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Two problems of theory of gravitation

Abstract This paper aims to discuss two issues that can have a significant impact on the foundations of the theory of gravitation:

1. The existence of relativity of space-time geometry with respect to the properties of used reference frame, which is a manifestation of the long-known fact of relativity of geometry of space and time with respect to properties of measuring instruments (Henri Poincare).

2. Lack of invariance of Einstein's equations with respect to the geodetic transformations preserving unchanged the equations of motion of test particles. Because of this, the physically equivalent states are described, generally speaking, by means of different solutions of these equations. In other words, there is no one-to-one correspondence between the solutions of these equations and the set of admissible physical states

1 Relativity of Space-Time

Einstein's theory of gravity is a realization of the idea of the relativity of the properties of space-time with respect to the distribution of matter. However, it is well known that before the advent of Einstein's theory, Henri Poincaré showed that the properties of space and time are also relative to the properties of the used measuring instruments [1]. Of course now it can be said also about the properties of space-time too. However, these convincing arguments have never been implemented in physical theory.

We can make a step towards the realization of this idea, if we will pay attention that the properties of measuring instruments are one of the characteristics of the used reference frame. We can, therefore, susapect that we

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deal with the fact of space-time relativity with respect to the used reference frame.

By a non-inertial frame of reference (NIRF) we mean the frame, the body of reference of which is formed by point masses moving in an IFR under the effect of a given force field. At that, we postulated, according to Special Relativity, that space-time in inertial reference frame (IFR) is pseudo-Euclidean. On this basis one can find the line element of space-time in the NIFR from the viewpoint of observers located in the NIFR and proceeded from relativity of space and time in the Bercley-Leibnitz-Mach-Poincaré (BLMP) meaning.

The reference body (RB) of a reference frame is supposed to be formed by identical point masses m. If an observer in the reference frame is at rest , his world line coincides with the world line of some point of the reference body. It is obvious for such an observer in an IFR that the accelerations of the point masses forming his reference body are equal to zero. This fact takes place in relativistic meaning, too. That is, if the line element of space-time in the IFR is denoted by $d\sigma$ and $u^\alpha = dx^\alpha/d\sigma$ is the field of 4- velocity of the point masses forming the reference body, then the absolute derivative of u^α is equal to zero: [1]

$$Du^\alpha/d\sigma = 0. \tag{1}$$

(We mean that an arbitrary coordinate system is used.)

Does it occur for an observer in the NIFR ? That is, if the differential metric form of space-time in a NIFR is denoted by ds, does the 4-velocity vector $\zeta^\alpha = dx^\alpha/ds$ of the point-masses forming the reference body of the NIFR satisfy the equation

$$D\zeta^\alpha/ds = 0 ? \tag{2}$$

If the space-time is absolute, equation (2) holds for only $ds = d\sigma$. However, if space and time are relative in the BLMP sense, then for both observers, located in some IFR and NIFR, the motion of the point masses, forming their reference bodies, which are kinematically equivalent, must be dynamically equivalent, too (both in non-relativistic and relativistic sense). Any observer in the NIFR, isolated from the external world and proceeded from relativity of space-time in BLMP meaning, consider points of the reference body as the ones of his physical space, and space of events as his space-time. Therefore, from his viewpoint point masses forming the reference body of his frame are not under action of any forces (the same as for the observer in IFR), and their 4-velocity must be equal to zero. In other words, since for the observer in the IFR world lines of the reference body are, according to (1), some geodesic lines, for the observer in the NIFR the world lines of the of his BR also must be geodesic lines in his space-time, which can be expressed by (2).

The equation (2) uniquely determines the fundamental metric form in the above NIRF. Indeed, the differential equations of these world lines are at the same time Lagrange equations describing in Minkowski space-time the motion of the point masses forming the reference bodies of the NIFR. The last equations can be obtained from a Lagrange action S by the principle of the least action. Therefore, the equations of the geodesic lines can be obtained from a differential metric form $ds = k\ dS$, where k is constant,

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[1] We use notations and definitions, following the Landau and Lifshitz book [7].

$dS = \mathcal{L}(x, \dot{x})dt$, and $\mathcal{L}(x, \dot{x})$ is a Lagrange function describing in Minkowski space-time the motion of identical point masses m forming the body reference of the NIFR. . The constant k is equal to $-(mc)^{-1}$, as it follows from the analysis of the case when the frame of reference is inertial.

The above NIFR can be named a proper reference frame (PRF) of the force field given in a IFR Thus, if we proceed from relativity of space and time in the BLMP sense, then the line element of space-time in PRFs can be expected to have the following form:

$$ds = -(mc)^{-1} \, dS(x, dx). \tag{3}$$

Therefore, properties of space-time in PRFs are entirely determined by properties of used frames in accordance with the BLMP idea of relativity of space and time.

2 Examples

The above NIFR can be named a proper reference frame (PRF) of the force field given in a IFR. Consider examples of PRFs.

1. The reference body is formed by noninteracting electric charges, moving in a constant homogeneous electric field $\mathcal{E}$. The motion of the charges in an IFR is described in the Cartesian coordinates system by a Lagrangian [7]

$$L = -m \, c^2 \, (1 - v^2/c^2)^{1/2} + \mathcal{E} \, e \, x, \tag{4}$$

where v is the speed of a particle. According to (3) the space-time metric differential form in this frame is given by

$$ds = d\sigma - (wx/c^2)dx^0, \tag{5}$$

where

$$d\sigma = [\eta_{\alpha\beta}dx^\alpha dx^\beta]^{1/2}$$

is the differential metric form of the Minkowski space-time in the IFR in the coordinate system being used, and $w = e\,\mathcal{E}/m$ is the acceleration of the charges.

2. The reference body consists of noninteracting electric charges in a constant homogeneous magnetic field H directed along the axis z. The Lagrangian describing the motion of the particles can be written as follows [7]:

$$L = -mc^2(1 - v^2/c^2)^{1/2} - (m\Omega_0/2)(\dot{x}y - x\dot{y}), \tag{6}$$

where $\dot{x} = dx/dt$, $\dot{y} = dx/dt$, and $\Omega_0 = eH/2mc$.

The points of such a system rotate in the plane xy around the axis z with the angular frequency

$$\omega = \Omega_0[1 + (\Omega_0 r/c)^2]^{-1/2}, \tag{7}$$

where $r = (x^2 + y^2)^{1/2}$. The linear velocity of the BR points tends to c when $r \to \infty$.

For the given NIFR

$$ds = d\sigma + (\Omega_0/2c)\,(ydx - xdy).\tag{8}$$

Thus in the above examples space-time is a Finslerian.

3. Another, rather unexpected example, give the recent results on the motion of small elements of a perfect isentropic fluid [5].

Instead of the traditional continuum assumption, the behavior of the fluid flow can be considered as the motion of a finite number of particles under the influence of interparticles forces which mimic effects of pressure, viscosity, etc. [6]. Owing to replacement of integration by summation over a number of particles, continual derivatives become simply time derivatives along the particles trajectories. The velocity of the fluid at a given point is the velocity of the particle at this point. The continuity equation is always fulfilled and can consequently be omitted. Owing to such discretization the motion of particles is governed by means of solutions of ordinary differential equations of classical or relativistic dynamics.

In [5] it was shown that the Lagrangian described the motion of macroscopically small elements ("particles") of a perfect isentropic fluid is given by

$$L = -mc \left(G_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{1/2} d\lambda,\tag{9}$$

In this equation m are the masses of the "particles", c is speed of light, $G_{\alpha\beta} = \varkappa^2 \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the metric tensor of the Minkowski space-time E,

$$\varkappa = \frac{w}{nmc^2} = 1 + \frac{\varepsilon}{\rho c^2} + \frac{P}{\rho c^2},,\tag{10}$$

where w is the fluid enthalpy per unit of volume, ε is the fluid density energy, $\rho = mn$, n is the particles number density, P is the pressure i the fluid, λ is a parameter along 4-paths of particles.

In an inertial reference frame (i.e. in Minkowski space-time E) we can set the parameter $\lambda = \sigma$ which yields the following Lagrange equations which does not contain the mass m:

$$\frac{d}{d\sigma}\,(\varkappa u_\alpha) - \frac{\partial \varkappa}{\partial x^\alpha} = 0\tag{11}$$

where $u_\alpha = \eta_{\alpha\beta} u^\beta$, and $u^\alpha = dx^\alpha/d\sigma$. For adiabatic processes [7]

$$\frac{\partial}{\partial x^\alpha}\left(\frac{w}{n}\right) = \frac{1}{n}\frac{\partial P}{\partial x^\alpha},\tag{12}$$

and we arrive at the equations of the motion of the set of the particles in the form

$$w\frac{du_\alpha}{d\sigma} + u_\alpha u^\beta \frac{\partial P}{\partial x^\beta} - \frac{\partial P}{\partial x^\alpha} = 0.\tag{13}$$

where $du_\alpha/d\sigma = (\partial u_\alpha/\partial x^\epsilon)\,u^\epsilon$. It is the general accepted relativistic equations of the motion of fluid [7].

In a comoving reference frame the space-time the line element is of the form

$$ds^2 = G_{\alpha\beta}dx^\alpha dx^\beta. \tag{14}$$

In this case the element of the proper time is ds. After the setting $\lambda = s$, the Lagrangian equation of the motion takes the standard form of a congruence of geodesic lines :

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0, \tag{15}$$

where $du_\alpha/ds = (\partial u_\alpha/\partial x^\epsilon)\, u^\epsilon$, $u^\alpha = du^\alpha/ds$, and

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}G^{\alpha\epsilon}\left(\frac{\partial G_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial G_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial G_{\beta\gamma}}{\partial x^\epsilon}\right). \tag{16}$$

In the Cartesian coordinates

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{\varkappa}\left(\frac{\partial \varkappa}{\partial x^\gamma}\delta^\alpha_\beta + \frac{\partial \varkappa}{\partial x^\beta}\delta^\alpha_\gamma - \eta^{\alpha\epsilon}\frac{\partial \varkappa}{\partial x^\epsilon}\eta_{\beta\gamma}\right), \tag{17}$$

so that

$$\Gamma^1_{00} = -\frac{1}{\rho c^2}\frac{\partial P}{\partial x^1} \tag{18}$$

In the spherical coordinates the scalar curvature R is given by

$$R = \frac{6}{\varkappa^3 r^2}(r^2\varkappa')', \tag{19}$$

where the prime denotes a derivative with respect to r.

Therefore, the motion of small elements of the fluid in a comoving reference frame can be viewed as the motion in a Riemannian space-time with a nonzero curvature.

4. Suppose that in the Minkowski space-time gravitation can be described as a tensor field $\psi_{\alpha\beta}(x)$ in E, and the Lagrangian, describing the motion of a test particle with the mass m in E is of the form

$$\mathcal{L} = -mc[g_{\alpha\beta}(\psi)\,\dot{x}^\alpha\,\dot{x}^\beta]^{1/2}, \tag{20}$$

where $\dot{x}^\alpha = dx^\alpha/dt$ and $g_{\alpha\beta}$ is a symmetric tensor whose components are functions of $\psi_{\alpha\beta}$ [3]. If particles move under influence of the force field $\psi_{\alpha\beta}(x)$, then according to (3) the space-time line element in PFRs of this field takes the form

$$ds^2 = g_{\alpha\beta}(\psi)\,dx^\alpha\,dx^\beta \tag{21}$$

Consequently, the space-time in such PRFs is Riemannian V with curvature other than zero. The tensor $g_{\alpha\beta}(\psi)$ is a space-time metric tensor in the PRFs.

Viewed by an observer located in the IRF, the motion of the particles, forming the reference body of the PRF, is affected by the force field $\psi_{\alpha\beta}$. Let $x^i(t,\chi)$ be a set of the particles paths, depending on the parameter χ. Then, for the observer located in the IRF the relative motion of a pair of particles

from the set is described in non-relativistic limit by the differential equations
[4]

$$\frac{\partial^2 n^i}{\partial t^2} + \frac{\partial^2 U}{\partial x^i \partial x^k} n^k = 0, \tag{22}$$

where $n^k = \partial x^k / \partial \chi$ and U is the gravitational potential.

However, the observer in a PRF of this field will not feel the existence of
the field. The presence of the field $\psi_{\alpha\beta}$ will be displayed for him differently —
as a space-time curvature which manifests itself as a deviation of the world
lines of nearby points of the reference body.

For a quantitative description of this fact it is natural for him to use the
Riemannian normal coordinates. [2] In these coordinates spatial components
of the deviation equations of geodesic lines are

$$\frac{\partial n^i}{\partial t^2} + R^i_{0k0} n^k = 0, \tag{23}$$

where R^i_{0k0} are the components of the Riemann tensor. In the Newtonian
limit these equations coincide with (22).

Thus, in two above frames of reference we have two different descriptions
of particles motion — as moving under the action of a force field in the
Mankowski space-time, and as moving along the geodesic line in a Riemann
space-time with the curvature other than zero.

Of course, (3) refers to any classical field $\mathcal{F}$. In particular, space-time in
PRFs of an electromagnetic field is Finslerian [13]. However, since ds, in this
case, depends on the mass and charge of the particles forming the reference
body, this fact is not of great significance.

Thus any force field can be considered based on the aggregate "IRF +
Minkowski space", and based on the aggregate "PRF + non-Euclidean space-
time with metric (3)". From this point of view of geometrization of gravity
is the second possibility, which was discovered by Einstein's intuition.

*It is important to realize that the relativity of space-time geometry to
the frame of reference is the same important and fundamental property of
physical relativity as relativity to act of measurement, the physical realization
of which is quantum mechanics.* Full implementation of these ideas can have
far-reaching implications for fundamental physics.

3 Gravity equations and gauge-invariance

In the theory of gravitation the equations of motion of test particles play
a fundamental role. Notion of "gravitational field" emerged as something
necessary to correctly describe the motion of bodies. The magnitudes that
appear in the equations of motion, become the main characteristic of the field.
The field equations have emerged as a tool for finding these magnitudes for
a given distribution of masses.

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[2] This and the above consideration does not depend on the used coordinate sys-
tem, it can be performed by a covariant method.

All this is very similar to classical electrodynamics. In this case the equations of motion of test charges are invariant under gauge transformations of 4-potentials. For this reason, all 4-potentials, obtained from a given by a gauge transformation, describe the same field. That is why the field equations of classical electrodynamics are invariant under gauge transformations.

Einstein's equations of the motions of test particles in gravitational field are also invariant with respect to some class of transformations of the field variables in any given coordinate system — with respect to geodesic transformations of Christoffel symbols (or metric tensor) [8]. Such transformations for the Christoffel symbols are of the form

$$\overline{\Gamma}^{\alpha}_{\beta\gamma}(x) = \Gamma^{\alpha}_{\beta\gamma}(x) + \delta^{\alpha}_{\beta}\ \phi_{\gamma}(x) + \delta^{\alpha}_{\gamma}\phi_{\beta}(x), \tag{24}$$

where $\phi_{\alpha}(x)$ is a continuously differentiable vector field. (The transformations for the metric tensor are solutions of some complicate partial differential equations).

Consequently, all Christoffel symbols obtained from a given by geodesic transformations, describe the same gravitational field. The equations for determining the gravitational field must be invariant under such transformations, and the physical meaning can only have values which are invariant under geodesic transformations.

However, Einstein's gravitational equations are not consistent completely with the requirement which imposes on them the hypothesis of the motion of test particles along geodesics, because they are not geodesically invariant [9].

Therefore, we can assume that in a fully correct theory of gravity, based on the hypothesis of the motion of test particles along geodesics, geodesic transformations should play the role of gauge transformations, and coordinate transformations should play the same role as in electrodynamics.

Einstein equations are in good agreement with observations in weak and moderately strong fields. Therefore, if there are more correct equation of gravitation, then deriving from them physical results should differ observably from Einstein's equations only in strong fields.

Simplest vacuum equation of this kind were first proposed (from a different point of view) in [10], and discussed in greater detail in [13], their physical implications discussed in [11] - [13], and the equations in the presence of matter - in [14]. They are some geodesic-invariant modification of Einstein's equations.

From a theoretical point of view, the most satisfactory are the vacuum equations.

They predict some fundamentally new physical consequences which can be tested experimentally.

Under geodesic transformations the Ricci tensor $R_{\alpha\beta}$ of space-time V in PRFs of gravitational field transforms as follows:

$$\overline{R}_{\alpha\beta} = R_{\alpha\beta} + (n-1)\psi_{\alpha\beta}, \tag{25}$$

where

$$\psi_{\alpha\beta} = \psi_{\alpha;\beta} - \psi_{\alpha}\psi_{\beta}, \tag{26}$$

and a semicolon denotes a covariant differentiation in V. Therefore, the simplest generalization of the Einstein equations is of the form

$$R_{\alpha\beta} + (n-1)\Gamma_{\alpha\beta} = 0, \tag{27}$$

where $\Gamma_{\alpha\beta}$ is a tensor transformed under geodesic transformations as follows

$$\overline{\Gamma}_{\alpha\beta} = \Gamma_{\alpha\beta} - \psi_{\alpha\beta}. \tag{28}$$

Due to the fact that our space-time is a bimetric, there exists a vector field

$$Q_\alpha = \Gamma_\alpha - \overset{\circ}{\Gamma}_\alpha \tag{29}$$

where $\Gamma_\alpha = \Gamma_{\alpha\beta}^\beta$, $\overset{\circ}{\Gamma}_\alpha = \overset{\circ}{\Gamma}_{\alpha\beta}^\beta$, $\Gamma_{\alpha\beta}^\gamma$ and $\overset{\circ}{\Gamma}_{\alpha\beta}^\gamma$ are the Christoffel symbols in V and E, respectively.

Under geodesic transformations in V the quantities Γ_α are transformed as follows:

$$\overline{\Gamma}_\alpha = \Gamma_\alpha + (n+1)\,\psi_\alpha \tag{30}$$

For this reason, a tensor object

$$A_{\alpha\beta} = Q_{\alpha;\beta} - Q_\alpha Q_\beta, \tag{31}$$

where $Q_{\alpha;\beta}$ is a covariant derivative of Q_α in V, has the same transformation properties under geodesic transformations as must have the above vector field $\Gamma_{\alpha\beta}$.

The line element of space-time in PRFs was obtained from the Lagrangian motion of test particles in the Minkowski space-time E. If we want to find the equation of gravity in space-time E, you must realize that in this space, the Christoffel symbols $\Gamma_{\alpha\beta}^\gamma$ can be regarded as components of a tensor $\Gamma_{\alpha\beta}^\gamma - \overset{\circ}{\Gamma}_{\alpha\beta}^\gamma$ in the Cartesian coordinate system, i.e. as components of $\Gamma_{\alpha\beta}^\gamma$, where the ordinary derivatives replaced by covariant in the metric of space-time E. (Just as in bimetric Rosen's theory [15]).

Given this, we arrive at the conclusion that the equations

$$R_{\alpha\beta} - A_{\alpha\beta} = 0 \tag{32}$$

are a simplest geodesic invariant modification of the vacuum Einstein equations, considered from the point of view of flat space-time.

These equations can be written in another form. The simplest geodesic-invariant object in V is a Thomas symbols:

$$\Pi_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma - \frac{1}{n+1}\left(\delta_\alpha^\eta \Gamma_\beta + \delta_\beta^\eta \Gamma_\alpha\right). \tag{33}$$

It is not a tensor. However, from point of view of flat space-time E, they can be considered as components of the tensor $B_{\alpha\beta}^\gamma = \Pi_{\alpha\beta}^\gamma - \overset{\circ}{\Pi}_{\alpha\beta}^\gamma$, where $\overset{\circ}{\Pi}_{\alpha\beta}^\gamma$ is the Thomas symbols in E. In another words, $B_{\alpha\beta}^\gamma$ can be considered as the Thomas symbols where derivatives replaced by the covariant ones with

respect to the metric $\eta_{\alpha\beta}$. This geodesic-invariant tensor can be named by strength tensor of gravitational field.

The above gravitation equation can be written by tensor $B_{\alpha\beta}^{\gamma}$ as follows:

$$\bigtriangledown_{\gamma} B_{\alpha\beta}^{\gamma} - B_{\alpha\delta}^{\gamma} B_{\beta\gamma}^{\delta} = 0. \tag{34}$$

where $\bigtriangledown$ denotes a covariant derivative in E.

The physical consequences following from these equations do not contradict any observational data, however, lead to some unexpected results, which allow to us to test the theory. The first result is that they predict the existence of supermassive compact objects without event horizon which are an alternative to supermassive black holes in the centers of galaxies.[13] The second result is that they provide a simple and natural explanation for the fact of an acceleration of the universe as of a consequence of the gravity properties [2].

4 Remarks on the equations inside matter

We can not claim that the particles inside any material medium move along geodesics. Consequently, it is unclear whether the field equations inside the matter to be a generalization of the geodesic equations of Einstein. However, such equations have been proposed in the work [14]. Comparison of the results obtained from them with observations of the binary pulsar PSR 1913+16 shows good agreement with observations. Despite this, doubts as to their correctness are still remain. The problem is that the writing of generalization of the equations in the matter requires significantly narrow the class of admissible geodesic transformations of the metric tensor of space-time V. It is not clear whether such space-time is Riemannian. It is possible, geodesic invariance is violated in a material medium. For this reason, we do not consider these equations here in more detail, assuming that this is still a subject for further research.

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