Thermodynamics of third order Lovelock anti-de Sitter black holes revisited

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We compute the mass and the temperature of third order Lovelock black holes with negative Gauss-Bonnet coefficient $\alpha_2 < 0$ in anti-de Sitter space and perform the stability analysis of topological black holes. When $k = -1$, the third order Lovelock black holes are thermodynamically stable for the whole range r_+. When $k = 1$, we found that the black hole has an intermediate unstable phase for $D = 7$. In eight dimensional spacetimes, however, a new phase of thermodynamically unstable small black holes appears if the coefficient $\tilde{\alpha}$ is under a critical value. For $D \geq 9$, black holes have similar the distributions of thermodynamically stable regions to the case where the coefficient $\tilde{\alpha}$ is under a critical value for $D = 8$. It is worth to mention that all the thermodynamic and conserved quantities of the black holes with flat horizon don't depend on the Lovelock coefficients and are the same as those of black holes in general gravity.

I. INTRODUCTION

In the last decade anti-de Sitter (AdS) black holes and especially their thermodynamics have attracted considerable interest due to the AdS/CFT duality. According to the AdS/CFT conjecture the thermodynamics of the AdS black holes is related to the thermodynamics of the dual CFT residing on the boundary of the AdS space. It is well-know that the AdS Schwarzschild black hole is thermodynamically unstable when the horizon radius is small, while it is stable for large radius; there is a phase transition, named Hawking-Page transition[1], between the large stable black hole and a thermal AdS space. This phase transition is explained by Witten as the confinement/deconfinement transition of Yang-Mills

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theory in the AdS-CFT correspondence [2].

Besides Einstein-Hilbert action, in their low-energy limit string theories give rise to effective models of gravity in higher dimensions which involve the higher powers of curvature terms [3]. However, the higher powers of curvature could in common give rise to a fourth or even higher order differential equation for the metric, and it would introduce ghosts and violate unitarity, therefore, the higher derivative terms may be a source of inconsistencies. Among the gravity theories with higher derivative terms, the so-called Lovelock gravity [4] is quite special. Its equations of motion contain the most symmetric conserved tensor with no more than two derivative of the metric and it has been proven to be free of ghosts when expanding about the flat space, evading any problem with unitarity [5]. In this paper, we restrict ourself to explore the first four terms of the Lovelock gravity. Among these terms, the first term is cosmological term, the second term is Einstein term, and the third and the fourth terms are the second order Lovelock(Gauss-Bonnet) and third order Lovelock terms, respectively.

In third order Lovelock gravity, the static spherically symmetric black hole solutions were firstly found in [6]. Then, they also showed that the asymptotically flat uncharged black hole of third order Lovelock gravity may has two horizons, a fact that does not happen in lower order Lovelock gravity. The thermodynamics of the uncharged static black hole solutions with negative cosmological constant has been considered in [7]. We note that the Gauss-Bonnet coefficient always holds on positive in these papers. Here we will study third order Lovelock black hole solutions with the negative Gauss-Bonnet coefficient in anti-de Sitter space.

The outline of this paper is as follows. Considering the coefficient of Gauss-Bonnet term $\alpha_2 < 0$, we present static solution with special values of α_2 and α_3 in section II. Then, we discuss some related thermodynamic properties of black holes in D-dimensional spacetimes. According to the classification of horizon structures, $k = 0$ and $k \pm 1$, we will analyze the stability of black holes in section IV. Section III devotes to conclusions.

II.   BLACK HOLES IN ADS SPACE

The action of third order Lovelock gravity is given by

$$\mathcal{I} = \frac{1}{16\pi G} \int d^D x \sqrt{-g} (R - 2\Lambda + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3), \tag{1}$$

where $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$ is a negative cosmological constant, α_2 and α_3 are Gauss-Bonnet and third order Lovelock coefficients, respectively. In Eq. (1), the Gauss-Bonnet Lagrangian is

$$\mathcal{L}_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

and the third order Lovelock Lagrangian is

$$\begin{aligned}
\mathcal{L}_3 =\ & R^3 + 2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}{}_{\mu\nu} + 8R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\kappa}{}_{\nu\tau}R^{\rho\tau}{}_{\mu\kappa} \\
& + 24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}_{\mu} + 3RR^{\mu\nu\sigma\kappa}R_{\mu\nu\sigma\kappa} \\
& + 24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu} + 16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}{}_{\mu} - 12RR^{\mu\nu}R_{\mu\nu}.
\end{aligned}$$

We assume the metric being of the following form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij}dx^i dx^j, \tag{2}$$

where $h_{ij}dx^i dx^j$ represents the metric of a $(d-2)$-dimensional hypersurfaces with constant curvature scalar $(D-2)(D-3)k$ and volume Σ_k, here k is a constant. Without loss of the generality, one may take $k = 0$ or ± 1.

If we choose

$$\alpha_2 = \frac{\alpha}{(D-3)(D-4)}, \quad \alpha_3 = \frac{\alpha^2}{72\binom{D-3}{4}}, \tag{3}$$

the uncharged black hole solution in D-dimensions is described by [6, 7]

$$f(r) = k + \frac{r^2}{\alpha}[1 - (1 - \frac{3\alpha}{l^2} + \frac{3\alpha m}{r^{D-1}})^{\frac{1}{3}}], \tag{4}$$

This type of action Eq. (1) is derived in the low-energy limit of heterotic superstring theory. Thus, the coefficients α_2 and α_3 are regarded as inverse string tension and positive definite. While, the case for $\alpha_2 < 0$ is also available to study their black holes. Meanwhile, the corresponding one of the third order Lovelock term maintains positive. Here, we consider this case $\alpha = -\tilde{\alpha}$ with $\tilde{\alpha} > 0$. Therefore, the black hole solution Eq. (4) becomes

$$f(r) = k - \frac{r^2}{\tilde{\alpha}}[1 - (1 + \frac{3\tilde{\alpha}}{l^2} - \frac{3\tilde{\alpha}m}{r^{D-1}})^{\frac{1}{3}}], \tag{5}$$

where the gravitational mass M is expressed as $\frac{(D-2)\Sigma_k}{16\pi G}m$. Since the third order Lovelock term in Eq. (1) has no contribution to the field equation in six or less dimensional spacetimes, we consider D-dimensional spacetimes with $D \geq 7$.

On the other hand, the metric Eq. (2) goes to AdS space asymptotically. In the limit of $r \to +\infty$, we only hold back the first two terms of the taylor expansion $(1 + \frac{3\tilde{\alpha}}{l^2} - \frac{3\tilde{\alpha}m}{r^{D-1}})^{\frac{1}{3}}$. The asymptotic form for $f(r)$ is expressed as

$$f_\infty(r) = k - \frac{r^2}{\tilde{\alpha}}[1 - (1 + \frac{3\tilde{\alpha}}{l^2})^{1/3}] - \frac{m}{r^{D-3}(1 + \frac{3\tilde{\alpha}}{l^2})^{2/3}}. \tag{6}$$

In general relativity, the Schwarzchild-AdS black hole solution in D-dimensional spacetimes is [8]

$$f(r) = k + \frac{r^2}{l^2} - \frac{m}{r^{D-3}}. \tag{7}$$

Hence, we can read off the effective cosmological constant and effective gravitational mass

$$\frac{1}{l_{eff}^2} = \frac{1}{\tilde{\alpha}}[(1 + \frac{3\tilde{\alpha}}{l^2})^{1/3} - 1], \quad M_{eff} = \frac{M}{(1 + \frac{3\tilde{\alpha}}{l^2})^{2/3}}. \tag{8}$$

Note that the gravitational mass of black hole is determined by $f(r_+) = 0$. From Eq. (5), the mass can be expressed as in terms of the horizon radius r_+

$$M = \frac{(D-2)\Sigma_k r_+^{D-3}}{16\pi G}(k + \frac{r_+^2}{l^2} - \frac{\tilde{\alpha}k^2}{r_+^2} + \frac{\tilde{\alpha}^2 k}{3r_+^4}). \tag{9}$$

The Hawking temperature associated with the black hole horizon can be obtained by requirement of the absence of conical singularity at the horizon in the Euclidean section of the third order Lovelock black hole solution in AdS spacetimes. According to $T = \frac{f'(r_+)}{4\pi}$, the temperature of black hole is given by

$$T = \frac{1}{12\pi r_+(r_+^2 - k\tilde{\alpha})^2}[\frac{3(D-1)r_+^6}{l^2} + 3(D-3)r_+^4 k$$
$$- 3(D-5)r_+^2\tilde{\alpha}k^2 + (D-7)\tilde{\alpha}^2 k]. \tag{10}$$

Another important thermodynamic quantity is the entropy of black hole. In general relativity, the entropy of black hole satisfy the so-called area formula, namely entropy equals to one quarter of horizon area [9]. However, the area law of entropy is not satisfied in general in higher derivative gravity [7, 10]. While, as a thermodynamic system, the black hole must obey the first law of black hole thermodynamics, $dM = TdS$ and then we have

$$S = \int_0^{r_+} T^{-1}\frac{\partial M}{\partial r_+}dr_+$$
$$= \frac{\Sigma_k r_+^{D-2}}{4G}[1 - \frac{2(D-2)k\tilde{\alpha}}{(D-4)r_+^2} + \frac{(D-2)k^2\tilde{\alpha}^2}{(D-6)r_+^4}], \tag{11}$$

where

$$\frac{\partial M}{\partial r_+} = \frac{(D-2)\Sigma_k r_+^{D-8}}{48\pi G}[\frac{3(D-1)r_+^6}{l^2} + 3k(D-3)r_+^4$$
$$- 3(D-5)r_+^2\tilde{\alpha}k^2 + (D-7)\tilde{\alpha}^2 k]$$
$$= \frac{(D-2)\Sigma_k r_+^{D-7}}{4G}(r_+^2 - k\tilde{\alpha})^2 T. \tag{12}$$

Here, we would like to explore the problem of negative entropy. In Gauss-Bonnet gravity, it has been extensively studies in [11]. The entropy of third order Lovelock black holes Eq. (11) can be rewritten as

$$S = \frac{\Sigma_k r_+^{D-6}}{4G}[r_+^4 - \frac{2(D-2)k\tilde{\alpha}r_+^2}{(D-4)} + \frac{(D-2)k^2\tilde{\alpha}^2}{(D-6)}]. \tag{13}$$

We here note that the sign of entropy is determined by

$$S_{Pa} = r_+^4 - \frac{2(D-2)k\tilde{\alpha}r_+^2}{(D-4)} + \frac{(D-2)k^2\tilde{\alpha}^2}{(D-6)}. \tag{14}$$

Clearly, if $k = 1$, $\tilde{\alpha} < 0$ or $k = -1$, $\tilde{\alpha} > 0$, the function S_{Pa} is always positive. For $k = 1$, $\tilde{\alpha} > 0$ or $k = -1$, $\tilde{\alpha} < 0$, we obtain

$$S_{Pa} = r_+^4 - \frac{2(D-2)\tilde{\alpha}r_+^2}{(D-4)} + \frac{(D-2)\tilde{\alpha}^2}{(D-6)}.$$

It is interesting to mention that the function S_{Pa} is also positive for $r_+ > 0$. As a result, unlike the entropy in Gauss-Bonnet gravity, the entropy of black holes with special coefficient is always positive in third order Lovelock gravity.

III. STABILITY OF TOPOLOGICAL BLACK HOLES

In this section, we perform the stability analysis of topological black holes. The local thermodynamic stability of black hole is determined by the sign of its heat capacity. If the heat capacity is positive, we have that the black hole is locally stable to thermal fluctuations. When the heat capacity is negative, the black hole is said to be locally unstable.

The heat capacity of black holes is

$$C = \frac{\partial M}{\partial T} = (\frac{\partial M}{\partial r_+})(\frac{\partial r_+}{\partial T}), \tag{15}$$

where

$$\frac{\partial T}{\partial r_+} = \frac{1}{12\pi l^2 r_+^2 (r_+^2 - k\tilde{\alpha})^3}[3(D-1)r_+^8 - 3k(5(D-1)\tilde{\alpha} + (D-3)l^2)r_+^6$$
$$- 18\tilde{\alpha}k^2 r_+^4 - 2k(D-10)\tilde{\alpha}^2 r_+^2 + (D-7)\tilde{\alpha}^3 k^2]. \tag{16}$$

Then the expression of heat capacity indicating the local stability of the black hole is obtained

$$
\begin{aligned}
C &= \frac{(D-2)\Sigma_k r_+^{D-6}(r_+^2 - k\tilde{\alpha})^3 J(r)}{4G\Gamma(r)} \\
&= \frac{3(D-2)\pi\Sigma_k l^2 r_+^{D-5}(r_+^2 - k\tilde{\alpha})^5 T/G}{\Gamma(r)},
\end{aligned}
\tag{17}
$$

where $J(r)$ and $\Gamma(r)$ are expressed as $3(D-1)r_+^6 + 3k(D-3)r_+^4 l^2 - 3(D-5)r_+^2 \tilde{\alpha} k^2 l^2 + (D-7)\tilde{\alpha}^2 k l^2$ and $3(D-1)r_+^8 - 3k(5(D-1)\tilde{\alpha} + (D-3)l^2)r_+^6 - 18\tilde{\alpha}k^2 r_+^4 l^2 - 2k(D-10)\tilde{\alpha}^2 r_+^2 l^2 + (D-7)\tilde{\alpha}^3 k^2 l^2$, respectively.

It is clear that those physical properties depend on the horizon structure k and the dimensions of spacetime. Below, we will discuss each case according to the classification of horizon structure $k = 0$ and $k = \pm 1$, respectively.

A. Flat black hole

In case of $k = 0$, we have

$$
\begin{aligned}
M &= \frac{(D-2)\Sigma_k r_+^{D-1}}{16\pi G l^2}, \quad T = \frac{(D-1)r_+}{4\pi l^2}, S = \frac{\Sigma_k r_+^{D-2}}{4G}, \\
C &= \frac{(D-2)\Sigma_k r_+^{D-2}}{4G}, \quad F = -\frac{\Sigma_k r_+^{D-1}}{16\pi G l^2}.
\end{aligned}
\tag{18}
$$

One can find that these thermodynamic quantities are independent of the the Lovelock coefficients, and have the completely same expressions as those in [8]. We conclude that the higher order derivative terms do not affect the thermodynamic properties of black holes although they have different black hole solutions.

B. Hyperbolic black hole

Now, we turn to the case the horizon is a negative constant curvature hypersurface. For $D = 7$, the temperature of black holes Eq. (10) reduces to a simple form

$$
T = \frac{r_+}{2\pi(r_+^2 + \tilde{\alpha})^2}(3r_+^4/l^2 - 2r_+^2 - \tilde{\alpha}).
\tag{19}
$$

The existence of extremal black holes depend on the existence of positive roots for temperature $T = 0$, which reduces to

$$
r_+(3r_+^4/l^2 - 2r_+^2 - \tilde{\alpha}) = 0.
\tag{20}
$$

Then, the largest positive root of this equation which corresponds to radius of extremal black hole is obtained

$$r_{ext} = l[(1 + \sqrt{1 + 3\tilde{\alpha}/l^2})/3]^{1/2}. \tag{21}$$

The black holes solution presents a black hole provided $r_+ > r_{ext}$.

Substituting r_{ext} into the gravitational mass Eq. (9), the mass of extremal black hole is given by

$$
\begin{aligned}
M_{ext} &= \frac{(D-2)\Sigma_k r_{ext}^4}{16\pi G}\left(\frac{r_{ext}^2}{l^2} - 1 - \frac{\tilde{\alpha}}{r_{ext}^2} - \frac{\tilde{\alpha}^2}{3 r_{ext}^4}\right) \\
&= \frac{-(D-2)l^8 \Sigma_k}{16\pi G}\frac{(3\tilde{\alpha}/l^2 + 1)(1 + \sqrt{1 + 3\tilde{\alpha}/l^2})^4}{3^5}.
\end{aligned}
$$

In Fig. 1, we show the temperature T with $\tilde{\alpha} = 0.1$ and 0.2 in seven dimensional spacetimes, respectively. The temperature T always starts from zero at $r_+ = r_{ext}$ and then goes to positive infinity as r_+ increases. Obviously, $\frac{\partial T_H}{\partial r_H}$ Eq. (16) is positive for $r_+ > r_{ext}$. The heat capacity C Eq. (15) is the product of $\frac{\partial M}{\partial r_H}$ and $\frac{\partial r_H}{\partial T_H}$. We note that $\frac{\partial M}{\partial r_H}$ Eq. (12) is always positive for $T > 0$. In the $T = 0$ case, one can see from Eq. (16) that besides $r_+ = 0$, the heat capacity C also vanishes at $r_+ = r_{ext}$. In Fig. 2, we show the graph of heat capacity C versus horizon radius r_+ and the larger root of C corresponds to r_{ext}. Therefore, the heat capacity C is always positive and then black holes are thermodynamically stable.
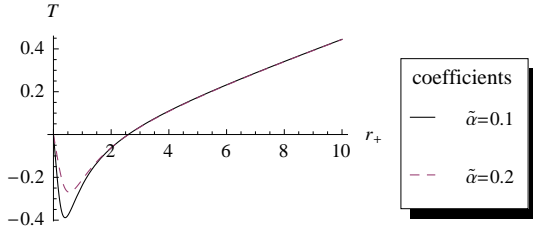

FIG. 1: Temperature T versus horizon radius r_+ with $k = -1$, $l^2 = 10$ and $D = 7$.
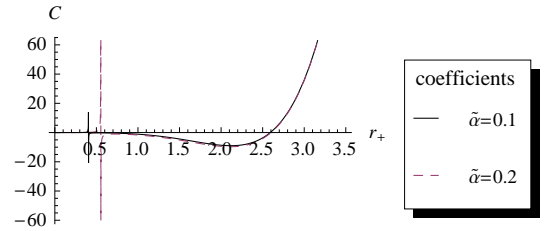
FIG. 2: Heat capacity C versus horizon radius r_+ with $k = -1$, $l^2 = 10$ and $D = 7$.

We can easily extend all of the discussions of the previous subsubsection to D-dimensional solutions. When $D \geq 8$, the existence of extremal black hole depends on equation $T = 0$

$$\frac{3(D-1)r_{ext}^6}{l^2} - 3(D-3)r_{ext}^4 - 3(D-5)r_{ext}^2\tilde{\alpha} - (D-7)\tilde{\alpha}^2 = 0. \tag{22}$$

If ordering $r_{ext}^2 = R_{ext} > 0$, we have a cubic equation

$$R_{ext}^3 - \frac{(D-3)l^2}{(D-1)}R_{ext}^2 - \frac{(D-5)l^2}{(D-1)}\tilde{\alpha}R_{ext} - \frac{(D-7)l^2}{3(D-1)}\tilde{\alpha}^2 = 0. \tag{23}$$

In order to find the exact solution, let

$$a_1 = -\frac{(D-3)l^2}{(D-1)}, \quad a_2 = -\frac{(D-5)l^2}{(D-1)}\tilde{\alpha}, \quad a_3 = -\frac{(D-7)l^2}{3(D-1)}\tilde{\alpha}^2 \qquad (24)$$

and we have

$$\begin{aligned}
Q &= \frac{3a_2 - a_1^2}{9} = -\frac{l^2[3\tilde{\alpha}(D-1)(D-5) + (D-3)^2 l^2]}{9(D-1)^2} \\
P &= \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54} \\
&= l^2 \big[\frac{\tilde{\alpha}^2(D-7)}{6(D-1)} + \frac{\tilde{\alpha}(D-3)(D-5)l^2}{6(D-1)^2} + \frac{(D-3)^3 l^4}{27(D-1)^3} \big].
\end{aligned} \qquad (25)$$

Then, we obtain the discriminant of this cubic equation $\Delta = Q^3 + P^2$

$$\begin{aligned}
\Delta &= \frac{\tilde{\alpha}^2(D-1)}{4l^4}[9(D-7)^2(D-1)\tilde{\alpha}^2 + 6\tilde{\alpha}(D-5)(13 - 10D + D^2)l^2 \\
&+ (D-9)(D-3)^2 l^4] \\
&= \frac{\tilde{\alpha}^2(D-1)}{4l^4}[\tilde{\alpha} + \frac{l^2}{3}][\tilde{\alpha} + \frac{(D-9)(D-3)^2 l^2}{3(D-1)(D-7)^2}].
\end{aligned} \qquad (26)$$

Depending on the choice of the parameter $\tilde{\alpha}$, the discriminant Δ has different signs and disappears at

$$\tilde{\alpha}^{(1)} = -\frac{l^2}{3}, \qquad \tilde{\alpha}^{(2)} = -\frac{(D-9)(D-3)^2 l^2}{3(D-1)(D-7)^2}. \qquad (27)$$

For $\Delta \geq 0$, the solution of the cubic equation Eq. (23) can be written down as

$$\begin{cases}
R_1 = S + T - \frac{a_1}{3}, \\
R_2 = -\frac{1}{2}(S+T) - \frac{a_1}{3} + \frac{1}{2}i\sqrt{3}(S-T), \\
R_3 = -\frac{1}{2}(S+T) - \frac{a_1}{3} - \frac{1}{2}i\sqrt{3}(S-T),
\end{cases}$$

where $S = \sqrt[3]{P + \sqrt{\Delta}}$ and $T = \sqrt[3]{P - \sqrt{\Delta}}$. If $\Delta < 0$, the solution is

$$\begin{cases}
\tilde{R}_1 = 2\sqrt{-Q}\cos(\theta/3) - \frac{a_1}{3}, \\
\tilde{R}_2 = 2\sqrt{-Q}[\cos(\theta/3) + 120°] - \frac{a_1}{3}, \\
\tilde{R}_3 = 2\sqrt{-Q}[\cos(\theta/3) + 240°] - \frac{a_1}{3},
\end{cases}$$

where $\theta = \arccos P/\sqrt{-Q^3}$.

For $D = 8$, $\tilde{\alpha}^{(2)}$ is positive and so there exist two case $0 < \tilde{\alpha} < \tilde{\alpha}^{(2)}$ and $\tilde{\alpha} > \tilde{\alpha}^{(2)}$. As to the latter case $\tilde{\alpha} > \tilde{\alpha}^{(2)}$, we have only one positive real root $r_{ext} = \sqrt{R_1}$ of the cubic equation Eq. (23). For $0 < \tilde{\alpha} < \tilde{\alpha}^{(2)}$, the discriminant Δ is negative and then the cubic

equation has one positive $r_{ext} = \sqrt{\tilde{R}_1}$ and two others negative roots. In case of $D \geq 9$, $\tilde{\alpha}^{(2)}$ is non-positive. Then, the discriminant Δ is positive for arbitrary value of coefficient $\tilde{\alpha}$ and the cubic equation only has one positive root $r_{ext} = \sqrt{\tilde{R}_1}$. Hence, in eight or higher dimensional spacetimes, the cubic equation has a positive root which corresponding to the radius of extremal black hole and the temperature T only vanishes at $r_+ = r_{ext}$. In Fig. 3, the temperature T is shown for $D = 8$ and 9. Taking $l^2 = 10$ and $D = 8$, we have $\tilde{\alpha}^{(2)} \approx 11.90$ and $r_{ext} \approx 2.70$. The graph of heat capacity C is plotted with different parameters $\tilde{\alpha}$ in Fig. 4. Since the equations $\frac{\partial r_H}{\partial T_H}$ and $\frac{\partial M}{\partial r_H}$ always maintain positive for $T > 0$, the heat capacity C is always positive for the temperature $T > 0$ and the black holes are thermodynamically stable.
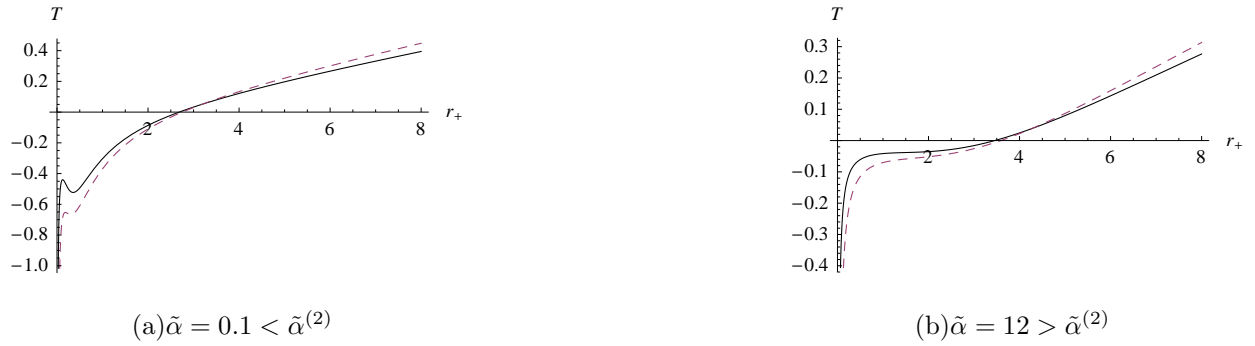

(a)$\tilde{\alpha} = 0.1 < \tilde{\alpha}^{(2)}$ (b)$\tilde{\alpha} = 12 > \tilde{\alpha}^{(2)}$

FIG. 3: Temperature T versus horizon radius r_+ for $k = -1$, $l^2 = 10$, $D = 8$ (solid line) and $D = 9$ (dashed line).

(a)$\tilde{\alpha} = 0.1 < \tilde{\alpha}^{(2)}$ (b)$\tilde{\alpha} = 12 > \tilde{\alpha}^{(2)}$
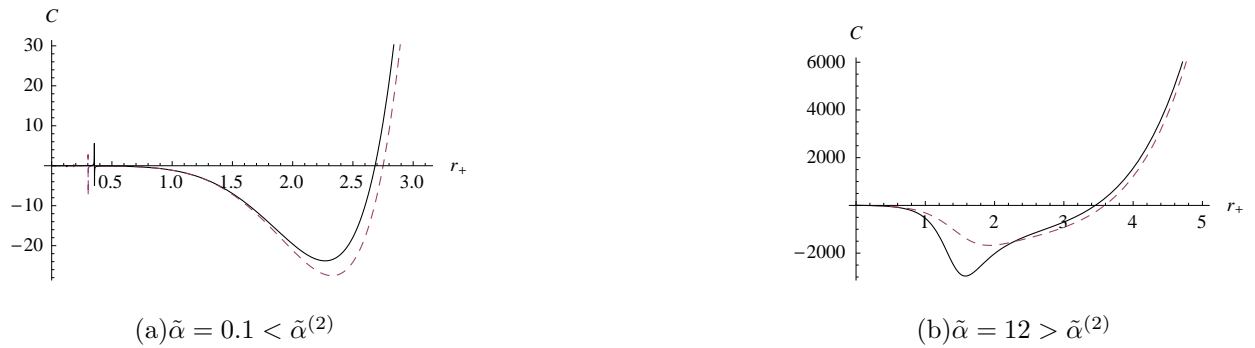
FIG. 4: Heat capacity C versus horizon radius r_+ for $k = -1$, $l^2 = 10$, $D = 8$ (solid line) and $D = 9$ (dashed line).

C.   Spherical black hole

In this subsection we shall explore some physical aspects of the black holes with positive constant curvature hypersurface horizon. From Eq. (8), one can see that there maybe exist a extremal black hole for $T = 0$, which is expressed as

$$r_+(3r_+^4/l^2 + 2r_+^2 - \tilde{\alpha}) = 0, \tag{28}$$

We notice that temperature T has a singularity at $r_+ = \sqrt{\tilde{\alpha}}$. Then, largest root which denotes the horizon radius of the extremal black hole is obtain

$$r_{ext} = l[(-1 + \sqrt{1 + 3\tilde{\alpha}/l^2})/3]^{1/2}. \tag{29}$$

Indeed the black hole solutions present a black hole for $r_+ > r_{ext}$, an extremal black hole if $r_+ = r_{ext}$, and a naked singularity for $r_+ < r_{ext}$. Hence, the gravitational mass of extremal black hole is given by

$$
\begin{aligned}
M_{ext} &= \frac{(D-2)\Sigma_k r_{ext}^4}{16\pi G}\left(1 + \frac{r_{ext}^2}{l^2} - \frac{\tilde{\alpha}}{r_{ext}^2} + \frac{\tilde{\alpha}^2}{3r_{ext}^4}\right) \\
&= \frac{(D-2)\Sigma_k}{3^5 \times 16\pi G}(27\tilde{\alpha}^3 + 9\tilde{\alpha}^2 l^2 + 96\tilde{\alpha}l^4 + 40l^6 \\
&\quad + 36l^4\tilde{\alpha}\sqrt{1 + 3\tilde{\alpha}} + 40l^6\sqrt{1 + 3\tilde{\alpha}}).
\end{aligned}
\tag{30}
$$

(a)$\tilde{\alpha} = 0.1$ (b)$\tilde{\alpha} = 0.2$

FIG. 5:   Temperature T versus horizon radius r_+ with $k = 1$, $l^2 = 10$ and $D = 7$.

In Fig. 5, we show the temperature T versus horizon radius r_+ with different parameters $\tilde{\alpha}$ in seven dimensional spacetimes. For $\tilde{\alpha} = 0.1$, we can obtain the horizon radius $r_{ext} \approx 0.22$. In Fig.6, heat capacity C is plotted with $\tilde{\alpha} = 0.1$. The heat capacity C vanishes at $r_{ext} \approx 0.22$ which corresponding to $T = 0$ and $\sqrt{\tilde{\alpha}} \approx 0.32$. Later, it blows up at $r_c \approx 2.70$ and then changes sign so that C becomes positive. Finally, C gradually goes to positive infinity as $r_+ \to \infty$. Therefore, the black holes in the regions $r_{ext} < r_+ < \sqrt{\tilde{\alpha}}$ and $r_+ > r_c$ are locally stable. For $\sqrt{\tilde{\alpha}} < r_+ < r_c$, black holes are locally unstable.

FIG. 6: Heat capacity C versus horizon radius r_+ with $k = 1$, $l^2 = 10$, $D = 7$, $\tilde{\alpha} = 0.1$ (solid line) and $\tilde{\alpha} = 0.2$ (dashed line).

For $D \geq 8$, the existence of extremal black hole is determined by

$$0 = 3(D-1)r_+^6/l^2 + 3(D-3)r_+^4 - 3(D-5)\tilde{\alpha}r_+^2 + (D-7)\tilde{\alpha}^2. \tag{31}$$

Here we also rewrite it to the form

$$R_+^3 + b_1 R_+^2 + b_2 R_+ + b_3 = 0, \tag{32}$$

where $R_+ = r_+^2$ and

$$b_1 = \frac{(D-3)l^2}{(D-1)}, \quad b_2 = -\frac{(D-5)l^2}{(D-1)}\tilde{\alpha}, \quad b_3 = \frac{(D-7)l^2}{3(D-1)}\tilde{\alpha}^2. \tag{33}$$

Adopted the same approach above, we have

$$Q = \frac{3b_2 - b_1^2}{9} = -\frac{l^2[3\tilde{\alpha}(D-1)(D-5) + (D-3)^2 l^2]}{9(D-1)^2}$$

$$P = \frac{9b_1 b_2 - 27b_3 - 2b_1^3}{54}$$

$$= -l^2\left[\frac{\tilde{\alpha}^2(D-7)}{6(D-1)} + \frac{\tilde{\alpha}(D-3)(D-5)l^2}{6(D-1)^2} + \frac{(D-3)^3 l^4}{27(D-1)^3}\right]. \tag{34}$$

Thus, we can obtain the discriminant of this cubic equation $\Delta = Q^3 + P^2$

$$\Delta = \frac{\tilde{\alpha}^2(D-1)}{4l^4}[9(D-7)^2(D-1)\tilde{\alpha}^2 + 6\tilde{\alpha}(D-5)(13 - 10D + D^2)l^2 + (D-9)(D-3)^2 l^4]$$

$$= \frac{\tilde{\alpha}^2(D-1)}{4l^4}[\tilde{\alpha} + \frac{l^2}{3}][\tilde{\alpha} + \frac{(D-9)(D-3)^2 l^2}{3(D-1)(D-7)^2}]. \tag{35}$$

We find that it is the same as Eq. (26). Therefore, the discriminant Δ vanishes at

$$\tilde{\alpha}^{(1)} = -\frac{l^2}{3}, \qquad \tilde{\alpha}^{(2)} = -\frac{(D-9)(D-3)^2 l^2}{3(D-1)(D-7)^2}. \tag{36}$$

For $\Delta \geq 0$, the solution of the cubic equation Eq. (23) can be written down as

$$
\begin{cases}
R_1 = S + T - \frac{b_1}{3}, \\
R_2 = -\frac{1}{2}(S+T) - \frac{a_1}{3} + \frac{1}{2}i\sqrt{3}(S-T), \\
R_3 = -\frac{1}{2}(S+T) - \frac{a_1}{3} - \frac{1}{2}i\sqrt{3}(S-T),
\end{cases}
$$

where $S = \sqrt[3]{P + \sqrt{\Delta}}$ and $T = \sqrt[3]{P - \sqrt{\Delta}}$. If $\Delta < 0$, the solution is

$$
\begin{cases}
\tilde{R}_1 = 2\sqrt{-Q}\cos(\theta/3) - \frac{b_1}{3}, \\
\tilde{R}_2 = 2\sqrt{-Q}[\cos(\theta/3) + 120°] - \frac{b_1}{3}, \\
\tilde{R}_3 = 2\sqrt{-Q}[\cos(\theta/3) + 240°] - \frac{b_1}{3},
\end{cases}
$$

where $\theta = \arccos P/\sqrt{-Q^3}$.

For $D = 8$, there are two case $0 < \tilde{\alpha} < \tilde{\alpha}^{(2)}$ and $\tilde{\alpha} > \tilde{\alpha}^{(2)}$. For $0 < \tilde{\alpha} < \tilde{\alpha}^{(2)}$, the discriminant Δ is negative and then this cubic equation has three roots: two positive $r_> = \sqrt{\tilde{R}_1}$, $r_< = \sqrt{\tilde{R}_2}$ and one negative $\sqrt{\tilde{R}_3}$. The larger positive one $r_>$ corresponds to the horizon radius of extremal black hole r_{ext}. However, for $\tilde{\alpha} > \tilde{\alpha}^{(2)}$, the discriminant Δ is positive and we also have a real which is negative. It is suggested that the temperature T is always positive for the whole range $r_+ \geq 0$ and there is no extremal black hole. In case of $D \geq 9$, the discriminant Δ is always positive. We find that this real one is negative and the temperature of black hole is always positive for $D \geq 9$. Hence, the extremal black holes only exist for $0 < \tilde{\alpha} < \tilde{\alpha}^{(2)}$ in eight dimensional spacetimes. In Fig.7, we plot the temperature versus horizon radius r_+ for $D = 8$, 9 and 10, respectively and the details are shown in Fig. 8. For $l^2 = 10$ and $D = 8$, we have $\tilde{\alpha}^{(2)} = 11.90$.
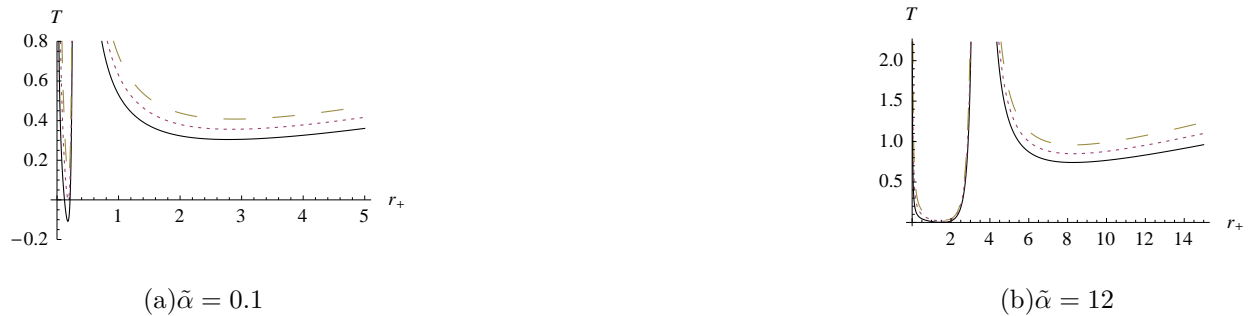

(a)$\tilde{\alpha} = 0.1$ (b)$\tilde{\alpha} = 12$

FIG. 7: Temperature T versus horizon radius r_+ with $l^2 = 10$ and $k = 1$. These three curves from up to down correspond to $D = 10$, $D = 9$ and $D = 8$, respectively.

Here, let us perform the stability analysis of black holes in eight dimensional spacetimes. First we consider the black holes with coefficient $\tilde{\alpha} < \tilde{\alpha}^{(2)}$. The heat capacity C Eq. (17)
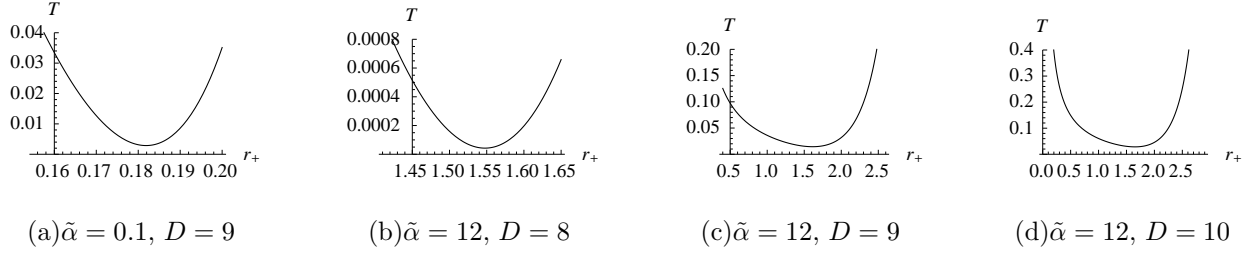
(a)$\tilde{\alpha} = 0.1$, $D = 9$ (b)$\tilde{\alpha} = 12$, $D = 8$ (c)$\tilde{\alpha} = 12$, $D = 9$ (d)$\tilde{\alpha} = 12$, $D = 10$

FIG. 8: Temperature T versus horizon radius r_+ with $l^2 = 10$ and $k = 1$.

(a)$0 < r_+ < 0.39$ (b)$0.39 < r_+ < 3.4$

FIG. 9: Heat capacity C versus horizon radius r_+ with $\tilde{\alpha} = 0.1$, $l^2 = 10$ and $D = 8$.

disappears at $r_+ = r_{ext}$ and $r_+ = \sqrt{\tilde{\alpha}}$. In Fig.9, we show heat capacity C versus horizon radius r_+ with $\sqrt{\tilde{\alpha}} = 0.1$. It shows that these two larger roots correspond to extremal horizon r_{ext} and $\sqrt{\tilde{\alpha}}$ and C is always positive in the region $r_{ext} < r_+ < \sqrt{\tilde{\alpha}}$ and the black holes are always local stable. If taking $l^2 = 10$ and $\tilde{\alpha} = 0.1$, we can obtain the extremal radius $r_{ext} \approx 0.22$. For $r_+ > \sqrt{\tilde{\alpha}}$, the heat capacity C becomes negative in the region $r_+ > \sqrt{\tilde{\alpha}}$ and gradually blows up at $r_+ = r_{cs}$ (say) and then changes sign becomes positive. Hence, black holes are local unstable in the domain $\sqrt{\tilde{\alpha}} < r_+ < r_{cs}$ and are stable for $r_+ > r_{cs}$. For $\tilde{\alpha} > \tilde{\alpha}^{(2)}$, the heat capacity C versus horizon radius r_+ is plotted with $\tilde{\alpha} = 12$ in Fig.10. The graph shows that C starts from zero, for small r_+, C is negative and gradually blows

(a)$0 < r_+ < 5$ (b)$5 < r_+ < 11$

FIG. 10: Heat capacity C versus horizon radius r_+ with $\tilde{\alpha} = 12$, $l^2 = 10$ and $D = 8$.

up at $r_{\tilde{c}1}$ and then becomes positive. Again C vanishes at $r_+ = \sqrt{\tilde{\alpha}}$ and becomes negative. Then it blows up at $r_{\tilde{c}2}$ and change sign so that becomes positive. The black holes are local stable for $r_{\tilde{c}1} < r_+ < \sqrt{\tilde{\alpha}}$ and $r_+ > r_{\tilde{c}2}$, but are unstable in the domain $r_+ < r_{\tilde{c}1}$ and $\sqrt{\tilde{\alpha}} < r_+ < r_{\tilde{c}2}$.

For $D \geq 9$, the temperature is always positive and there don't exist extremal black hole. In Fig 11, heat capacity C demonstrates that for small r_+, C is negative and gradually blows up at $r_+ = r_{c_*1}$ (say) and then change sign so that C becomes positive. Then, it vanishes at $r_+ = \sqrt{\tilde{\alpha}}$ and becomes negative. Again it blows up at $r_+ = r_{c_*2}$ and then becomes positive. Hence, in the region $0 < r_+ < r_{c_*1}$ and $\sqrt{\tilde{\alpha}} < r_+ < r_{c_*2}$, black holes are locally unstable. For $r_{c_*1} < r_+ < \sqrt{\tilde{\alpha}}$ and $r_{c_*2} < r_+$, black holes are locally stable. If taking $\tilde{\alpha} = 0.1$ and $D = 9$, we obtain $r_{c_*1} \approx 0.18$, $\sqrt{\tilde{\alpha}} \approx 0.31$ and $r_{c_*2} \approx 2.90$.
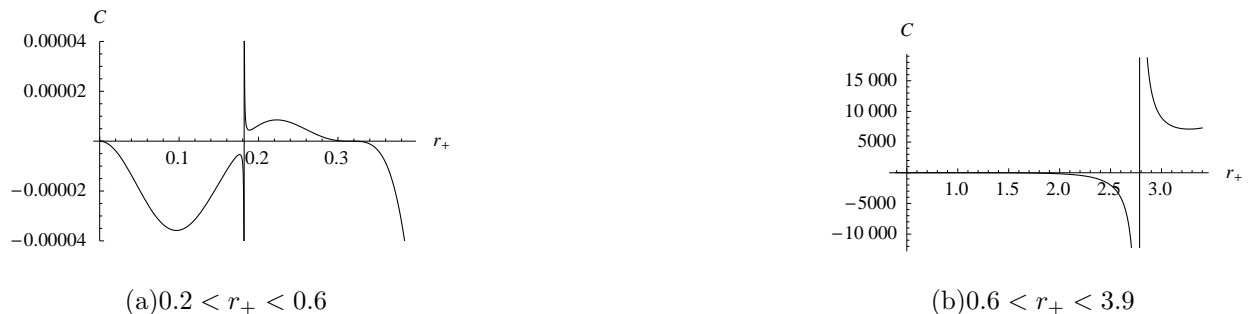

(a)$0.2 < r_+ < 0.6$ (b)$0.6 < r_+ < 3.9$

FIG. 11: Heat capacity C versus horizon radius r_+ with $l^2 = 10$, $D = 9$ and $\tilde{\alpha} = 0.1$

IV. CONCLUSIONS

By considering the coefficient $\hat{\alpha}_2 = \alpha$ and $\hat{\alpha}_3 = \alpha^2/3$, we studied the case $\hat{\alpha}_2 = \alpha < 0$ and $\hat{\alpha}_3 = \alpha^2/3 > 0$ and then presented the asymptotically AdS black hole solutions in third order Lovelock gravity. Later, we discussed the thermodynamic properties of black holes including gravitational mass, Hawking temperature and entropy of black holes and performed the stability analysis of these topological black holes.

For $k = 0$, all the thermodynamic and conserved quantities of the black holes don't depend on the Lovelock coefficients and are the same as those of black holes in Einstein gravity although the two black hole solutions are quit different. For the horizon is negative constant hypersurface, the thermodynamics of the black holes with Gauss-Bonnet and third order Lovelock terms are qualitatively similar to those of black holes without these higher derivative terms. The third order Lovelock black holes are thermodynamically stable for

the whole range r_+. For the positive constant hypersurface horizon, when $D = 7$, there exist extremal black holes and we found that the black hole has an intermediate unstable phase. In eight dimensional spacetimes, however, a new phase of thermodynamically unstable small black holes appears if the coefficient $\tilde{\alpha}$ is under a critical value. Simultaneously, the extremal black holes also vanishes. For $D \geq 9$, black holes have similar the distributions of thermodynamically stable regions to the case $D = 8$ when the coefficient $\tilde{\alpha}$ is under a critical value.

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