BACKWARD STOCHASTIC PDEs RELATED TO THE UTILITY MAXIMIZATION PROBLEM

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Abstract. We study utility maximization problem for general utility functions using dynamic programming approach. We consider an incomplete financial market model, where the dynamics of asset prices are described by an R^d-valued continuous semimartingale. Under some regularity assumptions we derive backward stochastic partial differential equation (BSPDE) related directly to the primal problem and show that the strategy is optimal if and only if the corresponding wealth process satisfies a certain forward-SDE. As examples the cases of power, exponential and logarithmic utilities are considered.

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1. Introduction

Portfolio optimization, hedging and derivative pricing are fundamental problems in mathematical finance, which are closely related to each other. A basic optimization problem of mathematical finance, such as optimal portfolio choice or hedging, is to optimize

$$E[U(X_T^{x,\pi})] \quad \text{over all} \quad \pi \quad \text{from a class } \Pi \quad \text{of strategies,} \tag{1.1}$$

where $X_t^{x,\pi} = x + \int_0^t \pi_u dS_u$ is the wealth process starting from initial capital x, determined by the self-financing trading strategy π and Π is some class of admissible strategies. U is an objective function which can be depended also on ω. It can be interpreted as a utility function or a function which measures a hedging error.

If we assume that $U(x)$ is strictly convex (for each ω) then one can interpret U as a function which measures a hedging error and consider the problem

$$\text{to minimize} \quad E[U(X_T^{x,\pi})] \quad \text{over all} \quad \pi \quad \text{from} \quad \Pi. \tag{1.2}$$

In [31] a backward stochastic PDE for value function

$$V(t,x) = \operatorname*{ess\,inf}_{\pi\in\Pi} E(U(x + \int_t^T \pi_u dS_u)/\mathcal{F}_t) \tag{1.3}$$

of (1.2) was derived and in terms of solutions of this equation a characterization of optimal strategies was given. We shall use the same approach to the case when the objective function U is strictly concave. In particular, if U is a utility function, then (1.1) corresponds to the utility maximization problem

$$\text{to maximize} \quad E[U(X_T^{x,\pi})] \quad \text{over all} \quad \pi\in\Pi, \tag{1.4}$$

i.e., for a given initial capital $x > 0$, the goal is to maximize the expected value from the terminal wealth.

The utility maximization problem was first studied by Merton (1971) in the classical Black-Scholes model. Using the Markov structure of the model he derived the Bellman equation for the value function of the problem and produced the closed-form solution of this equation in cases of power, logarithmic and exponential utility functions.

For general complete market models, it was shown by Pliska (1986), Cox and Huang (1989) and Karatzas et al (1987) that the optimal portfolio of the utility maximization problem is (up to a constant) equal to the density of the martingale measure, which is unique for complete markets. As shown by He and Pearson (1991) and Karatzas et al (1991), for incomplete markets described by Ito-processes, this method gives a duality characterization of optimal portfolios provided by the set of martingale measures. Their idea was to solve the dual problem of finding the suitable optimal martingale measure and then to express the solution of the primal problem by convex duality. Extending the domain of the dual problem the approach has been generalized to semimartingale models and under weaker conditions on the utility functions by Kramkov and Schachermayer (1999). See also more recent papers [12], [35], [6], [33], [13], [7], [36], [24], [1].

These approaches mainly give a reduction of the basic primal problem to the solution of the dual problem, but the constructive solution of the dual problem for general models of incomplete markets is itself demanding task.

Our goal is to derive a semimartingale Bellman equation (a stochastic version of the Bellman equation) related directly to the basic (or primal) optimization problem and to give constructions of optimal strategies. Applying the dynamic programming approach directly to the primal optimization problem may in many cases represent a valuable alternative to the commonly used convex duality approach.

Let S be an R^d-valued continuous semimartingale, defined on a filtered probability space satisfying the usual conditions. The process S describes the discounted price evolution of d risky assets in a financial market containing also a riskless bond with a constant price. To exclude arbitrage opportunities, we suppose that the set $\mathcal{M}^e$ of equivalent martingale measures for S is not empty. Since S is continuous, the existence of an equivalent martingale measure implies that the structure condition is satisfied, i.e., S admits the decomposition

$$S_t = M_t + \int_0^t d\langle M\rangle_s \lambda_s, \quad \int_0^t \lambda'_s d\langle M\rangle_s \lambda_s < \infty \ \text{ for all } \ t \ \ a.s., \tag{1.5}$$

where M is a continuous local martingale and λ is a predictable R^d-valued process.

We consider utility function U mapping $R_+ \equiv (0,\infty)$ into R. It is assumed to be continuously differentiable, strictly increasing, strictly concave and to satisfy the Inada conditions:

$$U^{'}(0) = \lim_{x\to 0} U^{'}(x) = \infty,$$

$$U^{'}(\infty) = \lim_{x\to\infty} U^{'}(x) = 0.$$

We also set $U(0) = \lim_{x\to 0} U(x)$ and $U(x) = -\infty$ for all $x < 0$.

Denote by $\mathcal{M}^e$ the set of martingale measures for S. Throughout the paper we assume that

$$\mathcal{M}^e \neq \emptyset.$$

For any $x \in R_+$, we denote by Π_x the class of predictable S-integrable processes π such that the corresponding wealth process is nonnegative at any instant, that is $X_t^{x,\pi} = x + \int_0^t \pi_u dS_u \geq 0$ for all $t \in [0,T]$.

For simplicity in introduction we consider the case with one risky asset.

Let us introduce the dynamical value function of the problem (1.4) defined as

$$V(t,x) = \operatorname*{esssup}_{\pi\in\Pi_x} E\big(U(x + \int_t^T \pi_u dS_u)/\mathcal{F}_t\big). \tag{1.6}$$

The classical Itô formula (or its generalization by Krylov 1980) plays a crucial role to derivation of the Bellman equation for the value function of controlled diffusion processes. For our purposes the Itô formula is not sufficient since the function V depends also on ω, even if U is deterministic. Therefore the Itô-Ventzel formula should be used.

Under some regularity assumptions on the value function (sufficient for the application of the Itô–Ventzell formula) we show in Theorem 3.1 that value function defined by (1.6) satisfies the following backward stochastic partial differential equation (BSPDE)

$$V(t,x) = V(0,x) + \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x)+\lambda(s)V_x(s,x))^2}{V_{xx}(s,x)} d\langle M\rangle_s \qquad (1.7)$$

$$+ \int_0^t \varphi(s,x)dM_s + L(t,x)$$

with the boundary condition

$$V(T,x) = U(x),$$

where $\int_0^t \varphi(s,x)dM_s + L(t,x)$ is the martingale part of $V(t,x)$, $L(t,x)$ is strongly orthogonal to M for all x and subscripts φ_x, V_x, V_{xx} stand for the partial derivatives. Moreover, the strategy π^* is optimal if and only if the corresponding wealth process X^{π^*} is a solution of the following forward SDE

$$X_t^{\pi^*} = X_0^{\pi^*} - \int_0^t \frac{\varphi_x(u, X_u^{\pi^*}) + \lambda(u)V_x(u, X_u^{\pi^*})}{V_{xx}(s, X_u^{\pi^*})} dS_u. \qquad (1.8)$$

Thus, to give the construction of the optimal strategy one should:
1) first solve the backward equation (1.7) (which determines V and φ simultaneously) and substitute corresponding derivatives of V and φ in equation (1.8), then
2) solve the forward equation (1.8) with respect to X^{π^*} and, finally,
3) reproduce the optimal strategy π^* from the corresponding wealth process X^{π^*}.

Theorem 3.1 is a verification theorem, since we require conditions directly on the value function V and not only on the basic objects (on the model and on the objective function U). Therefore we can't state that the solution of equation (1.7) exists, but for standard utility functions (e.g., for power, exponential, logarithmic and quadratic utilities) all conditions of Theorem 3.1 are satisfied and in these cases the existence of a unique solutions of corresponding backward equations follows from this theorem.

If $U(x) = x^p, p \in (0,1)$, then (1.4) corresponds to power utility maximization problem

$$\text{to maximize} \quad E(x + \int_0^T \pi_u dS_u)^p \quad \text{over all} \quad \pi \in \Pi_x. \qquad (1.9)$$

In this case $V(t,x) = x^p V_t$, where V_t is a semimartingale and all condition of Theorem 3.1 are satisfied. This theorem implies that the process V_t satisfies the following backward stochastic differential equation (BSDE)

$$V_t = V_0 + \frac{q}{2}\int_0^t \frac{(\varphi_s + \lambda_s V_s)^2}{V_s} d\langle M\rangle_s$$

$$+ \int_0^t \varphi_s dM_s + L_t, \quad V_T = 1, \qquad (1.10)$$

where $q = \frac{p}{p-1}$ and L is a local martingale strongly orthogonal to M.
Besides, equation (1.8) is transformed into a linear equation

$$X_t^* = x + (1-q)\int_0^t \frac{\varphi_u + \lambda_u V_u}{V_u} X_u^* dS_u \qquad (1.11)$$

for the optimal wealth process.

Therefore,

$$X_t^* = x\mathcal{E}_t((1-q)(\frac{\varphi}{V}+\lambda)\cdot S)$$

and the optimal strategy is of the form

$$\pi_t^* = x(1-q)(\frac{\varphi_t}{V_t}+\lambda_t)\mathcal{E}_t\big((1-q)(\frac{\varphi}{V}+\lambda)\cdot S\big).$$

Equations of type (1.10) was derived in [30] in relation to utility maximization problem and in [16] for constrained utility maximization problem. In comparison to the work [16] our results are at the same time more and less general. In [16] diffusion market model is considered and the boundedness of model coefficients is assumed. We are working with a general right-continuous filtration and under weaker boundedness conditions, but we have not included constraints on our strategies.

We consider also utility functions which take finite values on all real line, such as the exponential utility $U(x) = 1 - e^{-\gamma x}$. For this case Theorem 3.1 is not directly applicable. It needs a special choice of a class of trading strategies and additional assumption of the existence of the dual optimizer (see Schachermayer 2003). Exponential utility maximization problem we consider in section 4 and distinguish cases when this problem admits an explicit solution. In section 4 we consider also the case of quadratic utility.

The main tools of the work - Backward Stochastic Differential Equations, have been introduced by J. M. Bismut in [2] for the linear case as the equations for the adjoint process in the stochastic maximum principle. In [3] and [34] the well-posedness results for BSDEs with more general generators was obtained (see also [10] for references and related results). The semimartingale backward equation, as a stochastic version of the Bellman equation in an optimal control problem, was first derived in [3] by R. Chitashvili.

The main results of this paper are based on the papers of authors [31], [30].

2. Basic assumptions and some auxiliary facts

We consider an incomplete financial market model, where the dynamics of asset prices are described by an R^d-valued continuous semimartingale S defined on a filtered probability space $(\Omega, F, \mathcal{F} = (\mathcal{F}_t, t \in [0,T]), P)$ satisfying the usual conditions, where $F = \mathcal{F}_T$ and $T < \infty$ is a fixed time horizon. For all unexplained notations concerning the martingale theory used below we refer the reader to [17],[8],[28].

Denote by $\mathcal{M}^e$ the set of martingale measures, i.e., the set of measures Q equivalent to P on $\mathcal{F}_T$ such that S is a local martingale under Q. Let $Z_t(Q)$ be the density process of Q with respect to the basic measure P, which is a strictly positive uniformly integrable martingale. For any $Q \in \mathcal{M}^e$ there is a P-local martingale M^Q such that $Z(Q) = \mathcal{E}(M^Q) = (\mathcal{E}_t(M^Q), t \in [0,T])$, where $\mathcal{E}(M)$ is the Doleans-Dade exponential of M.

We recall the definition of BMO-martingales and the Muckenhoupt condition.

The square integrable continuous martingale M belongs to the class BMO if there is a constant $C > 0$ such that

$$E(\langle M\rangle_T - \langle M\rangle_\tau | F_\tau) \le C, \quad P-\text{a.}s.$$

for every stopping time τ.

A strictly positive uniformly integrable martingale Z satisfies the Muckenhoupt inequality denoted by $A_\alpha(P)$ for some $1 < \alpha < \infty$, iff there is a constant C such that

$$E((\frac{Z_\tau}{Z_T})^{\frac{1}{\alpha-1}} | F_\tau) \le C, \quad P-\text{a.}s.$$

for every stopping time τ.

Note that, if the mean variance tradeof $\langle \lambda\cdot M\rangle_T$ is bounded, then the density process $\mathcal{E}(-\lambda\cdot M)$ of the minimal martingale measure satisfies the Muckenhoupt inequality for any $\alpha > 1$.

The following assertion relates BMO and the Muckenhoupt condition.

Proposition 2.1. *(*[9], [21]*). Let M be a local martingale and $\mathcal{E}(\mathcal{M})$ its Doléans Exponential. The following assertions are equivalent:*

(i) M belongs to the class BMO,

(ii) $\mathcal{E}(\mathcal{M})$ is a uniformly integrable martingale satisfying the Muckenhoupt inequality $A_\alpha(P)$ for some $\alpha > 1$.

Let Π_x be the space of all predictable S-integrable processes π such that the corresponding wealth process is nonnegative at any instant, that is $x + \int_0^t \pi_u dS_u \geq 0$ for all $t \in [0, T]$.

In the sequel sometimes we shall use the notation $(\pi \cdot S)_t$ for the stochastic integral $\int_0^t \pi_u dS_u$.

By μ^κ we shall denote the Dolean's measure of an increasing process κ.

Suppose that the objective function $U(x) = U(\omega, x)$ satisfies the following conditions:

B1) $V(0, x) < \infty$ for some x,

B2) $U(\omega, x)$ is twice continuously differentiable and strictly concave for each ω,

B3) optimization problem (1.4) admits a solution, i.e., for any t and x there is a strategy $\pi^*(t, x)$ such that

$$V(t, x) = E(U(x + \int_t^T \pi_s^*(t, x) dS_s)/\mathcal{F}_t) \tag{2.1}$$

Remark 2.1. As shown by Kramkov and Schachermayer (1999), sufficient condition for B3), when U does not depend on ω, is that utility function $U(x)$ has asymptotic elasticity strictly less than 1, i.e.,

$$AE(U) = \limsup_{x \to \infty} \frac{x U_x(x)}{U(x)} < 1. \tag{2.2}$$

It follows from Kramkov and Schachermayer (2003) that for B3) the finitness of the dual value function is also sufficient.

Remark 2.2. The strict concavity of U implies that the optimal strategy is unique if it exists. Indeed, if there exist two optimal strategies π^1 and π^2, then by concavity of U the strategy $\bar{\pi} = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ is also optimal. Therefore,

$$\frac{1}{2} E[U(x + \int_t^T \pi_s^1 dS_s)|\mathcal{F}_t] + \frac{1}{2} E[U(x + \int_t^T \pi_s^2 dS_s)|\mathcal{F}_t]$$

$$= E[U(x + \int_t^T \bar{\pi}_s dS_s)|\mathcal{F}_t]$$

and

$$\frac{1}{2} U(x + \int_t^T \pi_s^1 dS_s) + \frac{1}{2} U(x + \int_t^T \pi_s^2 dS_s)$$

$$= U(x + \int_t^T \bar{\pi}_s dS_s) \quad P - a.s.$$

Now strict concavity of U leads to the equality $\int_t^T \pi_s^1 dS_s = \int_t^T \pi_s^2 dS_s$.

For convenience we give the proof of the following known assertion.

Lemma 2.1. *Under conditions B1)-B3) the value function $V(t, x)$ is a strictly concave function with respect to x.*

Proof. The concavity of $V(t, x)$ follows from B2) and B3), since for any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ and any $x_1, x_2 \in R$ we have

$$\alpha V(t, x_1) + \beta V(t, x_2)$$

$$= \alpha E[U(x_1 + \int_t^T \pi_u^*(t, x_1) dS_u)|\mathcal{F}_t] + \beta E[U(x_2 + \int_t^T \pi_u^*(t, x_2) dS_u)|\mathcal{F}_t]$$

$$\geq E[U(\alpha x_1+\beta x_2+\int_t^T(\alpha\pi^*_u(t,x_1)+\beta\pi^*_u(t,x_2))dS_u|\mathcal{F}_t]$$

$$\geq V(t,\alpha x_1+\beta x_2). \tag{2.3}$$

To show that $V(t,x)$ is strictly concave we must verify that if the equality

$$\alpha V(t,x_1)+\beta V(t,x_2)=V(t,\alpha x_1+\beta x_2) \tag{2.4}$$

is valid for some $\alpha,\beta\in(0,1)$ with $\alpha+\beta=1$, then $x_1=x_2$.

Indeed, if equality (2.4) holds, then from (2.3) and the strict convexity of U follows that P-a.s.

$$x_1+\int_t^T\pi^*_u(t,x_1)dS_u=x_2+\int_t^T\pi^*_u(t,x_2)dS_u,$$

which implies that $x_1=x_2$.

Remark 2.3. The concavity of $V(0,x)$ and condition $B1$) imply that $V(0,x)<\infty$ for all $x\in R_+$.

Ito-Ventzell's formula.

Let $(Y(t,x),\ \ t\in[0,T],\ \ x\in R)$ be a family of special semimartingales with the decomposition

$$Y(t,x)=Y(0,x)+B(t,x)+N(t,x), \tag{2.5}$$

where $B(\cdot,x)\in\mathcal{A}_{\rm loc}$ and $N(\cdot,x)\in\mathcal{M}_{\rm loc}$ for any $x\in R$. By the Galtchouk–Kunita–Watanabe (G-K-W) decomposition of $N(\cdot,x)$ with respect to M a parametrized family of semimartingales Y admits the representation

$$Y(t,x)=Y(0,x)+B(t,x)+\int_0^t\psi(s,x)dM_s+L(t,x), \tag{2.6}$$

where $L(\cdot,x)$ is a local martingale strongly orthogonal to M for all $x\in R$.

Assume that:

C1) there exists a predictable increasing process $(K_t,t\in[0,T])$ such that $B(\cdot,x)$ and $\langle M\rangle$ are absolutely continuous with respect to K, i.e., there is a measurable function $b(t,x)$ predictable for every x and a matrix-valued predictable process ν_t such that

$$B(t,x)=\int_0^t b(s,x)dK_s,\ \ \langle M\rangle_t=\int_0^t\nu_s dK_s.$$

Note that, by continuity of M the square characteristic $\langle M\rangle$ is absolutely continuous w.r.t. the continuous part K^c of the process K and

$$\langle M\rangle_t=\int_0^t\nu_s dK^c_s=\int_0^t\nu_s dK_s.$$

Without loss of generality one can assume that ν is bounded and the scalar product $u'\nu_t v$ for $u,v\in R^d$ we denote by $(u,v)_{\nu_t}$.

C2) the mapping $x\to Y(t,x)$ is twice continuously differentiable for all (ω,t),

C3) the first derivative $Y_x(t,x)$ is a special semimartingale, admitting the decomposition

$$Y_x(t,x)=Y_x(0,x)+B_{(x)}(t,x)+\int_0^t\psi_x(s,x)dM_s+L_{(x)}(t,x), \tag{2.7}$$

where $B_{(x)}(\cdot,x)\in\mathcal{A}_{\rm loc}$, $L_{(x)}(\cdot,x)$ is a local martingale orthogonal to M for all $x\in R$ and ψ_x is the partial derivative of ψ at x (note that $A_{(x)}$ and $L_{(x)}$ are not assumed to be derivatives of A and L respectively, whose existence does not necessarily follow from condition C2)),

C4) $Y_{xx}(t,x)$ is RCLL process for every $x\in R$,

C5) the functions $b(s,\cdot)$, $\psi(s,\cdot)$ and $\psi_x(s,\cdot)$ are continuous at x μ^K-a.e., C6) for any $c>0$

$$E\int_0^T \sup_{|x|\le c} g(s,x)dK_s<\infty$$

for g equal to $|b|$, $|\psi|^2$ and $|\psi|_x^2$.

In what follows we shall need the following version of Ito-Ventzell's formula

Proposition 2.2. *Let $(Y(\cdot,x),\ \ x\in R)$ be a family of special semimartingales satisfying conditions C1)-C6) and $X^\pi=x+\pi\cdot S$. Then the transformed process $Y(t,X_t^\pi),\ t\in[0,T]$ is a special semimartingale with the decomposition*

$$Y(t,X_t^\pi)=Y(0,c)+B_t+N_t,$$

where

$$B_t=\int_0^t\big[Y_x(s,X_s^\pi)\lambda_s' d\langle M\rangle_s\pi_s+\psi_x(s,X_s^\pi)'d\langle M\rangle_s\pi_s+$$

$$+\frac{1}{2}Y_{xx}(s,X_s^\pi)\pi_s' d\langle M\rangle_s\pi_s\big]+\int_0^t b(s,X_s^\pi)dK_s \tag{2.8}$$

and N is a continuous local martingale.

One can derive this assertion from Theorem 1.1 of [26] or from Theorem 2 of [4]. Here we don't require any conditions on $L(t,x)$ imposed in [26] and [4], since the martingale part of substituted process X^π is orthogonal to $L(\cdot,x)$ and since we don't give an explicit expression of martingale part N, which is not necessary for our purposes.

Remark 2.4. Since the semimartingale S is assumed to be continuous and is of the form (1.5), only the latter term of (2.8) may have the jumps, i.e., the process K is not continuous in general.

3. The BSPDE for the value function

In this section we derive a backward stochastic PDE for the value function related to the utility maximization problem.

Denote by $\mathcal{V}^{1,2}$ the class of functions $Y:\Omega\times[0,T]\times R\to R$ satisfying conditions C1)-C6).

Let us consider the following backward stochastic partial differential equation (BSPDE)

$$\begin{aligned}Y(t,x)&=Y(0,x)\\&+\frac{1}{2}\int_0^t\frac{(\psi_x(s,x)+\lambda(s)Y_x(s,x))'}{Y_{xx}(s,x)}d\langle M\rangle_s(\psi_x(s,x)+\lambda(s)Y_x(s,x))\\&+\int_0^t\psi(s,x)dM_s+L(t,x),\qquad\qquad L(\cdot,x)\perp M,\end{aligned}\tag{3.1}$$

with the boundary condition

$$Y(T,x)=U(x).\tag{3.2}$$

We shall say that Y solves equation (3.1),(3.2) if:

(i) $Y(\omega,t,x)$ is twice continuously differentiable for each (ω,t) and satisfies the boundary condition (3.2),

(ii) $Y(t,x)$ and $Y_x(t,x)$ are special semimartingales admitting decompositions (2.6) and (2.7) respectively, where ψ_x is the partial derivative of ψ at x and

(iii) $P-$ a.s. for all $x \in R$

$$B(t,x) = \frac{1}{2}\int_0^t \frac{(\psi_x(s,x)+\lambda(s)Y_x(s,x))'}{Y_{xx}(s,x)} d\langle M\rangle_s(\psi_x(s,x)+\lambda(s)Y_x(s,x)) \tag{3.3}$$

Remark 3.1. If we substitute expression of $B(t,x)$, given by equality (3.3), in the canonical decomposition (2.6) for Y we obtain equation (3.1).

Remark 3.2. A sufficient condition for twice differentiability of the value function $V(0,x)$ is given in Kramkov and Sirbu [22].

According to Proposition A1 the value process $V(t,x)$ is a supermartingale for any $x \in R$, which admits the canonical decomposition

$$V(t,x) = V(0,x) + A(t,x) + \int_0^t \varphi(s,x)dM_s + m(t,x), \tag{3.4}$$

where $-A(\cdot,x) \in \mathcal{A}^+$ and $m(\cdot,x)$ is a local martingale strongly orthogonal to M for all $x \in R_+$.

Assume that $V \in \mathcal{V}^{1,2}$. This implies that $V_x(t,x)$ is a special semimartingale with the decomposition

$$V_x(t,x) = V_x(0,x) + A_{(x)}(t,x) + \int_0^t \varphi_x(s,x)dM_s + m_{(x)}(t,x), \tag{3.5}$$

where $A_{(x)}(\cdot,x) \in \mathcal{A}_{\text{loc}}$, $m_{(x)}(\cdot,x)$ is a local martingale orthogonal to M for all $x \in R_+$ and φ_x coincides with the partial derivative of φ (μ^K-a.e.). Besides

$$A(t,x) = \int_0^t a(s,x)dK_s,$$

for a measurable function $a(t,x)$.

Recall that the scalar product $u'\nu_t v$ for $u, v \in R^d$ we denote by $(u,v)_{\nu_t}$.

Proposition 3.1. *Assume that conditions B1), B2) are satisfied and the value function $V(t,x)$ belongs to the class $\mathcal{V}^{1,2}$. Then the following inequality holds*

$$a(s,x) \le \frac{1}{2}\frac{|\varphi_x(s,x)+\lambda(s)V_x(s-,x)|^2_{\nu_s}}{V_{xx}(s-,x)} \tag{3.6}$$

for all $x \in R_+$ $\mu^K-a.e.$ Moreover, if the strategy π^* is optimal then the corresponding wealth process X^{π^*} is a solution of the following forward SDE

$$X_t^{\pi^*} = X_0^{\pi^*} - \int_0^t \frac{\varphi_x(s,X_s^{\pi^*})+\lambda(s)V_x(s,X_s^{\pi^*})}{V_{xx}(s,X_s^{\pi^*})}dS_s. \tag{3.7}$$

Proof. Using Ito-Ventzell's formula (Proposition 2.2) for the function $V(t,x,\omega) \in \mathcal{V}^{1,2}$ and for the process $(x + \int_s^t \pi_u dS_u,\ \ s \le t \le T)$ we have

$$\begin{aligned} &V(t, x + \int_s^t \pi_u dS_u) \\ &= V(s,x) + \int_s^t a(u, x + \int_s^u \pi_v dS_v)dK_u \\ &+ \int_s^t G(u,\pi_u, x + \int_s^u \pi_v dS_v)dK_u + N_t - N_s, \end{aligned} \tag{3.8}$$

where

$$G(t,p,x,\omega) = V_x(t-,x)p'\nu_t\lambda(t) + p'\nu_t\varphi_x(t,x) + \\ + \frac{1}{2}V_{xx}(t-,x)p'\nu_t p \tag{3.9}$$

and N is a martingale. Since by Proposition A1 of Appendix the process $(V(t, x + \int_s^t \pi_u dS_u), t \in [s,T])$ is a supermartingale for all $s \geq 0$ and $\pi \in \Pi_x$, the process

$$-\int_s^t \left[G(u,\pi_u, x + \int_s^u \pi_v dS_v) + a(u, x + \int_s^u \pi_v dS_v)\right] dK_u,$$

is increasing for any $s \geq 0$. Hence, the process

$$-\int_s^t \left[G(u,\pi_u, x + \int_s^u \pi_v dS_v) + a(u, x + \int_s^u \pi_v dS_v)\right] dK_u^c,$$

is also increasing for any $s \geq 0$, where $K = K^c + K^d$ is a decomposition of K into continuous and purely discontinuous increasing processes. Therefore, taking $\tau_s(\varepsilon) = \inf\{t \geq s : K_t^c - K_s^c \geq \varepsilon\}$ instead of t we have that for any $\varepsilon > 0$ and $s \geq 0$

$$\frac{1}{\varepsilon}\int_s^{\tau_s(\varepsilon)} a(u, x + \int_s^u \pi_v dS_v) dK_u^c \\ \leq -\frac{1}{\varepsilon}\int_s^{\tau_s(\varepsilon)} G(u,\pi_u, x + \int_s^u \pi_v dS_v) dK_u^c. \tag{3.10}$$

Passing to the limit in (3.10) as $\varepsilon \to 0$, from Lemma B of [31] we obtain that

$$a(s,x) \leq -G(s,\pi_s,x) \qquad \mu^{K^c} - a.e.$$

for all $\pi \in \Pi$. Thus

$$a(t,x) \leq \operatorname*{ess\,inf}_{\pi\in\Pi}\big(-G(t,\pi_t,x)\big) \qquad \mu^{K^c} - a.e. \tag{3.11}$$

On the other hand

$$\operatorname*{ess\,inf}_{\pi\in\Pi}\big(-G(t,\pi_t,x)\big) = \frac{|V_x(t-,x)\lambda(t) + \varphi_x(t,x)|^2_{\nu_t}}{2V_{xx}(t-,x)} \\ + \operatorname*{ess\,inf}_{\pi\in\Pi}\left(-\frac{1}{2}V_{xx}(t-,x)\Big|\pi_t + \frac{V_x(t-,x)\lambda(t) + \varphi_x(t,x)}{V_{xx}(t-,x)}\Big|^2_{\nu_t}\right) \\ = \frac{|V_x(t-,x)\lambda(t) + \varphi_x(t,x)|^2_{\nu_t}}{2V_{xx}(t-,x)}. \tag{3.12}$$

Indeed, since $V_{xx} < 0$ equality (3.12) follows from Lemma A.1. Thus, from (3.11) and (3.12) we have that for every $x \in R_+$

$$a(t,x) \leq \frac{|V_x(t-,x)\lambda(t) + \varphi_x(t,x))|^2_{\nu_t}}{2V_{xx}(t-,x)}, \quad \mu^{K^c} \quad a.e.$$

Since μ^K- a.e. $a(t,x) \geq 0$ and $\mu^{K^d}\{\nu \neq 0\} = 0$ we obtain that

$$a(t,x) \leq \frac{|V_x(t-,x)\lambda(t) + \varphi_x(t,x)|^2_{\nu_t}}{2V_{xx}(t-,x)}, \quad \mu^K \quad a.e. \tag{3.13}$$

Conditions C2) and C5) imply that inequality (3.13) holds μ^K-a.e. for all $x \in R$.

Let us show now that if the strategy π^* is optimal then the corresponding wealth process X^{π^*} is a solution of equation (3.7). Let $\pi^*(s,x)$ be the optimal strategy and denote by $X_t^*(s,x) = x + \int_s^t \pi_u^*(s,x) dS_u$ the corresponding wealth process.

By the optimality principle the process $V(t,x+\int_s^t \pi_u^*(s,x)dS_u)$ is a martingale on the time interval $[s,T]$ and the Ito-Ventzell formula implies that μ^K-a.s.

$$a(t,X_t^*(s,x))+(\lambda_t,\pi_t(s,x))_{\nu_t}V_x(t-,X_t^*(s,x))+$$

$$(\varphi_x(t,X_t^*(s,x)),\pi_t^*(s,x))_{\nu_t}+\frac{1}{2}|\pi_t^*(s,x)|_{\nu_t}^2V_{xx}(t-,X_t^*(s,x))=0. \tag{3.14}$$

It follows from (3.13) and (3.14) that μ^K-a.e.

$$V_{xx}(t-,X_t^*(s,x))\big|\pi_t^*(s,x)+\frac{\varphi_x(t,X_t^*(s,x))+\lambda(t)V_x(t-,X_t^*(s,x))}{V_{xx}(t-,X_t^*(s,x))}\big|_{\nu_t}^2\geq 0.$$

Since $V_{xx}<0$, integrating the latter relation by dK_u we obtain that

$$\int_s^t\big(\pi_u^*(s,x)+\frac{\varphi_x(u,X_u^*(s,x))+\lambda(u)V_x(u,X_u^*(s,x))}{V_{xx}(u,X_u^*(s,x))}\big)'d\langle M\rangle_u\times$$

$$\times\big(\pi_u^*(s,x)+\frac{\varphi_x(u,X_u^*(s,x))+\lambda(u)V_x(u,X_u^*(s,x))}{V_{xx}(u,X_u^*(s,x))}\big)=0. \tag{3.15}$$

The Kunita–Watanabe inequality and (3.15) imply that the semimartingale

$$\int_s^t\big(\pi_u^*(s,x)+\frac{\varphi_x(u,X_u^*(s,x))+\lambda(u)V_x(u,X_u^*(s,x))}{V_{xx}(u,X_u^*(s,x))}\big)dS_u$$

is indistinguishable from zero (since its $\mathcal{S}^2$-norm is zero) and we obtain that the wealth process of π^* satisfies equation

$$X_t^*(s,x)=x-\int_s^t\frac{\varphi_x(u,X_u^*(s,x))+\lambda(u)V_x(u,X_u^*(s,x))}{V_{xx}(u,X_u^*(s,x))}dS_u \tag{3.16}$$

which gives equation (3.7) for $s=0$. □

Recall that the process Z belongs to the class D if the family of random variables $Z_\tau I_{(\tau\leq T)}$ for all stopping times τ is uniformly integrable.

Under additional condition

C*) $(X_t^*(s,x),\ t\geq s)$ is a continuous function of (s,x) $P-$a.s. for each $t\in[s,T]$,

we shall show that the value function V satisfies equation (3.1)-(3.2).

This condition is satisfied, e.g., if the optimal wealth process $(X_t^*(s,x),\ t\geq s)$ does not depend on s and x, which we have in cases of power, logarithmic and exponential utility functions.

Theorem 3.1. *Let $V\in\mathcal{V}^{1,2}$ and assume that conditions B1)-B3), C*) are satisfied. Then the value function is a solution of BSPDE (3.1)-(3.2), i.e.,*

$$V(t,x)=V(0,x)+\frac{1}{2}\int_0^t\frac{(\varphi_x(s,x)+\lambda(s)V_x(s,x))'}{V_{xx}(s,x)}d\langle M\rangle_s(\varphi_x(s,x)+\lambda(s)V_x(s,x))$$

$$+\int_0^t\varphi(s,x)dM_s+m(t,x),\quad V(T,x)=U(x). \tag{3.17}$$

Moreover, the strategy π^* is optimal if and only if the corresponding wealth process X^{π^*} is a solution of the forward SDE (3.7), such that the process $V(t,X^{\pi^*})$ is from the class D.

Proof. Let $\pi^*(s,x)$ be the optimal strategy. By optimality principle $(V(t,X_t^*(s,x)), t \geq s)$ is a martingale. Therefore, using Ito-Ventzell's formula, taking (3.15) in mind, we have

$$\int_s^t \big[a(u,X_u^*(s,x)) - g(u,X_u^*(s,x)) +$$

$$+\big|\pi_u^*(s,x) + \frac{V_x(u,X_u^*(s,x))\lambda(u) + \varphi_x(u,X_u^*(s,x))}{V_{xx}(u,X_u^*(s,x))}\big|^2_{\nu_u}\big] dK_u = 0,$$
$$\text{for all } \ t \geq s \ \ P-a.s.,$$

where

$$g(s,x) = \frac{1}{2}\frac{|\varphi_x(s,x) + \lambda(s)V_x(s,x)|^2_{\nu_s}}{V_{xx}(s,x)}.$$

It follows from (3.15) that $\mu^K - a.e.$

$$\big|\pi_u^*(s,x) + \frac{V_x(u,X_u^*(s,x))\lambda(u) + \varphi_x(u,X_u^*(s,x))}{V_{xx}(u,X_u^*(s,x))}\big|^2_{\nu_u} = 0$$

and by (3.6)

$$a(s,x) \leq g(s,x) \quad \mu^K - a.e. \tag{3.18}$$

Thus,

$$\int_s^t [a(u,X_u^*(s,x)) - g(u,X_u^*(s,x))]dK_u = 0, \ t \geq s \quad\quad P-a.s.$$

This implies that $(a(s,x) - g(s,x))(K_s - K_{s-}) = 0$ for any $s \in [0,T]$. Therefore,

$$a(s,x) = g(s,x) \quad \mu^{K^d} - a.e. \tag{3.19}$$

On the other hand

$$\int_0^T \frac{1}{\varepsilon}\int_s^{\tau_s^\varepsilon} [a(u,X_u^*(s,x)) - g(u,X_u^*(s,x))]dK_u^c dK_s^c = 0, \quad P- \ a.s.$$

and by Proposition B of [31] we obtain that

$$\int_0^T [a(s,x) - g(s,x)]dK_s^c = 0, \ \ P-a.s.$$

Now (3.18), (3.19) and the latter relation result equality $a(s,x) = g(s,x)$ $\mu^K - a.e.$, hence

$$A(t,x) = \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x) + \lambda(s)V_x(s,x))'}{V_{xx}(s,x)} d\langle M\rangle_s(\varphi_x(s,x) + \lambda(s)V_x(s,x))$$

and $V(t,x)$ satisfies (3.1)-(3.2).

If $\hat{\pi}$ is a strategy such that the corresponding wealth process $X^{\hat{\pi}}$ satisfies equation (3.7) and $V(t,X_t^{\hat{\pi}})$ is from the class D, then $\hat{\pi}$ is optimal. Indeed, using the Ito-Ventzell formula and equations (3.7) and (3.17) we obtain that $V(t,X_t^{\hat{\pi}})$ is a local martingale, hence a martingale, since it belongs to the class D. Therefore $\hat{\pi}$ is optimal by optimality principle. □

Remark 3.3. For the utility functions which take finite values on all real line Proposition 3.1 and Theorem 3.1 are also true if we choose a suitable class of trading strategies. E.g., let Π_x be one of the class introduced by Schachermayer (2003)($\mathcal{H}_i(x)$ or $\mathcal{H}_i^{'}(x)$, for $i = 1,2$ or 3). The proof of abovementioned assertions is the same, there is a minor difference only in the proof of equality (3.12), where instead of Lemma A.1 the following arguments should be used: Conditions (C3), (C4) and (C6) imply that for each x the process

$$\frac{V_x(t-,x)\lambda(t) + \varphi_x(t,x)}{V_{xx}(t-,x)}$$

is predictable and S-integrable. Therefore, there exists a sequence of stopping times $(\tau_n(x), n \ge 1)$ with $\tau_n(x) \uparrow T$ for all $x \in R$ such that the wealth process corresponding to the strategy

$$\pi_t^n = -I_{[0,\tau_n]} \frac{V_x(t-,x)\lambda(t) + \varphi_x(t,x)}{V_{xx}(t-,x)}$$

is bounded and hence $\pi^n \in \Pi$ for each n. Therefore,

$$0 \le \operatorname*{ess\,inf}_{\pi \in \Pi} \left(-\frac{1}{2} V_{xx}(t-,x) \big| \pi_t + \frac{V_x(t-,x)\lambda(t) + \varphi_x(t,x)}{V_{xx}(t-,x)} \big|^2_{\nu_t} \right)$$

$$\le \left(-\frac{1}{2} V_{xx}(t-,x) \big| \frac{V_x(t-,x)\lambda(t) + \varphi_x(t,x)}{V_{xx}(t-,x)} \big|^2_{\nu_t} \right) I_{(\tau_n(x) \le t)} \to 0, \quad \mu^{K^c} \quad \text{a.s.},$$

which implies equality (3.12).

Theorem 3.2. *Let conditions B1)-B3) be satisfied. If the pair $(Y, \mathcal{X})$ is a solution of the Forward-Backward Equation*

$$\begin{aligned} Y(t,x) = U(x) & \\ -\frac{1}{2} \int_t^T & \frac{((\psi_x(s,x) + \lambda(s) Y_x(s,x))'}{Y_{xx}(s,x)} d\langle M \rangle_s (\psi_x(s,x) + \lambda(s) V_x(s,x)) \\ -\int_t^T & \psi(s,x) dM_s + L(T,x) - L(t,x) \end{aligned} \tag{3.20}$$

$$\mathcal{X}_t = x - \int_0^t \frac{\psi_x'(s,\mathcal{X}_s) + Y_x(s,\mathcal{X}_s)\lambda(s)}{Y_{xx}(s,\mathcal{X}_s)} dS_s, \tag{3.21}$$

$\mathcal{X} \ge 0$, $Y \in \mathcal{V}^{1,2}$ and $Y(t, \mathcal{X}_t)$ belongs to the class D, then such solution is unique.

Proof. Using the Ito-Ventzell's formula for $Y(t, x + \int_s^t \pi_u dS_u)$ we have

$$\begin{aligned} & Y(t, x + \int_s^t \pi_u dS_u) \\ & = Y(s,x) + \int_s^t b(u, x + \int_s^u \pi_v dS_v) dK_u \\ & + \int_s^t G(u, \pi_u, c + \int_s^u \pi_v dS_v) dK_u + N_t - N_s, \end{aligned} \tag{3.22}$$

where

$$G(t,p,x,\omega) = Y_x(t-,x) p' \nu_t \lambda(t) + p' \nu_t \psi_x(t,x) + \frac{1}{2} Y_{xx}(t-,x) p' \nu_t p$$

and N is a local martingale.

Since Y solves (3.20), then equality (3.3) is valid, which implies that $Y(t, x + \int_s^t \pi_u dS_u)$ is a local supermartingale for each $\pi \in \Pi$.

Let $\tau_n = \inf\{t : Y(t, x + \int_s^t \pi_u dS_u) \ge n\} \wedge T$. [1]) Then by supermartingale property and the monotone convergence theorem we have

$$Y(s,x) \ge E\big(Y(\tau_n, x + \int_s^{\tau_n} \pi_u dS_u) | \mathcal{F}_s\big)$$

---

[1]) It is assumed that $\inf \emptyset = \infty$ and $a \wedge b$ denotes $\min\{a, b\}$

$$\geq E\big(n\wedge U(x+\int_s^T \pi_u dS_u)|\mathcal{F}_s\big) \overset{n\to\infty}{\longrightarrow} E\big(U(x+\int_s^T \pi_u dS_u)|\mathcal{F}_s\big).$$

i.e.

$$Y(s,x)\geq E\big(U(x+\int_s^T \pi_u dS_u)|\mathcal{F}_s\big),\ \ \forall\pi\in\Pi_x,$$

which implies that

$$Y(s,x)\geq V(s,x). \tag{3.23}$$

Using now the Ito-Ventzell's formula for $Y(t,\mathcal{X}_t)$ taking into account that Y satisfies (3.20) and $\mathcal{X}$ solves (3.21) we obtain that $Y(t,\mathcal{X}_t)$ is a local martingale and, hence, it is a martingale, since $Y(t,\mathcal{X}_t)$ is from the class D.

Therefore, since $\mathcal{X}_0=x,\ Y(T,x)=U(x)$ we have that

$$Y(t,x)=E\big(U(x-\int_t^T \frac{Y_x(u,\mathcal{X}_u)\lambda_u+\psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)}dS_u)/\mathcal{F}_t\big). \tag{3.24}$$

Since $-\frac{\lambda(u)Y_x(u,\mathcal{X}_u)+\psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u))}\in\Pi_x$, from (3.23) and (3.24) we obtain that

$$Y(t,x)=V(t,x), \tag{3.25}$$

hence solution of (3.20) is unique if it exists and coincides with the value function. This implies that under conditions of theorem $V\in\mathcal{V}^{1,2}$.

Therefore, it follows from (3.25) and (3.21) that $\mathcal{X}$ satisfies equation (3.7). Besides, according to Proposition 3.1 the solution of (3.7) is the optimal wealth process, hence $\mathcal{X}=\mathcal{X}^{\pi*}$ by the uniqueness of the optimal strategy for the problem (1.4) (see Remark 2.2).

4. Utility maximization problem for power, logarithmic and exponential utility functions

In this section we calculate the value function and give constructions of optimal strategies for the utility maximization problem corresponding to the cases of power, logarithmic and exponential utility functions.

Power Utility.

Let $U(x)=x^p,\ p\in(0,1)$. Then (1.4) corresponds to power utility maximization problem

$$\text{to maximize}\quad E(x+\int_0^T \pi_u dS_u)^p\quad\text{over all}\quad \pi\in\Pi_x \tag{4.1}$$

where Π_x is a class of admissible strategies.

In this case the value function $V(t,x)$ is of the form x^pV_t, where V_t is a special semimartingale. Indeed, since Π_x is a cone (for any $x>0$ the strategy π belongs to Π_x iff $\frac{\pi}{x}\in\Pi_1$) , we have

$$V(t,x)=\operatorname*{ess\,sup}_{\pi\in\Pi_x}E\big((x+\int_t^T \pi_u dS_u)^p/\mathcal{F}_t\big)$$

$$=x^p\operatorname*{ess\,sup}_{\pi\in\Pi_x}E\big((1+\int_t^T \frac{\pi_u}{x}dS_u)^p/\mathcal{F}_t\big)=x^pV_t,$$

where

$$V_t = \operatorname*{ess\,sup}_{\pi\in\Pi_1} E\big((1+\int_t^T \pi_u dS_u)^p/\mathcal{F}_t\big)$$

is a supermartingale by optimality principle.

Let $V_t = V_0 + A_t + N_t$ be the canonical decomposition of V_t, where A is a decreasing process and N is a local martingale. Using the G–K–W decomposition we have that

$$V_t = V_0 + A_t + \int_0^t \varphi_s dM_s + L_t, \tag{4.2}$$

where L is a local martingale with $< L, M >= 0$.

It is evident that for $U(x) = x^p$ the condition (2.2) is satisfied and the optimal strategy for the problem (4.1) exists. Since in this case $V(t,x) = x^p V_t$ it is also evident that $V(t,x) \in \mathcal{V}^{1,2}$ and all conditions of Theorem 3.1 are satisfied (note that one can take $-A + \langle M\rangle$ as a dominated process K).

Therefore we have the following corollary of Theorem 3.1

Theorem 4.1. *If $U(x) = x^p, p \in (0,1)$, then the value function $V(t,x)$ is of the form $x^p V_t$, where V_t satisfies the following backward stochastic differential equation (BSDE)*

$$V_t = V_0 + \frac{q}{2}\int_0^t \frac{(\varphi_s+\lambda_s V_s)'}{V_s} d\langle M\rangle_s (\varphi_s + \lambda_s V_s)$$

$$+\int_0^t \varphi_s dM_s + L_t, \quad V_T = 1, \tag{4.3}$$

where $q = \frac{p}{p-1}$ and L is a local martingale strongly orthogonal to M.

Besides, the optimal wealth process is a solution of the linear equation

$$X^*_t = x - (q-1)\int_0^t \frac{\varphi_u + \lambda_u V_u}{V_u} X^*_u dS_u \tag{4.4}$$

Therefore,

$$X^*_t = x\mathcal{E}_t(-(q-1)(\frac{\varphi}{V}+\lambda)\cdot S)$$

and the optimal strategy is of the form

$$\pi^*_t = -x(q-1)\mathcal{E}_t(-(q-1)(\frac{\varphi}{V}+\lambda)\cdot S)(\frac{\varphi_t}{V_t}+\lambda_t).$$

Remark 4.1. If there is a martingale measure Q that satisfies the Muckenhoupt condition $A_\alpha(P)$ for $\alpha = \frac{1}{p}$ then the process V is bounded. Indeed, by the Hölder inequality for any $\pi \in \Pi$

$$E\big((1+\int_t^T \pi_u dS_u)^p/\mathcal{F}_t\big) = E^Q\big((1+\int_t^T \pi_u dS_u)^p \frac{Z^Q_t}{Z^Q_T}|\mathcal{F}_t\big) \le$$

$$\le \big(E^Q((1+\int_t^T \pi_u dS_u)|\mathcal{F}_t)\big)^p \big(E^Q\big((Z^Q_t/Z^Q_T)^{\frac{1}{1-p}}|\mathcal{F}_t\big)\big)^{1-p} \le$$

$$\le \big(E\big((Z^Q_t/Z^Q_T)^{\frac{1}{\frac{1}{p}-1}})|\mathcal{F}_t\big)\big)^{1-p} \le C^{1-p}.$$

Under condition $A_{\frac{1}{p}}(P)$ equation (4.3) admits a unique bounded strictly positive solution. This follows from Theorem 3.2, since in this case the process $\mathcal{E}_t(-(q-1)(\frac{\varphi}{V}+\lambda)\cdot S)$ belongs to the class D.

Now we shall consider two cases when equation (4.3) admits an explicit solution

Case 1.

Let $S_t(q) = M_t + q\int_0^t d\langle M\rangle_s\lambda_s$ and let $Q(q)$ be a measure defined by $dQ(q) = \mathcal{E}_T(-q\lambda\cdot M)dP$. Note that $S(q)$ is a local martingale under $Q(q)$ by Girsanov's theorem.

Assume that

$$e^{\frac{q(q-1)}{2}\langle\lambda\cdot M\rangle_T} = c + \int\limits_0^T h_u dS_u(q), \tag{4.5}$$

where c is a constant and h is a predictable $S(q)$-integrable process such that $h\cdot S(q)$ is a $Q(q)$-martingale.

This condition is satisfied iff the q-optimal martingale measure coincides with the minimal martingale measure. For diffusion market models this condition is fulfilled for so called "almost complete" models, i.e., when the market price of risk is measurable with respect to the filtration generated by price processes of basic securities.

Let condition (4.5) be satisfied. Let us consider the process

$$Y_t = \big(E(\mathcal{E}^q_{t,T}(-\lambda\cdot M)/F_t)\big)^{\frac{1}{1-q}}. \tag{4.6}$$

Since

$$\mathcal{E}^q_t(-\lambda\cdot M) = \mathcal{E}_t(-q\lambda\cdot M)e^{\frac{q(q-1)}{2}\langle\lambda\cdot M\rangle_t},$$

condition (4.5) implies that

$$Y_t = \big(E^{Q(q)}(e^{\frac{q(q-1)}{2}(\langle\lambda\cdot M\rangle_T-\langle\lambda\cdot M\rangle_t}/F_t)\big)^{\frac{1}{1-q}} =$$

$$= e^{\frac{q}{2}\langle\lambda\cdot M\rangle_t}\big(c + \int\limits_0^t h_u dS_u(q)\big)^{\frac{1}{1-q}}.$$

By the Itô formula

$$Y_t = Y_0 + \frac{q}{2}\int\limits_0^t Y_s\lambda'_s d\langle M\rangle_s\lambda_s + \frac{q}{1-q}\int\limits_0^t \frac{Y_s\lambda'_s}{c+(h\cdot S(q))_s}d\langle M\rangle_s h_s$$

$$+\frac{q}{2}\frac{1}{(1-q)^2}\int\limits_0^t \frac{Y_s h'_s}{(c+(h\cdot S(q))_s)^2}d\langle M\rangle_s h_s + \frac{1}{1-q}\int\limits_0^t \frac{Y_s h_s}{c+(h\cdot S(q))_s}dM_s \tag{4.7}$$

and denoting $\frac{1}{q-1}\frac{Y_s h_s}{c+(h\cdot S(q))_s}$ by ψ_s we obtain that

$$Y_t = Y_0 + \frac{q}{2}\int\limits_0^t \frac{(\psi_s+\lambda_s Y_s)'}{Y_s}d\langle M\rangle_s(\psi_s+\lambda_s Y_s) + \int\limits_0^t \psi_s dM_s.$$

It is evident from (4.6) that $Y_T = 1$. Thus the triple (Y,ψ,L), where $\psi = \frac{1}{q-1}\frac{Yh}{c+h\cdot S(q)}$, $L=0$ and Y defined by (4.6), satisfies equation (4.3).

Case 2.

Assume that

$$e^{-\frac{q}{2}\langle\lambda\cdot M\rangle_T} = c + m_T, \tag{4.8}$$

where c is a constant and m is a martingale strongly orthogonal to M.

For diffusion market models this condition is satisfied when the market price of risk is measurable with respect to the filtration independent relative to the asset price process.

Let us consider the process

$$Y_t = E(e^{-\frac{q}{2}(\langle\lambda\cdot M\rangle_T-\langle\lambda\cdot M\rangle_t)}/F_t). \tag{4.9}$$

Condition (4.8) implies that

$$Y_t = e^{\frac{q}{2}\langle\lambda\cdot M\rangle_t}(c+m_t)$$

and by the Itô formula

$$Y_t = Y_0 + \frac{q}{2}\int_0^t Y_s d\langle\lambda\cdot M\rangle_s + \int_0^t e^{\frac{q}{2}\langle\lambda\cdot M\rangle_s} dm_s.$$

It follows from here that the triple (Y,ψ,L), where $\psi=0$, $L_t=\int_0^t e^{\frac{q}{2}\langle\lambda\cdot M\rangle_s}dm_s$ and Y defined by (4.9), satisfies equation (4.3) and the optimal strategy is

$$\pi_t^* = x(1-q)\lambda_t\mathcal{E}_t((1-q)\lambda\cdot S).$$

Logarithmic Utility

For the logarithmic utility

$$U(x)=\log x,\quad x>0$$

the value function of corresponding utility maximization problem takes the form

$$V(t,x)=\log x + V_t,$$

where V_t is a special semimartingale.

Indeed, since for any $x>0$ the strategy π belongs to Π_x iff $\frac{\pi}{x}\in\Pi_1$, we have

$$V(t,x) = \operatorname*{esssup}_{\pi\in\Pi_x} E\big(\log\big(x+\int_t^T \pi_u dS_u\big)/\mathcal{F}_t\big)$$

$$= \operatorname*{esssup}_{\pi\in\Pi_x} E\big(\log x\big(1+\int_t^T \frac{\pi_u}{x} dS_u\big)/\mathcal{F}_t\big)$$

$$= \log x + \operatorname*{esssup}_{\pi\in\Pi_x} E\big(\log\big(1+\int_t^T \frac{\pi_u}{x} dS_u\big)/\mathcal{F}_t\big) = \log x + V_t,$$

where

$$V_t = \operatorname*{esssup}_{\pi\in\Pi_1} E\big(\log\big(1+\int_t^T \pi_u dS_u\big)/\mathcal{F}_t\big)$$

is a supermartingale by the optimality principle.

It is also evident that all conditions of Theorem 3.1 are satisfied. In this case $\varphi_x(t,x)=0, V_x(t,x)=\frac{1}{x}, V_{xx}(t,x)=-\frac{1}{x^2}$ and equation (3.7) gives the following expression for V_t

$$V_t = V_0 - \frac{1}{2}\langle\lambda\cdot M\rangle_t + \int_0^t \varphi_s dM_s + L_t,$$

$$V_T = 0, \tag{4.10}$$

which admits an explicit solution

$$V_t = -\frac{1}{2}E(\langle\lambda\cdot M\rangle_T - \langle\lambda\cdot M\rangle_t/F_t).$$

Thus, we have the following corollary of Theorem 3.1

Theorem 4.2. *If $U(x)=\log x$, then the value function of the problem is represented as*

$$V(t,x) = \log x - \frac{1}{2}E(\langle\lambda\cdot M\rangle_T - \langle\lambda\cdot M\rangle_t/F_t).$$

Besides, the optimal wealth process is a solution of the linear equation

$$X_t^* = x + \int_0^t \lambda_u X_u^* dS_u. \tag{4.11}$$

Thus,

$$X_t^* = x\mathcal{E}_t(\lambda \cdot S)$$

and the optimal strategy is of the form

$$\pi_t^* = \lambda_t X_t^* = x\lambda_t \mathcal{E}_t(\lambda \cdot S).$$

Exponential Utility

Let us consider the case of exponential utility function

$$U(x) = -e^{-\gamma(x-H)}$$

with risk aversion parameter $\gamma \in (0, \infty)$, where H is a bounded contingent claim describing a random payoff at time T. We assume that H is bounded F_T-measurable random variable.

For any $Q \in \mathcal{M}^e$ let $(Z_t^Q, t \in [0,T])$ be the density process of Q with respect to P and assume that

$$\mathcal{M}_{\ln}^e = \{Q \in \mathcal{M}^e : EZ_T^Q \ln Z_T^Q < \infty\} \neq \emptyset.$$

We define the space of trading strategies Π as the space of all predictable S-integrable processes π such that the corresponding wealth process X^π is a martingale relative to any $Q \in \mathcal{M}_{\ln}^e$. So, Π is the space Θ_2 from Delbaen et al. (2002) and the space $\mathcal{H}_2$ from Schachermayer (2003).

Let us consider the maximization problem

$$\max_{\pi \in \Pi} E(-e^{-\gamma(x + \int_0^T \pi_u dS_u - H)}), \tag{4.12}$$

the maximal expected utility we can achieve by starting with initial capital x, using some strategy $\pi \in \Pi$ and paying out H at time T.

The corresponding value function

$$V(t,x) = \operatorname*{esssup}_{\pi \in \Pi_x} E(-e^{-\gamma(x + \int_t^T \pi_u dS_u - H)}/\mathcal{F}_t) \tag{4.13}$$

is of the form $V(t,x) = -e^{-\gamma x} V_t$, where

$$V_t = \operatorname*{essinf}_{\pi \in \Pi_x} E(e^{-\gamma(\int_t^T \pi_u dS_u - H)} | \mathcal{F}_t) \tag{4.14}$$

is a special semimartingale.

Let $V_t = V_0 + A_t + N_t$ be the canonical decomposition of V_t, where A is an increasing process and N is a local matingale. Using the G–K–W decomposition we have that

$$V_t = V_0 + A_t + \int_0^t \varphi_s dM_s + L_t, \tag{4.15}$$

where L is a local martingale with $\langle L, M \rangle = 0$.

Since $\mathcal{M}_{\ln}^e \neq \emptyset$, the optimal strategy in the class Π exists and $V_t > 0$ for all t (see, e.g., Delbaen at al. (2002) and Yu. Kabanov and Ch. Stricker (2002)). It is evident that $V(t,x) = -e^{-\gamma x} V_t \in \mathcal{V}^{1,2}$ and all conditions of Theorem 3.1 are satisfied.

Therefore, Theorem 3.1 and Remark 3.3 imply the validity of the following

Theorem 4.3. *The value function (4.13) is of the form $-e^{-\gamma x} V_t$, where V_t satisfies the BSDE*

$$V_t = V_0 + \frac{1}{2} \int_0^t \frac{(\varphi_s + \lambda_s V_s)^2}{V_s} d\langle M \rangle_s + \int_0^t \varphi_s dM_s + L_t \tag{4.16}$$

with the boundary condition

$$V_T = e^{\gamma H}.$$

where L is a local martingale strongly orthogonal to M.

Besides, the optimal wealth process is expressed as

$$X_t^* = x + \int_0^t \frac{\varphi_u + \lambda_u V_u}{\gamma V_u} dS_u \tag{4.17}$$

and the optimal strategy is of the form

$$\pi_t^* = \frac{\varphi_t + \lambda_t V_t}{\gamma V_t}.$$

Remark 4.2 It is evident that $V_t \le E(e^{\gamma H}|F_t) \le const.$ If there exists a martingale measure Q that satisfies the Reverse Hölder R_{LlogL} condition, i.e., if

$$E(\frac{Z_T^Q}{Z_t^Q} \ln \frac{Z_T^Q}{Z_t^Q}|F_t) \le C$$

for all t, then there is a constant $c > 0$ such that $V_t \ge c$ for all t (see [7], [29]). Under R_{LlogL} condition the value process V is the unique bounded strictly positive solution of BSDE (4.16).

Now we shall give explicit solutions of equation (4.16) in two extreme cases.

Case 1.

Assume that

$$\gamma H - \frac{1}{2}\langle \lambda \cdot M \rangle_T = c + \int_0^T h_u dS_u, \tag{4.18}$$

where c is a constant and h is a predictable S-integrable process such that $h \cdot S$ is a martingale with respect to the minimal martingale measure.

This condition is satisfied iff the minimal entropy martingale measure coincides with the minimal martingale measure and H is attainable. For diffusion market models this condition is fulfilled for so called "almost complete" models, i.e., when the market price of risk is measurable with respect to the filtration generated by price processes of basic securities.

Similarly to the case of power utility one can show that the triple (Y, ψ, L), where

$$Y_t = e^{E^{Q^{min}}(\gamma H - \frac{1}{2} <\lambda \cdot M>_{tT}/F_t)}, \quad \psi_t = Y_t h_t, \quad L_t = 0$$

satisfies equation (4.16) and the optimal strategy is

$$\pi_t^* = \frac{1}{\gamma}(\lambda_t + h_t).$$

Case 2.

Assume that

$$e^{\gamma H - \frac{1}{2}\langle \lambda \cdot M \rangle_T} = c + m_T, \tag{4.19}$$

where c is a constant and m is a martingale strongly orthogonal to M.

For diffusion market models this condition is satisfied when the market price of risk and H are measurable with respect to the filtration independent relative to the asset price process.

One can show that the triple (Y, ψ, L), where

$$Y_t = e^{E(\gamma H - \frac{1}{2}\langle \lambda \cdot M \rangle_{tT}/F_t)}, \quad \psi_t = 0, \quad L_t = \int_0^t e^{\frac{1}{2}\langle \lambda \cdot M \rangle_s} dm_s$$

satisfies equation (4.16) and the optimal strategy is $\pi_t^* = \frac{1}{\gamma}\lambda_t$.

Quadratic Utility.

Let $U(x) = 2bx - x^2$, where b is a positive constant.

Assume that

$$\mathcal{M}_2^e = \{Q \in \mathcal{M}^e : E(Z_T^Q)^2 < \infty\} \ne \emptyset.$$

and let Π be the space of all predictable S-integrable processes π such that $\int_0^T \pi_u dS_u$ is in $L^2(P)$ and the stochastic integral $\int_0^t \pi_u dS_u$ is a martingale relative to any $Q \in \mathcal{M}_2^e$.

In this case (1.4) corresponds to the utility maximization problem

$$\text{to maximize} \quad E[x + 2b\int_0^T \pi_u dS_u - (\int_0^T \pi_u dS_u)^2] \quad \text{over all} \quad \pi \in \Pi, \tag{4.20}$$

which is equivalent to the problem

$$\text{to minimize} \quad E(x + \int_0^T \pi_u dS_u - b)^2 \quad \text{over all} \quad \pi \in \Pi. \tag{4.21}$$

This is the mean variance hedging problem with a constant contingent claim.

In this case the value function of (4.20) is of the form $V(t,x) = b^2 - (x-b)^2 V_t$, where

$$V_t = \operatorname*{ess\,inf}_{\pi \in \Pi} E\big((1 + \int_t^T \pi_u dS_u)^2 / \mathcal{F}_t\big)$$

is a supermartingale by optimality principle.

Let $V_t = V_0 + A_t + N_t$ be the canonical decomposition of V_t, where A is an increasing process and N is a local martingale. Using the G–K–W decomposition we have that

$$V_t = V_0 + A_t + \int_0^t \varphi_s dM_s + L_t, \tag{4.22}$$

where L is a local martingale with $\langle L, M \rangle = 0$.

Since $\mathcal{M}_2^e \neq \emptyset$, the optimal strategy in the class Π exists and $V_t > 0$ for all t (see, e.g., Gourieroux et al. 1998 or Heath et al. 2001) . It is evident that $V(t,x) = b^2 - (x-b)^2 V_t$ belongs to the class $\mathcal{V}^{1,2}$ and all conditions of Theorem 3.1 are satisfied (again one can take $A + \langle M \rangle$ as a dominated process K)

Therefore, Theorem 3.1 and Remark 3.3 imply the followin assertion

Theorem 4.4. *If $U(x) = 2bx - x^2, b \geq 0$, then the value function $V(t,x)$ is of the form $b^2 - (x-b)^2 V_t$, where V_t satisfies the BSDE*

$$V_t = V_0 + \int_0^t \frac{(\varphi_s + \lambda_s V_s)'}{V_s} d\langle M \rangle_s (\varphi_s + \lambda_s V_s)$$

$$+ \int_0^t \varphi_s dM_s + L_t, \quad V_T = 1. \tag{4.23}$$

Besides, the optimal wealth process is a solution of the linear equation

$$X_t^* = x - \int_0^t \frac{\varphi_u + \lambda_u V_u}{V_u} X_u^* dS_u \tag{4.24}$$

Therefore,

$$X_t^* = x\mathcal{E}_t(-(\frac{\varphi}{V} + \lambda) \cdot S)$$

and the optimal strategy is of the form

$$\pi_t^* = -x\mathcal{E}_t(-(\frac{\varphi}{V} + \lambda) \cdot S)(\frac{\varphi_t}{V_t} + \lambda_t).$$

5. Diffusion market models

The main task of this section is to establish a connection between the semimartingale backward equation for the value process and the classical Bellman equation for the value function related to the utility maximization problem in the case of Markov diffusion processes. For Markov diffusion models the value process can be represented as a space-transformation of an asset price process by the value function. The problem is to establish the differentiability properties of the value function from the fact that the value process satisfies the corresponding BSDE. The role of the bridge between these equations is played by the statements describing all invariant space-transformations of diffusion processes, studied in Chitashvili and Mania (1996) and formulated here in the appendix, in a suitable case adapted to financial market models. This approach enables us to prove that there exists a solution (in a certain sense) of the Bellman equation and that this solution is differentiable (in a generalized sense) under mild assumptions on the model coefficients. Although, in our case, the generalized derivative at t and the second order generalized derivatives at x do not exist separately in general (we prove an existence of a generalized L-operator), these derivatives do not enter in the construction of optimal strategies which are explicitly given in terms of the first order derivatives of the value function. It should be noted that the theory of viscosity solutions is usually applied to such problems (see, e.g., El Karoui et al (1997)), but differentiability of the value function is in general beyond the reach of this method.

We assume that the dynamics of the asset price process is determined by the following system of stochastic differential equations

$$dS_t = \mathrm{diag}(S_t)(\mu(t,S_t,R_t)dt + \sigma^l(t,S_t,R_t)dW_t^l) \tag{5.1}$$

$$dR_t = b(t,S_t,R_t)dt + \delta(t,S_t,R_t)dW_t^l + \sigma^\perp(t,S_t,R_t)dW_t^\perp \tag{5.2}$$

Here $W=(W^1,...,W^n)$ be an n-dimensional standard Brownian motion defined on a complete probability space (Ω,F,P) equipped with the P-augmentated filtration generated by W, $F=(F_t, t\in[0,T])$. By $W^l=(W^1,...,W^d)$ and $W^\perp=(W^{d+1},...,W^n)$ are denoted the d and $n-d$ dimensional Brownian motions respectively.

Assume that

S1) the coefficients $\mu, b, \delta, \sigma^l, \sigma^\perp$ are measurable and bounded;

S2) $n\times n-$ matrix function $\sigma\sigma'$ is uniformly elliptic, i.e., there is a constant $c>0$ such that

$$(\sigma(t,s,r)\lambda, \sigma(t,s,r)\lambda) \geq c|\lambda|^2$$

for all $t\in[0,T], s\in R_+^d, r\in R^{n-d}$ and $\lambda\in R^n$, where σ is defined by

$$\sigma(t,s,r) = \begin{pmatrix} \sigma^l(t,s,r) & 0 \\ \delta(t,s,r) & \sigma^\perp(t,s,r), \end{pmatrix}.$$

In addition we assume that

S3) the system (5.1), (5.2) admits a unique strong solution.

Straightforward calculations yield that in this case

$$\lambda = \mathrm{diag}(S)^{-1}(\sigma^l\sigma^{l'})^{-1}\mu,$$

where $\sigma^{l'}$ denotes the transpose of σ^l,

$$\frac{d\langle M\rangle_t}{dt} = diag(S_t)(\sigma^l\sigma^{l'})(t,S_t,R_t)diag(S_t)$$

is the ν_t process, $\theta=(\sigma^l)^{-1}\mu$ is the market price of risk and

$$\langle\lambda\cdot M\rangle_t = \int_0^t ||\theta_s||^2 ds$$

is the mean variance tradeoff.

By results of Krylov (1980) for sufficiently smooth coefficients μ,σ,b,δ the value function $V(t,x)$ can be represented as $v(t,x,S_t,R_t)$ with sufficiently smooth function $v(t,x,s,r),\ t\in[0,T],\ x\in R_+,\ s\in$

$R^d_+,\ r\in R^{n-d}$. Hence by the equation (3.1) and the Itô formula we obtain that $v(t,x,s,r)$ satisfies the PDE

$$\mathcal{L}v(t,x,s,r)+v_s(t,x,s,r)'\operatorname{diag}(s)\mu(t,s,r)+v_r(t,x,s,r)'b(t,s,r) \tag{5.3}$$

$$=\frac{1}{2}\frac{|v_{sx}(t,x,s,r)+diag(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_{rx}(t,x,s,r)+\lambda'(t,s,r)v_x(t,x,s,r)|^2_{\nu_t}}{v_{xx}(t,x,s,r)},$$

$$v(T,x,s,r)=U(x), \tag{5.4}$$

which coincides with the Bellman equation of optimization problem (1.4), (5.1),(5.2) for controlled Markov process. Moreover the optimal strategy is

$$\pi^*(t,x,s,r)=$$
$$\frac{v_{sx}(t,x,s,r)+\operatorname{diag}(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_{rx}(t,x,s,r)+\lambda'(t,s,r)v_x(t,x,s,r)}{v_{xx}(t,x,s,r)},$$

In this section we study the solvability of (5.3), (5.4) in the particular cases of utility functions but with weaker conditions on coefficients.

First consider the case of power utility.

Theorem 5.1. *Let condition S1), S2) and S3) be satisfied. Then the value function $v(t,s,r)$ admits all first order generalized derivatives v_s and v_r, a generalized L-operator*

$$\mathcal{L}v=v_t+\frac{1}{2}tr(diag(s)\sigma^l\sigma^{l'}(t,s,r)\operatorname{diag}(s)v_{ss}+tr(\delta\sigma^{l'}(t,r,s)diag(s)v_{sr})$$
$$+\frac{1}{2}tr((\delta\delta'(t,s,r)+\sigma^{\perp}\sigma^{\perp'}(t,s,r))v_{rr})$$

(in the sense of Definition D of the Appendix) and is the unique bounded solution of equation

$$\mathcal{L}v(t,s,r)+v_s(t,s,r)'diag(s)\mu(t,s,r)+v_r(t,s,r)'b(t,s,r)$$
$$=\frac{q}{2}\frac{|v_s(t,s,r)+diag(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r)+\lambda(t,s,r)v(t,s,r)|^2_{\nu_t}}{v(t,s,r)}$$
$$dtdsdr-a.e. \tag{5.5}$$

with the boundary condition

$$v(T,s,r)=1. \tag{5.6}$$

Moreover, the optimal strategy is defined as

$$\pi^*(t,x,s,r)=(1-q)(\lambda(t,s,r)+\frac{\varphi(t,s,r)}{v(t,s,r)})x$$

and the optimal wealth process is of the form

$$X^*_t=x\mathcal{E}_t((1-q)(\frac{\varphi}{v}+\lambda)\cdot S),$$

where $\varphi(t,s,r)=v_s(t,s,r)+\operatorname{diag}(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r)$.

Proof. Existence. Since (S,R) is a Markov process, the feedback controls are sufficient and the value process is expressed by

$$V_t=v(t,S_t,R_t)\quad a.s. \tag{5.7}$$

where

$$v(t,s,r)=\sup_{\pi\in\Pi_1}E\big((1+\int_t^T\pi_u ds_u)^p|S_t=s,R_t=r\big).$$

(one can show this fact, e.g., similarly to [5]).

Since the value process satisfies equation (4.3), it is an Itô process. From the equality $\mathcal{E}(-\lambda\cdot M)=\mathcal{E}(-\int_0^\cdot \theta_u dw_u^l)$ and boundedness of θ follows that $\mathcal{E}(-\lambda\cdot M)$ satisfies The Muckenhoupt inequality. Thus $V_t=\operatorname*{esssup}_{\pi\in\Pi_1} E((1+\int_t^T\pi_u dS_u)^p/\mathcal{F}_t)$ is bounded (see Remark 4.1)and the martingale part of V is in BMO by Proposition 7 from [29]. Hence the finite variation part of V_t is of integrable variation and from (5.7) we have that $v(t,S_t,R_t)$ is an Itô process of the form (B.1) (Appendix). Therefore, Proposition B of the Appendix implies that the function $v(t,s,r)$ admits a generalized L-operator, all first order generalized derivatives and can be represented as

$$
\begin{aligned}
v(t,S_t,R_t)=v_0+\int_0^t(v_s(u,S_u,R_u)'\operatorname{diag}(S_u)\sigma^l(u,S_u,R_u)\\
+v_r(u,S_u,R_u)'\delta(u,S_u,R_u))dW_s^l+\int_0^t v_r(u,S_u,R_u)'\sigma^\perp(u,S_u,R_u)dW_s^\perp\\
+\int_0^t\mathcal{L}v(u,S_u,R_u)ds+\int_0^t\big(v_s(u,S_u,R_u)'\operatorname{diag}(X_s)\mu(u,S_u,R_u)\\
+v_r(u,S_u,R_u)b(u,S_u,R_u)\big)du, \qquad (5.8)
\end{aligned}
$$

where $\mathcal{L}V$ is the generalized $\mathcal{L}$-operator.

On the other side the value process is a solution of (4.3) and by the uniqueness of the canonical decomposition of semimartingales, comparing the martingale parts of (5.8) and (4.3), we have that $dt\times dP$- a.e.

$$
\varphi_t=v_s(t,S_t,R_t)+\operatorname{diag}(S_t)^{-1}\sigma^{l'}(t,S_t,R_t)^{-1}\delta'(t,S_t,R_t)v_r(t,S_t,R_t), \qquad (5.9)
$$

$$
\varphi_t^\perp=\sigma^{\perp'}(t,S_t,R_t)v_r(t,S_t,R_t). \qquad (5.10)
$$

Then, equating the processes of bounded variation of the same equations, taking into account (5.8) and (5.9), we derive

$$
\begin{aligned}
\int_0^t\big(\mathcal{L}v(u,S_u,R_u)+v_s(u,S_u,R_u)'\operatorname{diag}(S_u)\mu(u,S_u,R_u)\\
+v_r(u,S_u,R_u)b(u,S_u,R_u)\big)du\\
=\frac{q}{2}\int_0^t\frac{|\varphi_u+\lambda(u,S_u,R_u)v(u,S_u,R_u)|^2_{\nu_u}}{v(u,S_u,R_u)}du \qquad (5.11)
\end{aligned}
$$

which gives that $v(t,s,r)$ solves the Bellman equation (5.5).

Unicity. Let $\tilde v(t,s,r)$ be a bounded positive solution of (5.5), (5.6), from the class V^L. Then using the generalized Itô formula (Proposition B of Appendix) and equation (5.5) we obtain that $\tilde v(t,S_t,R_t)$ is a solution of (4.3), hence $\tilde v(t,S_t,R_t)$ coincides with the value process v by Theorem 4.1. Therefore $\tilde v(t,S_t,R_t)=v(t,S_t,R_t)$ a.s. and $\tilde v=v$, $dtdxdy$ a.e. □

Now we consider extreme cases for the stochastic volatility models. Let first assume that coefficients μ,σ^l does not contain the variable r. Hence equation (5.1) takes the form

$$
dS_t=\operatorname{diag}(S_t)(\mu(t,S_t)dt+\sigma^l(t,S_t)dW_t^l). \qquad (5.12)
$$

Let $S(q)$ be the Itô process governed by SDE

$$
dS_t(q)=\operatorname{diag}(S_t(q))\sigma^l(t,S_t(q))(dW_t^l+q\theta(t,S_t(q))dt), \qquad (5.13)
$$

where $dW_t^l+q\theta(t,S_t)dt$ is Brownian motion w.r.t. measure $dQ(q)=\mathcal{E}_T(-q\int_0^\cdot\theta_u dw_u^l)dP$. Thus by 4.6 the value process is represented as

$$
V_t=v(t,S_t(q))=(\tilde v(t,S_t(q))^{\frac{1}{1-q}},
$$

where $\tilde{v}(t,s)=E^{Q(q)}\big(e^{\frac{q(q-1)}{2}\int_t^T|\theta_u|^2du}|S_t(q)=s\big)$. Therefore we have

Corollary 5.1. *Let conditions S1), S2) and S3) be satisfied for the coefficients of system (5.13). Then the value process can be represented as* $(\tilde{v}(t,S_t(q))^{\frac{1}{1-q}}$*, where* $\tilde{v}(t,s)$ *is the classical solution of the linear PDE*

$$\tilde{v}_t(t,s)+\frac{1}{2}tr(\mathrm{diag}(s)\sigma^l\sigma^{l'}(t,s)\,\mathrm{diag}(s)\tilde{v}_{ss}(t,s))$$

$$+\frac{q(q-1)}{2}|\theta(t,s)|^2\tilde{v}(t,s)=0, \tag{5.14}$$

$$\tilde{v}(T,s)=1. \tag{5.15}$$

The second extreme case corresponds to the stochastic volatility model of the form

$$dS_t=\mathrm{diag}(S_t)(\mu(t,S_t,R_t)dt+\sigma^l(t,S_t,R_t)dW_t^l)$$

$$dR_t=b(t,R_t)dt+\sigma^{\perp}(t,R_t)dW_t^{\perp}. \tag{5.16}$$

Corollary 5.2. *Let conditions S1), S2) and S3) be satisfied for the coefficients of the system (5.16) and* θ *does not depend on the variable s. Then the value process of the optimization problem (4.1) is of the form* $V_t=v(t,R_t)$*, where* $v(t,r)=E(e^{-\frac{q}{2}\int_t^T|\theta(u,R_u)|^2du}|R_t=r)$ *satisfies the linear PDE*

$$v_t(t,r)+\frac{1}{2}tr(\sigma^{\perp}\sigma^{\perp'}(t,r)v_{rr}(t,r))$$

$$+v_r(t,r)'b(t,r)-\frac{q}{2}|\theta(t,r)|^2v(t,r)=0, \tag{5.17}$$

$$v(T,r)=1. \tag{5.18}$$

Similar results can be obtained for exponential utility function.

Proposition 5.1. *Let conditions S1), S2) and S3) be satisfied and* $H=g(S_T,R_T)$ *for a continuous bounded function* $g(s,r)$*. Then the value function* $v(t,s,r)$ *for the problem (4.12) admits all first order generalized derivatives* v_s *and* v_r*, a generalized L-operator and is the unique bounded solution of equation*

$$\mathcal{L}v(t,s,r)+v_s(t,s,r)'\,\mathrm{diag}(s)\mu(t,s,r)+v_r(t,s,r)'b(t,s,r)$$

$$=\frac{1}{2}\frac{|v_s(t,s,r)+diag(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r)+\lambda(t,s,r)v(t,s,r)|^2_{\nu_t}}{v(t,s,r)}$$

$$dtdsdr-a.e. \tag{5.19}$$

with the boundary condition

$$v(T,s,r)=e^{-\gamma g(s,r)}. \tag{5.20}$$

Moreover, the optimal strategy is defined as

$$\pi^*(t,x,s,r)=\frac{1}{\gamma}(\lambda(t,s,r)+\frac{\varphi(t,s,r)}{v(t,s,r)})x,$$

where $\varphi(t,s,r)=v_s(t,s,r)+diag(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r)$ *and the optimal wealth process is defined by (4.17).*

The following assertion, for the case of logarithmic utility, follows immediately from Theorem 4.3 and the Feynmann-Kac formula:

Proposition 5.2. *Let condition S1), S2) and S3) be satisfied and* $U(x)=\log x$*. Then the value function can be represented as* $v(t,S_t,R_t)$*, where* $v(t,s,r)$ *is unique solution of linear PDE*

$$\mathcal{L}v(t,s,r)+v_s(t,s,r)'\,\mathrm{diag}(s)\mu(t,s,r)$$

$$+v_r(t,s,r)'b(t,s,r)+|\theta(t,s,r)|^2v(t,s,r)=0, \tag{5.21}$$

$$v(T,s,r)=1 \tag{5.22}$$

and the optimal strategy is $\pi^*(t,x,s,r)=\lambda(t,s,r)x$.

Appendix A

Let us show that the family

$$\Lambda_t^\pi = E(U(x+\int_0^T \pi_u dS_u)|\mathcal{F}_t), \quad \pi \in \Pi_x(\tilde{\pi}, t, T) \tag{A.1}$$

satisfies the ε-lattice property (with $\varepsilon = 0$) for any $t \in [0,T]$ and $\tilde{\pi}$. $\Pi(\tilde{\pi}, t, T)$ is a set of predictable S-integrable processes π from Π_x such that

$$\pi_s = \tilde{\pi}_s I_{(0\le s<t)}.$$

We shall write $\Pi(t,T)$ instead of $\Pi(0,t,T)$ for the class of strategies corresponding to $\tilde{\pi}=0$ up to time t.

We must show that for any $\pi^1,\ \pi^2 \in \Pi(\tilde{\pi}, t, T)$ there exists a strategy $\pi \in \Pi(\tilde{\pi}, t, T)$ such that

$$\Lambda_t^\pi = \max(\Lambda_t^{\pi^1}, \Lambda_t^{\pi^2}). \tag{A.2}$$

For any π^1 and π^2 let us define the set

$$B = \{\omega : \Lambda_t^{\pi^1} \le \Lambda_t^{\pi^2}\}$$

and let

$$\pi_s = \tilde{\pi}_s I_{(0\le s<t)} + \pi_s^1 I_B I_{(s\ge t)} + \pi_s^2 I_{B^c} I_{(s\ge t)}$$

It is evident that

$$\text{if} \quad \tilde{\pi}, \pi^1, \pi^2 \in \Pi_x, \quad \text{then} \quad \pi \in \Pi_x. \tag{A.3}$$

Since B is $\mathcal{F}_t-$measurable we have

$$\Lambda_t^\pi = E(U(x+\int_0^T \pi_u dS_u)|\mathcal{F}_t)$$

$$= E(U(x+\int_0^t \tilde{\pi}_u dS_u + I_B\int_t^T \pi_u^1 dS_u + I_{B^c}\int_t^T \pi_u^2 dS_u)|\mathcal{F}_t)$$

$$= I_B E(U(x+\int_0^t \tilde{\pi}_u dS_u + \int_t^T \pi_u^1 dS_u)|\mathcal{F}_t) + I_{B^c} E(U(x+\int_\tau^t \tilde{\pi}_u dS_u + \int_t^T \pi_u^2 dS_u)|\mathcal{F}_t)$$

$$= I_B E(U(x+\int_0^T \pi_u^1 dS_u)|\mathcal{F}_t) + I_{B^c} E(U(x+\int_0^T \pi_u^2 dS_u)|\mathcal{F}_t)$$

$$= E(U(x+\int_0^T \pi_u^1 dS_u)|\mathcal{F}_t) \vee E(U(x+\int_0^T \pi_u^2 dS_u)|\mathcal{F}_t),$$

hence (A.2) is satisfied.

Proposition A1) (Optimality principle). Let condition B1) be satisfied.

a) For all $x \in R$, $\pi \in \Pi$ and $s \in [0,T]$ the process $(V(t, x+\int_s^t \pi_u dS_u), t \ge s)$ is a supermartingale, admitting an RCLL modification.

b) $\pi^*(s,x)$ is optimal iff $(V(t, x+\int_s^t \pi_u^* dS_u), t \ge s)$ is a martingale.

c) for all $s < t$

$$V(s,x) = \operatorname*{ess\,sup}_{\pi\in\Pi(s,T)} E(V(t, x+\int_s^t \pi_u dS_u)|\mathcal{F}_s). \tag{A.4}$$

Proof. a) For simplicity we shall take s equal to zero. Let us show that $Y_t = V(t, x + \int_0^t \tilde{\pi}_u dS_u)$ is supermartingale for all x and $\tilde{\pi}$. Since

$$Y_t = \operatorname*{ess\,sup}_{\pi\in\Pi(t,T)} E(U(x + \int_0^t \tilde{\pi}_u dS_u + \int_t^T \pi_u dS_u)|\mathcal{F}_t)$$

using the lattice property of the family (A.1) from Lemma 16.A.5 of [11] we have

$$E(Y_t|\mathcal{F}_s) = E(\operatorname*{ess\,sup}_{\pi\in\Pi(t,T)} E(U(x + \int_0^t \tilde{\pi}_u dS_u + \int_t^T \pi_u dS_u)|\mathcal{F}_t)|\mathcal{F}_s)$$

$$= E(\operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},t,T)} E(U(x + \int_0^T \pi_u dS_u)|\mathcal{F}_t)|\mathcal{F}_s)$$

$$= \operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},t,T)} E(U(x + \int_0^T \pi_u dS_u)|\mathcal{F}_s). \tag{A.5}$$

It is evident that $\Pi(\tilde{\pi}, t, T) \subseteq \Pi(\tilde{\pi}, s, T)$ for $s \le t$, which implies the inequality

$$\operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},t,T)} E(U(x + \int_0^T \pi_u dS_u)|\mathcal{F}_s)$$

$$\le \operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},s,T)} E(U(x + \int_0^T \pi_u dS_u)|\mathcal{F}_t)$$

$$= V(s, x + \int_0^s \tilde{\pi}_u dS_u). \tag{A.6}$$

Thus (A.4) and (A.5) imply that $E(Y_t/F_s) \le Y_s$.

b) If $V(t, x + \int_0^t \pi^*_u dS_u))$ is a martingale, then

$$\inf_{\pi\in\Pi} EU(x + \int_0^T \pi_u dS_u) = V(0, x) = EV(0, x)$$

$$= EV(T, x + \int_0^T \pi^*_u dS_u) = EU(x + \int_0^T \pi^*_u dS_u),$$

hence, π^* is optimal.

Conversely, if π^* is optimal, then

$$EV(0, x) = \sup_{\pi\in\Pi} EU(x + \int_0^T \pi_u dS_u)$$

$$= EU(x + \int_0^T \pi^*_u dS_u) = EV(T, x + \int_0^T \pi^*_u dS_u).$$

Since $V(t, x+\int_0^t \pi^*_u dS_u)$ is a supermartingale, the latter equality implies that this process is a martingale (it follows from Lemma 6.6 of [27]).

c) Since $Y_t = V(t, x + \int_s^t \tilde{\pi}_u dS_u)$ is a supermartingale for any $\tilde{\pi} \in \Pi(s,T)$, $x \in R$ and $t \geq s$ we have

$$V(s,x) \geq E(V(t, x + \int_s^t \tilde{\pi}_u dS_u)|\mathcal{F}_s),$$

hence

$$V(s,x) \leq \operatorname*{ess\,sup}_{\tilde{\pi}\in\Pi(s,T)} E(V(t, x + \int_s^t \tilde{\pi}_u dS_u)|\mathcal{F}_s). \tag{A.7}$$

On the other hand for any $\tilde{\pi}$

$$E(V(t, x + \int_s^t \tilde{\pi}_u dS_u)|\mathcal{F}_s) =$$

$$E(\operatorname*{ess\,sup}_{\pi\in\Pi(t,T)} E(U(x + \int_s^t \tilde{\pi}_u dS_u + \int_t^T \pi_u dS_u)|\mathcal{F}_t)\mathcal{F}_s) \geq$$

$$E(E(U(x + \int_s^T \tilde{\pi}_u dS_u)|\mathcal{F}_t)\mathcal{F}_s) = E(U(x + \int_s^T \tilde{\pi}_u dS_u)|\mathcal{F}_s).$$

Taking esssup of the both parts we obtain

$$\operatorname*{ess\,sup}_{\tilde{\pi}\in\Pi(s,T)} E(V(t, x + \int_s^t \tilde{\pi}_u dS_u)|\mathcal{F}_s) \geq$$

$$\operatorname*{ess\,sup}_{\tilde{\pi}\in\Pi(s,T)} E(U(x + \int_s^T \tilde{\pi}_u dS_u)|\mathcal{F}_s) = V(s,x). \tag{A.8}$$

Thus the equality (A.3) follows from (A.6) and (A.7).

Let us show now that the process $\tilde{V}(t, x + \int_0^t \tilde{\pi}_u dS_u)$ admits an RCLL modification for each $x \in R$ and $\pi \in \tilde{\Pi}$. According to Theorem 3.1 of [27] it is sufficient to prove that the function $E\tilde{V}(t, x + \int_0^t \tilde{\pi}_u dS_u)), t \in [0,T])$ is right-continuous for every $x \in R$.

Let $(t_n, n \geq 1)$ be a sequence of positive numbers such that $t_n \downarrow t$, as $n \to \infty$. Since $\tilde{V}(t, x + \int_0^t \tilde{\pi}_u dS_u)$ is a supermartingale, we have

$$E\tilde{V}(t, x + \int_0^t \tilde{\pi}_u dS_u) \geq \lim_{n\to\infty} E\tilde{V}(t_n, x + \int_0^{t_n} \tilde{\pi}_u dS_u). \tag{A.9}$$

Let us show the inverse inequality. For $s = 0$ equality (A.4) takes the form

$$E\tilde{V}(t, x + \int_0^t \tilde{\pi}_u dS_u) = \max_{\pi\in\tilde{\Pi}(\tilde{\pi},t,T)} E(U(x + \int_0^T \pi_u dS_u). \tag{A.10}$$

Therefore, for any $\varepsilon > 0$ there exists a strategy π^ε such that

$$E\tilde{V}(t, x + \int_0^t \tilde{\pi}_u dS_u) \leq E(U(x + \int_0^t \tilde{\pi}_u dS_u + \int_t^T \pi^\varepsilon_u dS_u) + \varepsilon. \tag{A.11}$$

Let us define a sequence $(\pi^n, n \geq 1)$ of strategies

$$\pi^n_s = \tilde{\pi}_s I_{(s<t_n)} + \pi^\varepsilon_s I_{(s\geq t_n)}.$$

Using inequality (A.11), the continuity of U (it follows from B1) and B2), the convergence of the stochastic integrals and Fatou's lemma, we have

$$E\tilde{V}(t,x+\int_0^t\tilde{\pi}_u dS_u)\le E(U(x+\int_0^t\tilde{\pi}_u dS_u+\int_t^T\pi^\varepsilon_u dS_u)+\varepsilon=$$

$$=E(\lim_n U(x+\int_0^{t_n}\tilde{\pi}_u dS_u+\int_{t_n}^T\pi^\varepsilon_u dS_u))+\varepsilon\ge$$

$$\ge\underline{\lim}_n E(E(U(x+\int_0^{t_n}\tilde{\pi}_u dS_u+\int_{t_n}^T\pi^\varepsilon_u dS_u)/\mathcal{F}_{t_n}))+\varepsilon\ge$$

$$\ge\underline{\lim}_n E(\operatorname*{ess\,sup}_{\pi\in\tilde{\Pi}(\tilde{\pi},t_n,T)}E(U(x+\int_0^{t_n}\tilde{\pi}_u dS_u+\int_{t_n}^T\pi_u dS_u)/\mathcal{F}_{t_n}))+\varepsilon=$$

$$=\underline{\lim}_{n\to\infty}E(\tilde{V}(t_n,x+\int_0^{t_n}\tilde{\pi}_u dS_u)+\varepsilon. \tag{A.12}$$

Since ε is an arbitrary positive number, from (A.12) we obtain that

$$E\tilde{V}(t,x+\int_0^t\tilde{\pi}_u dS_u))\le\underline{\lim}_{n\to\infty}E\tilde{V}(t_n,x+\int_0^{t_n}\tilde{\pi}_u dS_u)), \tag{A.13}$$

which together with (A.9) implies that the function $(E\tilde{V}(t,x+\int_0^t\tilde{\pi}_u dS_u)),t\in[0,T])$ is right-continuous.

Lemma A.1. *Let b_t be a predictable process and S a continuous semimartingale. Suppose that $\mathcal{K}$ is an adapted continuous increasing process and $\mu^{\mathcal{K}}$ the corresponding Dolean's measure. Denote by Π_x the space of all predictable S-integrable processes π such that for all $t\in[0,T]$*

$$x+\int_0^t\pi_u dS_u\ge 0.$$

Then $\mu^{\mathcal{K}}$ a.e.

$$\operatorname*{ess\,inf}_{\pi\in\Pi_x}\big|\pi_t-b_t\big|=0$$

Proof. Taking a bounded continuous approximation $b^{n,m}_t$ of $b^n_t=b_tI_{(|b_t|\le n)}$ in the sense of $\mu^{\mathcal{K}}$-a.e. convergence we have that

$$\operatorname*{ess\,inf}_{\pi\in\Pi_x}\big|\pi_t-b_t\big|\le\operatorname*{ess\,inf}_{\pi\in\Pi_x}\big|\pi_t-b^{n,m}_t\big|+|b^{n,m}_t-b_t|.$$

Therefore, without loss of generality we may assume that b is continuous and S-integrable. Let us denote by

$$\pi^{r,n}_t=b_t\psi^{r,n}_t\mathcal{E}_t(\frac{1}{x}(b\psi^{r,n})\cdot S),$$

for $r\in[0,T],n\in N$, where $\psi^{r,n}_t=I_{(r-\frac{1}{n},r+\frac{1}{n})}(t)(1-n|t-r|)$. It is evident that π^n belongs to Π_x for all $r,n\ge1$. Indeed,

$$x+\int_0^t\pi^n_u dS_u=x+x\int_0^t\mathcal{E}_u(\frac{1}{x}b\psi^{r,n}\cdot S)\frac{1}{x}b_u\psi^{r,n}(u)dS_u$$

$$=x\mathcal{E}_t(\frac{1}{x}b\psi^{r,n}\cdot S)\ge 0.$$

Denote by γ_t the expression $\operatorname*{ess\,inf}_{\pi\in\Pi_x}\big|\pi_t - b_t\big|$. By definition $\gamma_t \le |\pi_t - b_t|$, $\mu^{\mathcal{K}}$-a.e. for all $\pi \in \Pi_x$. Therefore $\gamma_t \le |\pi_t^{r,n} - b_t|$ on the set B with $\mu^{\mathcal{K}}(B^c) = 0$ for all rational $r \in [0,T]$ and integer n. Let

$$\widetilde{\gamma}_t = \gamma_t, \;\text{ if }\; (\omega,t)\in B \;\text{ and }\; \widetilde{\gamma}_t = 0, \;\text{ if }\; (\omega,t)\in B^c.$$

Then $\widetilde{\gamma}_t \le |\pi_t^{r,n} - b_t|$ for all t,ω,r,n. It is easy to see that $\pi_t^{r,n}$ is continuous function of variables s,t for each n, since $\psi_t^{s,n} \equiv \int_0^t (1_{(s-\frac{1}{n},s)}(u) - 1_{(s,s+\frac{1}{n})}(u))du$ and

$$\int_0^t \psi_u^{s,n} b_u dS_u \equiv \psi_t^{s,n}\int_0^t b_u dS_u - \int_0^t \left(\int_0^v b_u dS_u\right)\left(1_{(s-\frac{1}{n},s)}(v) - 1_{(s,s+\frac{1}{n})}(v)\right)dv$$

$$\equiv \psi_t^{s,n}\int_0^t b_u dS_u - \int_{(s-\frac{1}{n})\vee 0}^{s\wedge t} b_u dS_u + \int_{s\vee 0}^{(s+\frac{1}{n})\wedge t} b_u dS_u$$

are continuous. Hence $\pi_t^{r,n} \to \pi_t^{s,n}$, as $r \to s$ uniformly in $t\in[0,T]$. Passing to the limit as $r\to s$ we have

$$\widetilde{\gamma}_t \le |\pi_t^{s,n} - b_t|, \;\text{ for all }\; t,s \;\; P-a.s.$$

Since $P-a.s.$

$$\pi_s^{s,n} = b_s\mathcal{E}_s(\frac{1}{x}(b\psi^{s,n})\cdot S) \to b_s, \quad \text{as }\; n\to\infty,$$

we can conclude that $\widetilde{\gamma}_s = 0$ for all s $P-$a.s.. This implies that $\gamma_s = 0$, $\mu^{\mathcal{K}}-$a.e..

Appendix B

Now we introduce some notions which enable us to present an application of Theorem 1 to the Markov case.

Consider the system of stochastic differential equations (5.1), (5.2) and assume that conditions S1) and S2) are satisfied. Under these conditions there exists a unique weak solution of (5.1), (5.2), which is a Markov process and its transition probability function admits a density $p(s,(x_0,y_0),t,(x,y))$ with respect to the Lebesgue measure. We shall use the notation $p(t,x,y) = p(0,(x_0,y_0),t,(x,y))$ for the fixed initial condition $S_0 = x_0, R_0 = y_0$.
Introduce the measure μ on the space $([0,T]\times R_+^d \times R^{n-d}, \mathcal{B}([0,T]\times R_+^d\times R^{n-d}))$ defined by

$$\mu(dt,dx,dy) = p(t,x,y)dtdxdy.$$

Let $C^{1,2}$ be the class of functions f continuously differentiable at t and twice differentiable at x,y on $[0,T]\times R_+^d\times R^{n-d}$. For functions $f\in C^{1,2}$ the L operator is defined as

$$\begin{aligned} Lf =& f_t + tr(\frac{1}{2}diag(x)\sigma^l\sigma^{l'}diag(x)f_{xx}) + tr(\delta\sigma^{l'}diag(x)f_{xy}) \\ &+ tr(\frac{1}{2}(\delta\delta' + \sigma^{\perp}\sigma^{\perp'})f_{yy}) \end{aligned}$$

where f_t, f_{xx}, f_{xy} and f_{yy} are partial derivatives of the function f, for which we use the matrix notations.

Definition B. We shall say that a function $f = (f(t,x,y), t\ge 0, x\in R_+^d, y\in R^{n-d})$ belongs to the class V_μ^L if there exists a sequence of functions $(f^n, n\ge 1)$ from $C^{1,2}$ and measurable μ-integrable functions $f_{x_i}(i\le d)$, $f_{y_j}(d<j\le n)$ and (Lf) such that

$$E\sup_{s\le T}|f^n(s,S_s,R_s) - f(u,S_u,R_u)| \to 0, \text{ as } n\to\infty,$$

$$\iint_{[0,T]\times R_+^d\times R^{n-d}} (f_{x_i}^n(s,x,y) - f_{x_i}(s,x,y))^2 x_i^2 \mu(ds,dx,dy) \to 0, \;\; i\le d,$$

$$\iint_{[0,T]\times R_+^d\times R^{n-d}} (f_{y_j}^n(s,x,y) - f_{y_j}(s,x,y))^2 \mu(ds,dx,dy) \to 0, \;\; d<j\le n,$$

$$\iint\limits_{[0,T]\times R^d_+\times R^{n-d}} |Lf^n(s,x,y)-(Lf)(s,x,y)|\mu(ds,dx,dy)\to 0,$$

$$\text{as } \ n\to\infty.$$

Now we formulate the statement proved in Chitashvili and Mania (1996) in the case convenient for our purposes.

Proposition B. Let conditions S1)-S2) be satisfied and let $f(t,S_t,R_t)$ be a bounded process. Then the process $(f(t,S_t,R_t), t\in[0,T])$ is an Itô process of the form

$$f(t,S_t,R_t)=f(0,S_0,R_0)+\int\limits_0^t g(s,\omega)dW_s+\int\limits_0^t a(s,\omega)ds, \quad \text{a.s.}$$

with

$$E\int\limits_0^t g^2(s,\omega)ds<\infty,\ \ E\int\limits_0^t |a(s,\omega)|ds<\infty \tag{B.1}$$

if and only if f belongs to V^L_μ. Moreover the process $f(t,S_t,R_t)$ admits the decomposition

$$f(t,S_t,R_t)=f(0,S_0,R_0)+\sum_{i=1}^d\int\limits_0^t f_{x_i}(s,S_s,R_s)dS^i_s+$$

$$\sum_{j=d+1}^n\int\limits_0^t f_{y_j}(s,S_s,R_s)dR^j_s+\int\limits_0^t(Lf)(s,S_s,R_s)ds. \tag{B.2}$$

Remark. For continuous functions $f\in V^L_\mu$ the condition

$$\sup_{(t,x,y)\in D}|f^n(t,x,y)-f(t,x,y)|\to 0,\ \text{as } n\to\infty \tag{B.3}$$

for every compact $D\in[0,T]\times R^d_+\times R^{n-d}$, can be used instead of the first relation of Definition B.

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