Inducing Barbero-Immirzi Connections along SU(2)-reductions of Bundles on Spacetime*

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Abstract: We shall present here a general apt technique to induce connections along bundle reductions which is different from the standard *restriction*. The technique is a generalization of the mechanism presented in [**1**] to define at spacetime level the Barbero-Immirzi (BI) connection used in LQG.

The general prescription to define such a *reduced connection* is interesting from a mathematical viewpoint and it allows a general and direct control on transformation laws of the induced object. Moreover, unlike what happens by using standard restriction, we shall show that once a bundle reduction is given, then *any* connection induces a *reduced connection* with no constraint on the original holonomy as it happens when connections are simply *restricted*.

1. Introduction

Barbero-Immirzi (BI) connection is used in LQG to describe gravitational field on space; see [**2**], [**3**]. In standard literature it is obtained by a canonical transformation on the phase space of the spatial Hamiltonian system describing classical GR; see [**4**].

Samuel argued that there is no spacetime connection which *restricts* to BI connection due to holonomy considerations; see [**5**]. Thiemann claimed (see [**6**]) that all it is needed for the theory to make sense is the definition of the connection on space, while Samuel and others would privilege spacetime objects. Despite we partially agree with Thiemann's point of view, we have to remark that even when bundle topologies are assumed to be trivial and there is no issue about objects' globality, still transformation laws are essential for the interpretation of the theory. In these trivial situations transformation laws are not used to obtain globality, but they are used for *covariance*. For the object defined to be called BI-connection it must trasform as a SU(2)-connection, though transformation laws are inherited by the original spin connections and cannot be *imposed* at will. Moreover, one has to define the SU(2)-gauge transformations as a subgroup of the original Spin(3, 1)-gauge transformations and such a subgroup must be defined canonically, i.e. in a gauge and observer-independent fashion.

This is particularly evident when one considers that the BI connections are then described by means of their holonomy; of course holonomies are motivated and meaningful only for connections and one could not be satisfied with a generic spatial field which resembles a SU(2)-connection but has different transformation laws. If the action of the gauge group is modified then the holonomies are not necessarily gauge covariant quantities any longer. On the other hand, if gauge covariance is abandoned the hole argument (see [**4**]) is compromised and the

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physical observables of the theory (together with its interpretation) are compromized, too.

For these reasons we have investigated a possible construction to define BI connection keeping gauge covariance under full control; see **[1]**. The construction is based on the existence of a SU(2)-reduction of the original principal spin bundle P. A *SU(2)-reduction* is a pair $({}^+P, \iota)$ where ${}^+P$ is a SU(2)-bundle and $\iota : {}^+P \to P$ a (vertical) principal morphism with respect to the canonical group embedding $i : \mathrm{SU}(2) \to \mathrm{Spin}(3,1)$:

$$\begin{tikzcd} {}^+P \arrow[r, "\iota"] \arrow[d] & P \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{1.1}$$

In standard situations, when spacetimes are required to allow global Lorentzian metrics and global spinors (that is equivalent to require that first and second Stiefel-Whitney classes vanish) such a reduction can be shown to exist always (see **[7]**) with no further topological obstruction. For a simplified situation, when we can imagine the spin bundle P to be trivial, the reduction always exists and the reduced bundle ${}^+P$ is also trivial.

The SU(2)-reduction defines a *canonical* embedding of SU(2)-gauge transformations (namely, $\mathrm{Aut}({}^+P)$) into the Spin(3,1)-gauge transformations (namely, $\mathrm{Aut}(P)$). One can now consider a spin connection ω on P. This cannot always be *restricted* to ${}^+P$. To be able to restrict the connection ω to the sub-bundle $\iota({}^+P) \subset P$, ω-horizontal spaces must happen to be tangent to the sub-bundle itself. Of course, this is a condition on ω for it being restrictable; a trivial necessary condition for this is that the holonomy of the original connection ω happens to get value in the subgroup $\mathrm{SU}(2) \subset \mathrm{Spin}(3,1)$ in the first place. Hence there are spin connections that cannot be restricted (see **[5]**; we thank Smirnov for addressing our attention on this point **[8]**).

In **[1]** we proposed a *different* prescription to induce a SU(2)-connection A on ${}^+P$ out of the spin connection ω on P; despite this presciption is not canonical (and below we shall describe exactly in which sense it is not) it is *generic*; all spin connections ω induce a *reduced* connection A on ${}^+P$, in particular with no restriction on holonomies.

To summarize, we showed in **[1]** that one can define a SU(2)-connection on ${}^+P$, i.e. over spacetime. This connection can be then restricted to space to obtain the standard BI connection. However, the spacetime reduced SU(2)-connection is not the restriction of a spin connection on spacetime and its holonomy is not necessarily dictated by the original spin connection (which therefore is not required to be in SU(2) as argued instead in **[5]** and **[8]**).

This paper is organized as follows: in Section 2 we shall define the reduction prescription from a more general point of view with respect to what we did in **[1]**. In Section 3 we shall obtain the BI prescription defined in **[1]** from our new and more general point of view. In Section 4 we shall start discussing BI connection in any dimension $m > 2$.

Hoping it could help readers who are approaching these issues for the first time we add a detailed derivation of some of the algebraic facts. If the reader wishes to skip these details, recompile this TEX sourcefile uncommenting (just above the title) the command `\CollapseAllCNotes`.
We shall use below the following typographic conventions: paired color terms cancel out, underlined terms are similar (or to be collected together), framed terms are zero.

2. Induced Connections along Reductions and Reductive Algebras

In this Section we shall consider the algebraic structures that enable us to reduce the connections. Let us consider a principal bundle P with group G and a subgroup $i : H \to G$. Let us then assume and fix any *H-reduction* (Q, ι) given by

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & P \\ \downarrow & & \downarrow \\ M & = & M \end{array} \tag{2.1}$$

The group embedding $i : H \to G$ induces an algebra embedding $T_e i : \mathfrak{h} \to \mathfrak{g}$. Let us define the vector space $V = \mathfrak{g}/\mathfrak{h}$ so to have the short squence of vector spaces

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r, "T_e i"] & \mathfrak{g} \arrow[r, "p"] & V \arrow[r] \arrow[l, bend left, dashed, "\Phi"] & 0 \end{tikzcd} \tag{2.2}$$

where $\Phi : V \to \mathfrak{g}$ is a sequence splitting (i.e. $p \circ \Phi = \mathrm{id}_V$) which always exists for sequences of vector spaces. Accordingly, one has $\mathfrak{g} = \mathfrak{h} \oplus \Phi(V)$.

We say that H is *reductive* in G if there is an action $\lambda : H \times V \to V$ such that $\mathrm{ad}(h)(\Phi(v)) \equiv \Phi \circ \lambda(h, v)$ where $\mathrm{ad} : H \times \mathfrak{g} \to \mathfrak{g}$ is the restriction to the subgroup H of the *adjoint action* of G on its algebra $\mathfrak{g}$; see **[9]**, **[10]**, **[11]**. In other words, the subspace $\Phi(V) \subset \mathfrak{g}$ is invariant with respect to the adjoint action of $H \subset G$ on the algebra $\mathfrak{g}$.

Let us stress that the vector subspace $\Phi(V) \subset \mathfrak{g}$ is not required to be (and often it is not) a subalgebra; accordingly, one is not choosing any group splitting $G = H \times K$ (as for example it happens (incidentally) in the case of the (anti)selfdual decomposition $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$). A group splitting (and the corresponding projection) is not at all used; one just needs the group embedding $i : H \to G$.

We shall show hereafter that a bundle H-reduction $\iota : Q \to P$ with respect to a subgroup H reductive in G is enough to allow that *each* G-connection ω on P induces an H-connection on Q, which will be called the *reduced connection*.

Let us consider a G-connection ω on P locally given by

$$\omega = dx^\mu \otimes \left(\partial_\mu - \omega^A_\mu(x)\rho_A\right) \tag{2.3}$$

where ρ_A is the pointwise basis for vertical right invariant vector fields on P associated to a basis T_A of the Lie algebra $\mathfrak{g}$; see **[12]** for notation.

Resorting to the algebra splitting one can consider an *adapted basis* $T_A = (T_i, T_\alpha)$, T_i being a basis of $\mathfrak{h}$ and T_α a basis of $\Phi(V)$. The corresponding basis of vertical right invariant vector fields on P splits as $\rho_A = (\rho_i, \rho_\alpha)$.

In view of the reductive splitting of the algebras, for any H-gauge transformation $\varphi : U \to H$, one has

$$(\rho'_i, \rho'_\alpha) \equiv \rho'_A = \mathrm{ad}^B_A(\varphi)\rho_B \equiv (\mathrm{ad}^j_i(\varphi)\rho_j, \lambda^\beta_\alpha(\varphi)\rho_\beta) \tag{2.4}$$

Accordingly, the G-connection can be splitted as

$$\omega = dx^\mu \otimes \left(\partial_\mu - \omega^i_\mu(x)\rho_i\right) \oplus (-\omega^\alpha_\mu(x)dx^\mu \otimes \rho_\alpha) \tag{2.5}$$

Since ρ_i transform with respect to the adjoint representation of H and ρ_α transform wrt to the representation λ, then the quantities

$$A = dx^\mu \otimes \left(\partial_\mu - \omega^i_\mu(x)\rho_i\right) \qquad K = -\omega^\alpha_\mu(x)dx^\mu \otimes \rho_\alpha \tag{2.6}$$

are (modulo trivial and canonical isomorphisms) an H-connection on Q and a vector valued 1-form on Q, respectively.

In the following Section we shall show how the standard BI connection can be obtained in this framework as done in [**1**].

Let us stress that, once the H-reduction is assumed and the corresponding splitting is shown to be reductive, then *all* connections ω of P induce a H-connection A on Q, in particular with no holonomy constraints.

As argued in [**8**], torsionless connections obey severe constraints on possible holonomies they can have; see [**13**], [**14**]. These results do not directly apply to gauge connections (and spin connections in particular); however, when a frame is considered, as it is done in LQG, spin connections induce also spacetime connections which *are* in fact constrained in their possible holonomies, so that one could eventually consider this as a constraint on the holonomy of the original spin connection. Since GR field equations imply torsionless spin connections, then a potential issue can be considered:

> *can torsionless Spin*$(1,3)$*-connections (among which all solutions of GR) induce Spin*(3)*-connections when the holonomy group Spin*(3) *is forbidden by the classification?*

The answer is in the negative if Spin(3)-connections are induced by *restriction*. But it is in the positive if Spin(3)-connections are induced by *reduction* as above.

Of course, one could argue that the existence of bundle reduction and the reductive splitting is a constraint equivalent to the one on the holonomies. However, this is not the case; one can consider the subgroup $i : \mathrm{SU}(2) \to \mathrm{Spin}(3,1)$ which is in fact *reductive* (as we shall show below). If the spin bundle P considered is trivial then there is no topological obstruction to the existence of the reduction $\iota : {}^+P \to P$. In this situation all hypotheses about the prescription for reduced connections are satisfied and *each* $\mathrm{Spin}(3,1)$-connection induces a reduced SU(2)-connection, included the torsionless connections which cannot be *restricted* in view of the constraints on holonomy.

3. An Example: i : SU(2) $\to$ Spin(3, 1)

The group $\mathrm{Spin}(3,1)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$ which is a sort of *complexification* of SU(2) that is identified accordingly as a real section $i : \mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C})$.

The corresponding algebra of $\mathfrak{sl}(2,\mathbb{C})$ is spanned (on $\mathbb{R}$) by (τ_i, σ_i) where σ_i are standard Pauli matrices and $\tau_i = i\sigma_i$. An element of $\mathfrak{sl}(2,\mathbb{C})$ is thence in the form $\xi = \xi^i_{(1)}\tau_i + \xi^i_{(2)}\sigma_i$ and the algebra embedding is given by

$$T_e i : \mathfrak{su}(2) \to \mathfrak{sl}(2,\mathbb{C}) : \xi^i\tau_i \mapsto \xi^i\tau_i \tag{3.1}$$

The quotient V is spanned by σ_i and the splitting of the algebra sequence can be fixed as

$$\Phi : V \to \mathfrak{sl}(2, C) : \sigma_i \mapsto \sigma_i + \gamma\tau_i \qquad (\gamma \in \mathbb{R}) \tag{3.2}$$

which is in fact always transverse to $\mathfrak{su}(2) \subset \mathfrak{sl}(2, \mathbb{C})$.

One can easily show that such a splitting is reductive and the representation $\lambda : \mathrm{SU}(2) \to \mathrm{SO}(3)$ coincides with the standard covering map exhibiting the group $\mathrm{SU}(2)$ as the double covering of the orthogonal group $\mathrm{SO}(3)$ on space.

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Let us consider $S = a_0\mathbb{I} + a^i\tau_i \in \mathrm{SU}(2)$, which is obtained by $a_0, a^i \in \mathbb{R}$ with $(a_0)^2 + |\vec{a}|^2 = 1$ and set $\gamma\tau_k + \sigma_k = e_k$. We have to compute

$$\begin{aligned}
Se_kS^{-1} =&(a_0\mathbb{I} + a^i\tau_i)(\gamma\tau_k + \sigma_k)(a_0\mathbb{I} - a^j\tau_j) = \\
=&(a_0\mathbb{I} + a^i\tau_i)(\gamma a_0\tau_k + a_0\sigma_k - \gamma a^j(-\epsilon_{kj}{}^l\tau_l - \delta_{kj}\mathbb{I}) - a^j(-\epsilon_{kj}{}^l\sigma_l + i\delta_{kj}\mathbb{I})) = \\
=&(a_0\mathbb{I} + a^i\tau_i)(\gamma(a_0\delta^l_k + a^j\epsilon_{kj}{}^l)\tau_l + (a_0\delta^l_k + a^j\epsilon_{kj}{}^l)\sigma_l + (\gamma - i)a^{\cdot}_{k}\mathbb{I}) = \\
=&\gamma(a_0a_0\delta^l_k + a_0a^j\epsilon_{kj}{}^l)\tau_l + (a_0a_0\delta^l_k + a_0a^j\epsilon_{kj}{}^l)\sigma_l + (\gamma - i)a_0a^{\cdot}_{k}\mathbb{I}+ \\
&+\gamma(a_0a^i\delta^l_k + a^ia^j\epsilon_{kj}{}^l)(-\epsilon_{il}{}^n\tau_n - \delta_{il}\mathbb{I}) + i(a_0a^i\delta^l_k + a^ia^j\epsilon_{kj}{}^l)(i\epsilon_{il}{}^n\sigma_n + \delta_{il}\mathbb{I}) + (\gamma - i)a^ia^{\cdot}_{k}\tau_i = \\
=&\gamma((a_0)^2\delta^l_k + a_0a^j\epsilon_{kj}{}^l)\tau_l + ((a_0)^2\delta^l_k + a_0a^j\epsilon_{kj}{}^l)\sigma_l + (\gamma - i)a_0a^{\cdot}_{k}\mathbb{I}+ \\
&-\gamma a_0a^i\epsilon_{ik}{}^n\tau_n - \gamma a_0a^{\cdot}_{k}\mathbb{I} + \gamma a^ia^j(\delta_{ki}\delta^n_j - \delta^n_k\delta_{ji})\tau_n - \boxed{\gamma a^ia^j\epsilon_{kji}\mathbb{I}}+ \\
&-a_0a^i\epsilon_{ik}{}^n\sigma_n + ia_0a^{\cdot}_{k}\mathbb{I} + a^ia^j(\delta_{ki}\delta^n_j - \delta^n_k\delta_{ji})\sigma_n + \boxed{ia^ia^j\epsilon_{kji}\mathbb{I}} + (\gamma - i)a^{\cdot}_{k}a^i\tau_i = \\
=&(\gamma - i)((a_0)^2\delta^l_k + \underline{\underline{a_0a^j\epsilon_{kj}{}^l}})\tau_l + (\gamma - i)a_0a^{\cdot}_{k}\mathbb{I}+ \\
&-\underline{\underline{(\gamma - i)a_0a^i\epsilon_{ik}{}^n\tau_n}} - (\gamma - i)a_0a^{\cdot}_{k}\mathbb{I} + \underline{(\gamma - i)a^{\cdot}_{k}a^j\tau_j} - (\gamma - i)|\vec{a}|^2\tau_k + \underline{(\gamma - i)a^{\cdot}_{k}a^i\tau_i} = \\
=&\left[\left((a_0)^2 - |\vec{a}|^2\right)\delta^j_k - 2a_0a^i\epsilon_{ik}{}^j + 2a^{\cdot}_{k}a^j\right](\gamma\tau_j + \sigma_j) = \lambda^l_k(S)e_l
\end{aligned} \tag{3.3}$$

This shows how $S(\gamma\tau_k + \sigma_k)S^{-1} \in V$, hence the splitting is reductive and the representation λ is given by

$$\lambda : \mathrm{SU}(2) \times V \to V : (S, e_k) \mapsto \lambda^l_k(S)e_l \tag{3.4}$$

where in view of (3.3) one has

$$\lambda^l_k(S) = \begin{pmatrix} (a_0)^2 + (a^1)^2 - (a^2)^2 - (a^3)^2 & 2(a^{\cdot}_{1}a^2 - a_0a^3) & 2(a^{\cdot}_{1}a^3 + a_0a^2) \\ 2(a^{\cdot}_{1}a^2 + a_0a^3) & (a_0)^2 - (a^1)^2 + (a^2)^2 - (a^3)^2 & 2(a^{\cdot}_{2}a^3 - a_0a^1) \\ 2(a^{\cdot}_{1}a^3 - a_0a^2) & 2(a^{\cdot}_{2}a^3 + a_0a^1) & (a_0)^2 - (a^1)^2 - (a^2)^2 + (a^3)^2 \end{pmatrix} \tag{3.5}$$

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Let us also stress that $\Phi(V)$ in this case is not a subalgebra.

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It is sufficient to show that V is not closed with respect to commutators. For example (assuming $\gamma \neq 0$) one has:

$$\begin{aligned}
[\sigma_1 + \gamma\tau_1, \sigma_2 + \gamma\tau_2] =&[\sigma_1, \sigma_2] + 2\gamma[\tau_1, \sigma_2] + \gamma^2[\tau_1, \tau_2] = \\
&=2(\gamma^2 + 1)\tau_3 - 4\gamma\sigma_3 = -2\gamma\left(\tfrac{-1}{\gamma}\tau_3 + \sigma_3\right) - 2\gamma(\sigma_3 + \gamma\tau_3)
\end{aligned} \tag{3.6}$$

The result is not in $\Phi(V)$ unless one has $-\frac{1}{\gamma} = \gamma$ (i.e. $\gamma^2 + 1 = 0$).

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The basis σ_{ab} of vertical right invariant vector fields is given by the following identification with the algebra

$$\begin{array}{ccc} -4\sigma_{12} = \tau_3 & 4\sigma_{13} = \tau_2 & -4\sigma_{23} = \tau_1 \\ 4\sigma_{01} = \sigma_1 & 4\sigma_{02} = \sigma_2 & 4\sigma_{03} = \sigma_3 \end{array} \tag{3.7}$$

as one can check by computing commutators of fields σ_{ab} (see Appendix A for notation). Hence the basis of $\Phi(V)$ is $e_k = 4\left(\sigma_{0k} + \frac{\gamma}{2}\epsilon_k{}^{ij}\sigma_{ik}\right)$, i.e.

$$e_1 = 4\left(\sigma_{01} + \gamma\sigma_{23}\right) \quad e_2 = 4\left(\sigma_{02} - \gamma\sigma_{13}\right) \quad e_3 = 4\left(\sigma_{03} + \gamma\sigma_{12}\right) \tag{3.8}$$

Then we can split a generic connection

$$\begin{aligned} \omega =& dx^\mu \otimes \left(\partial_\mu - \omega^{ab}_\mu \sigma_{ab}\right) = dx^\mu \otimes \left(\partial_\mu - \omega^{ij}_\mu\sigma_{ij} - 2\omega^{0i}_\mu \sigma_{0i}\right) = \\ =& dx^\mu \otimes \left(\partial_\mu - \omega^{ij}_\mu\sigma_{ij} - 2\omega^{0i}_\mu(\sigma_{0i} \pm \tfrac{\gamma}{2}\epsilon_i{}^{jk}\sigma_{jk})\right) = \\ =& dx^\mu \otimes \left(\partial_\mu - (\omega^{jk}_\mu + \gamma\omega^{0i}_\mu\epsilon_i{}^{jk})\sigma_{jk}\right) - \tfrac{1}{2}\omega^{0i}_\mu e_i \end{aligned} \tag{3.9}$$

Hence one can define

$$A^i_\mu = \tfrac{1}{2}\epsilon^i{}_{jk}A^{jk}_\mu = \tfrac{1}{2}\epsilon^i{}_{jk}\omega^{jk}_\mu + \gamma\omega^{0i}_\mu \qquad\qquad K^i_\mu = -\tfrac{1}{2}\omega^{0i}_\mu \tag{3.10}$$

According to the general theory, A^i_μ is a SU(2)-connection and K^i_μ is a Lie algebra valued 1-form; this can be easily seen by a direct calculation.

Now that we have reproduced the results of [1], we are ready to show that the ones considered are the *only* reductive splittings. A generic splitting is in fact $\Phi: V \to \mathfrak{sl}(2,\mathbb{C}) : \sigma_i \mapsto \sigma_i + \beta^j_i\tau_j$. If one imposes reductivity one easily finds the condition

$$\beta^m_i\delta_{jk} = \delta_{lj}\delta^m_i\beta^l_k \tag{3.11}$$

that is satisfied if and only if $\beta^j_i = \gamma\delta^j_i$.

Let us set $e_k = \sigma_k + \beta^i_k\tau_i$. Following the line of the proof of reductivity given above one can easily show that

$$S\cdot e_k \cdot S^{-1} = \lambda^j_k(S)e_j + 2a^ia^j(\delta^m_i\beta^l_k\delta_{lj} - \beta^m_j\delta_{ki})\tau_m + 2a_0a^j(\beta^l_k\epsilon_{lj}{}^m - \epsilon_{kj}{}^n\beta^m_n)\tau_m \tag{3.12}$$

Since the span of $(\tau_n, n=1,2,3)$ is transverse to $\Phi(V)$ which is spanned by $(e_k : k = 1,2,3)$ the extra terms must vanish for all $S \in \mathrm{SU}(2)$.
Hence one must have

$$\begin{cases} \delta^m_{(i}\delta_{j)l}\beta^l_k = \beta^m_{(j}\delta_{i)k} \\ \beta^l_k\epsilon_{lj}{}^m = \epsilon_{kj}{}^n\beta^m_n \quad \Rightarrow \epsilon^i{}_h{}^j\left(\beta^m_{[j}\delta_{i]k} - \delta^m_{[j}\delta_{i]l}\beta^l_k\right) = 0 \end{cases} \qquad \Rightarrow \beta^m_j\delta_{ik} = \delta^m_j\delta_{il}\beta^l_k \tag{3.13}$$

which proves equation (3.11). By tracing (3.11) wrt the indices (im) one has

$$\beta\delta_{jk} = 3\delta_{lj}\beta^l_k \quad \Rightarrow \beta^j_k = \tfrac{\beta}{3}\delta^j_k \tag{3.14}$$

4. Barbero-Immirzi Connections in Dimension $m > 2$

Let us consider here spacetimes with dimension $m \equiv n+1 > 2$; the relevant spin groups are $\mathrm{Spin}(n)$ for space and $\mathrm{Spin}(n,1)$ for spacetime. Here both the groups are thought as embedded within their relevant Clifford algebra; see **[12]**. The even Clifford algebras (where the groups' Lie algebras are embedded) are spanned by even products of Dirac matrices, here denoted by $\mathbb{I}, E_{\alpha\beta}, E_{\alpha\beta\gamma\delta}, \dots$ with $\alpha, \beta \dots = 0..n$. The Clifford algebras are suitably embedded one into the other by

$$i_0 : \mathcal{C}(n) \to \mathcal{C}(n,1) : E_{i_1 \dots i_{2l}} \mapsto E_{i_1 \dots i_{2l}} \tag{4.1}$$

with $i_1, i_2 \dots = 1..n$. In other words, the lower dimensional Clifford algebra $\mathcal{C}(n)$ is realized within the higher dimensional one $\mathcal{C}(n,1)$ by means of even products of Dirac matrices, except E_0. Such an algebra embedding restricts to a group embedding

$$i : \mathrm{Spin}(n) \to \mathrm{Spin}(n,1) \tag{4.2}$$

The corresponding covering maps allow to define the embedding of $i : \mathrm{SO}(n) \to \mathrm{SO}(n,1)$ which corresponds to rotations that fix the time axes, i.e.

$$\ell_i : \mathrm{SO}(n) \to \mathrm{SO}(n,1) : \lambda \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \tag{4.3}$$

We have to show that such an embedding is reductive. For this, let us consider the sequence

$$\begin{tikzcd} 0 \arrow[r] & so(n) \arrow[r, "T_e i"] & so(n,1) \arrow[r, "p"] & V \arrow[l, bend left, dashed, "\Phi"] \arrow[r] & 0 \end{tikzcd} \tag{4.4}$$

The complement vector space V is spanned by E_{0i} and we fix the splitting by setting

$$\Phi : V \to so(n,1) : E_{0i} \mapsto E_{0i} + \tfrac{1}{2}\beta_i{}^{jk} E_{jk} \tag{4.5}$$

One can write down the condition for which such a splitting is reductive, i.e.

$$\lambda_i^l \beta_l{}^{jk} = \beta_i{}^{lm} \lambda_l^j \lambda_m^k \tag{4.6}$$

which must hold true for any $\lambda \in \mathrm{SO}(n)$. Then one can consider a 1-parameter subgroup $\lambda(t)$ based at the identity (i.e. $\lambda(0) = \mathbb{I}$) and the corresponding Lie algebra element $\dot\lambda = \dot\lambda(0)$; the infinitesimal form of (4.6) is then

$$\dot\lambda_i^l \beta_l{}^{jk} = \beta_i{}^{lk} \dot\lambda_l^j + \beta_i{}^{jm} \dot\lambda_m^k \tag{4.7}$$

which must hold for any $\dot\lambda \in so(n)$, i.e. for any skew–symmetric matrix.

Then one should try to look for solutions of condition (4.7) that correspond to reductive splittings. We shall here provide explicit solutions for $2 \le n \le 5$ (i.e. spacetime dimension $3 \le m \le 6$).

For $n = 2$, Latin indices range in $i, j, \dots = 1, 2$. The condition (4.7) specifies to

$$\begin{cases} \beta_1{}^{12} = \beta_2{}^{12} \\ \beta_2{}^{12} = -\beta_1{}^{12} \end{cases} \tag{4.8}$$

Hence one has ${\beta_1}^{12} = {\beta_2}^{12} = 0$, so that there is no reductive splitting other then ${\beta_i}^{jk} = 0$.

For $n = 3$ (i.e. $m = 4$), Latin indices range in $i, j, \ldots = 1, 2, 3$. The condition (4.7) is now equivalent to (3.11) as one can show by setting $\beta^l_i = \frac{1}{2}\epsilon^l{}_{mn}{\beta_i}^{mn}$. Hence the only solution is ${\beta_i}^{jk} = \gamma{\epsilon_i}^{jk}$ which span reductive splittings.

For $n = 4$ and $n = 5$ (i.e $m = 5$ and $m = 6$, respectively), one can check directly (using Maple tensor package; see **[15]**) and show again that the only solution is the trivial one: ${\beta_i}^{jk} = 0$.

5. Conclusions and Perspectives

We showed that BI-connections can be properly understood in terms of bundle reductions along reductive group splittings. As a side effect this overcomes any objection about the holonomy constraints since the holonomy of the reduced connection is not identical to the holonomy of the original connection.

To summarize, the standard BI connection is not the spatial restriction of a spacetime spin connection. It is instead the restriction *of the reduction* of a spacetime spin connection and the restricted spacetime connection is the spacetime counterpart of the spatial BI connection, though it is not a $\mathrm{Spin}(n, 1)$-connection.

Further investigation will be devoted to see whether non-trivial reductive splittings exist in higher dimension ($m \geq 7$) or different signatures. If they exist, then one will be able to study the dynamics of higher-dimensional gravity along the lines of **[16]**. This would be possible using the Holst dynamics as written in **[17]** or the modified dynamics (the ones equivalent to $f(R)$ models) as in **[18]**.

Appendix A. Commutators of σ_{ab}

A pointwise right-invariant basis for vertical vector fields on a principal $\mathrm{Spin}(\eta)$-bundle P is induced by a frame $e : P \to L(M)$ locally represented by the matrices e^μ_a in the form (see **[12]**)

$$\sigma_{ab} = \eta_{c[b}e^\mu_{a]}\frac{\partial}{\partial e^\mu_c} \tag{A.1}$$

One can easily prove that the commutators are

$$[\sigma_{ab}, \sigma_{cd}] = \tfrac{1}{2}\left(\eta_{ac}\delta^e_b\delta^f_d + \eta_{bd}\delta^e_a\delta^f_c - \eta_{ad}\delta^e_b\delta^f_c - \eta_{cb}\delta^e_a\delta^f_d\right)\sigma_{ef} \tag{A.2}$$

In dimension 4 the indices run in the range $a, b = 0, ..3$ and one can set

$$\hat\sigma_i := 4\sigma_{0i} \qquad\qquad \hat\tau_i = -2{\epsilon_i}^{jk}\sigma_{jk} \quad (\Rightarrow \sigma_{jk} = -\tfrac{1}{4}{\epsilon_{jk}}^i\hat\tau_i) \tag{A.3}$$

The commutators (A.2) specify to

$$[\hat\sigma_i, \hat\sigma_j] = 2{\epsilon_{ij}}^k\hat\tau_k \qquad [\hat\sigma_i, \hat\tau_j] = -2{\epsilon_{ij}}^k\hat\sigma_k \qquad [\hat\tau_i, \hat\tau_j] = -2{\epsilon_{ij}}^k\hat\tau_k \tag{A.4}$$

which accounts for the identification of vertical vector fields with algebra generators given by (3.7).

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