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A note on wealth in a volatile economy

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Abstract

I show that if the capital accumulation dynamics is stochastic a new term, in addition to that given by accounting prices, has to be introduced in order to derive a correct estimate of the genuine wealth of an economy. In a simple model with multiplicative accumulation dynamics I show that: 1) the value function is always a decreasing function of volatility 2) the accounting prices are affected by volatility 3) the new term always gives a negative contribution to wealth changes. I discuss results for models with constant elasticity utility functions. When the elasticity of marginal utility is larger than one, accounting prices increase with volatility whereas when it is less than one accounting prices decrease with volatility. These conclusions are not altered when adopting optimal saving rates.

Estimating genuine wealth for an economy has been the subject of much recent interest as it is a key instrument to assess sustainability. In particular Arrow, Dasgupta and Maler [1] – hereafter ADM – have broadly discussed the issue in a recent paper. In brief, ADM show that wealth changes over time can be computed from the knowledge of accounting prices of the various resources in the economy. Their approach is mainly deterministic, apart from the case of a discrete time stochastic model. Here I generalize the treatment to the case of continuous time stochastic processes and show that the effect of volatility cannot be expressed in terms of accounting prices. I discuss a simple model and its application to constant elasticity utility functions in the next section. I discuss both the case of imperfect economies and of efficient ones, where the saving rate is adjusted to its optimal value, as discussed in [2].

Let

$$V(k_0, t_0) = E \int_0^\infty d\tau e^{-\delta\tau} u(c_{t_0+\tau}) \tag{1}$$

be the wealth function, where $c_{t_0+\tau}$ is the consumption flow at time $t_0 + \tau$.

We imagine that $k(t)$ follows a stochastic dynamics

$$dk = a(k,t)dt + b(k,t)dW, \qquad k(t_0) = k_0 \tag{2}$$

where dW is the increment of the Wiener process: $E[dW] = 0$, $dW^2 = E[dW^2] = dt$. If the process is time homogeneous a and b do not depend explicitly on t. Most results we are going to use are standard in stochastic processes (see e.g. [3]).

Because the situation at time $t_0+\tau$ can no more be deterministically related to the capital stock k_0 at time t_0, we can no more assume that $c_{t_0+\tau}$ is given by a resource allocation mechanism $\alpha(t_0, t_0+\tau, k_0)$. We assume that $c_{t_0+\tau} = C[k(t_0+\tau)]$ is a function of capital stock at the same time, and hence it is itself a random quantity.

Let us consider the change in V in a time interval dt. In that time interval k changes by an amount dk which is random. So

$$\begin{aligned} dV(k_0, t_0) &= E[V(k_0+dk, t_0)] - V(k_0, t_0) \\ &= \frac{\partial V(k_0,t_0)}{\partial t_0}dt + \frac{\partial V(k_0,t_0)}{\partial k_0}E[dk] + \frac{1}{2}\frac{\partial^2 V(k_0,t_0)}{\partial k_0^2}E[dk^2] + \ldots \end{aligned}$$

Now, in order to compute dV/dt in the limit $dt \to 0$ we observe that $E[dk^2] = b^2(k_0,t_0)E[dW^2] + \ldots = b^2(k_0,t_0)dt + \ldots$ and then

$$\frac{dV}{dt_0} = \frac{\partial V(k_0,t_0)}{\partial t_0} + \frac{\partial V(k_0,t_0)}{\partial k_0}a(k,t) + \frac{1}{2}\frac{\partial^2 V(k_0,t_0)}{\partial k_0^2}b^2(k_0,t_0) \tag{3}$$

The first two terms in the r.h.s. are the usual ones, the last – which we shall call the *Ito term* – is new.

1 A simple model

Let us take a simple economy with a single capital good whose stock is $k(t)$. In a time interval dt the production is

$$dY = (\mu dt + \sigma dW)k \tag{4}$$

which means that the production process is subject to stochastic fluctuations. The resource allocation mechanism is defined by assuming that in a time interval dt a fraction νdt of the capital $k(t)$ is consumed: $dC = \nu k dt$ which gives an utility $dU = u(\nu k)dt$. The parameter ν is the consumption rate, which is easily related to the saving rate $1-\nu$, and it may be fixed by economic policy to an optimal value. We shall discuss our results in terms of the consumption rate below, rather than in terms of saving rates. Capital accumulates according to[1]

$$dk = dY - dC = (\mu - \nu)kdt + \sigma k dW, \qquad t > t_0, \qquad k(t_0) = k_0 \tag{5}$$

[1]Capital depreciation can be incorporated into dC. More precisely, capital depreciates at a rate γ is equivalent to having $dC = (\nu+\gamma)kdt$, no depreciation as far as k is concerned. Depreciation can be set back in by replacing $\nu \to \nu+\gamma$ and $u(x) \to u[(1+\gamma/\nu)x]$ in the following.

The wealth function is then given by

$$V(k_0,t_0) = E\int_{t_0}^{\infty} e^{-\delta(t-t_0)}u(\nu k)dt = \int_{t_0}^{\infty} dte^{-\delta(t-t_0)}\int_0^{\infty} P(k,t|k_0,t_0)u(\nu k)dk$$

Let us compute the distribution $P(k,t|k_0,t_0)$ of $k(t)$ conditional on $k(t_o) = k_0$. Let us first derive the distribution of $x = \ln k$. We have

$$\begin{aligned} dx = x(t+dt) - x(t) &= log[1 + dk/k(t)] = \frac{dk}{k(t)} - \frac{1}{2}\left(\frac{dk}{k(t)}\right)^2 + \dots \\ &= \left(\mu - \nu - \frac{\sigma^2}{2}\right)dt + \sigma dW \end{aligned}$$

where we have used the fact that $dW^2 = dt$ and we have neglected all terms of order higher than dt. Notice that the effect of stochastic fluctuations is to reduce the rate of growth by a term $-\sigma^2/2$.

It is clear that $x(t)$ satisfies a biased random walk:

$$x(t) = \ln k_0 + \left(\mu - \nu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma W(t) \tag{6}$$

where $W(t)$ is a gaussian variable with zero average and variance $t - t_0$. I introduce, for convenience the parameter

$$v = \mu - \nu - \frac{\sigma^2}{2} \tag{7}$$

which allows us to write

$$V(k_0,t_0) = \int_0^{\infty} d\tau e^{-\delta\tau}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi\tau}\sigma}\exp\left[-\frac{(x - \ln k_0 - v\tau)^2}{2\sigma^2\tau}\right]u\left(\mu e^x\right) \tag{8}$$

or, changing variables to $z = \ln k_0 + v\tau + \sigma\sqrt{\tau}x$,

$$V(k_0,t_0) = \int_0^{\infty} d\tau e^{-\delta\tau}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}u\left(\nu k_0 e^{(\mu-\nu-\sigma^2/2)\tau+\sigma\sqrt{\tau}z}\right) \tag{9}$$

Proposition: If $u"(c) < 0$ then V decreases with σ.
Proof: Let us introduce the shorthand

$$C = \nu k_0 e^{(\mu-\nu-\sigma^2/2)\tau+\sigma\sqrt{\tau}z} \tag{10}$$

Then

$$\frac{\partial V}{\partial\sigma} = \int_0^{\infty} d\tau e^{-\delta\tau}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}u'(C)C[\sqrt{\tau}z - \sigma\tau] \tag{11}$$

If we take a partial integration in z of the term proportional to $\sqrt{\tau}z$ we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} u'(C) C \sqrt{\tau} z &= -\sqrt{\tau} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \left[\frac{\partial}{\partial z} e^{-z^2/2} \right] u'(C) C \\
&= -\sqrt{\tau} \left| e^{-z^2/2} u'(C) C \right|_{-\infty}^{\infty} + \sqrt{\tau} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \frac{\partial}{\partial z} u'(C) C \\
&= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} [u''(C) C + u'(C)] C \sigma \tau
\end{aligned}$$

The boundary term in the integration by parts vanishes for any convex utility function $u(C)$. If we combine this with the previous formula, we get

$$\frac{\partial V}{\partial \sigma} = \sigma \int_0^{\infty} d\tau \tau e^{-\delta\tau} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} u''(C) C^2 \tag{12}$$

If $u''(C) < 0$, as assumed, then V decreases with σ.

Note in particular that the time integration kernel $\tau e^{-\delta\tau}$ has an extra τ factor which reflects the fact that uncertainty increases with time. This kernel has a maximal weight at $\tau = 1/\delta$.

If we use the shorthand $V = \int \ldots u(C)$ then accounting price p is given by

$$k_0 p \equiv k_0 \frac{\partial V}{\partial k_0} = \int \ldots u'(C) C. \tag{13}$$

We have seen that

$$\frac{\partial V}{\partial \sigma} = \sigma \int \ldots \tau u''(C) C^2 \tag{14}$$

then symbolic calculus tells us that

$$k_0 \frac{\partial p}{\partial \sigma} = \sigma \int \ldots \tau C^2 \frac{d^2}{dC^2} u'(C) C = \sigma \int \ldots \tau C^2 [2u''(C) + C u'''(C)] \tag{15}$$

depending on the sign of $2u''(C) + Cu'''(C)$ accounting price will decreases or increase with σ.

Proposition If $u"(c) < 0$ then the Ito term is negative.

Proof: By direct calculation, it is easy to see that

$$\frac{\partial^2 V}{\partial k_0^2} = k_0^{-2} \int_0^{\infty} d\tau e^{-\delta\tau} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} u''(C) C^2 \tag{16}$$

If $u''(C) < 0$, as assumed, then the Ito term gives a *negative* contribution in Eq. (3).

The change of the value function for this simple model, form Eq. (3), is given by:

$$\frac{dV}{dt} = \int_0^{\infty} d\tau e^{-\delta\tau} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[(\mu - \nu) u'(C) C + \frac{\sigma^2}{2} u''(C) C^2 \right] \tag{17}$$

Note the similarity of the Ito term $\frac{1}{2}\sigma^2 k_0^2 \frac{\partial^2 V}{\partial k_0^2}$ with

$$\sigma^2 \frac{\partial V}{\partial \sigma^2} = \frac{\sigma^2}{2} \int_0^\infty d\tau \tau e^{-\delta\tau} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} u''(C) C^2$$

where it not for the factor τ in the time integration kernel, it would be possible to express the Ito term as an accounting price of volatility.

Finally we remark that optimal consumption rate ν^* is given by the solution of

$$\nu^* = \frac{\int \ldots u'(C)C}{\int \ldots \tau u'(C)C} = -\left[\frac{\partial \ln p}{\partial \delta}\right]^{-1}. \tag{18}$$

The effect of volatility in the case where the optimal consumption rate is implemented can be appreciated from the equation

$$\frac{dV}{d\sigma} = \frac{\partial V}{\partial \sigma} + \left.\frac{\partial V}{\partial \nu}\right|_{\nu^*} \frac{d\nu^*}{d\sigma} \tag{19}$$

Since $\frac{\partial V}{\partial \nu} = -\int \ldots \tau u'(C)C < 0$, if $\frac{d\nu^*}{d\sigma} < 0$ the adjustment of the optimal policy will mitigate the effect of volatility, whereas if $\frac{d\nu^*}{d\sigma} > 0$ it will accentuate it.

In order to get more specific results we have to specialize to specific forms of the utility function.

1.1 The case $u(C) = -C^{-\gamma}$, $\gamma > 0$

If we take $u(C) = -C^{-\gamma}$ then, because of Eq. (13), we have $\frac{\partial V}{\partial k_0} = -\gamma V$. The behavior of accounting price is clearly related to that of the value function.

Integrals can be performed easily and we get:

$$V(k_0) = -\frac{(\nu k_0)^{-\gamma}}{\delta + \gamma(\mu - \nu) - \gamma(1+\gamma)\sigma^2/2} \tag{20}$$

This diverges when $\sigma \to \sigma_c$ from below where the critical volatility is

$$\sigma_c^2 = 2\frac{\delta + \gamma(\mu - \nu)}{\gamma(1+\gamma)} \tag{21}$$

The fact that $V \to -\infty$ means that the integral which defines V is dominated by the region $k(t) \approx 0$. This means that the economy is going bankrupt ($k = 0$) after some time.

It is easy to see that V decreases with σ but the accounting price p increases with σ. Indeed $2u''(C) + Cu'''(C) = \gamma(1+\gamma)^2 c^{-\gamma-2} > 0$ in this case.

The wealth change is given by:

$$\frac{dV}{dt} = \frac{\gamma[\mu - \nu - (1+\gamma)\sigma^2/2](\nu k_0)^{-\gamma}}{\delta + \gamma(\mu - \nu) - \gamma(1+\gamma)\sigma^2/2} \tag{22}$$

which becomes negative if $\sigma^2 > 2(\mu - \nu)/(1+\gamma)$. Note that only accounting for the change in capital stock without taking into account the Ito term would overestimate wealth:

$$k_0 \frac{\partial V}{\partial k_0} = \frac{\gamma(\mu-\nu)(\nu k_0)^{-\gamma}}{\delta + \gamma(\mu-\nu) - \gamma(1+\gamma)\sigma^2/2} \tag{23}$$

This is always positive (provided $\mu > \nu$). The optimal consumption rate ν^* is, in this case

$$\nu^* = \frac{\delta + \gamma\mu}{1+\gamma} - \gamma\frac{\sigma^2}{2} \tag{24}$$

and it decreases with ν. Note that $\nu^* \to 0$ as $\sigma \to \sigma_c$. The adjustment in the optimal consumption rate compensates the effect of volatility, but it is insufficient to change the main conclusions, i.e. V still decreases with σ and p increases with it.

1.2 The case $u(C) = C^\beta$, $0 < \beta < 1$

When the utility function is $u(C) = C^\beta$ we have $u'(C)C = \beta u(C)$ so the behavior of the value function V and of accounting prices $p = V/\beta$ is the same. In particular we find:

$$V(k_0) = \frac{(\nu k_0)^\beta}{\delta - \beta(\mu-\nu) + \beta(1-\beta)\sigma^2/2} \tag{25}$$

Concavity requires $\beta < 1$. This may diverge if $\delta + \nu < \mu$ because capital grows very fast end the integral is dominated by the $C \to \infty$ region. In this case volatility smooths the divergence.

Again V decreases with σ but now also the accounting price decreases with σ. Indeed $2u''(C) + Cu'''(C) = -\beta^2(1-\beta)c^{\beta-2} < 0$ in this case.

Wealth change is given by:

$$\frac{dV}{dt} = \frac{\beta[\mu - \nu - (1-\beta)\sigma^2/2](\nu k_0)^\beta}{\delta - \beta(\mu-\nu) + \beta(1-\beta)\sigma^2/2} \tag{26}$$

again the Ito term can turn this negative if $\sigma^2 > 2(\mu-\nu)/(1-\beta)$. Again neglecting the Ito term overestimates wealth changes.

The optimal consumption rate is

$$\nu^* = \frac{\delta - \beta\mu}{1-\beta} + \beta\frac{\sigma^2}{2}$$

which is clearly increasing with σ. Accounting prices, on the optimal policy, decrease with σ even more strongly than with ν fixed.

1.3 The case $u(C) = \ln C$

In this case it is easy to check that

$$V(k_0) = \frac{\ln(\nu k_0)}{\delta} + \frac{\mu - \nu - \sigma^2/2}{\delta^2} \tag{27}$$

and again V decreases with σ but accounting prices are independent of σ as well as the optimal consumption rate. Now

$$\frac{dV}{dt} = \mu - \nu - \frac{\sigma^2}{2} \tag{28}$$

and the same considerations as above apply.

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References

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