^+)$ from (1).

We now discuss the interpretation of solutions to these equations via horizons, singularities and the Bondi mass. Note first that in analogous 4-dimensional (4D) spherically symmetric reductions, Φ is related to the radius r by $\Phi = \kappa^2 r^2$ [2, 4]. Therefore, a point in the CGHS space-time (M, g) is said to be *future marginally trapped* if $\partial_+ \Phi$ vanishes and $\partial_- \Phi$ is negative there [2, 7]. The quantum corrected ‘‘area’’ of a trapped point is given by $\mathbf{a} := (\Phi - 2\bar{N}G\hbar)$. Note that it is dimensionless because in D space-time dimensions the area of spatial spheres has dimensions $[L]^{D-2}$. The world-line of these marginally trapped points forms a *generalized dynamical horizon* (GDH). As time evolves, this area *shrinks* because of quantum radiation (hence ‘generalized’: the world-line is time-like rather than space-like). The area finally shrinks to zero. The MFA equations are formally singular where $\Phi = 2\bar{N}G\hbar$, thus at the end-point of evaporation the GDH meets a space-like singularity. The ‘last ray’ — the null geodesic from this point to \mathcal{I}_R^+ — is the future Cauchy horizon of the semi-classical space-time. See Fig. 1.

Next, let us discuss the structure at null infinity [3, 6]. As in the classical theory, we assume (and this is borne out by the simulations) that the semi-classical space-time is asymptotically flat at \mathcal{I}_R^+ in the sense that, as one takes the limit $z^+ \rightarrow \infty$ along the lines $z^- = \text{const}$, the field Φ has the following behavior

$$\Phi = A(z^-) e^{\kappa z^+} + B(z^-) + O(e^{-\kappa z^+}), \quad (4)$$

where A and B are smooth functions of z^- . A similar expansion holds for Θ . The metric g_{ab} admits an *asymptotic* time translation t^a . The function $A(z^-)$ determines the affine parameter y^- of t^a via $e^{-\kappa y^-} = A(z^-)$. Thus y^- can be regarded as the unique asymptotic time parameter with respect to g_{ab} (up to an additive constant). The MFA equations imply that there is a balance law at \mathcal{I}_R^+ [3, 6], motivating new definitions of a Bondi mass $M_{\text{Bondi}}^{\text{ATV}}$ and a manifestly positive energy flux F^{ATV}

$$M_{\text{Bondi}}^{\text{ATV}} = \frac{dB}{dy^-} + \kappa B + \bar{N}\hbar G \left(\frac{d^2 y^-}{dz^{-2}} \left(\frac{dy^-}{dz^-} \right)^{-2} \right) \quad (5)$$

$$F^{\text{ATV}} = \frac{\bar{N}\hbar G}{2} \left[\frac{d^2 y^-}{dz^{-2}} \left(\frac{dy^-}{dz^-} \right)^{-2} \right]^2 \quad (6)$$

so that $d(M_{\text{Bondi}}^{\text{ATV}})/dy^- = -F^{\text{ATV}}$. In the classical theory ($\hbar = 0$), there is no energy flux at \mathcal{I}_R^+ , and $M_{\text{Bondi}}^{\text{ATV}}$ reduces to the standard Bondi mass formula, which includes only the first two terms in (5). Previous literature [1, 2, 7–9] on the CGHS model used this classical expression also in the semi-classical theory. We will see that this traditionally used Bondi mass, $M_{\text{Bondi}}^{\text{Trad}}$, is physically unsatisfactory.

III. Scaling and the Planck Regime. It turns out that the mean field theory admits a scaling symmetry. To express it explicitly, let us fix z^\pm and regard all fields as functions of z^\pm . Then, given any solution (Θ, Φ, N, f_+) to all the field equations and a positive number λ , $(\lambda\Theta, \lambda\Phi, \lambda N, f_+)$ is also a solution (once z^- is shifted to $z^- + (\ln \lambda)/\kappa$) [4, 11]. Under this transformation, we have

$$\begin{aligned} g^{ab} &\rightarrow g^{ab}, \quad \kappa y^- \rightarrow \kappa y^- - \ln \lambda \\ (M, F^{\text{ATV}}, \mathbf{a}_{\text{GDH}}) &\rightarrow \lambda(M, F^{\text{ATV}}, \mathbf{a}_{\text{GDH}}) \end{aligned} \quad (7)$$

where \mathbf{a}_{GDH} denotes the area of the GDH, and M is either the Bondi mass $M_{\text{Bondi}}^{\text{ATV}}$ or the ADM mass M_{ADM} . This symmetry implies that, *as far as space-time geometry and energetics are concerned, only the ratios M/N matter*. Thus, whether a black hole is ‘macroscopic’ or ‘Planck size’ depends on the ratios M/N and $\mathbf{a}_{\text{GDH}}/N$ rather than on the values of M or \mathbf{a}_{GDH} themselves. This fact, which has important consequences, was only partly appreciated in work before [4, 11] (for example, in [12] it was noted that N could be ‘scaled out’ of the problem and that the results are ‘qualitatively independent of N ’). We thus define

$$\begin{aligned} (M^*, M_{\text{Bondi}}^*, F^*) &= (M_{\text{ADM}}, M_{\text{Bondi}}^{\text{ATV}}, F^{\text{ATV}})/\bar{N}, \quad \text{and} \\ m^* &= M_{\text{Bondi}}^* |_{\text{last ray}} \end{aligned} \quad (8)$$

To compare these quantities with the Planck mass, we first note that in 2D G , \hbar and c do not suffice to determine the Planck scale uniquely because $G\hbar$ is dimensionless. This ambiguity can be removed by regarding the 2D theory as a spherical symmetric reduction from 4D. The result is: $M_{\text{Pl}}^2 = \hbar\kappa^2/G$, and $\tau_{\text{Pl}}^2 = G\hbar/\kappa^2$. We can now regard a black hole as macroscopic if its evaporation time is much larger than the Planck time. Using the fact that, in the external field approximation, the energy flux is given by $F_{\text{Haw}} = (\bar{N}\hbar\kappa^2/2)$, this condition leads us to say that *a black hole is macroscopic if $M^* \gg G\hbar M_{\text{Pl}}$* . Note that the relevant quantity is M^* rather than M . (For details, see [4]).

IV. Results. Here we describe some key results from numerical solution of the CGHS equations (1)-(2). We consider two families of initial data, most conveniently described in a ‘‘Kruskal-like’’ coordinate $\kappa x^+ = e^{\kappa z^+}$. The first is a collapsing shell used throughout the CGHS literature so far,

$$(\partial f_+ / \partial x^+)^2 = \frac{M^*}{12} \delta(x^+ - 1/\kappa), \quad (9)$$

parameterized by M^* . The other is a smooth ($f_+(x^+)$ is \mathcal{C}^4), two parameter (\tilde{M}^*, w) profile defined by

$$\int_0^{x^+} d\bar{x}^+ \left(\frac{\partial f_+}{\partial \bar{x}^+} \right)^2 = \frac{\tilde{M}^*}{12} \left(1 - e^{(\kappa x^+ - 1)^2 / w^2} \right)^4 \theta(x^+ - 1/\kappa), \quad (10)$$

where θ is the unit step function, w characterizes the width of the matter distribution, and \tilde{M}^* is related to the ADM mass via $M^* \approx \tilde{M}^*(1 + 1.39w)$. Unraveling of the unforeseen behavior required high precision numerics (both in terms of requiring small truncation error and using the full range of 16-digit double precision floating point arithmetic [5]), especially in the macroscopic mass limit which is of primary importance. Numerical solutions from both classes of initial data were obtained for a range of masses M^* from 2^{-10} to 16, a range of widths from $w = 0$ to $w = 4$, and \bar{N} varying from 0.5 to 1000. Since we are primarily interested in black holes which are initially macroscopic, here we will focus on $M^* \geq 1$ and, since the computations did bear out the scaling behavior, on $\bar{N} = 1$. We set $\hbar = G = \kappa = 1$.

Our numerical simulations show that, as expected, the semi-classical space-time is asymptotically flat at \mathcal{I}_R^+ but, *in contrast to the classical theory*, \mathcal{I}_R^+ is incomplete, i.e., y^- has a finite value at the last ray. However, several other expectations in the standard paradigm turned out to be incorrect, or open to new interpretations. In addition, for macroscopic black holes formed in a prompt collapse, unforeseen universalities emerged. These surprising features can be summarized as follows.

First, the traditionally used Bondi mass $M_{\text{Bondi}}^{\text{Trad}}$ can become *negative and large even when the GDH is macroscopic*. In the standard paradigm, by contrast, it has been generally assumed that $M_{\text{Bondi}}^{\text{Trad}}$ is positive and tends to zero as the GDH shrinks (though see [10]). Second, while the improved Bondi mass, $M_{\text{Bondi}}^{\text{ATV}}$, does remain positive throughout evolution, at the last ray it can be large. In fact this ‘end state’ exhibits a universality shown in Fig 2 where m^* , the final value of (M_{Bondi}^*) , is plotted against the rescaled ADM mass M^* for a range of initial data. It is clear from the plot that there is a qualitative difference between $M^* \gtrsim 4$ and $M^* \lesssim 4$. In the first case the value of the end point Bondi mass is universal, $m^* \approx 0.864$. For $M^* < 4$ on the other hand, the value of m^* depends sensitively on M^* . Thus in the MFA it is natural to regard CGHS black holes with $M^* \gtrsim 4$ as *macroscopic*, and those with $M^* \lesssim 4$ as *microscopic*. Numerical studies have been used in the past to clarify a number of properties of the CGHS model [9, 10, 12, 13], such as dynamics of the GDH. However, they could not uncover universal behavior because, in the present terminology, they covered only *microscopic* cases ($M^* \leq 2.5$ in all prior studies). This limitation was not recognized because scaling was not properly understood.

Third, for macroscopic ($M^* \gtrsim 4$) black holes that form *promptly*, after some early transient behavior, dynamics of physical quantities at the GDH and at \mathcal{I}_R^+ approach *universal curves*. By promptly, we mean the characteristic width of the ingoing pulse is less than that of the initial GDH (more precisely, $w/M^* \lesssim 0.1$). This is most clearly demonstrated in the behavior of the flux F^* , or equivalently the Bondi mass M_{Bondi}^* , measured at \mathcal{I}_R^+ . An appropriately shifted affine parameter $y_{\text{sh}}^- = y^- + \text{const}$ provides an invariantly defined time coordinate and Fig. 3 shows the universality of evolution of F^* and M_{Bondi}^* with respect to it. The shift aligns the y^- coordinates amongst the solutions, which we are free to do as y^- is only uniquely defined to within an arbitrary (physically

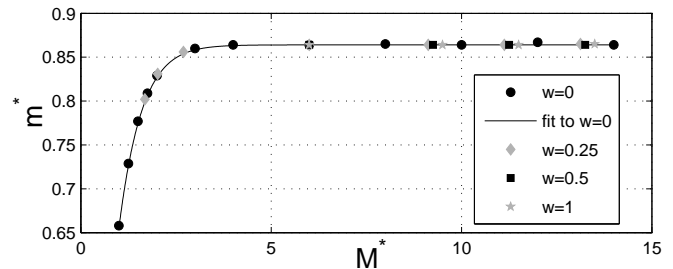


Figure 2. The final mass m^* versus the initial mass M^* (8) for a variety of initial data (9-10). The curve fit to the data is $m^* = \alpha(1 - e^{-\beta(M^*)^\gamma})$, with $\alpha \approx 0.864$, $\beta \approx 1.42$, and $\gamma \approx 1.15$.

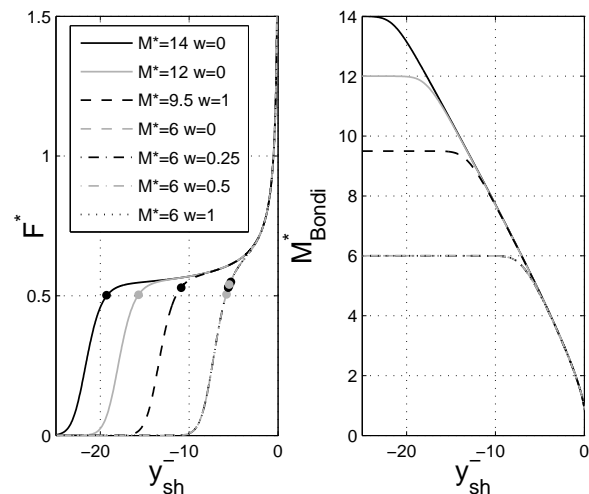


Figure 3. F^* and M_{Bondi}^* (8) plotted against y_{sh}^- from solutions with several values of parameters M^* and w of Eqs (9)-(10). In all cases F^* starts at 0 in the distant past ($\kappa y_{\text{sh}}^- \ll -1$), and then joins a universal curve at a time that depends on the initial mass. The time when the dynamical horizon first forms is marked on each flux curve (which is later for larger w , though note that the mass and flux curves for all the $M^* = 6$ cases are indistinguishable in the figure). We have not yet found an extrapolation of the flux to the last ray $y_{\text{sh}}^- = 0$ that conclusively answers whether it is finite there. However, all functions we tried that fit the data well have a *finite integrated flux*. Moreover, when the flux starts rising rapidly, we are still well within the regime where the numerical solution converges, and we can follow the solution clearly into a regime where the mass has reached its final value of $m^* \approx 0.864$.

irrelevant) additive constant. Fourth, the relation between the asymptotic time coordinates y_{sh}^- on \mathcal{I}_R^+ and z^- on \mathcal{I}_L^- also exhibits universality: $y_{\text{sh}}^-(z^-)$ depends only on M^* and not on the parameter w characterizing the matter profile.

The overall situation bares some parallels to the discovery of critical phenomena at the threshold of gravitational collapse in classical general relativity [14] where universal properties were discovered in a system that, at the time, seemed to have been already explored exhaustively. Of course, numerical in-

vestigations cannot *prove* universality; here we only studied two families of initial data. However, since these families, in particular the distribution, are not ‘special’ in any way, we believe this is strong evidence that universality will hold for all classes of CGHS initial data with large M^* where black holes form promptly —i.e., it is a feature of the ‘pure’ quantum decay of a GDH, pure in the sense that the decay is not contaminated by a continued infall from \mathcal{I}_R^- .

Finally, along the last ray, our simulations showed that curvature remains finite. Thus, contrary to wide spread belief, based in part on [13], there is no ‘thunderbolt singularity’ in the metric.

V. Conclusions. In the external field approximation, the energy flux is initially zero and, after the transient phase, quickly asymptotes to the Hawking value $F_{\text{Haw}} = \bar{N}\hbar\kappa^2/2$, which for the constants used in the simulations shown here is $F_{\text{Haw}} = 0.5$. In the MFA calculation, on the other hand, at the end of the transient phase the energy flux is *higher* than this value, keeps monotonically increasing and is about 70% greater than F_{Haw} when $M_{\text{Bondi}} \sim 2\bar{N}M_{\text{Pl}}$ (see Fig 3). One might first think that the increase is because, as in 4D, the black hole gets hotter as it evaporates. This is *not* so: For CGHS black holes, $T_{\text{Haw}} = \kappa\hbar/2\pi$ and κ is an absolute constant. Rather, the departure from $F_{\text{Haw}} = 0.5$ shows that, once the back reaction is included, the flux fails to be thermal at the late stage of evaporation, even while the black hole is macroscopic. This removes a widely quoted obstacle against the possibility that the outgoing quantum state is pure in the full theory.

In the classical solution, \mathcal{I}_R^+ is *complete* and its causal past covers only a part of space-time; there is an event horizon. But \mathcal{I}_R^+ is smaller than \mathcal{I}_L^- in a precise sense: z^- , the affine parameter along \mathcal{I}_L^- , is finite at the future end of \mathcal{I}_R^+ . This is why pure states on \mathcal{I}_L^- of a *test* quantum field \hat{f}_- on the classical solution evolve to mixed states on \mathcal{I}_R^+ , i.e., why the S matrix is non-unitary [3, 6]. In the MFA, by contrast, our analysis shows that as expected y^- is *finite* at the last ray on \mathcal{I}_R^+ . Thus, \mathcal{I}_R^+ is incomplete whence we cannot even ask if the semi-classical space-time admits an event horizon; *what forms and evaporates is, rather, the GDH*. However, this incompleteness also opens the possibility that $\bar{\mathcal{I}}_R^+$, the right null infinity of the full quantum space-time, may be larger than \mathcal{I}_R^+ and unitarity may be restored. Indeed, since there is no thunderbolt, space-time can be continued beyond the last ray. In the mean field theory the extension is ambiguous. But it

is reasonable to expect that the ambiguities will be removed by full quantum gravity [15]. Indeed, since we only have $(0.864/24)M_{\text{Pl}}$ of Bondi mass left over at the last ray *per evaporation channel* (i.e., per scalar field), it is reasonable to assume that this remainder will quickly evaporate after the last ray and $M_{\text{Bondi}}^{\text{ATV}}$ and F^{ATV} will continue to be zero along the quantum extension $\bar{\mathcal{I}}_R^+$ of \mathcal{I}_R^+ . This implies that $\bar{\mathcal{I}}_R^+$ is ‘as long as’ \mathcal{I}_L^- and hence the S -matrix is unitary: The vacuum state on \mathcal{I}_L^- evolves to a many-particle state with finite norm on $\bar{\mathcal{I}}_R^+$ [3, 6]. Thus unitarity of the S matrix follows from rather mild assumptions on what transpires beyond the last ray.

Note, however, this unitarity of the S -matrix from \mathcal{I}_R^- to the extended \mathcal{I}_R^+ does *not* imply that all the information in the infalling matter on \mathcal{I}_R^- is imprinted in the outgoing state on $\bar{\mathcal{I}}_R^+$. Indeed, the outgoing quantum state is completely determined by the function $y^-(z^-)$ and our universality results imply that, on \mathcal{I}_R^+ , this function only depends on M_{ADM} and not on further details of the matter profile [4]. Since only a tiny fraction of Planck mass is radiated per channel in the portion of $\bar{\mathcal{I}}_R^+$ that is not already in \mathcal{I}_R^+ , it seems highly unlikely that the remaining information can be encoded in the functional form of $y^-(z^-)$ in that portion. Thus, information in the matter profile on \mathcal{I}_R^- will not all be recovered at $\bar{\mathcal{I}}_R^+$ even in the full quantum theory of the CGHS model. This contradicts a general belief; indeed, because the importance of $y^-(z^-)$ was not appreciated and its universality was not even suspected, there have been attempts at constructing mechanisms for recovery of this information [8].

To summarize, in 2D there are two distinct issues: i) unitarity of the S -matrix from \mathcal{I}_L^- to $\bar{\mathcal{I}}_R^+$; and ii) recovery of the infalling information on \mathcal{I}_R^- at $\bar{\mathcal{I}}_R^+$. The distinction arises because right and left pieces of \mathcal{I}^\pm do not talk to each other. In 4D, by contrast, we only have one \mathcal{I}^- and only one \mathcal{I}^+ . Therefore if the S -matrix from \mathcal{I}^- to \mathcal{I}^+ is unitary, all information in the ingoing state at \mathcal{I}^- is automatically recovered in the outgoing state at \mathcal{I}^+ . To the extent that the CGHS analysis provides guidance for the 4D case, it suggests that unitarity of the S -matrix should continue to hold also in 4D [6].

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