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SUPERSYMMETRY APPROACH TO WISHART CORRELATION MATRICES: EXACT RESULTS

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We calculate the marginal probability density of real and complex Wishart correlation matrices. For deep mathematical reasons, no explicit expression could be obtained for the real case so far. We circumvent these problems by using a supersymmetry approach. This allows us to derive an exact expression for the marginal probability density of real Wishart correlation matrices in terms of twofold integrals. Within this approach the result for the marginal probability density of complex Wishart correlation matrices is rederived as a test case.

1. Introduction. Complex systems in a rich variety of fields are in the focus of modern research [1, 2, 3, 4]. Correlation matrices obtained from data sampling are a key tool for studying such systems [5, 6]. Multivariate statistics allows one to model correlation matrices by using random matrix theory (RMT) [7]. In the framework of RMT, we calculate the marginal probability density function (p.d.f.) for real and complex correlation matrices. For the case of complex correlation matrices, an explicit expression is known [8]. Real correlation matrices are more frequently encountered, but unfortunately, their marginal p.d.f. and related quantities are up to now not known in closed. This is so, because a certain integral over the orthogonal group is not available in closed form. Sophisticated power series techniques have been developed in order to tackle this problem [7]. However, the resulting expressions suffer from the disadvantage that a resummation of the infinite series has not been possible so far. For large dimension of the correlation matrices, asymptotic results were derived [9]. Recently some new results for the marginal p.d.f. and the two-point correlation function in the asymptotic regime have been found [10].

Here we provide exact results for the marginal p.d.f. of real and complex correlation matrices. We use an alternative approach which circumvents the problems mentioned above. The approach relies on the supersymmetry method [11] which is nowadays an indispensable tool for RMT applications in physics [12]. We derive an exact expression for the marginal p.d.f. of

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real Wishart correlation matrices in terms of a twofold integral. As a test case, we rederive the result for the marginal p.d.f. of complex Wishart correlation matrices. For a physics audience, we published our main results in Ref. [13]. Here, we present the material for an audience of mathematicians and statisticians. In doing so, we review the salient, necessary features of Supersymmetry.

The article is organized as follows: We introduce our notation and some basic quantities in Sec. 2. In Sec. 3, we map the problem onto superspace. We explicitly evaluate the expression for the marginal p.d.f. in Sec. 4. In Sec. 5 we numerically integrate the results and compare them with a Monte-Carlo simulation of the marginal p.d.f. We summarize and conclude in Sec. 6.

2. Formulating the Problem.

We define the proper ensemble of random matrices in Sec. 2.1. In Sec. 2.2, we introduce a generating function for the marginal p.d.f. which serves as starting point for the supersymmetry approach.

2.1. *Ensemble of Wishart correlation matrices and marginal p.d.f.*

We briefly sketch the RMT approach to correlation matrices as set up in Ref. [7]. We consider real and complex Wishart correlation matrices. The building block for those are rectangular $p\times n$ matrices which we denote by $W = [W_{jk}]$, with $j = 1, \ldots, p, k = 1, \ldots, n$. We always assume that $p \leq n$. For the case $p > n$ the $p \times p$ matrix $WW^\dagger$ has $p - n$ generic zero eigenvalues. The entries of W are either real or complex random variables. These cases are labeled by the Dyson index β which takes the values $\beta = 1$ for real entries, i.e. $W_{jk} \in \mathbb{R}$, and $\beta = 2$ for complex entries, i.e. $W_{jk} \in \mathbb{C}$. For the joint probability distribution of the entries of W we consider a multivariate Gaussian weight

$$P_\beta(W, C) = D_\beta \exp\left(-\frac{\beta}{2}\mathrm{Tr}W^\dagger C^{-1}W\right), \tag{2.1}$$

where C is the correlation matrix. The full measure is then $P_\beta(W, C)d[W]$ where

$$d[W] = \begin{cases} \displaystyle\prod_{j=1}^{p}\prod_{k=1}^{n} dW_{jk} & , \text{ for } \beta = 1\,, \\ \displaystyle\prod_{j=1}^{p}\prod_{k=1}^{n} d\mathrm{Re}W_{jk}d\mathrm{Im}W_{jk} & , \text{ for } \beta = 2\,, \end{cases} \tag{2.2}$$

is the corresponding volume element. This measure fulfills the invariance condition

$$P_\beta(W, C)d[W] = P_\beta(UW, UCU^\dagger)d[UW] \tag{2.3}$$

for an arbitrary orthogonal ($\beta = 1$) and unitary ($\beta = 2$) $p \times p$ matrix U. Since the domain of W ($\mathbb{R}^{p\times n}$ for $\beta = 1$, $\mathbb{C}^{p\times n}$ for $\beta = 2$) is invariant under rotations, we may replace C by the diagonal matrix of its eigenvalues if invariant quantities such as the marginal p.d.f. are to be studied. Hence we use $C = U^\dagger \Lambda U$ where $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_p)$ is a positive diagonal $p \times p$ matrix, i.e. $\Lambda_j > 0$. The constant D_β in Eq. (2.1) ensures the normalisation of $P_\beta(W, C)d[W]$ to unity and is given by

$$D_\beta = \left[\left(\frac{2\pi}{\beta}\right)^p \det\Lambda\right]^{-\beta n/2}. \tag{2.4}$$

The set of random matrices $WW^\dagger$ with entries of W distributed according to Eq. (2.1) is referred to as the ensemble of Wishart correlation matrices (sometimes also as correlated Wishart ensemble). We notice that for the choice $\Lambda = \mathbb{1}_p$, where $\mathbb{1}_p$ denotes the $p \times p$ unit matrix, the ensemble defined by Eq. (2.1) is equivalent to the Gaussian chiral random matrix ensemble. This ensemble is employed to study generic features in the theory of Quantum Chromo dynamics (QCD) [14].

The ensemble averaged marginal p.d.f. for the eigenvalues of the matrix $WW^\dagger$ can be expressed in terms of the resolvent and reads

$$S_\beta(x) = -\frac{1}{p\pi}\lim_{\varepsilon\to 0}\mathrm{Im}\int d[W]P_\beta(W,\Lambda)\mathrm{Tr}\frac{\mathbb{1}_{4/\beta}}{x^+\mathbb{1}_{4/\beta} - WW^\dagger}. \tag{2.5}$$

The limit $\varepsilon \to 0$ is a weak one which means that we have to integrate first and then take the limit. With the notation x^+ we indicate that x carries a small positive imaginary increment, i.e. $x^+ = x + \imath\varepsilon$ with $\varepsilon > 0$. Due to the definition in Eq. (2.5) the marginal p.d.f. is a function of the $p+1$ parameters $x, \Lambda_1, \ldots, \Lambda_p$. We drop the dependence on Λ by writing $S_\beta(x)$. The integration in Eq. (2.5) extends over the whole domain of W, respectively.

2.2. *Generating function.* The starting point for the supersymmetry approach is the generating function

$$Z_\beta(J) = \int d[W]P_\beta(W,\Lambda)\frac{\det(x^+\mathbb{1}_p + J\mathbb{1}_p - WW^\dagger)}{\det(x^+\mathbb{1}_p - WW^\dagger)}, \tag{2.6}$$

where J is a source variable. The marginal marginal p.d.f. is given as derivative

$$S_\beta(x) = -\frac{1}{\pi p}\lim_{\varepsilon\to 0}\mathrm{Im}\frac{\partial Z_\beta(J)}{\partial J}\bigg|_{J=0} \tag{2.7}$$

of the generating function at $J = 0$. By construction, the generating function is normalized to unity at $J = 0$, i.e. $Z_\beta(0) = 1$. In the following we derive a compact representation for the generating function by mapping it onto superspace.

3. Map onto superspace. In Sec. 3.1, we write the ratio of determinants in the generating function as a Gaussian integral over a supervector. Then we carry out the ensemble average. To the reader not experienced with anticommuting variables, we recommend the introductory parts of Refs. [15, 11, 16]. In Sec. 3.2, we use a duality between ordinary and supermatrices to express the result of the ensemble average as a supermatrix integral. Afterwards we integrate over the supervector in Sec. 3.3. We take the derivative with respect to the source variable and evaluate the resulting expression in Sec. 3.4.

3.1. *Ensemble average.* The determinant in the denominator of Eq. (2.6) can be expressed as a Gaussian integral over a vector comprising ordinary commuting variables. The determinant in the numerator can be expressed as a Gaussian integral over a vector with anticommuting entries [15, 11]. Combining both expressions we obtain a representation for the ratio of determinants in Eq. (2.6) in terms of a Gaussian integral over a supervector Ψ

$$\frac{\det\left((x^+ + J)\mathbb{1}_p - WW^\dagger\right)}{\det\left(x^+\mathbb{1}_p - WW^\dagger\right)} =$$
$$= \int d[\Psi] \exp\left\{\frac{\imath\beta}{2}\Psi^\dagger\left(\mathbb{1}_p \otimes (x^+\mathbb{1}_{4/\beta} + J\gamma) - WW^\dagger \otimes \mathbb{1}_{4/\beta}\right)\Psi\right\}, \tag{3.1}$$

where we have introduced the matrix

$$\gamma = \begin{bmatrix} 0_{2/\beta} & 0_{2/\beta} \\ 0_{2/\beta} & \mathbb{1}_{2/\beta} \end{bmatrix} \tag{3.2}$$

and the supervector

$$\begin{aligned} \Psi &= \left[u_1, \ldots, u_p, \quad v_1, \ldots, v_p, \quad \zeta_1, \ldots, \zeta_p, \quad \zeta_1^*, \ldots, \zeta_p^*\right]^T \quad &&, \text{for}\beta = 1, \\ \Psi &= \left[z_1, \ldots, z_p, \quad \zeta_1, \ldots, \zeta_p\right]^T &&, \text{for}\beta = 2. \end{aligned} \tag{3.3}$$

The symbol $0_{2/\beta}$ denotes the $2/\beta \times 2/\beta$ zero matrix. Here, $u_j, v_j \in \mathbb{R}$ and $z_j \in \mathbb{C}$ are ordinary real or complex variables while ζ_j, ζ_j^* are anticommuting variables, also referred to as Grassmann variables, which by definition satisfy

$$\zeta_j\zeta_i = -\zeta_i\zeta_j \quad , \quad \zeta_j^*\zeta_i^* = -\zeta_i^*\zeta_j^* \quad , \quad \zeta_j\zeta_i^* = -\zeta_i^*\zeta_j\,. \tag{3.4}$$

These relations imply that the Grassmann variables are nilpotent. Hence, each function depending on them can be represented as a finite power series in these variables. Grassmann variables can be multiplied and added. Thus, they build an algebra $\mathfrak{G}$ referred to as Grassmann algebra. The complex conjugation "$*$" is extended to the Grassmann algebra by the three relations

$$(\zeta_j)^* = \zeta_j^* \quad , \quad \left(\zeta_j^*\right)^* = -\zeta_j \quad , \quad (\chi_1\chi_2)^* = \chi_1^*\chi_2^* \,, \tag{3.5}$$

where $\chi_1, \chi_2 \in \mathfrak{G}$, i.e. they can be linear combinations of the Grassmann variables. This definition is referred to as conjugation of the second kind [15]. In Eq. (3.1), $d[\Psi]$ denotes the product of all differentials of elements comprised in the supervector Ψ, i.e.

$$d[\Psi] = \begin{cases} \prod\limits_{j=1}^{p} du_j dv_j d\zeta_j d\zeta_j^* & , \text{ for } \beta = 1 \,, \\ \prod\limits_{j=1}^{p} d\mathrm{Re}z_j d\mathrm{Im}z_j d\zeta_j d\zeta_j^* & , \text{ for } \beta = 2 \,. \end{cases} \tag{3.6}$$

For the integration over Grassmann variables we employ the same conventions as in Ref. [15],

$$\int \zeta_j^n d\zeta_j = \int \left(\zeta_j^*\right)^n d\zeta_j^* = \frac{\delta_{n1}}{\sqrt{2\pi}} \,, \, n \in \{0, 1\} \,. \tag{3.7}$$

The differentials $d\zeta_j$ and $d\zeta_j^*$ anticommute with each other and with ζ_j and ζ_j^*, too.

We introduce the p-component vectors

$$\begin{aligned} u &= \left[u_1, \quad \dots \quad , u_p\right]^T, \qquad z = \left[z_1, \quad \dots \quad , z_p\right]^T, \\ v &= \left[v_1, \quad \dots \quad , v_p\right]^T, \qquad \zeta = \left[\zeta_1, \quad \dots \quad , \zeta_p\right]^T. \end{aligned} \tag{3.8}$$

Plugging the representation (3.1) into the generating function (2.6) and changing the order of integrations we find

$$\begin{aligned} Z_\beta(J) = &\int d[\Psi] \exp\left\{\frac{\imath\beta}{2}\Psi^\dagger\left(\mathbb{1}_p \otimes (x^+\mathbb{1}_{4/\beta} + J\gamma)\right)\Psi\right\} \\ &\times D_\beta \int d[W] \exp\left\{-\frac{\beta}{2}\left(\mathrm{Tr}W^\dagger\Lambda^{-1}W + \imath\Psi^\dagger(WW^\dagger \otimes \mathbb{1}_{4/\beta})\Psi\right)\right\}. \end{aligned} \tag{3.9}$$

We write the last term in the second exponential as

$$\Psi^\dagger(WW^\dagger \otimes \mathbb{1}_{4/\beta})\Psi = \mathrm{Tr}KWW^\dagger, \tag{3.10}$$

where the matrix K has a dyadic structure. It contains the elements of the supervector and reads

$$K = uu^T + vv^T - \left(\zeta\zeta^\dagger - \zeta^*\zeta^T\right) \quad \text{, for} \quad \beta = 1, \tag{3.11}$$
$$K = zz^\dagger - \zeta\zeta^\dagger \quad \text{, for} \quad \beta = 2.$$

We notice that the matrix K reflects the symmetry of the matrix ensemble, i.e. $K = K^* = K^T$ is real symmetric for $\beta = 1$ and $K = K^\dagger$ is Hermitian for $\beta = 2$. We also notice that K is a $p \times p$ matrix in ordinary space, even though it contains Grassmann variables. They only appear in combinations such as $\zeta_j\zeta_k$ which obviously are commuting objects. In Eq. (3.9) the matrix W appears quadratically in the exponent. Hence the integral over W has a Gaussian form. Using the notations (3.10) and (3.11) we readily obtain

$$D_\beta \int d[W] \exp\left\{-\frac{\beta}{2}\mathrm{Tr}\left(\Lambda^{-1} + \imath K\right) W W^\dagger \right\} = \tag{3.12}$$
$$= \det{}^{-n\beta/2}\left(\mathbb{1}_p + \imath\Lambda^{1/2}K\Lambda^{1/2}\right)$$

as the result for the ensemble average.

3.2. *Duality between ordinary and superspace.* We now rewrite the result (3.12) in terms of a superdeterminant. This is possible due to a duality relation between ordinary dyadic matrices and dyadic supermatrices [17, 18]. We introduce the rectangular supermatrices

$$A = \begin{bmatrix} u & v & \zeta & -\zeta^* \end{bmatrix} \quad \text{, for} \quad \beta = 1\,, \tag{3.13}$$
$$A = \begin{bmatrix} z & \zeta \end{bmatrix} \quad \text{, for} \quad \beta = 2$$

and the corresponding adjoint matrices $A^\dagger$. The operator "†" differs for supermatrices from the one for ordinary matrices. The adjoint of a $(a+b) \times (c+d)$ supermatrix

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{3.14}$$

is

$$\Sigma^\dagger = \begin{bmatrix} \Sigma_{11}^\dagger & \Sigma_{21}^\dagger \\ -\Sigma_{12}^\dagger & \Sigma_{22}^\dagger \end{bmatrix} . \tag{3.15}$$

The $a \times c$ block Σ_{11} and $b \times d$ block Σ_{22} consist of commuting variables and are referred to as Boson-Boson and Fermion-Fermion block, respectively, whereas the $a \times d$ block Σ_{12} and the $b \times c$ block Σ_{21} comprise anticommuting variables and are known as Boson-Fermion and Fermion-Boson block. This terminology stems from high energy physics where, in contrast to our context, the "Bosons" and " Fermions" represent physical particles.

With the supermatrix A we may write

$$K = AA^\dagger. \tag{3.16}$$

The duality is born out in the relation [19, 18]

$$\det\left(\mathbb{1}_p + \imath\Lambda^{1/2}AA^\dagger\Lambda^{1/2}\right) = \mathrm{sdet}\left(\mathbb{1}_{4/\beta} + \imath A^\dagger\Lambda A\right). \tag{3.17}$$

Here "sdet" denotes the analog of the determinant for supermatrices. For this and other quantities to appear later on we employ the same definitions and conventions as in Ref. [15]. For $a = c$ and $b = d$ the superdeterminant of the supermatrix in Eq. (3.14) is defined as

$$\mathrm{sdet}\Sigma = \frac{\det(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})}{\det\Sigma_{22}}. \tag{3.18}$$

In particular we have $\mathrm{sdet}\Sigma\Sigma' = \mathrm{sdet}\Sigma\ \mathrm{sdet}\Sigma'$ for two supermatrices. The superdeterminant in Eq. (3.18) is only well defined if Σ_{22} is invertible. The supertrace is defined by

$$\mathrm{str}\Sigma = \mathrm{Tr}\Sigma_{11} - \mathrm{Tr}\Sigma_{22}. \tag{3.19}$$

This ensures the cyclic invariance $\mathrm{str}\Sigma\Sigma' = \mathrm{str}\Sigma'\Sigma$ of the supertrace for two matrices of the form (3.14) with $a = c$ and $b = d$.

The matrix

$$B \equiv A^\dagger\Lambda A \tag{3.20}$$

on the right hand side of Eq. (3.17) is the dyadic supermatrix dual to K. In contrast to the $p \times p$ matrix K the matrix B has dimension 4×4 and 2×2 for $\beta = 1$ and $\beta = 2$, respectively. This dimensional reduction is the crucial advantage of the supersymmetry method. It originates from the fact that the entries of B are bilinear forms. Importantly, the symmetries of B carry over to the representation of the generating function in superspace. We have $B = B^\dagger$ for $\beta = 1, 2$. In the complex case, B has no further symmetries. However in the real case B has an additional symmetry which can be expressed by

$$B^* = YBY^T, \quad \text{with} \quad Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{for} \quad \beta = 1, \tag{3.21}$$

reflecting the fact that the dual matrix K is real.

We proceed by expressing the superdeterminant on the right hand side of Eq. (3.17) as a supermatrix integral. There are several ways to do this. For our purpose an appropriate Dirac-distribution is most convenient. Symbolically, we write

$$\operatorname{sdet}^{-n\beta/2}(\mathbb{1}_{4/\beta}+\imath B)=\int d[\sigma]\operatorname{sdet}^{-n\beta/2}(\mathbb{1}_{4/\beta}+\imath\sigma)\delta(\sigma-B). \tag{3.22}$$

For this integral representation to be consistent, the supermatrix σ on the right hand side has to have the same symmetries as B. Hence σ is a Hermitian supermatrix with dimension 4×4 for $\beta=1$ and 2×2 for $\beta=2$. We write the supermatrix σ as

$$\sigma=\begin{bmatrix}\sigma_1 & \eta\\ -\eta^\dagger & \imath\sigma_2\end{bmatrix}, \tag{3.23}$$

where the entries are ordinary block matrices with dimension $2/\beta$. The diagonal blocks σ_1 and σ_2 are the Boson-Boson and the Fermion-Fermion block, respectively. The off-diagonal blocks η and $\eta^\dagger$ contain all anticommuting variables of σ. For $\beta=2$ the diagonal blocks σ_1 and σ_2 are simply real numbers whereas η and $\eta^\dagger=\eta^*$ are two Grassmann variables. For $\beta=1$ the diagonal blocks σ_1 and σ_2 are real symmetric 2×2 matrices. The additional symmetry (3.21) in the real case enforces that σ_2 is proportional to the unit matrix. The factor $\imath$ in front of the entry σ_2 is due to the Wick-rotation and ensures the convergence of the integral (3.22). To be mathematically consistent it has to be ignored in all algebraic manipulations [20, 15]. The off-diagonal block matrices η have for $\beta=1$ the following structure

$$\eta=\begin{bmatrix}\chi & \chi^*\\ \xi & \xi^*\end{bmatrix},\qquad \eta^\dagger=\begin{bmatrix}\chi^* & \xi^*\\ -\chi & -\xi\end{bmatrix}, \tag{3.24}$$

where χ and ξ denote Grassmann variables.

The measure of the supermatrix σ in Eq. (3.22) is flat and reads explicitly

$$d[\sigma]=\begin{cases}d\sigma_{111}d\sigma_{121}d\sigma_{221}d\sigma_2 d\chi d\chi^* d\xi d\xi^* & ,\ \text{for}\quad \beta=1,\\ d\sigma_1 d\sigma_2 d\eta d\eta^* & ,\ \text{for}\quad \beta=2,\end{cases} \tag{3.25}$$

where σ_{111} and σ_{221} are the diagonal elements and σ_{121} is the off-diagonal element of the real symmetric matrix σ_1 in the real case.

To render the expression (3.22) meaningful, we have to define the Dirac-distribution for supermatrices in Eq. (3.22). We use the analog of the Fourier representation for the Dirac-distribution [21]

$$\delta(\sigma-B)=\frac{\beta^2}{4}\int d[\varrho]\exp(\imath\operatorname{str}(\sigma-B)\varrho), \tag{3.26}$$

where the integration extends over another supermatrix ϱ which has the same structure as σ. The measure $d[\varrho]$ is similar to the one of σ, see Eq. (3.25). Using the definition (3.26) in Eq. (3.22) yields the ensemble average (3.12) in terms of a supermatrix integral

$$\det{}^{-n\beta/2}\left(\mathbb{1}_p + \imath\Lambda^{1/2}K\Lambda^{1/2}\right) = \\ = \frac{\beta^2}{4}\int d[\sigma]\int d[\varrho]\mathrm{sdet}^{-n\beta/2}(\mathbb{1}_{4/\beta} + \imath\sigma)\exp\left(\imath\mathrm{str}(\sigma - B)\varrho\right). \tag{3.27}$$

Obviously, the integral over σ does neither contain the elements of the supervector nor the eigenvalues Λ_j. Hence it is a function of ϱ only. It is the supersymmetric version of the Ingham-Siegel integral [17, 18]

$$I_\beta(\varrho) = \int d[\sigma]\mathrm{sdet}^{-n\beta/2}(\mathbb{1}_{4/\beta} + \imath\sigma)\exp(\imath\mathrm{str}\sigma\varrho). \tag{3.28}$$

In Appendix A we evaluate it explicitly for the case of a diagonal supermatrix ϱ. Since the Ingham-Siegel integral as defined in Eq. (3.28) is apparently not convergent, one has to understand the integral as a distribution. The ϱ integral over the Ingham-Siegel integral with a sufficiently integrable function has to exist. Later on we will see that this is indeed the case for our calculations.

Changing the order of integration in Eq. (3.27) and using the notation (3.28), we cast the ensemble average into the form

$$\det{}^{-n\beta/2}\left(\mathbb{1}_p + \imath\Lambda^{1/2}K\Lambda^{1/2}\right) = \frac{\beta^2}{4}\int d[\varrho]I_\beta(\varrho)\exp(-\imath\mathrm{str}B\varrho) \tag{3.29}$$

and obtain

$$Z_\beta(J) = \frac{\beta^2}{4}\int d[\varrho]I_\beta(\varrho) \\ \times\int d[\Psi]\exp\left\{\frac{\imath\beta}{2}\Psi^\dagger\left(\mathbb{1}_p\otimes(x^+\mathbb{1}_{4/\beta} + J\gamma)\right)\Psi - \imath\mathrm{str}\varrho B\right\} \tag{3.30}$$

for the generating function in Eq. (3.9).

3.3. *Integrating over the supervector.* We rewrite the second term in the exponent of Eq. (3.30) containing B. Since the entries are bilinear forms we can express B as a sum of dyadic matrices

$$B = \sum_{j=1}^{p}\Lambda_j\Psi_j^\dagger\Psi_j, \tag{3.31}$$

where we have introduced the supervectors

$$\begin{aligned} \Psi_j &= \left[u_j, \quad v_j, \quad -\zeta_j^*, \quad \zeta_j\right]^T &&, \text{for} \quad \beta = 1, \\ \Psi_j &= \left[z_j^*, \quad -\zeta_j^*\right]^T &&, \text{for} \quad \beta = 2 \end{aligned} \tag{3.32}$$

with $j = 1, \ldots, p$. For $\beta = 1, 2$, Ψ is four- and two component, respectively. Hence we can write

$$\mathrm{str}\varrho B = \sum_{j=1}^{p} \Lambda_j \mathrm{str}\Psi_j\Psi_j^\dagger\varrho = \sum_{j=1}^{p} \Lambda_j \Psi_j^\dagger \varrho \Psi_j. \tag{3.33}$$

The first term in the exponent of the integrand in Eq. (3.30) can be reexpressed in a similar form

$$\Psi^\dagger \left(\mathbb{1}_p \otimes (x^+ \mathbb{1}_{4/\beta} + J\gamma)\right) \Psi = \sum_{j=1}^{p} \Psi_j^\dagger (x^+ \mathbb{1}_{4/\beta} + J\gamma) \Psi_j. \tag{3.34}$$

Using Eqs. (3.33) and (3.34), the integral over the supervector Ψ factorizes into a p-fold product of Gaussian integrals extending over the supervectors Ψ_j. Similarly to Refs. [15, 11], we find

$$\begin{aligned} &\int d[\Psi] \exp\left(\imath \sum_{j=1}^{p} \Psi_j^\dagger \left(\frac{\beta}{2}(x^+ \mathbb{1}_{4/\beta} + J\gamma) - \Lambda_j \varrho\right) \Psi_j\right) = \\ &= \prod_{j=1}^{p} \int d[\Psi_j] \exp\left(\imath \Psi_j^\dagger \left(\frac{\beta}{2}(x^+ \mathbb{1}_{4/\beta} + J\gamma) - \Lambda_j \varrho\right) \Psi_j\right) \\ &= \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2} \left(\frac{\beta}{2}(x^+ \mathbb{1}_{4/\beta} + J\gamma) - \Lambda_j \varrho\right). \end{aligned} \tag{3.35}$$

Plugging the result into the expression (3.30), we obtain the desired representation of the generating function in superspace

$$Z_\beta(J) = \frac{\beta^2}{4} \int d[\varrho] I_\beta(\varrho) \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2} \left(\frac{\beta}{2}(x^+ \mathbb{1}_{4/\beta} + J\gamma) - \Lambda_j \varrho\right). \tag{3.36}$$

Originally the generating function was an integral over ordinary $p \times n$ matrices. The representation (3.36) is an integral over supermatrices with dimension $4{\times}4$ for $\beta = 1$ and 2×2 for $\beta = 2$. This drastically reduces the number of degrees of freedom, that is, the number of integrals to be evaluated.

3.4. *Representation of the marginal p.d.f. as eigenvalue integral.* To obtain from the generating function the marginal p.d.f. we use the relation (2.7). The source variable J appears in the generating function (3.36) only in the product of superdeterminants. We find

$$\frac{\partial}{\partial J}\prod_{j=1}^{p}\mathrm{sdet}^{-\beta/2}\left(\frac{\beta}{2}(x^+\mathbb{1}_{4/\beta}+J\gamma)-\Lambda_j\varrho\right)\Bigg|_{J=0} = \tag{3.37}$$

$$= -\left(\frac{\beta}{2}\right)^2\sum_{k=1}^{p}\mathrm{str}\frac{\mathbb{1}_{4/\beta}}{\beta x^+\mathbb{1}_{4/\beta}/2-\Lambda_k\varrho}\gamma \times\prod_{j=1}^{p}\mathrm{sdet}^{-\beta/2}\left(\frac{\beta}{2}x^+\mathbb{1}_{4/\beta}-\Lambda_j\varrho\right).$$

Thus, we arrive at the expression

$$S_\beta(x) = \frac{\beta^4}{16\pi p}\sum_{k=1}^{p}\lim_{\varepsilon\to 0}\mathrm{Im}\int d[\varrho]I_\beta(\varrho)\mathrm{str}\frac{\mathbb{1}_{4/\beta}}{\beta x^+\mathbb{1}_{4/\beta}/2-\Lambda_k\varrho}\gamma \times\prod_{j=1}^{p}\mathrm{sdet}^{-\beta/2}\left(\frac{\beta}{2}x^+\mathbb{1}_{4/\beta}-\Lambda_j\varrho\right) \tag{3.38}$$

for the marginal p.d.f. The natural way to proceed from Eq. (3.38) is to diagonalise the supermatrix ϱ. For the real case ϱ is diagonalised by the supergroup UOSp(2/2). In the complex case, the diagonalising supergroup is U(1/1). The diagonalisation reads [22]

$$\varrho = u^{-1}Ru,\quad R=\begin{cases}\mathrm{diag}(r_1,r_2,\imath R_2,\imath R_2) & ,\text{for}\quad \beta=1,\\ \mathrm{diag}(r_1,\imath R_2) & ,\text{for}\quad \beta=2,\end{cases} \tag{3.39}$$

where the matrix u is in the corresponding supergroup. Here $r_1, r_2, R_2 \in \mathbb{R}$ and $r_1, R_2 \in \mathbb{R}$ are the eigenvalues of ϱ for $\beta = 1$ and $\beta = 2$ respectively. With $R_1 = \mathrm{diag}(r_1, r_2)$ and $R_1 = r_1$, we denote the Boson-Boson blocks of the diagonal supermatrix R. We notice that, for $\beta = 1$ the third and the fourth eigenvalue R_2 are degenerated in accordance with the discussion following Eq. (3.23).

The measure transforms as

$$d[\varrho] = d[R]d\mu(u)\mathrm{Ber}_\beta(R), \tag{3.40}$$

see also Ref. [22]. Here $d\mu(u)$ denotes the Haar-measure of the supergroup corresponding to the metric $\mathrm{str}\, du du^\dagger$. The Jacobian of the diagonalisation, also referred to as Berezinian, reads [15, 20]

$$\mathrm{Ber}_\beta(R) = \begin{cases} \dfrac{|r_1 - r_2|}{(r_1 - \imath R_2)^2 (r_2 - \imath R_2)^2} & , \quad \text{for} \quad \beta = 1, \\ \dfrac{1}{(r_1 - \imath R_2)^2} & , \quad \text{for} \quad \beta = 2. \end{cases} \tag{3.41}$$

The Ingham-Siegel integral (3.28) is invariant under the diagonalisation of ϱ, i.e. $I_\beta(\varrho) = I_\beta(R)$. Moreover, the superdeterminants in Eq. (3.38) remain unaffected by the diagonalisation. The supertrace of the resolvent contains the symmetry breaking matrix γ and, therefore, the unitary supermatrices u do not drop out. Rather, they appear in the form

$$\mathrm{str} \frac{\mathbb{1}_{4/\beta}}{\beta x^+ \mathbb{1}_{4/\beta}/2 - \Lambda_k \varrho} \gamma = \mathrm{str} \frac{\mathbb{1}_{4/\beta}}{\beta x^+ \mathbb{1}_{4/\beta}/2 - \Lambda_k R} u \gamma u^{-1}. \tag{3.42}$$

Collecting everything the diagonalisation of ϱ yields

$$\begin{aligned} S_\beta(x) = \frac{\beta^4}{16\pi p} \sum_{k=1}^{p} \lim_{\varepsilon \to 0} \mathrm{Im} \int d[R] \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2} \left(\frac{\beta}{2} x^+ \mathbb{1}_{4/\beta} - \Lambda_j R \right) \\ \times \mathrm{Ber}_\beta(R) I_\beta(R) \int d\mu(U) \mathrm{str} \frac{1}{\beta x^+ \mathbb{1}_{4/\beta}/2 - \Lambda_k R} u \gamma u^{-1} + \mathrm{B.T.} \end{aligned} \tag{3.43}$$

There is an additional term which we denote by B.T. In physics it is known as Efetov-Wegner term [20], in mathematics as Rothstein contribution [23]. It arises due to the definition (3.7) of the integration for Grassmann variables which is intimately related to differential operators. An integration in superspace can be viewed as derivation which mixes commuting and anti-commuting variables under a change of coordinates. Hence B.T. can be considered as boundary terms arising by partial integrations [24] and thus has no counterpart in ordinary analysis. Here a crucial difference between the real and the complex case emerges. In the complex case this additional contribution causes no problem because we can evaluate it explicitly, in the real case this is not so.

LEMMA 3.1. *The Efetov-Wegner term arising from the diagonalisation of the marginal p.d.f.* (3.38) *is for the complex case ($\beta = 2$) given by the Dirac-distribution*

$$\mathrm{B.T.} = \delta(\mathrm{x}). \tag{3.44}$$

We proof this in Appendix B. In the real case there appear terms in a addition to the Dirac-distribution which are much more involved. Due to their complicated structure the method of calculating the marginal p.d.f. via a diagonalisation of ϱ is inconvenient. Apart from the boundary terms, the structures which appear for the real and the complex case are analog. To show that, we proceed with the diagonalisation for both cases. The knowledge of the integral over the supergroup in Eq. (3.43) is essential.

LEMMA 3.2. *Let τ and λ be Hermitian supermatrices diagonalised by the supergroups* $\mathrm{U}(N/N)$ *or* $\mathrm{UOSp}(2N/2N)$*. Let further $u \in \mathrm{U}(N/N)$ or $u \in \mathrm{UOSp}(2N/2N)$ be an unitary supermatrix and let $d\mu(u)$ be the Haar-measure of the corresponding supergroup. Then we have*

$$\int d\mu(u) \mathrm{Str}\, \tau u \lambda u^{-1} \propto \mathrm{Str}\, \tau\, \mathrm{Str}\, \lambda. \tag{3.45}$$

The proof is given in Appendix C. With the appropriate choice for τ and λ and with $N = 1$ we apply Lemma 3.2 to the group integral in Eq. (3.43). We notice that there is an ambiguity for the choice of the over all constant in Eq. (3.45) arising from the ambiguity for the normalisation of the Haar-measure for the supergroups $\mathrm{U}(N/N)$ and $\mathrm{UOSp}(2N/2N)$. In our context, no problem arises, because the normalisation of $S_\beta(x)$ to unity fixes the constant. For $N = 1$ it is $-1/(2\pi)$ in both cases.

Applying Lemma (3.2) to Eq. (3.43) we have

$$\begin{aligned} S_\beta(x) = -\frac{\beta^3}{16\pi^2 p} \sum_{k=1}^{p} \lim_{\varepsilon \to 0} \mathrm{Im} \int d[R] \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2} \left(\frac{\beta}{2} x^+ \mathbb{1}_{4/\beta} - \Lambda_j R \right) \\ \times \mathrm{Ber}_\beta(R) I_\beta(R) \mathrm{str} \frac{\mathbb{1}_{4/\beta}}{\beta x^+ \mathbb{1}_{4/\beta}/2 - \Lambda_k R} + \mathrm{B.T.} \end{aligned} \tag{3.46}$$

Removing the summation in Eq. (3.46) over k with help of the following relation

$$\begin{aligned} \frac{\partial}{\partial x} \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2} \left(\frac{\beta}{2} x^+ \mathbb{1}_{4/\beta} - \Lambda_j R \right) = \\ = -\frac{\beta^2}{4} \sum_{k=1}^{p} \mathrm{str} \frac{\mathbb{1}_{4/\beta}}{\beta x^+ \mathbb{1}_{4/\beta}/2 - \Lambda_k R} \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2} \left(\frac{\beta}{2} x^+ \mathbb{1}_{4/\beta} - \Lambda_j R \right), \end{aligned} \tag{3.47}$$

and scaling R according to $R \to \beta x R/2$ yield

$$\begin{aligned} S_\beta(x) = -\frac{\beta}{4\pi^2 p} \lim_{\varepsilon \to 0} \mathrm{Im} \frac{\partial}{\partial x} \int d[R] \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2} \left(R^- - \Lambda_j^{-1} \mathbb{1}_{4/\beta} \right) \\ \times \mathrm{Ber}_\beta(R) I_\beta \left(\frac{\beta x R}{2} \right) + \mathrm{B.T.} \quad , \end{aligned} \tag{3.48}$$

where we define $R^- = R - \imath\varepsilon 1\!\!1_{4/\beta}$. We employ the normalisation of the marginal p.d.f.

$$1 = \int_0^\infty dx S_\beta(x). \tag{3.49}$$

Inserting Eq. (3.48) into Eq. (3.49) the integration over x becomes trivial. The upper boundary yields no contribution since $I_\beta(\beta x R/2)$ is identically zero for $x \to +\infty$. From the lower boundary we obtain the identity.

$$1 = \frac{\beta}{4\pi^2 p} \lim_{\varepsilon\to 0} \lim_{x\to 0^+} \mathrm{Im} \int d[R] \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2}\left(R^- - \Lambda_j^{-1} 1\!\!1_{4/\beta}\right) \times \mathrm{Ber}_\beta(R) I_\beta\left(\frac{\beta x R}{2}\right). \tag{3.50}$$

The above mentioned ambiguity for the normalisation of the Haar-measure is removed by using Eq. (3.50).

We now clarify how the derivative with respect to x in Eq. (3.48) has to be understood. We note that, in Eq. (3.48), x is only contained in the Ingham-Siegel integral. Using the explicit form of I_β (see Appendix A) we notice that the integral is of the following type

$$Y = \frac{\partial}{\partial x} \int d[R] G(R) \Theta(xR) \exp(-x \mathrm{str} R), \tag{3.51}$$

where $G(R)$ is independent of x. The Heaviside-distribution for a matrix R is defined by

$$\Theta(R) = \begin{cases} 1 & \text{, if} \quad R \quad \text{is a positive definit matrix,} \\ 0 & \text{, else.} \end{cases} \tag{3.52}$$

In Eq. (3.51) we write it as

$$\Theta(xR) = \Theta(x)\Theta(R) + \Theta(-x)\Theta(-R). \tag{3.53}$$

According to the definition of the marginal p.d.f. we always have $x \geqslant 0$. The derivative with respect to x is the differentiation from the right at $x = 0$. Hence we can omit the second term in Eq. (3.53) and obtain for Eq. (3.51)

$$Y = \delta(x) \lim_{x\to 0^+} \int d[R] \Theta(R) G(R) \exp(-x \mathrm{str} R) - \int d[R] \Theta(R) G(R) \exp(-x \mathrm{str} R) \mathrm{str} R. \tag{3.54}$$

Applying the discussion above to Eq. (3.48) we obtain for the marginal p.d.f.

$$\begin{aligned} S_\beta(x) &= \frac{\beta^2}{8\pi^2 p}\lim_{\varepsilon\to 0}\mathrm{Im}\int d[R]\mathrm{Ber}_\beta(R) I_\beta\left(\frac{\beta x R}{2}\right) \\ &\qquad \times\prod_{j=1}^{p}\mathrm{sdet}^{-\beta/2}\left(R^- - \Lambda_j^{-1}\mathbb{1}_{4/\beta}\right)\mathrm{str}R \\ &\quad -\delta(x)\frac{\beta}{4\pi^2 p}\lim_{\varepsilon\to 0}\lim_{x\to 0^+}\mathrm{Im}\int d[R]\mathrm{Ber}_\beta(R) I_\beta\left(\frac{\beta x R}{2}\right) \\ &\qquad \times\prod_{j=1}^{p}\mathrm{sdet}^{-\beta/2}\left(R^- - \Lambda_j^{-1}\mathbb{1}_{4/\beta}\right) \\ &\quad +\mathrm{B.T.} \quad . \end{aligned} \tag{3.55}$$

The relation (3.50) determines the integral in the second line of Eq. (3.55) and reveals that this contribution is given by $-\delta(x)$. We obtain

$$\begin{aligned} S_\beta(x) = \frac{\beta^2}{8\pi^2 p}\lim_{\varepsilon\to 0}\mathrm{Im}\int d[R]\mathrm{Ber}_\beta(R) I_\beta\left(\frac{\beta x R}{2}\right) \\ \times\prod_{j=1}^{p}\mathrm{sdet}^{-\beta/2}\left(R^- - \Lambda_j^{-1}\mathbb{1}_{4/\beta}\right)\mathrm{str}R + \mathrm{B.T.} - \delta(\mathrm{x}). \end{aligned} \tag{3.56}$$

Due to Lemma 3.1 the last two terms cancel each other for the complex case. For the real case this is not true and the cumbersome structure of the boundary terms severely limit the usefulness of the formulae above.

However in Eq. (3.56) the marginal p.d.f. is expressed in terms of an eigenvalue integral over the supermatrix R with dimension 4×4 and 2×2 for $\beta = 1$ and $\beta = 2$, respectively. This means the problem of calculating the marginal p.d.f. is reduced to a double and a threefold integral for the complex and the real case, respectively.

4. Evaluation of the marginal p.d.f. We consider in Sec. 4.1 the complex case ($\beta = 2$) and rederive the result found in Ref. [8]. In Sec. 4.2 we study the real case and derive an expression in terms of a twofold integral.

4.1. *Complex case.* For $\beta = 2$ the supermatrix $R = \mathrm{diag}(r_1, \imath R_2)$ has dimension 2×2. The Ingham-Siegel integral then reads

$$\begin{aligned} I_2(xR) &= \frac{2\pi}{\Gamma(n)}\Theta(x)\Theta(r_1) r_1^n \\ &\quad \times\exp[-x(r_1+\imath R_2)]\left(-\frac{\partial}{\partial \imath R_2}\right)^{n-1}\delta(R_2), \end{aligned} \tag{4.1}$$

see Appendix A. Using that $\mathrm{str}R = 1/\sqrt{\mathrm{Ber}_2(R)}$, we obtain

$$S_2(x) = \frac{\Theta(x)}{2\pi^2 p} \lim_{\varepsilon\to 0} \mathrm{Im} \int d[R] \frac{I_2(xR)}{r_1 - \imath R_2} \prod_{j=1}^{p} \frac{\imath R_2^- - \Lambda_j^{-1}}{r_1^- - \Lambda_j^{-1}}. \tag{4.2}$$

THEOREM 4.1. *In the complex case the marginal p.d.f. has the form*

$$S_2(x) = \frac{(-1)^{p(p-1)/2}\Theta(x)}{p\Delta_p(\Lambda^{-1})} \det \begin{bmatrix} 0 & \left[\frac{\exp(-x/\Lambda_i)}{\Lambda_i^n}\right]_{i=1,\ldots,p} \\ \left[\frac{x^{n-j}}{(n-j)!}\right]_{j=1,\ldots,p} & \left[\Lambda_i^{-j+1}\right]_{i,j=1,\ldots,p} \end{bmatrix} \tag{4.3}$$

which is equivalent to the result found in Ref. [8]

$$S_2(x) = \frac{\sum_{i=1}^{p}\sum_{j=1}^{p} \mathcal{D}(i,j) x^{n-p+j-1} \exp(-x/\Lambda_i)}{p\left(\prod_{l=1}^{p}(n-l)!\right) \det^n \Lambda \Delta_p(\Lambda^{-1})} \Theta(x), \tag{4.4}$$

where the $\mathcal{D}(i,j)$ denotes the cofactor of the matrix with entries

$$D_{st} = (n-p+s-1)!\Lambda_t^{n-p+s}, \qquad s,t = 1,\ldots,p. \tag{4.5}$$

The Vandermonde determinant is defined by

$$\Delta_p(\Lambda^{-1}) = \prod_{i<j}^{p} \left(\frac{1}{\Lambda_j} - \frac{1}{\Lambda_i}\right). \tag{4.6}$$

The proof is given in Appendix D.

4.2. *Real case.* Due to the complicated boundary terms the representation of the marginal p.d.f. in Eq. (3.56) as eigenvalue integral is not useful for an explicit calculation. Hence we take the generating function in Eq. (3.36) as our starting point and proceed with the evaluation of the Ingham-Siegel integral in Eq. (3.28). Afterwards we carry out the integration over the Grassmann variables and differentiate with respect to J. As we do not make changes of variables involving the anticommuting ones, Efetov-Wegner boundary terms can a priori not occur [23].

We parametrise the matrices σ and ϱ in the form

$$\sigma = \begin{bmatrix} \sigma_1 & \eta \\ -\eta^\dagger & \imath\sigma_2 \mathbb{1}_2 \end{bmatrix}, \qquad \varrho = \begin{bmatrix} \varrho_1 & \Omega \\ -\Omega^\dagger & \imath\varrho_2 \mathbb{1}_2 \end{bmatrix}, \tag{4.7}$$

where each entry corresponds to a 2×2 block. In Appendix A we evaluate the Ingham-Siegel integral for a diagonal supermatrix ϱ. The transfer to the case of non-diagonal ϱ is straight forward due to the rotation invariance. We find

$$I_1(\varrho) = \frac{(-2)^{n+1}\pi}{(n-2)!} \det{}^{(n-1)/2} \varrho_1 \exp(-\mathrm{str}\,\varrho)\, \Theta(\varrho_1) \left(\frac{\partial}{2\imath\partial\varrho_2}\right)^{n-2} \delta(2\varrho_2 - \imath \mathrm{Tr}\,\varrho_1^{-1}\Omega\Omega^\dagger). \tag{4.8}$$

It is easy to check that the generating function remains invariant under a diagonalisation of the Boson-Boson block ϱ_1 if we transform simultaneously Ω according to $\Omega \to v\Omega$. Here v is the orthogonal matrix which diagonalises ϱ_1. The diagonalisation reads

$$\varrho_1 = v^T R_1 v, \quad R_1 = \mathrm{diag}(r_1, r_2), \quad v \in O(2) \tag{4.9}$$

and the measure transforms according to

$$d[\varrho_1] = d[R_1] d\mu(v) |\Delta_2(R_1)|, \qquad |\Delta_2(R_1)| = |r_1 - r_2|. \tag{4.10}$$

Hence we may replace in the generating function (3.36) the matrix ϱ_1 by the diagonal matrix R_1. The matrix Ω in Eq. (4.8) appears in the argument of the Dirac-distribtuion and contains anticommuting variables. To interpret this distribtuion in a meaningful way [22, 16], we have to formally expand it in the anticommuting variables. We write

$$\Omega = \begin{bmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{bmatrix}, \qquad \Omega^\dagger = \begin{bmatrix} \alpha^* & \beta^* \\ -\alpha & -\beta \end{bmatrix}. \tag{4.11}$$

The expansion yields

$$I_1(\varrho) = \frac{4\pi(-1)^n}{(n-2)!} \det{}^{(n-1)/2} R_1 \exp(-\mathrm{str}\,\varrho)\, \Theta(R_1) \left\{ \left(\frac{\partial}{\imath\partial\varrho_2}\right)^{n-2} - \left(\frac{\alpha\alpha^*}{r_1} + \frac{\beta\beta^*}{r_2}\right) \left(\frac{\partial}{\imath\partial\varrho_2}\right)^{n-1} + \frac{\alpha\alpha^*\beta\beta^*}{r_1 r_2} \left(\frac{\partial}{\imath\partial\varrho_2}\right)^{n} \right\} \delta(\varrho_2). \tag{4.12}$$

Expanding the product of superdeterminants in the generating function (3.36) with respect to $\alpha, \alpha^*, \beta, \beta^*$ we find

$$\prod_{j=1}^{p} \mathrm{sdet}^{-1/2}\left(x^+ 1\!\!1_4 + J\gamma - 2\Lambda_j \varrho\right) = \prod_{l=1}^{p} \frac{(x^+ + J - 2\Lambda_l \imath\varrho_2)}{\det^{1/2}(x^+ - 2\Lambda_l r_1)}$$

$$\times \Bigg\{ 1 - \sum_{j=1}^{p} \frac{(2\Lambda_j)^2}{(x^+ + J - 2\Lambda_j \imath\varrho_2)} \left(\frac{\alpha\alpha^*}{(x^+ - 2\Lambda_j r_1)} + \frac{\beta\beta^*}{(x^+ - 2\Lambda_j r_2)} \right)$$

$$+ \sum_{j\neq k}^{p} \frac{(2\Lambda_j)^2}{(x^+ - 2\Lambda_j r_1)(x^+ + J - 2\Lambda_j \imath\varrho_2)}$$

$$\times \frac{(2\Lambda_k)^2}{(x^+ - 2\Lambda_k r_2)(x^+ + J - 2\Lambda_k \imath\varrho_2)} \alpha\alpha^*\beta\beta^* \Bigg\}. \tag{4.13}$$

Combining the expressions (4.12) and (4.13) the integration over the anti-commuting variables $\alpha, \alpha^*, \beta, \beta^*$ yields

$$Z_1(J) = \frac{(-1)^n}{4(n-2)!} \int d[R_1] \int d\varrho_2 |\Delta_2(R_1)| \det{}^{(n-1)/2} R_1 \Theta(R_1)$$

$$\exp\Big(-(r_1 + r_2 - 2\imath\varrho_2) \Big) \prod_{l=1}^{p} \frac{(x^+ + J - 2\Lambda_l \imath\varrho_2)}{\det^{1/2}(x^+ - 2\Lambda_l R_1)}$$

$$\Bigg\{ \det{}^{-1} R_1 \left(\frac{\partial}{\imath\partial\varrho_2} \right)^n + \sum_{j=1}^{p} \frac{(2\Lambda_j)^2}{(x^+ + J - 2\Lambda_j \imath\varrho_2)} \left(\frac{1}{(x^+ - 2\Lambda_j r_1) r_2} \right.$$

$$\left. + \frac{1}{(x^+ - 2\Lambda_j r_2) r_1} \right) \left(\frac{\partial}{\imath\partial\varrho_2} \right)^{n-1}$$

$$+ \sum_{j\neq k}^{p} \Bigg(\frac{(2\Lambda_j)^2}{(x^+ + J - 2\Lambda_j \imath\varrho_2)(x^+ - 2\Lambda_j r_1)} \times$$

$$\frac{(2\Lambda_k)^2}{(x^+ + J - 2\Lambda_k \imath\varrho_2)(x^+ - 2\Lambda_k r_2)} \left(\frac{\partial}{\imath\partial\varrho_2} \right)^{n-2} \Bigg) \Bigg\} \delta(\varrho_2) \tag{4.14}$$

To obtain from Eq. (4.14) the marginal p.d.f. we employ the relation (2.7). We expand the product in the numerator in terms of elementary symmetric function of the variable $Q_j = x^+ - 2\imath\Lambda_j\varrho_2$. The following three relations hold

$$\begin{aligned} &\frac{\partial}{\partial J}\prod_{l=1}^{p}(J+Q_l)\Bigg|_{J=0} = E_{p-1}(Q) \\ &\frac{\partial}{\partial J}\frac{1}{J+Q_j}\prod_{l=1}^{p}(J+Q_l)\Bigg|_{J=0} = E_{p-2;j}(Q) \\ &\frac{\partial}{\partial J}\frac{1}{J+Q_j}\frac{1}{J+Q_k}\prod_{l=1}^{p}(J+Q_l)\Bigg|_{J=0} = E_{p-3;j,k}(Q), \end{aligned} \tag{4.15}$$

where $E_{n;j,k}(Q)$ denotes the elementary symmetric polynomial of order n in the variables $Q_l, \quad l = 1,\ldots,p \neq j,k$, with Q_j and Q_k omitted,

$$E_{n;j,k}(Q) = \sum_{\substack{1\leq i_1<\ldots<i_n\leq p \\ \neq i \neq k}} \left(x^+ - 2\imath\Lambda_{i_1}\varrho_2\right)\ldots\left(x^+ - 2\imath\Lambda_{i_n}\varrho_2\right), \tag{4.16}$$

and $E_{n;j}(Q)$ and $E_n(Q)$ are analogously defined. Employing the relations (4.15) we find

$$\begin{aligned} S_1(x) = &\frac{(-1)^{n+1}}{4\pi p(n-2)!}\lim_{\varepsilon\to 0}\mathrm{Im}\int d[R_1]\int d\varrho_2 \frac{|\Delta_2(R_1)|\det^{(n-1)/2}R_1\Theta(R_1)}{\prod_{l=1}^{p}\det^{1/2}(x^+ - 2\Lambda_l R_1)} \\ &\exp\Big(-(r_1+r_2-2\imath\varrho_2)\Big)\Bigg\{\det{}^{-1}R_1 E_{p-1}(Q)\left(\frac{\partial}{\imath\partial\varrho_2}\right)^{n} + \\ &+\sum_{j=1}^{p}(2\Lambda_j)^2\left(\frac{1}{(x^+-2\Lambda_j r_2)r_1}+\frac{1}{(x^+-2\Lambda_j r_1)r_2}\right)E_{p-2;j}(Q)\left(\frac{\partial}{\imath\partial\varrho_2}\right)^{n-1} \\ &+\sum_{j\neq k}^{p}\frac{(2\Lambda_j)^2(2\Lambda_k)^2}{(x^+-2\Lambda_j r_1)(x^+-2\Lambda_k r_2)}E_{p-3;j,k}(Q)\left(\frac{\partial}{\imath\partial\varrho_2}\right)^{n-2}\Bigg\}\delta(\varrho_2) \end{aligned} \tag{4.17}$$

for the marginal p.d.f. Since the integral over the fermionic variable ϱ_2 contains the derivative of the Dirac-distribution, there are no problems of convergence in the fermionic part. The problem of calculating the marginal p.d.f. is thus essentially reduced to a sum of twofold integrals.

5. Numerical treatment of the marginal p.d.f. On the hand, the square roots in the bosonic integrals (4.17) seem to be insurmountable obstacles for analytic calculations. On the other hand, the singularities at

$\kappa_j = x/2\Lambda_j$, $j = 1, \ldots, p$, are numerical obstacles but they are treatable. We show how to carry out such a numerical evaluation of Eq. (4.17).

We perform the integration over ρ_2. Then the integrals in Eq. (4.17) over the variables r_1 and r_2 can be cast in one of the following three forms

$$I_1 = \lim_{\varepsilon\to 0} \mathrm{Im} \int_0^\infty \int_0^\infty \frac{f(r_1,r_2)dr_1dr_2}{\prod\limits_{j=1}^{p} \sqrt{(r_1^- - \kappa_j)}\sqrt{(r_2^- - \kappa_j)}}, \tag{5.1}$$

$$I_{l2} = \lim_{\varepsilon\to 0} \mathrm{Im} \int_0^\infty \int_0^\infty \frac{f(r_1,r_2)dr_1dr_2}{\prod\limits_{j\neq l}^{p} \sqrt{(r_1^- - \kappa_j)}\sqrt{(r_2^- - \kappa_j)}} \times \frac{1}{(r_1^- - \kappa_l)^{3/2}\sqrt{(r_2^- - \kappa_l)}}, \tag{5.2}$$

$$I_{lm3} = \lim_{\varepsilon\to 0} \mathrm{Im} \int_0^\infty \int_0^\infty \frac{f(r_1,r_2)dr_1dr_2}{\prod\limits_{j\neq l,m}^{p} \sqrt{(r_1^- - \kappa_j)}\sqrt{(r_2^- - \kappa_j)}} \times \frac{1}{(r_1^- - \kappa_l)^{3/2}\sqrt{(r_2^- - \kappa_l)}\sqrt{(r_1^- - \kappa_m)}(r_2^- - \kappa_m)^{3/2}}, \tag{5.3}$$

where $f(r_1, r_2)$ denotes a function that does not have further singularities and is real at $\varepsilon = 0$. We remark that the choice for the square root of the determinants in Eq. (4.17) as products of the eigenvalues is the correct one since they result from Gaussian integrals, see. Eq. (3.12).

The eigenvalues Λ_j can always be ordered according to

$$0 < \Lambda_1 < \Lambda_2 < \ldots < \Lambda_p < \infty. \tag{5.4}$$

The variables κ_j inherit this order

$$\kappa_{p+1} = 0 < \kappa_p < \kappa_{p-1} < \ldots < \kappa_1 < \infty = \kappa_0. \tag{5.5}$$

We divide the positive real axis into $p + 1$ disjoint intervals,

$$[0, \infty[= \bigcup_{j=0}^{p} [\kappa_{j+1}, \kappa_j[. \tag{5.6}$$

In each of these intervals the following observation holds

$$r_1 \in [\kappa_{i+1}, \kappa_i[\Rightarrow (-1)^i \prod_{j=1}^{p} (r_1 - \kappa_j) > 0, \tag{5.7}$$

$$r_2 \in [\kappa_{k+1}, \kappa_k[\Rightarrow (-1)^k \prod_{j=1}^{p} (r_2 - \kappa_j) > 0. \tag{5.8}$$

Combining Eq. (5.7) with Eq. (5.8) we have

$$(-1)^{i+k} \prod_{j=1}^{p} (r_1 - \Lambda_j^{-1})(r_2 - \Lambda_j^{-1}) > 0. \tag{5.9}$$

The product in Eq. (5.9) is always negative if $i + k$ is odd. The imaginary part of the integral (5.1) yields a sum over all pairs of intervals such that Eq. (5.9) is fulfilled. Integrations over other domains do not contribute when taking the imaginary part. Collecting everything we arrive at

$$I_1 = \sum_{\substack{0 \leq i,j \leq p \\ (i+j) \in 2\mathbb{N}+1}} (-1)^{(i+j+1)/2} \int\limits_{\kappa_{i+1}}^{\kappa_i} \int\limits_{\kappa_{j+1}}^{\kappa_j} \frac{f(r_1, r_2) dr_1 dr_2}{\sqrt{\prod\limits_{k=1}^{p} \left| (r_1 - \Lambda_k^{-1})(r_2 - \Lambda_k^{-1}) \right|}}. \tag{5.10}$$

Hence the integrals over r_1 and r_2 extend over two disjoint intervals. The square root singularities in Eq. (5.1) are integrable.

The singularities to the power $3/2$ in the integrals (5.2) and (5.3) make a discussion similar to the one for Eq. (5.1) more involved. The integrations over $3/2$-singularities have to be understood as principal value integrals. We apply the identity

$$\begin{aligned} &\int\limits_{(\kappa_l+\kappa_{l+1})/2}^{(\kappa_l+\kappa_{l-1})/2} \frac{g(y)dy}{(y - \imath\varepsilon - \kappa_l)^{3/2}} \\ = \ & 2 \int\limits_{(\kappa_l+\kappa_{l+1})/2}^{(\kappa_l+\kappa_{l-1})/2} \frac{\partial_y g(y) dy}{(y - \imath\varepsilon - \kappa_l)^{1/2}} - 2 \left[\frac{g(y)}{(y - \imath\varepsilon - \kappa_l)^{1/2}} \right]_{(\kappa_l+\kappa_{l+1})/2}^{(\kappa_l+\kappa_{l-1})/2} \end{aligned} \tag{5.11}$$

to the integrals around the $3/2$-singularities. The function g has to be sufficiently integrable and differentiable. This is fulfilled in our case. With help of Eq. (5.11) we get expressions to which the former discussions also apply.

Using the commercial software MATHEMATICA® [25] we now can numerically calculate the integrals in Eq. (4.17) . In Figs. 1 and 2 we show an example for $p = 10$ and $n = 200$. We compare our analytical result (solid line) with a Monte-Carlo simulation (histogram) of the same ensemble consisting of 10^5 random matrices. The agreement between the Monte-Carlo

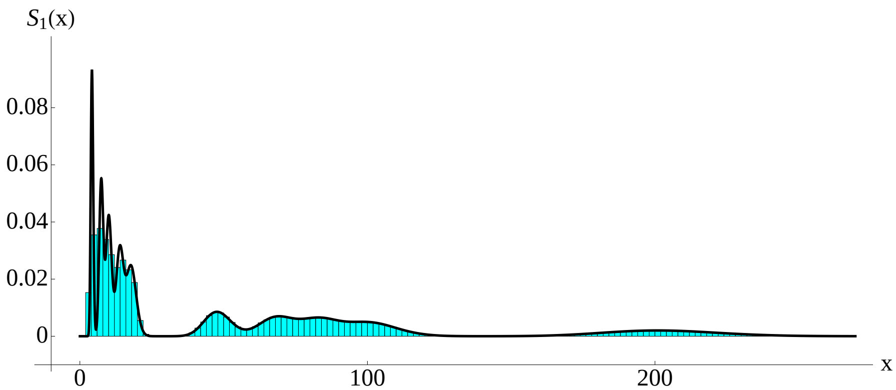

FIG 1. *Marginal p.d.f. for the values* $p = 10$ *and* $n = 200$. *The set of* $\{\Lambda_j\}$, $j = 1, \ldots, 10$, *is* $\{1, 0.49, 0.4225, 0.36, 0.25, 0.09, 0.729, 0.0529, 0.04, 0.0225\}$ *The solid line corresponds to the analytical result. The bin size of the Histogramm is 5.*

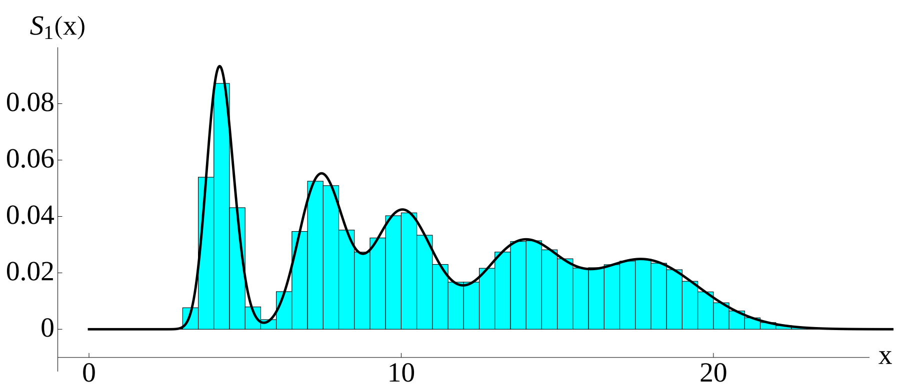

FIG 2. *Same as Fig. 1 zoomed into the interval* $x \in [0, 25]$. *The bin size of the Histogramm is 0.5 .*

simulation and the analytical result is perfect. This confirms that our formula is correct.

6. Summary. We developed a supersymmetry approach which allowed us to derive exact expressions for the marginal p.d.f. of real and complex

Wishart correlation matrices. As far as the marginal p.d.f. is concerned, the supersymmetry method circumvents the structural problems connected with real Wishart correlation matrices. Its crucial advantage is the drastic reduction of the numbers of integrals. For both cases we were able to express the marginal p.d.f. in terms of integrals over eigenvalues of supermatrices.

For the complex case the supermatrix has dimension 2×2. We have shown that the eigenvalue integral provides in this case a determinantal structure. Moreover, we carried out the remaining integrals and rederived the result of Ref. [8].

For the real case, the supermatrix over which the integration extends has dimension 4×4. The crucial difference to the complex case is that the Efetov-Wegner terms arising from the diagonalisation are much more complicated. Therefore we integrated over the matrix without changing the variables and ended up with integrals over the diagonal elements which are threefold integrals. Due to the Dirac-distribution in the fermionic integral we arrived at twofold integrals. This is a considerable progress compared to previous approaches. The remaining twofold integral seems not suited for further analytical calculations. However, we showed how to compute the marginal p.d.f. numerically. The comparison of our results with a Monte-Carlo simulation clearly verifies our formula for the marginal p.d.f.

We hope that we could demonstrate that the supersymmetry method is a powerful tool to tackle problems in multivariate statistics.

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APPENDIX A: INGHAM-SIEGEL INTEGRAL

Although the integral in Eq. (3.28) has also been calculated in Refs. [18, 17] we sketch how to evaluate it. We aim at calculating the following supermatrix integral

$$I_\beta(R) = \int d[\sigma] \mathrm{sdet}^{n\beta/2}(\sigma - \imath 1\!\!1_{4/\beta}) \exp(\imath \mathrm{str} \sigma R), \tag{A.1}$$

where the integral extends over a Hermitian supermatrix σ as given in Eq. (3.23) and R denotes a diagonal supermatrix. The expression (A.1) is the supersymmetric version of the ordinary Ingham-Siegel integral [26, 27] which reads

$$\text{(A.2)} \qquad \mathfrak{I}_\beta(Q) = \int d[X] \det{}^{-N}(X - \imath\epsilon 1\!\!1_M) \exp(\imath \mathrm{Tr} X Q) \\ \propto \det{}^{N-M} Q \Theta(Q) \exp(-\epsilon \mathrm{Tr} Q) \quad \text{with} \quad \epsilon > 0 \quad \text{and} \quad N > M - 1.$$

In Eq. (A.2) X and Q denote ordinary Hermitian $M \times M$ matrices. There is an analogous expression for real symmetric matrices. To evaluate $I_\beta(R)$ we explicitly write down the superdeterminant and the exponent in Eq. (A.1)

$$\text{(A.3)} \qquad \mathrm{sdet}(\sigma - \imath 1\!\!1_{4/\beta}) = \frac{\det\left(\sigma_1 - \imath 1\!\!1_{2/\beta} + \eta(\imath\sigma_2 - \imath 1\!\!1_{2/\beta})^{-1}\eta^\dagger\right)}{\det\left(\imath\sigma_2 - \imath 1\!\!1_{2/\beta}\right)}, \\ \mathrm{str}\sigma R = \mathrm{Tr}\sigma_1 R_1 + \mathrm{Tr}\sigma_2 R_2.$$

Shifting σ_1 according to $\sigma_1 \to \sigma_1 + \eta(\imath\sigma_2 - \imath 1\!\!1_{2/\beta})^{-1}\eta^\dagger$ we see that the integrals over σ_1 and σ_2 factorise and (A.1) acquires the form

$$\text{(A.4)} \qquad I_\beta(R) = \imath^n \int d[\sigma_1] \det{}^{-n\beta/2}(\sigma_1 - \imath 1\!\!1_{2/\beta}) \exp(\imath \mathrm{Tr}\sigma_1 R_1) \\ \times \int d\sigma_2 \det{}^{n\beta/2}(\sigma_2 - 1\!\!1_{2/\beta}) \exp(\imath \mathrm{Tr}\sigma_2 R_2) \\ \times \int d[\eta] \exp(-\mathrm{Tr} R_1 \eta(\sigma_2 - 1\!\!1_{2/\beta})^{-1}\eta^\dagger).$$

Obviously the integration over σ_1 corresponds to the ordinary Ingham-Siegel integral. Hence we apply the results in Refs. [26, 27]

$$\text{(A.5)} \qquad \int d[\sigma_1] \det{}^{-n\beta/2}(\sigma_1 - \imath 1\!\!1_{2/\beta}) \exp(\imath \mathrm{Tr}\sigma_1 R_1) = \\ = k_\beta \Theta(R_1) \det{}^{(n+1)\beta/2-2} R_1 \exp(-\mathrm{Tr} R_1),$$

where k_β is a constant

$$\text{(A.6)} \qquad k_1 = \sqrt{\pi} \frac{(2\pi)^2 \imath^n}{\Gamma(n/2)\Gamma((n-1)/2)}, \qquad k_2 = \frac{\imath^n 2\pi}{\Gamma(n)}.$$

Also the integral over η is easily done

$$\text{(A.7)} \qquad \int d[\eta] \exp(-\mathrm{Tr} R_1 \eta(\sigma_2 - 1\!\!1_{2/\beta})^{-1}\eta^\dagger) = \frac{(2/\beta)^2}{(2\pi)^{2/\beta}} \frac{\det R_1}{\det(\sigma_2 - 1\!\!1_{2/\beta})}.$$

Employing Eqs. (A.5) and (A.7) we obtain

$$I_\beta(R) = k_\beta \frac{\imath^n (2/\beta)^2}{(2\pi)^{2/\beta}} \Theta(R_1) \det{}^{(n+1)\beta/2-1} R_1 \exp(-\mathrm{Tr} R_1) \int d\sigma_2 \det{}^{n\beta/2-1}(\sigma_2 - \mathbb{1}_{2/\beta}) \exp(\imath \mathrm{Tr} \sigma_2 R_2). \tag{A.8}$$

We recall that σ_2 and R_2 are both proportional to the unit matrix. To do the integral over σ_2 we shift σ_2 according to $\sigma_2 \to \sigma_2 + \mathbb{1}_{2/\beta}$. This reveals that the integral is basically the $(n-2/\beta)$-fold derivative of the Dirac-distribution

$$\int d[\sigma_2] \det{}^{n\beta/2-1}(\sigma_2 - \mathbb{1}_{2/\beta}) \exp(\imath \mathrm{Tr} \sigma_2 R_2) = = 2\pi(\beta/2)^{n-2/\beta+1} \exp(\imath \mathrm{Tr} R_2) \left(\frac{\partial}{\partial \imath R_2}\right)^{n-2/\beta} \delta(R_2). \tag{A.9}$$

Plugging Eq. (A.9) into Eq. (A.8) we obtain

$$I_\beta(R) = K_\beta \Theta(R_1) \det{}^{(n+1)\beta/2-1} R_1 \exp(-\mathrm{str} R) \times \left(-\frac{\partial}{\partial \imath R_2}\right)^{n-2/\beta} \delta(R_2), \tag{A.10}$$

with

$$K_1 = \sqrt{\pi} \frac{2\pi}{2^{n-3}\Gamma(n/2)\Gamma((n-1)/2)}, \qquad K_2 = -\frac{2\pi}{\Gamma(n)} \tag{A.11}$$

as the constants.

APPENDIX B: PROOF OF LEMMA 3.1

As starting point we use the representation of the generating function in superspace and calculate the corresponding Efetov-Wegner term. The Efetov-Wegner term for the marginal p.d.f. is then obtained by using the relation (2.7). The generating function reads for $\beta = 2$

$$Z_2(J) = \int d[\varrho] I_2(\varrho) \prod_{j=1}^{p} \mathrm{sdet}^{-1}\left(x^+ \mathbb{1}_2 + J\gamma - \Lambda_j \varrho\right). \tag{B.1}$$

We recall that the Ingham-Siegel integral is rotation invariant. Hence it remains the same when diagonalising ϱ. We express the superdeterminant in Eq. (B.1) as a Gaussian integral over the p supervectors Ψ_j, $j=1,\ldots,p$,

$$\prod_{j=1}^{p} \mathrm{sdet}^{-1}\left(x^+\mathbb{1}_2 + J\gamma - \Lambda_j\varrho\right) = \tag{B.2}$$
$$= \imath^{-p}\int d[\Psi]\exp\left[\imath\sum_{j=1}^{p}\Psi_j^\dagger(x^+\mathbb{1}_2+J\gamma)\Psi_j + \imath\mathrm{str}\varrho B\right].$$

The supermatrix B and the supervectors Ψ_j are defined as in Eqs. (3.31) and (3.32). Then, we have

$$\begin{aligned} Z_2(J) &= \imath^{-p}\int d[\varrho]I_2(\varrho)\int d[\Psi]\exp\left[\imath\sum_{j=1}^{p}\Psi_j^\dagger(x^+\mathbb{1}_2+J\gamma)\Psi_j + \imath\mathrm{str}\varrho B\right] \\ &= \imath^{-p}\int d[\sigma]\int d[\Psi]\exp\left[\imath\sum_{j=1}^{p}\Psi_j^\dagger(x^+\mathbb{1}_2+J\gamma)\Psi_j\right]\delta(\sigma-B) \\ &\quad\times\int d[\varrho]I_2(\varrho)\exp(\imath\mathrm{str}\varrho\sigma). \end{aligned} \tag{B.3}$$

In the second line we have replaced the supermatrix B by a Hermitian supermatrix σ because B cannot be diagonalised in the Fermion-Fermion block. This is an immediate consequence of statement 4.1 in Ref. [18]. The matrix σ however has generic eigenvalues and hence there are no problems connected with a diagonalisation of σ.

We are interested in the diagonalisation of ϱ and consider

$$J(\sigma) = \int d[\varrho]I_2(\varrho)\exp(\imath\mathrm{str}\varrho\sigma). \tag{B.4}$$

This expression remains invariant under a diagonalisation of σ. Hence we write σ as $\sigma = USU^\dagger$. However we still consider the eigenvalues of σ as functions of the original cartesian coordinates. This means the diagonalisation of σ is a book keeping device which do not change the measure $d[\sigma]$ in Eq. (B.3). We have

$$J(\sigma) = \int d[\varrho]I_2(\varrho)\exp(\imath\mathrm{str}\varrho S). \tag{B.5}$$

The diagonalisation of $\varrho = VRV^\dagger$ yields

$$J(\sigma) = \int d[R]I_2(R)\varphi(R,S) + \imath \tag{B.6}$$

see also Ref. [24]. Here

$$\varphi(R,S) = \int d\mu(V) \exp\left(\imath \mathrm{str} V R V^\dagger S\right) \tag{B.7}$$

denotes the supermatrix Bessel function [28]. With $d\mu(U)$ we denote the Haar-measure of the supergroup diagonalising ϱ, i.e. U(1/1). The additional term $-\imath$ is the Efetov-Wegner term arising from the diagonalisation of ϱ in Eq. (B.5).

The supermatrix Bessel function is rotation invariant under U(1/1). Therefore, we are allowed to replace the diagonal matrix S by the full supermatrix σ. Inserting the resulting expression for $J(\sigma)$ into the generating function (B.4) we find

$$\begin{aligned} Z_2(J) = \imath^{-p} &\int d[\sigma] \int d[\Psi] \exp\left[\imath \sum_{j=1}^{p} \Psi_j^\dagger (x^+ 1\!\!1_2 + J\gamma)\Psi_j\right] \delta(\sigma - B) \\ &\times \int d[R] I_2(R) \int d\mu(V) \exp(\imath \mathrm{str} V R V^\dagger \sigma) \\ &- \imath^{-(p+1)} \int d[\sigma] \int d[\Psi] \exp\left[\imath \sum_{j=1}^{p} \Psi_j^\dagger (x^+ 1\!\!1_2 + J\gamma)\Psi_j\right] \delta(\sigma - B). \end{aligned} \tag{B.8}$$

We integrate over σ and Ψ_j in the first term and obtain

$$\begin{aligned} Z_2(J) &= \int d[R] I_2(R) \int d\mu(V) \prod_{j=1}^{p} \mathrm{sdet}^{-1}\left(x^+ 1\!\!1_2 + J\gamma - \Lambda_j V R V^\dagger\right) \\ &- \imath^{-(p+1)} \int d[\sigma] \int d[\Psi] \exp\left[\imath \sum_{j=1}^{p} \Psi_j^\dagger (x^+ 1\!\!1_2 + J\gamma)\Psi_j\right] \delta(\sigma - B). \end{aligned} \tag{B.9}$$

The first line in the expression above is exactly of the form what a diagonalisation of Eq. (B.1) would yield if the integral extends over an ordinary matrix. The additional term in Eq. (B.8) is a contribution which has no analog in ordinary analysis. It is the Efetov-Wegner term we are looking for. The integration over σ and Ψ_j yields

$$\begin{aligned} Z_2(J) &= \int d[R] I_2(R) \prod_{j=1}^{p} \mathrm{sdet}^{-1}\left(x^+ 1\!\!1_2 - J\gamma - \Lambda_j R\right) \\ &+ \mathrm{sdet}^{-p}(x^+ + J\gamma) \end{aligned} \tag{B.10}$$

We use the relation (2.7) between the generating function and the marginal p.d.f. and get

$$\text{B.T.} = -\frac{1}{p\pi}\lim_{\varepsilon\to 0}\text{Im}\frac{\partial}{\partial J}\text{sdet}^{-p}(x^{+}\mathbb{1}_2 + J\gamma)\bigg|_{J=0} = \delta(x) \tag{B.11}$$

for the Efetov-Wegner term which is the desired result.

APPENDIX C: PROOF OF LEMMA 3.2

We aim at calculating the group integral

$$\mathcal{K}_\beta(\tau,\lambda) = \int d\mu(u)\text{str}(\tau u\lambda u^{-1}), \tag{C.1}$$

where τ and λ are Hermitian supermatrices diagonalisable by the group of unitary supermatrices. For $\beta = 2$ one can easily write down a parametrisation of U(1/1) and evaluate (C.1) by an explicit calculation. However for the real case the diagonalising supermatrices u have dimension 4×4 and therefore using an explicit parametrisation of UOSp(2/2) becomes tedious. Thus a different line of reasoning is called for which actually applies independently of the size of the matrices u to both cases $\beta = 1$ and $\beta = 2$. As we will see the symmetries of $\mathcal{K}_\beta$ suffice to determine (C.1).

We observe four properties of $\mathcal{K}_\beta$:

1. Due to the cyclic invariance of the supertrace, $K_\beta(\tau,\lambda)$ is symmetric in τ and λ

   $$\mathcal{K}_\beta(\tau,\lambda) = \mathcal{K}_\beta(\lambda,\tau). \tag{C.2}$$

2. The rotation invariance of the Haar-measure implies that $\mathcal{K}_\beta(\tau,\lambda)$ can only depend on invariants of the matrices τ and λ. This means the property

   $$\mathcal{K}_\beta(\tau,\lambda) \quad = \quad \mathcal{K}_\beta(u\tau u^\dagger,\lambda) \tag{C.3}$$

   holds $\forall\ u \in \text{U}(N/N)$ for $\beta = 2$ and $\forall\ u \in \text{UOSp}(2N/2N)$ for $\beta = 1$. In combination with the first property the symmetry (C.2) is also true for λ.

3. The linearity of the supertrace imposes that the superfunction $\mathcal{K}_\beta$ is bilinear, i.e. for three supermatrices τ, Σ and λ it is

   $$\mathcal{K}_\beta(a\tau + b\Sigma,\lambda) = a\mathcal{K}_\beta(\tau,\lambda) + b\mathcal{K}_\beta(\Sigma,\lambda) \quad , \ a,b\in\mathbb{C}, \tag{C.4}$$

   and analogous for λ.

4. The superfunction $\mathcal{K}_\beta$ is a polynomial in the entries of supermatrices τ and λ.

The three properties (C.2), (C.3) and (C.4) suffice to determine $\mathcal{K}_\beta(\tau,\lambda)$ up to a constant. All polynomial invariants of the function $\mathcal{K}_\beta(\tau,\lambda)$ can be expressed in terms of supertraces of τ and λ because of the second and fourth property. We have

$$\mathcal{K}_\beta(\tau,\lambda)=\mathcal{K}_\beta(\mathrm{str}\tau,\mathrm{str}\tau^2,\dots,\mathrm{str}\lambda,\mathrm{str}\lambda^2,\dots). \tag{C.5}$$

The bilinearity of $\mathcal{K}_\beta$ leads to Eq. (3.45).

APPENDIX D: PROOF OF THEOREM 4.1

The expression (4.2) for the marginal p.d.f. in the complex case has a determinantal structure. To uncover it, we use the identity [29]

$$\frac{\prod\limits_{l<k}^{p+1}(\kappa_{k2}-\kappa_{l2})}{\prod\limits_{k=1}^{p+1}(\kappa_1-\kappa_{k2})}=(-1)^p\det\begin{bmatrix}\left[\dfrac{1}{\kappa_1-\kappa_{k2}}\right]_{k=1,\dots,(p+1)}\\ \left[\kappa_{k2}^{j-1}\right]_{\substack{k=1,\dots,(p+1)\\ j=1,\dots,p}}\end{bmatrix}. \tag{D.1}$$

We need this formula to rearrange the term

$$\frac{\Delta_p(\Lambda^{-1})\prod\limits_{j=1}^{p}(\imath R_2^- -\Lambda_j^{-1})}{\prod\limits_{j=1}^{p}(r_1^- -\Lambda_j^{-1})}. \tag{D.2}$$

Identifying $\kappa_1=r_1^-$, $\kappa_{k2}=\Lambda_k^{-1}$ for $k=1,\dots,p$ and $\kappa_{(p+1)2}=\imath R_2^-$ we obtain for Eq. (4.2)

$$\begin{aligned}S_2(x)&=\frac{(-1)^{p(p-1)/2}\Theta(x)}{2\pi^2 p\Delta_p(\Lambda^{-1})}\\ &\times\lim_{\varepsilon\to0}\mathrm{Im}\int d[R]I_2(xR)\det\begin{bmatrix}\dfrac{1}{r_1-\imath R_2} & \left[\dfrac{1}{r_1^- -\Lambda_k^{-1}}\right]_{k=1,\dots,p}\\ \left[(\imath R_2^-)^{j-1}\right]_{j=1,\dots,p} & \left[\Lambda_k^{-j+1}\right]_{j,k=1,\dots,p}\end{bmatrix}.\end{aligned} \tag{D.3}$$

Since the Ingham-Siegel integral (4.1) factorizes in functions of r_1 and $\imath R_2$, we cast Eq. (D.3) expression into the form

$$S_2(x)=\frac{\Theta(x)(-1)^{p(p-1)/2}}{p\Delta_p(\Lambda^{-1})\Gamma(n)}\det\begin{bmatrix}L_0 & [L_{1k}]_{k=1,\dots,p}\\ [L_{2j}]_{j=1,\dots,p} & \left[\Lambda_k^{-j+1}\right]_{j,k=1,\dots,p}\end{bmatrix}. \tag{D.4}$$

with

$$\text{(D.5)} \qquad L_0 = \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} \int_0^\infty dr_1 r_1^n e^{-xr_1}, \times \int_{-\infty}^{+\infty} dR_2 \frac{\exp(\imath x R_2)}{r_1 - \imath R_2} \left(-\frac{\partial}{\partial \imath R_2} \right)^{n-1} \delta(R_2) = 0$$

$$\text{(D.6)} \qquad L_{1k} = \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} \int_0^\infty dr_1 r_1^n \exp(-xr_1) \frac{1}{r_1^- - \Lambda_k^{-1}} = \frac{\exp(-x/\Lambda_k)}{\Lambda_k^n},$$

$$\text{(D.7)} \qquad L_{2j} = \int_{-\infty}^{+\infty} dR_2 (\imath R_2)^{j-1} e^{\imath x R_2} \left(-\frac{\partial}{\partial \imath R_2} \right)^{n-1} \delta(R_2) = x^{n-j} \frac{(n-1)!}{(n-j)!}.$$

The imaginary part of the integrals over the fermionic eigenvalue R_2 is always of order ε and hence vanishes in the limit $\varepsilon \to 0$. Thus we can ignore ε here and restrict the regularisation to the integrals over the bosonic eigenvalue r_1. However, ε does not appear in L_0, see Eq. (D.5). Thus the integral is real and its imaginary part is equal to zero. For the integral L_{1k} the regularisation yields a Dirac-distribution for r_1 at Λ_k^{-1} and hence the integration is trivial. The integrals over R_2, i.e. L_{2j}, are simple as well due to the presence of the Dirac-distribution in Eq. (D.7). Inserting the results (D.5), (D.6) and (D.7) we find for the marginal p.d.f. Eq. (4.3).

The second statement concerns the equivalence of Eq. (4.3) with the result found in Ref. [8]. To prove it we expand the determinant (4.3) with respect to the first row and the first column

$$\text{(D.8)} \qquad D = \det \begin{bmatrix} 0 & \left[\dfrac{\exp(-x/\Lambda_i)}{\Lambda_i^n} \right]_{i=1,\ldots,p} \\ \left[\dfrac{x^{n-j}}{(n-j)!} \right]_{j=1,\ldots,p} & \left[\Lambda_i^{-j+1} \right]_{i,j=1,\ldots,p} \end{bmatrix} = \sum_{i=1}^p \sum_{j=1}^p (-1)^{j+i} \frac{x^{n-j} \exp(-x/\Lambda_i)}{(n-j)! \Lambda_i^n} \det \left[\Lambda_s^{-t+1} \right]_{\substack{s,t=1,\ldots,p \\ s \neq i, t \neq j}}.$$

We reorder the summation over j by replacing this index according to $j \to p - j + 1$. Furthermore we write the factor Λ_i^n in the denominator as

$\det^{-n}\Lambda\prod_{l\neq i}^{p}\Lambda_l^n$ and obtain for the right hand side of Eq. (D.8)

$$\begin{aligned} D &= \frac{1}{\det^n\Lambda}\sum_{i=1}^{p}\sum_{j=1}^{p}(-1)^{i+l+p-1}\frac{x^{n-p+j-1}\exp(-x/\Lambda_i)}{(n-p-i-1)!} \\ &\quad\times\det\left[\Lambda_s^{n-t+1}\right]_{\substack{s,t=1,\ldots,p\\ s\neq i,t\neq p-j+1}}. \end{aligned} \tag{D.9}$$

Now we transpose the matrix in the determinant of Eq. (D.9) and permute the columns such that

$$\det\left[\Lambda_s^{n-t+1}\right]_{\substack{s,t=1,\ldots,p\\ s\neq i,t\neq p-j+1}} = (-1)^{(p-1)(p-2)/2}\det\left[\Lambda_t^{n-p+s}\right]_{\substack{t,s=1,\ldots,p\\ t\neq j,s\neq i}}. \tag{D.10}$$

Multiplying the rows with $(n-p+s-1)!$ and using the relation (D.10) yields for the determinant (D.8)

$$D = (-1)^{p(p-1)/2}\frac{\sum\limits_{i=1}^{p}\sum\limits_{j=1}^{p}x^{n-p+j-1}\exp(-x/\Lambda_i)\mathcal{D}(i,j)}{\left(\prod\limits_{l=1}^{p}(n-l)!\right)\det^n\Lambda}, \tag{D.11}$$

where

$$\mathcal{D}(i,j) = (-1)^{i+j}\det[(n-p+s-1)!\Lambda_s^{n-p+t}]_{\substack{s,t=1,\ldots,p\\ s\neq i,t\neq j}} \tag{D.12}$$

denotes the cofactor of the matrix (4.5). This shows that our result is equivalent to the one in Ref. [8].

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