

Polarization modes for strong-field gravitational waves

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Abstract. Strong-field gravitational plane waves are often represented in either the Rosen or Brinkmann forms. These forms are related by a coordinate transformation, so they should describe essentially the same physics, but the two forms treat polarization states quite differently. Both deal well with linear polarizations, but there is a qualitative difference in the way they deal with circular, elliptic, and more general polarization states. In this article we will describe a general algorithm for constructing arbitrary polarization states in the Rosen form.

1. Brinkmann form

Consider the general pp spacetime geometry [1, 2, 3, 4, 5]

$$ds^2 = -2 du dv + H(u, x, y) du^2 + dx^2 + dy^2. \quad (1)$$

It is then a standard result that the only nonzero component of the Ricci tensor is

$$R_{uu} = -\frac{1}{2} \left\{ \partial_x^2 H(u, x, y) + \partial_y^2 H(u, x, y) \right\}. \quad (2)$$

Restricting attention to vacuum plane waves [6] gives us the form:

$$ds^2 = -2 du dv + \left\{ [x^2 - y^2] H_+(u) + 2xy H_\times(u) \right\} du^2 + dx^2 + dy^2. \quad (3)$$

In this form of the metric the two polarization modes are explicitly seen to decouple. By choosing $H_+(u)$ and $H_\times(u)$ appropriately we can construct any general polarization state.

2. Rosen form

The “most general” form of the Rosen metric is [1, 7]

$$ds^2 = -2 du dv + g_{AB}(u) dx^A dx^B, \quad (4)$$

where $x^A = \{x, y\}$. The only non-zero component of the Ricci tensor is [1]:

$$R_{uu} = - \left\{ \frac{1}{2} g^{AB} g''_{AB} - \frac{1}{4} g^{AB} g'_{BC} g^{CD} g'_{DA} \right\}. \quad (5)$$

Though relatively compact, because of the implicit matrix inversions this is a grossly nonlinear function of the metric components. In particular, in this form of the metric the + and \times linear polarizations do not decouple in any obvious way [8].

3. Linear Polarization

Consider the strong-field gravity wave metric in the + linear polarization. That is, set $g_{xy} = 0$. It is found most useful to put the resulting metric in the form [8]

$$ds^2 = -2 du dv + S^2(u) \left\{ e^{+X(u)} dx^2 + e^{-X(u)} dy^2 \right\}. \quad (6)$$

Then

$$R_{uu} = -\frac{1}{2} \left\{ 4 \frac{S''}{S} + (X')^2 \right\}. \quad (7)$$

In vacuum we have the general vacuum wave for + polarization in the form

$$ds^2 = -2 du dv + S^2(u) \left\{ \exp \left(2 \int^u \sqrt{-S''/S} du \right) dx^2 + \exp \left(-2 \int^u \sqrt{-S''/S} du \right) dy^2 \right\}. \quad (8)$$

From the + polarization, by rotating the x - y plane through a fixed but arbitrary angle Θ_0 , we can easily deal with linear polarization modes along any desired axis.

4. Arbitrary Polarization

Now take an arbitrary, possibly u dependent, polarization and consider the following metric ansatz [8]:

$$ds^2 = -2 du dv + S^2(u) \left\{ [\cosh(X(u)) + \cos(\theta(u)) \sinh(X(u))] dx^2 + 2 \sin(\theta(u)) \sinh(X(u)) dx dy + [\cosh(X(u)) - \cos(\theta(u)) \sinh(X(u))] dy^2 \right\}. \quad (9)$$

Note setting $\theta(u) = \Theta_0$ corresponds to linear polarization. The vacuum field equations imply

$$4 \frac{S''}{S} + (X')^2 + \sinh^2(X(u)) (\theta')^2 = 0. \quad (10)$$

Let us introduce a dummy function $L(u)$ and split this into the two equations

$$4 \frac{S''}{S} + (L')^2 = 0, \quad (L')^2 = (X')^2 + \sinh^2(X(u)) (\theta')^2. \quad (11)$$

The first of these equations is just the equation you would have to solve for a pure + (or in fact any linear) polarization. The second of these equations can be rewritten as

$$dL^2 = dX^2 + \sinh^2(X) d\theta^2, \quad (12)$$

and is the statement that L can be interpreted as distance in the hyperbolic plane H_2 .

Compare this to Maxwell electromagnetism, where polarizations can be specified by

$$\vec{E}(u) = E_x(u) \hat{x} + E_y(u) \hat{y}, \quad (13)$$

with no additional constraints. Thus an electromagnetic wavepacket of arbitrary polarization can be viewed as an arbitrary “walk” in the (E_x, E_y) plane. We could also go to a magnitude-phase representation (E, θ) where

$$\vec{E}(u) = E(u) \cos \theta(u) \hat{x} + E(u) \sin \theta(u) \hat{y}. \quad (14)$$

So an electromagnetic wavepacket of arbitrary polarization can also be viewed as an arbitrary “walk” in the (E, θ) plane, where the (E, θ) plane is provided with the natural Euclidean metric

$$dL^2 = dE^2 + E^2 d\theta^2. \quad (15)$$

In contrast for gravitational waves in the Rosen form we are now dealing with an arbitrary “walk” in the hyperbolic plane, H_2 . Furthermore, because of the nonlinearity of general relativity, there is still one remaining differential equation to solve.

5. Circular Polarization

As an important example, we consider strong-field circular polarization. Circular polarization corresponds to a fixed distortion X_0 with a linear advancement of $\theta(u)$:

$$\theta(u) = \Omega_0 u; \quad X(u) = X_0. \quad (16)$$

Then [8]

$$ds^2 = -2 du dv + S^2(u) \left\{ [\cosh(X_0) + \cos(\Omega_0 u) \sinh(X_0)] dx^2 + 2 \sin(\Omega_0 u) \sinh(X_0) dx dy + [\cosh(X_0) - \cos(\Omega_0 u) \sinh(X_0)] dy^2 \right\}. \quad (17)$$

So the only nontrivial component of the Ricci tensor is

$$R_{uu} = -\frac{1}{2} \left\{ 4 \frac{S''}{S} + \sinh^2(X_0) \Omega_0^2 \right\}. \quad (18)$$

Solving the vacuum equations gives

$$S(u) = S_0 \cos \left\{ \frac{\sinh(X_0) \Omega_0 (u - u_0)}{2} \right\}. \quad (19)$$

This now describes a spacetime that has good reason to be called a strong-field circularly polarized gravity wave. Note the weak-field limit corresponds to $X_0 \ll 1$ so for an arbitrarily long interval in u we have $S \approx S_0$, and without loss of generality we can set $S \approx 1$. Then, as expected, we obtain [8]

$$ds^2 \approx -2 du dv + dx^2 + dy^2 + X_0 \left\{ \cos(\Omega_0 u) [dx^2 - dy^2] + 2 \sin(\Omega_0 u) dx dy \right\}. \quad (20)$$

6. Decoupling the general form

Let us return to considering the metric in general Rosen form

$$ds^2 = -2 du dv + g_{AB}(u) dx^A dx^B, \quad (21)$$

where x^A, x^B represent any arbitrary number of dimensions ($d_{\perp} \geq 2$) transverse to the (u, v) plane. It is easy to check that the only non-zero component of the Ricci tensor is still

$$R_{uu} = - \left\{ \frac{1}{2} g^{AB} g''_{AB} - \frac{1}{4} g^{AB} g'_{BC} g^{CD} g'_{DA} \right\}. \quad (22)$$

Let us now decompose the $d_{\perp} \times d_{\perp}$ matrix g_{AB} into an “envelope” $S(u)$ and a unit determinant related to the “direction of oscillation” [8]. That is, let us take

$$g_{AB}(u) = S^2(u) \hat{g}_{AB}(u), \quad (23)$$

where $\det(\hat{g}) \equiv 1$. (A related discussion can be found in [9].) Calculating the various terms of the Ricci tensor using the relations

$$[\hat{g}^{AB} \hat{g}'_{AB}] = 0, \quad [\hat{g}^{AB} \hat{g}''_{AB}] - [\hat{g}^{AB} \hat{g}'_{BC} \hat{g}^{CD} \hat{g}'_{DA}] = 0, \quad (24)$$

it is found that [8]

$$R_{uu} = -d_{\perp} \frac{S''}{S} - \frac{1}{2} [\hat{g}^{AB} \hat{g}'_{BC} \hat{g}^{CD} \hat{g}'_{DA}]. \quad (25)$$

Note that we have now succeeded in decoupling the determinant ($\det(g) = S^{2d_\perp}$; effectively the “envelope” $S(u)$ of the gravitational wave) from the unit-determinant matrix $\hat{g}(u)$.

Now consider the set $SS(\mathbb{R}, d_\perp)$ of all unit determinant real symmetric matrices, and on that set consider the Riemannian metric

$$dL^2 = \text{tr} \left\{ [\hat{g}]^{-1} d[\hat{g}] [\hat{g}]^{-1} d[\hat{g}] \right\}. \quad (26)$$

Then

$$R_{uu} = -\frac{1}{2} \left\{ 2d_\perp \frac{S''(u)}{S(u)} + \left(\frac{dL}{du} \right)^2 \right\}. \quad (27)$$

This means an arbitrary polarization vacuum Rosen wave is an arbitrary walk in $SS(\mathbb{R}, d_\perp)$, with distance along the walk $L(u)$ related to the envelope function $S(u)$ as in the discussion above.

7. Discussion

Arbitrary polarizations, while trivial in the Brinkmann form, are difficult to implement in the Rosen form. To address this puzzle we have re-analyzed the Rosen strong-field gravity wave in terms of an “envelope” function and two freely specifiable functions. The vacuum field equations can be interpreted in terms of a single differential equation governing the “envelope”, coupled with an arbitrary walk in polarization space. In particular we have indicated how to construct a circularly polarized Rosen form gravity wave, and how to generalize this central idea beyond (3+1) dimensions. Further detailed calculations and discussions of these Rosen form polarizations can be found in [8].

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